Contents

1. Axioms of probability: (1) $P(A) \geq$ ____, (2) $P(S) =$ ___, ...

2. $P(A \cap B) =$ ______. If A and B are ____________ then $P(A \cap B) = P(A)P(B)$.

3. $P($____$)=1 - P(A)$. $P(A \cup B) = P(A) + P(B)$ ____________.

   If A and B are ___________ then $P(A \cup B) = P(A) + P(B)$.

4. $X \sim bin(n, p)$: $f(i) =$ ____________ if $i \in \{0, ..., n\}$, $\mu =$ _____, $\sigma^2 =$ _____, where $q = 1-p$

5. $X \sim Pois(\lambda)$. $f(i) =$ ____________ if $i \geq 0$. $\mu =$ ______, $\sigma^2 =$ ______

6. $Y = g(X)$. $E(g(X)) = \begin{cases} \sum_y y f_Y(y) & dis \\ \int y f_Y(y)dy & cts \end{cases} = \begin{cases} \rule{4cm}{0.4pt} & dis \\ \\ \rule{4cm}{0.4pt} & cts \end{cases}$, $\mu_Y =$ ____, $\sigma_Y^2 =$ ____

7. The mgf of X is $M(t) =$ ________, $\frac{d^k M(t)}{dt^k}\Big|_{t=0} =$ ____________

8. A cdf $F(t)$ $(=$ ________$)$ satisfying (1) $F(-\infty) =$ ____, and $F(\infty) =$ ____, (2) $F(x+) =$ ____,

(3) $F(x)$ ____ . Moreover, $F(b) - F(a) = P($_______$)$

9. $F(t) = \begin{cases} \sum_{x \leq t} \rule{2cm}{0.4pt} & dis \\ \\ \int_{-\infty}^{t} \rule{2cm}{0.4pt} & cts \end{cases}$, $f(t) = \begin{cases} \rule{2cm}{0.4pt} & dis \\ \\ \rule{1.5cm}{0.4pt} & cts \end{cases}$

10. $E(aX + b) =$ ____________. $Var(aX + b) =$ ____________

11. $X \sim N(\mu, \sigma^2)$. $f(x) =$ ________________, $\frac{X-\mu}{\sigma} \sim$ ____________

12. $X \sim \mathcal{G}(\alpha, \beta)$. $f(x) =$ ____________ if $x >$ _, $\mu =$ _____, $\sigma^2 =$ _____, $\Gamma(\alpha + 1) =$ ______

13. $Exp(\lambda) =$ ____________, $\chi^2(\nu) =$ ____________,

14. $f_X(x) = \begin{cases} \int f(x, y) \rule{1.5cm}{0.4pt} & cts \\ \\ \rule{1.5cm}{0.4pt} f(x, y) & dis \end{cases}$ 15. $E(g(X, Y)) = \begin{cases} \rule{4cm}{0.4pt} & cts \\ \\ \rule{4cm}{0.4pt} & dis \end{cases}$

16. $f_{X|Y}(x|y) =$ ____________, $F_{X|Y}(x|y) = P\,($____________$)$

17. $E(c) =$ ____, $E(ag(X, Y) + bh(X, Y)) =$ ________________,

18. $Cov(X,Y) =$ _____________, $V(aX+bY) =$ _________________________, $\rho(X,Y) =$ _____

19. $E(X|Y=y) = \begin{cases} \rule{3cm}{0.4pt} & cts \\ \rule{3cm}{0.4pt} & dis \end{cases}$. $E(E(X|Y)) =$ ______, $E(V(X|Y))+V(E(X|Y)) =$ ______.

20. $U = h(Y)$, where h is ____________ and Y is cts. $f_U(u) = f_Y($______$) \cdot$ ______

21. Let Y_1, ..., Y_n be a random sample of Y. $\overline{Y} =$ __________________, $S^2 = S_Y^2 =$ ____________, $\mu_{\overline{Y}} =$ ____________, and $\sigma^2_{\overline{Y}} =$ ____________.

22. $F_{\overline{Y}}(t)$ ____$\Phi($_________$)$, where $\Phi(t)$ is the cdf of _______.

