Quiz on Friday: 447: [9]–[25], 448: [1]-[13]

Ex. 9.13. Assuming X_1, ..., $X_n \overset{i.i.d.}{\sim} Gamma(\alpha, \beta)$, find an MME of (α, β).

Sol. Note that $E(X) = \alpha\beta$ and $V(X) = \alpha\beta^2$.

Sketch: $\overline{X} = \alpha\beta$ and $\overline{X^2} = \alpha\beta^2 + (\alpha\beta)^2$

=> $\overline{X^2} = \overline{X}\beta + (\overline{X})^2$.

=> $\hat{\beta} = \frac{\overline{X^2} - (\overline{X})^2}{\overline{X}}$

and $\hat{\alpha} = \frac{(\overline{X})^2}{\overline{X^2} - (\overline{X})^2}$.

§9.7. The Method of Maximum Likelihood.

[13] Given a random sample X_1, ..., X_n from $f(x;\theta)$, their likelihood is $L(\theta) =$ __________, the MLE $\hat{\theta}$ of θ maximizes _________. If $g(\theta)$ is a _________ function of θ, the MLE of $g(\theta)$ is _________.

key: $\underline{\prod_i f(X_i;\theta)}$, $\underline{L(\theta)}$. $\underline{1-1}$, $\underline{g(\hat{\theta})}$,

Ex. 9.14. Given a random sample X_1, ..., X_n from $bin(1,p)$, find the MLE of p.

Sol. Two usual steps.

(1) solve $\frac{\partial \ln L}{\partial p} = 0$ to get $\hat{p}$;

(2) either check (2a) $\frac{\partial^2 \ln L}{\partial p^2} < 0$? or check

(2b) $\ln L$ at the boundary points: 0 and 1: whether $\ln L(a) < \ln L(\hat{p})$ and $\ln L(b) < \ln L(\hat{p})$.

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\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} f(X_i;p) \\
&= \prod_{i=1}^{n} (p^{X_i}(1-p)^{1-X_i}) \\
&= p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i} \\
&= p^Y(1-p)^{n-Y} \qquad\qquad \text{where } Y = \textstyle\sum_{i=1}^n X_i \sim bin(?,?) \\
\ln L &= Y\ln p + (n-Y)\ln(1-p) \\
(\ln L)'_p &= Y/p - (n-Y)/(1-p) = 0 \\
&=> \quad Y(1-p) - (n-Y)p = 0 \\
&=> \quad Y(1-p) - np + Yp = 0 \\
&=> \quad Y = np \quad => \begin{cases} p = Y/n = \overline{X} \ ? \\ \hat{p} = Y/n = \overline{X} \ ? \end{cases} \quad \text{which is correct ?}
\end{aligned}
$$

$(\ln L)''_p = -Y/p^2 - (n-Y)/(1-p)^2 < 0$. Thus $\hat{p} = \overline{X}$ is the MLE of p.

$\ln L(a) < \ln L(\hat{p})$?

$\ln L(b) < \ln L(\hat{p})$. ?

Ex. 9.15. Given a random sample Y_1, ..., Y_n from $N(\mu, \sigma^2)$, find the MLE of (μ, σ^2).

Sol.

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\begin{aligned}
L =& \prod_{i=1}^{n} f(Y_i; \mu, \sigma^2) \\
=& \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{Y_1-\mu}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{Y_n-\mu}{2\sigma^2}} \\
=& (\frac{1}{\sqrt{2\pi\sigma^2}})^n e^{-\sum_{i=1}^{n} \frac{Y_i-\mu}{2\sigma^2}} \\
\ln L =& -\frac{n}{2}\ln\sigma^2 - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu)^2 \\
\frac{\partial \ln L}{\partial \mu} =& \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i-\mu) = 0 \qquad => \qquad \hat{\mu} = \overline{Y} \\
\frac{\partial \ln L}{\partial \sigma^2} =& -\frac{n}{2}/\sigma^2 + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i-\mu)^2 = 0 \\
=> \quad \sigma^2 =& \frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu)^2 \\
=> \quad \sigma^2 =& \frac{1}{n}\sum_{i=1}^{n}(Y_i-\overline{Y})^2 \\
=& \frac{1}{n}\sum_{i=1}^{n}(Y_i^2 - 2\overline{Y}Y_i + (\overline{Y})^2)
\end{aligned}
$$

