2. (25 points). Let $X_1, ..., X_5$ denote a random sample of size 5 from a pdf $f(x;\theta) = \theta e^{-x\theta}$, $x, \theta > 0$.

2.1 (3 points) The likelihood function $L(\theta) =$?

2.2 (3 point) Derive the moment generating function of X_1.

2.3 (3 point) Derive the moment generating function of $\sum_{i=1}^{5} X_i$.

2.4 (3 points) Notice that $Y \sim \chi^2(v)$, which has mgf $(1-2t)^{-v/2}$. Then $v =$?.

2.5 (10 points) Find a most powerful test of size 0.1 for testing H_0: $\theta = 1$ v.s. H_a: $\theta = 2$, simplify it (see (2.4)) and give the RR explicitly using the χ^2 table **(Hint: use 2.3 and 2.4)**.

Sol. (1) The likelihood function $L(\theta) = \prod_{i=1}^{n} \theta exp(-X_i\theta) = \theta^n exp(-\sum_{i=1}^{n} X_i\theta)$

(2)
$$\begin{aligned} M_{X_1}(t) =& E(exp(X_1 t)) \\ =& \int_0^\infty e^{xt}\theta e^{-\theta x}dx = \int_0^\infty \theta e^{-(\theta-t)x}dx \\ =& \frac{\theta}{\theta - t} \end{aligned}$$

(3)
$$\begin{aligned} M_{\sum_i X_i}(t) =& E(exp(\sum_{i=1}^{5} X_i t)) \\ =& E(\prod_{i=1}^{5} exp(X_i t)) = \prod_{i=1}^{5} E(exp(X_i t) \\ =& (E(exp(X_1 t)))^5 \\ =& (\int_0^\infty e^{xt}\theta e^{-\theta x}dx)^5 = (\int_0^\infty \theta e^{-(\theta-t)x}dx)^5 \\ =& (\frac{\theta}{\theta - t})^5 \end{aligned}$$

(4) Notice that $Y \sim \chi^2(v)$, which has mgf $M_Y(t) =(1-2t)^{-v/2}$. Then $v = 10$. The reason is as follow:

$$M_{\sum_i X_i}(t) = (\frac{\theta}{\theta - t})^5 = (\frac{1}{1 - t/\theta})^5 = (1 - t/\theta)^{-5}$$

If $\theta = 1$, letting $W = \sum_{i=1}^{5} X_i$, then
$$\begin{aligned} M_{2\sum_i X_i}(t) =& E(\exp(2\sum_{i}^{n} X_i t)) \\ =& E(\exp(\sum_{i}^{5} X_i 2t)) \\ =& (1-2t)^{-5} \end{aligned}$$

$$\text{That is, } 2\sum_{i=1}^{5} X_i = 2W \sim \chi^2(10) \text{ if } \theta = 1.$$

(5) H_o: $\theta = 1$, H_1: $\theta = 2$. $L(\theta) = \prod_{i=1}^{5} \theta e^{-X_i\theta} = \theta^5 \exp(-\sum_{i=1}^{5} X_i\theta) = \theta^5 exp(-W\theta)$, where $W = \sum_{i=1}^{5} X_i$.

$$\begin{aligned} RR: \quad \frac{L(\theta_o)}{L(\theta_1)} =& \frac{\theta_o^5 e^{-\theta_o W}}{\theta_1^5 e^{-\theta_1 W}} \qquad\qquad f(t) = \frac{\theta^5 t^{5-1} e^{-\theta t}}{\Gamma(5)} \\ =& \frac{e^{(\theta_1 - \theta_o)W}}{(\theta_1/\theta_o)^5} \le k \qquad\qquad \theta_1 > \theta_o \\ W \le& \frac{\ln k(\theta_1/\theta_o)^5}{(\theta_1 - \theta_o)} = c \qquad\qquad 2W \sim \chi^2(10) \end{aligned}$$

$$RR: \quad 2W = 2\sum_{i=1}^{5} X_i = \chi^2(10) \le \chi^2_{0.90,10} \approx 4.865 \ or \ W \le 2.43$$

$$0.1 = P(W \le c) = P(\chi^2(10) \le 2c) = P(\chi^2(10) \le 4.865)$$