44. If X_1___X_2.

X_i's $\sim$:	$X_1 + X_2 \sim$:
$G(\alpha_i, \beta)$	__________
$\chi^2(v_i)$	__________
$Pois(\lambda_i)$	__________
$N(\mu_i, \sigma_i^2)$	__________
$bin(n_i, p)$	__________

Answer Keys can be found from:
http://people.math.binghamton.edu/qyu/ftp/form2.pdf

Notations in 450: $k, i, n \geq 0$ and $0 \leq m \leq n$. $P(X \geq 0) = 1$ and $x, t > 0$. $S_X(x) = s(x) = \mathbb{P}\{X > x\}$,

1. If $H(0) = 0$ and $\underline{H' \geq 0}$, then $\underline{E(H(X))} = \int_0^\infty s(t)H'(t)\,dt$, *e.g.*,
   $\underline{E(X)} = \int_0^\infty s(t)\,dt$, $\underline{E(X^p)} = \int_0^\infty s(t)pt^{p-1}\,dt$, $E[X \wedge a] = \int_0^a s(t)\,dt$.
2. If $P(X \in \{0, 1, 2, ...\}) = \underline{1}$ and $\underline{H \uparrow}$, then
   $E[\underline{H(X)}] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(H(k) - H(k-1))$, $E[\underline{X}] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}$,
   $E[\underline{X^2}] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(2k-1)$, $E[\min(X, a)] = \sum_{\underline{k=1}}^a \mathbb{P}\{X \geq k\}$.
3. $T(x) = T_x = \underline{(X - x)|(X > x)}$,
   $\underline{{}_tp_x} = S_{T(x)}(t) = \frac{s(x+t)}{s(x)}$, $\underline{{}_tq_x} = F_{T(x)}(t) = \frac{s(x)-s(x+t)}{s(x)}$,
   $\underline{{}_{s|t}q_x} = \mathbb{P}\{s < T(x) \leq s + t\} = {}_sp_x \cdot {}_tq_{x+s}$, $\underline{{}_{s|}q_x} = {}_{s|1}q_x$, $\underline{p_x} = {}_1p_x$, $\underline{q_x} = {}_1q_x$,
4. $\underline{{}_{m+n}p_x} = {}_mp_x \cdot {}_np_{x+m}$, $\underline{{}_np_x} = p_x p_{x+1} \cdots p_{x+n-1}$,
   $\underline{{}_{\sum_{j=1}^k n_j}p_x} = {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}$.
5. The force of mortality is $\mu_X(x) = \mu(x) = \mu_x = \underline{\frac{f_X(x)}{S_X(x-)}}$. $\mu_{T(x)}(t) = \underline{\mu_x(t)}$. If X is cts,
   $\underline{\mu(x)} = -\frac{d}{dx}\ln S_X(x)$, $\underline{S_X(x)} = \exp\left(-\int_0^x \mu(t)\,dt\right)$, $\underline{f_{T(x)}(t)} = {}_tp_x\mu(x+t)$. $\mu_x(t) = \underline{\mu(x+t)}$.
6. $\underline{\mathring{e}_x} = E[T(x)] = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$, $\underline{\mathring{e}_{x:\overline{n}|}} = E[T(x) \wedge n] = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$.
7. The central rate of failure on $(x, x+n]$ is $\underline{{}_nm_x} = \frac{\int_0^n {}_tp_x\mu_x(t)\,dt}{\int_0^n {}_tp_x\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$,
   $\underline{m_x} = {}_1m_x$, $\underline{{}_na(x)} = E(T(x)|T(x) \leq n) = \frac{\mathring{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$, $\underline{a(x)} = {}_1a(x)$.
8. $\underline{K_x} = \lceil T(x) \rceil$, $\lceil t \rceil = \underline{k}$ if $t \in (k-1, k]$, $K(x) = K_x \underline{-1}$,
   $\underline{f_{K_x}(k)} = {}_{k-1|}q_x = {}_{k-1}p_x \cdot q_{x+k-1} = (\prod_{j\geq 0}^{k-2} p_{x+j})q_{x+k-1}$
9. $\underline{e_x} = E[K(x)] = p_x(1 + e_{x+1}) = \sum_{k=1}^\infty {}_kp_x = e_{x:\overline{n}|} + {}_np_xe_{x+n}$,
   $\underline{e_{x:\overline{n}|}} = E(K(x) \wedge n) = \sum_{k=1}^n {}_kp_x$, $\underline{E[(K(x))^2]} = \sum_{k=1}^\infty (2k-1) \cdot {}_kp_x$.
10. The KME $\hat{S}_{pl}(t) = \prod_{t_k \leq t}\underline{(1 - \frac{d_k}{r_k})}$, and $\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \underline{\frac{1}{n}(\hat{S}_{pl}(t))^2} \sum_{k:\ t_k \leq t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}$.
    The Nelson-Aalen estimator: $\tilde{S}_{NA}(t) = \underline{e^{-H(t)}}$, where $H(t) = \sum_{t_k \leq t} \frac{d_k}{r_k}$.
    $\hat{\sigma}^2_{\tilde{S}_{NA}(t)} = (\tilde{S}_{NA}(t))^2\hat{\sigma}^2_{H(t)}$, where $\hat{\sigma}^2_{H(t)} = \underline{\sum_{t_j \leq t} \frac{(r_j - d_j)d_j}{(r_j - 1)r_j^2}}$.
11. $\underline{\ell_x} = \#$ of individuals alive at age x, $\underline{{}_1L_x} = L_x$. ${}_td_x = \underline{\ell_x - \ell_{x+t}}$, $d_x = \underline{{}_1d_x}$,
    $\underline{{}_tp_x} = \prod_{x \leq k < x+t}(1 - d_k/l_k)$. $\underline{T_x} = \ell_x\mathring{e}_x = \int_0^\infty \ell_{x+t}\,dt = \sum_{k=x}^\infty L_k$
    $=$ E(# of years lived beyong age x by the cohort group with l_0 members),
    $\underline{{}_nL_x} = \ell_x\mathring{e}_{x:\overline{n}|} = T_x - T_{x+n}$. $s(x) = \underline{\frac{\ell_x}{\ell_0}}$, ${}_tp_x = \underline{\frac{\ell_{x+t}}{\ell_x}}$, ($T_x$ in #3 differs from T_x in #11).
12.

