Math 454 - Spring 2025 - Homework 10

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Status - Reading Assignments:

Here are the reading assignments to be completed before the first one of this HW.

SCF2 (Shreve – Stoch. Calculus for Finance, II Textbook):
Ch. 1 – 4

MF454 lecture notes:
Ch.2 – 11

Other:
Nothing assigned yet

New reading assignments:

In the following: • MF = MF454 = my course lecture notes • SCF2 = Shreve: Stochastic Calculus for Finance II • WMS = Wackerly, et al = standard Math 447 Textbook

Reading assignment 1 - due Monday, March 24:

a. Study for the midterm!

Reading assignment 2 - due: Wednesday, March 26:

a. Carefully read MF ch.12. (Less than 8 pages!) The process Θ_t of Girsanov's theorem is assumed in continuous time finance models by the market price of risk process. It is obtained as the solution of the so–called market price of risk equation(s). Girsanov tells us how to use Θ_t to create the continuous time equivalent of the risk–neutral measure $\widetilde{P}$ you studied in the BAM model. Of pareticular importance in that respect is Remark 12.2.

Reading assignment 3 - due Friday, March 28:

a. Carefully read MF ch.13.1. Be sure to familiarize yourself with the generalized Black–Scholes market model of Definition 13.1 and the notation used in my lecture notes.

b. Study EXTRA CAREFULLY MF ch.13.2 and ch.13.3. Here you are reaping the benefits of knowing the fundamentals of stochastic calculus and having spent so much effort on the abstract math of ch.4-6.

Written assignments are on the next page.

Written assignments:

Prove the following assignments closed book (once you have looked up the proposition).

Written assignment 1: Re–write the following integral equations for the process U_t in differential form. Here $g = g_t(\omega)$ is adapted, and α, β are constants.

(a) $U_t \;=\; e^{\alpha t}$
(b) $U_t = \int_0^t g_u dW_u$. $\square$
(c) $U_t \;=\; e^{\alpha W_t}$
(d) $U_t \;=\; e^{\alpha X_t}$ where the Itô process X has the stochastic differential
$dX_t \;=\; \beta dt + \gamma dW_t$.
(e) $U_t \;=\; X_t^2$ where X has the stochastic differential
$dX_t \;=\; \alpha X_t dt + \sigma X_t dW_t$. $\square$

Solution to (a)

$$dU_t \;=\; \alpha e^{\alpha t}\, dt \;=\; \alpha U_t dt\,.$$

Solution to (b)

$$dU_t \;=\; g_t dW_t\,.$$

Solution to (c): Apply the Itô formula with $f(t,x) = e^{\alpha x}$. Then $f_t = 0$, $f_x = \alpha e^{\alpha x}$, $f_{xx} = \alpha^2 e^{\alpha x}$:

$$dU_t \;=\; df(t,W_t) \;=\; \alpha e^{\alpha x}\, dW_t \;+\; \frac{1}{2}\alpha^2 e^{\alpha x} dt\,.$$

Solution to (d): As in **(c)**, apply the Itô formula with $f(t,x) = e^{\alpha x}$. Since $dX_t dX_t = \gamma^2 dt$,

$$\begin{aligned} dU_t \;&= \alpha e^{\alpha x}\, dX_t \;+\; \frac{1}{2}\alpha^2 e^{\alpha x} dX_t dX_t \;=\; \alpha\beta e^{\alpha x}\, dt \;+\; \alpha\gamma e^{\alpha x}\, dW_t \;+\; \frac{1}{2}\alpha^2\gamma^2 e^{\alpha x} dt \\ &= \alpha U_t \left[\beta \;+\; \frac{\alpha\gamma^2}{2}\right]\, dt \;+\; \alpha\gamma U_t\, dW_t\,. \end{aligned}$$

Solution to (e): Apply the Itô formula with $f(t,x) = x^2$. Then $f_t = 0$, $f_x = 2x$, $f_{xx} = 2$.
Further, $dX_t dX_t = \sigma^2 X_t^2 dt$. Thus,

$$\begin{aligned} dU_t \;&= df(t,W_t) \;=\; 2X_t\, dX_t \;+\; \frac{1}{2}\cdot 2 \cdot dX_t dX_t\,. \\ &= 2X_t\big(\alpha X_t dt + \sigma X_t dW_t\big) \;+\; \sigma^2 X_t^2\, dt \\ &= \big(2\alpha X_t \;+\; \sigma^2 X_t^2\big) dt \;+\; \sigma X_t dW_t\,. \end{aligned}$$

Written assignment 2: Prove closed book Theorem 9.7 (SCF2 Theorem 4.4.9 - Itô integral of a deterministic integrand). You can find the proof in the SCF2 text.

Written assignment 3: Prove that $\displaystyle\int_0^t W_u dW_u \;=\; \frac{W_t^2}{2} - t$: Apply Itô's formula to $f(t,x) = x^2$.

Solution to #3: Since $f_t = 0,\ f_x = 2x,\ f_{xx} = 2,$

$$dW_t^2 \;=\; 2W_t dW_t \;+\; \frac{1}{2}\cdot 2dt \quad \text{thus,} \quad W_t dW_t \;=\; d\left(\frac{1}{2}W_t^2\right) \;-\; dt\,.$$

We indegrate that last equation and obtain

$$\int_0^t W_u dW_u \;=\; \frac{1}{2}(W_t^2 - W_0^2) \;-\; (t-0) \;=\; \frac{W_t^2}{2} \;-\; t\,.$$

Written assignment 4: Work open book(!) in the SCF2 text through the proof of Proposition 9.1 (SCF2 Example 4.4.10 - Vasicek interest rate model). You will not find problems of that computational complexity in the exams of this course, but you may need some skills in that area when going for a quant certification.

Written assignment 5: Work open book(!) in the SCF2 text through the proof of Proposition 9.2 (SCF2 Example 4.4.11 - Cox–Ingersoll–Ross (CIR) interest rate model) This one is tougher than #3.

Written assignment 6: Try to do Exercise 9.4 of the lecture notes closed book before you look at the solution which is given here.

Solution to #6: Define Y by $Y_t = e^{\alpha W_t}$. For $f(t,x) = e^{\alpha x}$ the Itô formula then yields

$$dY_t \;= \tfrac{1}{2}\alpha^2 e^{\alpha W_t}dt + \alpha e^{\alpha W_t}dW_t \;=\; \tfrac{1}{2}\alpha^2 Y_t dt + \alpha Y_t dW_t.$$

Moreover $Y_0 = e^{\alpha W_0} = e^0 = 1$. We integrate both sides and obtain

$$Y_t \;= 1 + \tfrac{1}{2}\alpha^2 \int_0^t Y_u du + \alpha \int_0^t Y_u dW_u.$$

Taking expected values will make the stochastic integral vanish. After moving expectation within the integral sign in the ds-integral and defining m by $m(t) = E[Y_t]$ we obtain the equation

$$m(t) \;= 1 + \tfrac{1}{2}\alpha^2 \int_0^t m(s)ds + 0.$$

This is an integral equation, but if we take the t-derivative we obtain the ODE

$$\dot{m}(t) \;= \tfrac{\alpha^2}{2} m(t), \qquad m(0) \;=\; 1.$$

Solving this standard equation gives us the answer

$$E[e^{\alpha W_t}] \;= E[Y_t] \;=\; m(t) \;=\; e^{\alpha^2 t/2}. \;\blacksquare$$