15.4. $B = \{(x,y):\ \sqrt{x^2+y^2} \in [a,b],\ arctan(y/x) \in [\alpha,\beta]\}$
$R = \{(r,\theta):\ r \in [a,b],\ \theta \in [\alpha,\beta]\}$.

$$\int\int_B f(x,y)dxdy = \int\int_B f(x,y)dA = \int_\alpha^\beta \int_a^b f(rsin\theta, rcox\theta)rdrd\theta$$

Example 1. $B = \{(x,y):\ x^2+y^2 \in [1,4],\ arctan(y/x) \in [0,\pi]\}$
$R = \{(r,\theta):\ r \in [1,2],\ \theta \in [0,\pi]\}$.

$$\begin{aligned}
&\int\int_R 3x+4y^2dA \\
=&\int\int_R 3x+4y^2dxdy \\
=&(\int\int_{R_1} + \int\int_{R_2} + \int\int_{R_3})(3x+4y^2)dxdy \\
=&(\int_0^1 \int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} + \int_0^1 \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} + \int_1^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}})(3x+4y^2)dxdy \\
=&\int_0^1 (\frac{3}{2}x^2+4xy^2)\Big|_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}}dy + \int_0^1 (\frac{3}{2}x^2+4xy^2)\Big|_{\sqrt{1-y^2}}^{\sqrt{4-y^2}}dy + \int_1^2 (\frac{3}{2}x^2+4xy^2)\Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}dy \\
=&....
\end{aligned}$$

$sin^2(x) = \frac{1-cos(2x)}{2}$.
$cos^2(x) = \frac{1+cos(2x)}{2}$.

$$\begin{aligned}
&\int\int_R 3x+4y^2dA \\
=&\int_0^\pi \int_1^2 (3rcos\theta + 4(rsin\theta)^2)rdrd\theta \\
=&\int_0^\pi \int_1^2 [3rcos\theta + 2r^2(1-cos(2\theta))]rdrd\theta \\
=&......
\end{aligned}$$

Example 2. Volumn of the solid bounded by $z = 0$ and paraboloid $z = 1 - x^2 - y^2$.

$$V = \int\int_B zdxdy = \int\int_B 1-x^2-y^2dxdy,$$

$B = \{(x,y):\ x^2+y^2 \in [0,1]\}$.

$$\begin{aligned}
&\int\int_B zdxdy \\
=&\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1-x^2-y^2dxdy \\
=&4\int_0^1 \int_0^{\sqrt{1-y^2}} 1-x^2-y^2dxdy
\end{aligned}$$

$R = \{(r, \theta): \ r \in [0,1], \ \theta \in [0, 2\pi]\}.$

$$\begin{aligned}
&\int\int_B z dx dy \\
=&\int_0^{2\pi}\int_o^1 1 - r^2 r dr d\theta \\
=&2\pi(\frac{r^2}{2} - \frac{r^4}{4})\Big|_0^1 \\
=&\pi/2
\end{aligned}$$

Example 3. Area of one of the four-leaved rose $B = \{(x, y) : 0 \le r \le cos(2\theta), \ |\theta| \le \pi/4\}$.

$$\begin{aligned}
Area =& \int\int_B 1 dx dy \\
=& \int_{-\pi/4}^{\pi/4}\int_0^{cos(2\theta)} r dr d\theta \\
=& \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\Big|_0^{cos(2\theta)} d\theta \\
=& \int_{-\pi/4}^{\pi/4} \frac{cos^2(2\theta)}{2} d\theta \\
=& \int_{-\pi/4}^{\pi/4} \frac{1 + cos(4\theta)}{4} d\theta \\
=& ......
\end{aligned}$$

Fubini's Theorem for Triple integral. If $f(x,y,z)$ is continuous on the subset B of $\mathcal{R}^3$, then

$$\int\int\int_B (x,y,z)dV = \int\int\int_B f(x,y,z)dxdydz = \int\int\int_B f(x,y,z)dzdydx =$$

$\cdots = \int\int\int_B f(x,y,z)dxdzdy$ **How many of them ?**

Example 5. If $f = xyz^2$ and $B = [0,1]\times[-1,2]\times[0,3]$, $\int\int\int_B (f(x,y,z)dV = ?$

Sol. $\int\int\int_B (f(x,y,z)dV = \int_0^3\int_{-1}^2\int_0^1 xyz^2 dxdydz = \int_0^1\int_{-1}^2\int_0^3 xyz^2 dzdydx$

$$= \int_0^1 xdx\int_{-1}^2 ydy\int_0^3 z^2dz = (x^2/2)\Big|_0^1 (y^2/2)\Big|_{-1}^2 (z^3/3)\Big|_0^3 = \cdots$$

Remark. B in Example 5 is a "3-dim rectangle" or a box. However, it is often that B is of the following three types:

Type I. $B = \{(x,y,z):\ (x,y)\in D, z\in[u_1(x,y),u_2(x,y)]\}$ then

$$\int\int\int_B f(x,y,z)dV = \int\int_D g(x,y)dxdy = \int\int_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dxdy$$

Type II. $B = \{(x,y,z):\ (x,z)\in D, y\in[u_1(x,z),u_2(x,z)]\}$ then

$$\int\int\int_B f(x,y,z)dV = \int\int_D g(x,z)dxdz = \int\int_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dxdz$$

Type III. $B = \{(x,y,z):\ (y,z)\in D, x\in[u_1(y,z),u_2(y,z)]\}$ then

$$\int\int\int_B f(x,y,z)dV = \int\int_D g(y,z)dydz = \int\int_D [\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx]dydz$$

Example 6. Compute $\int\int\int_B zdV$ where B is the solid bounded by the four planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. (See the figure).

Sol. Three ways: Types I, II, or III.