Interpolation	ℓ_{x+t}	${}_tp_x$
UDD		
exponential		
Balducci		

where _____.

key:

UDD	$(1-t)\ell_x + t\ell_{x+1}$ or $\ell_x - td_x$	$1-tq_x$
exponential	$(\ell_x)^{1-t}(\ell_{x+1})^t$ or $\ell_x p_x^t$	p_x^t
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$	$\frac{p_x}{t+(1-t)p_x}$

, $t \in [0,1]$.

14. **Life Insurance:** $Z = b_{T_x} v_{T_x}$,
Whole life ins: $Z_x = v^{K_x}$, $\overline{Z}_x = v^{T_x}$, $A_x = A_x(v) = E[v^{K_x}] = vq_x + vp_x A_{x+1}$, ${}^2A_x = E[Z_x^2] = A_x(v^2)$.

n–year term : $Z^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n)$, $\overline{Z}^1_{x:\overline{n}|} = v^{T_x} I(T_x \le n)$, $A^1_{x:\overline{n}|} = E[Z^1_{x:\overline{n}|}] = \sum_{k=1}^n v^k f_{K_x}(k) = vq_x + vp_x A^1_{x+1:\overline{n-1}|}$, ${}^2A^1_{x:\overline{n}|} = E((Z^1_{x:\overline{n}|})^2) = A^1_{x:\overline{n}|}(v^2)$.

n–year deferred : ${}_n|Z_x = v^{K_x} I(n < K_x)$, ${}_n|\overline{Z}_x = v^{T_x} I(n < T_x)$, ${}^2{}_n|A_x = {}_n|A_x\ (v^2)$, ${}_n|A_x = E[{}_n|Z_x] = \sum_{k=n+1}^\infty v^k f_{K_x}(k) = v^{n+1} f_{K_x}(n+1) + {}_{n+1}|A_x = vp_x \cdot {}_{n-1}|A_{x+1}$.