$\overline{Y^2} - (\overline{Y})^2$ $\qquad = \text{Can it be simplified ?}$

Need to check whether $(\mu, \sigma^2) = (\overline{Y}, \hat{\sigma}^2)$ is indeed the MLE:

(1) $\frac{\partial^2 \ln L}{\partial \mu^2}$, $\frac{\partial^2 \ln L}{\partial (\sigma^2)^2}\ln L$, ... or

(2) $\ln L$ at $\mu = \pm\infty$ and $\sigma^2 = 0$ and ∞.

It is more convenienct to check (2) here:

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\begin{aligned}
\ln L =& -\frac{n}{2}\ln\sigma^2 - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu)^2 \\
=& \begin{cases} -\infty & \text{if } \sigma^2 = \infty \\ -\frac{n}{2}\ln 0 - \frac{n}{2}\ln(2\pi) - \frac{1}{0+} = \infty - \infty \ \textbf{??} & \text{if } \sigma^2 = 0+ \quad (\ln 0+, \frac{1}{0+}) = \lim_{x\downarrow 0}(\ln x, \frac{1}{x}) = (-\infty, \infty) \\ -\infty & \text{if } \mu = \pm\infty. \end{cases}
\end{aligned}
$$

Since $\lim_{x\downarrow 0} \frac{\ln x}{x^{-1}} = \lim_{x\downarrow 0} \frac{(\ln x)'}{(x^{-1})'} = \lim_{x\downarrow 0} \frac{(1/x)}{-(x^{-2})} = 0$,

$$
-\ln(0+) - \frac{1}{0+} = -\infty.
$$

Thus the MLE of (μ, σ^2) is $(\overline{Y}, \hat{\sigma^2})$, where $\hat{\sigma^2} = \overline{Y^2} - (\overline{Y})^2$.

Ex. 9.16. Given a random sample Y_1, ..., Y_n from $U(0, \theta)$, find the MLE of θ.

Sol.

$$L = \prod_{i=1}^{n} f(Y_i; \theta)$$
$$= \begin{cases} \frac{1}{\theta} \times \cdots \times \frac{1}{\theta} & \text{if } 0 \le Y_i \le \theta,\ i = 1, ..., n \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \frac{1}{\theta^n} & \text{if } 0 \le Y_{(1)} \le Y_{(n)} \le \theta \\ 0 & \text{otherwise} \end{cases}$$
$$\le \frac{1}{(Y_{(n)})^n}.$$

Thus the MLE of θ is $\hat{\theta} = Y_{(n)}$.

Remark. The usual approach of taking $\frac{d\ln L}{d\theta} = 0$ solve for the MLE does not work, as $\frac{d}{d\theta}\ln L = \frac{d}{d\theta}(-n\ln\theta) = -\frac{n}{\theta} \neq 0$

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Invariance principle of the MLE: If g is a 1-1 function of θ and $\hat{\theta}$ is the MLE of θ then the MLE of $g(\theta)$ is $g(\hat{\theta})$.

Example 9.16 (continued) Find the MLE of $V(Y)$.

Sol. $V(Y) = \sigma_Y^2 = \theta^2/12$. Thus $\hat{\sigma}_Y^2 = Y_{(n)}^2/12$.

Example 9.15 (continued) Find the MLE of σ.

Sol. $V(Y) = \hat{\sigma}_Y^2 = \overline{Y^2} - (\overline{Y})^2$. Then the MLE of σ is $\hat{\sigma} = \sqrt{\overline{Y^2} - (\overline{Y})^2}$.