Type I. Integrate z first.

$$\int\int\int_B f(x,y,z)dV = \int\int_D g(x,y)dxdy = \int\int_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dxdy$$

Identify the two bounded hyperplanes: $z = u_1(x,y)$ and $z = u_2(x,y)$ from

$z=0$ and $x+y+z=1$ (yielding $z = 1-x-y$).

$\underbrace{0}_{u_1(x,y)} \le z \le \underbrace{1-x-y}_{u_2(x,y)}.$

$D = \{(x,y):\ 0\le x+y\le 1, x\ge 0, y\ge 0\}$ from $x+y+z=1$ **(how ??)** (See the figure).

Solve $\begin{cases} z=0 \\ x+y+z=1 \end{cases}$

The previous steps are the most important tricks !

$$
\begin{aligned}
&\int\int\int_B f(x,y,z)dV \\
=&\int\int_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dxdy \\
=&\int\int_D [\int_0^{1-x-y} zdz]dxdy \\
=&\int\int_D [z^2/2]_0^{1-x-y} dxdy \\
=&\int\int_D \frac{(1-x-y)^2}{2} dxdy \text{ (It becomes a double integral problem)} \\
=&\int_0^1\int_0^{1-y} \frac{(1-x-y)^2}{2} dxdy \quad (\text{as } D=\{(x,y):\ 0\le x+y\le 1, x\ge 0, y\ge 0\}) \\
=&\int_0^1\int_0^{1-y} -\frac{(1-x-y)^2}{2} d(1-x-y)dy \qquad \textbf{why ?} \\
=&\int_0^1 -\frac{(1-x-y)^3}{2*3}\Big|_{x=0}^{x=1-y} dy \\
=&\int_0^1 \frac{(1-y)^3}{6} dy \\
=&\int_0^1 -\frac{(1-y)^3}{6} d(1-y) \\
=&-\frac{(1-y)^4}{6*4}\Big|_0^1
\end{aligned}
$$

Can we try type II ? Integrate y first.

$$\int\int\int_B f(x,y,z)dV = \int\int_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dxdz$$

Identify the two bounded hyperplanes: $y=u_1(x,z)$ and $y=u_2(x,z)$ from

$y=0$ and $x+y+z=1$ (yielding $y=1-x-z$).

$\underbrace{0}_{u_1(x,z)} \le y \le \underbrace{1-x-z}_{u_2(x,z)}$.

$D=\{(x,z):\ 0\le x+z\le 1, x,z\ge 0\}$ from $x+y+z=1$

Solve $\begin{cases} y=0 \\ x+y+z=1 \end{cases}$

$$
\begin{aligned}
\int\int\int_B f(x,y,z)dV =& \int\int_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dxdz \\
=&\int\int_D [\int_0^{1-x-z} zdy]dxdz \\
=&\int\int_D z(1-x-z)dxdz \text{ (It becomes a double integral problem.)} \\
=&\int_0^1\int_0^{1-z} z(1-x-z)dxdz \\
&\text{(as } x \text{ is bounded by } x=0 \text{ and } x+z=1 \text{ (yielding } x=1-z)) \\
=&......
\end{aligned}
$$

Does Type III work ?

Example 7. Evaluate $\int\int\int_B \sqrt{x^2+z^2}dV$ where B is the solid bounded by $y = x^2+z^2$ and $y = 4$.

Sol. Several approaches: Types I, II or III.
Type I. $B = \{(x,y,z): (x,y) \in D, z \in [u_1(x,y), u_2(x,y)]\}$ then

$$\int\int\int_B f(x,y,z)dV = \int\int_D g(x,y)dxdy = \int\int_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dxdy$$

Identify the two bounded hyperplanes: $z = u_1(x,y)$ and $z = u_2(x,y)$ from
$B = \{(x,y,z): x^2+z^2 \le y \le 4\}$.
=> $|z| \le \sqrt{y-x^2}$.
$u_1(x,y) = ?$
$u_2(x,y) = ?$
For D: Solve $\begin{cases} z = 0 \\ y = x^2+z^2 \end{cases}$ and $\begin{cases} z = 0 \\ y = 4 \end{cases}$ => $y = x^2$ and $y = 4$.
=> $x^2 \le y \le 4$ and $|x| \le 2$.

$$\begin{aligned} &\int\int\int_B f(x,y,z)dV \\ =&\int\int_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dxdy \\ =&\int_{-2}^{2}\int_{x^2}^{4}[\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}dz]dydx \\ =&??? \end{aligned}$$

To solve $\int \sqrt{x^2+z^2}dz$, set $z = x\tan(\theta)$, and use $1+\tan^2\theta = \sec^2\theta$.

Type II. Integrate y first. $\int\int\int_B f(x,y,z)dV = \int\int_D[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dxdz$.
$B = \{(x,y,z): x^2+z^2 \le y \le 4\}$. =>
$\underbrace{x^2+z^2}_{u_1(x,z)} \le y \le \underbrace{4}_{u_1(x,z)}$ and $D = \{(x,z): x^2+z^2 \le 4\}$.

$$\begin{aligned} &\int\int\int_B f(x,y,z)dV \\ =&\int\int_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dxdz \\ =&\int\int_D [\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}dy]dxdz \\ =&\int\int_D (4-(x^2+z^2))\sqrt{x^2+z^2}dy]dxdz \end{aligned}$$

(It becomes a double integral problem.)

$$\begin{aligned} =&\int_0^{2\pi}\int_0^2 (4-r^2)r(rdrd\theta) \\ =&\int_0^{2\pi}\int_0^2 (4r^2-r^4)drd\theta \\ =&2\pi[\frac{4}{3}r^3 - r^5/5]\Big|_0^2 \end{aligned}$$

Which type you like better ?