n–year pure endowment : $Z_{x:\overline{n}|}^{\ \ 1} = v^n I(n < K_x)$, $\overline{Z}_{x:\overline{n}|}^{\ \ 1} = v^n I(n < T_x)$, $A_{x:\overline{n}|}^{\ \ 1} = {}_nE_x = v^n {}_np_x$, ${}^2A_{x:\overline{n}|}^{\ \ 1} = E((Z_{x:\overline{n}|}^{\ \ 1})^2) = A_{x:\overline{n}|}^{\ \ 1}(v^2)$,

n–year endowment : $Z_{x:\overline{n}|} = v^{K_x \wedge n}$, $\overline{Z}_{x:\overline{n}|} = v^{T_x \wedge n}$, $A_{x:\overline{n}|} = \sum_{k=1}^n v^k f_{K_x}(k) + v^n {}_np_x$, ${}^2A_{x:\overline{n}|} = E((Z_{x:\overline{n}|})^2) = A_{x:\overline{n}|}(v^2)$,

m–year defer n-year term : ${}_m|_nZ_x = v^{K_x} I(m < K_x \le n+m)$, ${}_m|_n\overline{Z}_x = v^{T_x} I(m < T_x \le n+m)$, ${}_m|_nA_x = {}_mE_x A^1_{x+m:\overline{n}|}$, ${}^2{}_m|_nA_{x:\overline{n}|} = E[({}_m|_nZ_x)^2] = \sum_{k=m+1}^{m+n} v^{2k} f_{K_x}(k)$.

$Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x$, $\quad Z^1_{x:\overline{n}|} \cdot {}_n|Z_x = 0$,
$Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, $Z^1_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ 1} = 0$,
${}_n|A_x = {}_nE_x A_{x+n}$, $\quad A_x = A^1_{x:\overline{n}|} + {}_nE_x A_{x+n}$.

15. $(IZ)_x = K_x v^{K_x}$. $(I\overline{Z})_x = T_x v^{T_x}$. $(I\overline{Z})_x = \lceil T_x \rceil v^{T_x}$. $(DZ)^1_{x:\overline{n}|} = (n+1-K_x) v^{K_x} I(K_x \le n)$. $(\overline{DZ})^1_{x:\overline{n}|} = (n - T_x) v^{T_x} I(T_x \le n)$. $(D\overline{Z})^1_{x:\overline{n}|} = \lceil n - T_x \rceil v^{T_x} I(T_x \le n)$.

16. $v = \frac{1}{i+1}$, $\delta = -\ln v$, $d = 1 - v$. $\sum_{k=1}^n kx^{k-1} = (\frac{1-x^{n+1}}{1-x})'_x$.

$a_{\overline{n}|} = \sum_{k=1}^n v^k = v(\frac{1-v^n}{1-v})$. $\sum_{k=0}^n v^k = \frac{1-v^{n+1}}{1-v}$.

17.

due	PV $\quad \ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v}$	APV
whole life	$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = \frac{1-Z_x}{1-v}$	$\ddot{a}_x = \sum_{k=0}^\infty v^k {}_kp_x$
n–y. def.	${}_n|\ddot{Y}_x = \sum_{k\ge n}^{K_x-1} v^k = \frac{v^n - v^{K_x}}{1-v} I(K_x > n)$	${}_n|\ddot{a}_x = \sum_{k=n}^\infty v^k {}_kp_x$
n–y. tem.	$\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{(K_x \wedge n)-1} v^k = \frac{1-Z_{x:\overline{n}|}}{d}$	$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$
n–y. cer.	$\ddot{Y}_{\overline{x:\overline{n}|}} = \sum_{k=0}^{(n \vee K_x)-1} v^k = \ddot{a}_{\overline{n}|} + {}_n|\ddot{Y}_x$	$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_n|\ddot{a}_x$

$i \ne 0$.

immediate: $Y_x = \sum_{k\ge 1}^{K_x-1} v^k = \ddot{Y}_x - 1$, $\quad {}_n|Y_x = \sum_{k>n}^{K_x-1} v^k = {}_{n+1}|\ddot{Y}_x$,
$Y_{x:\overline{n}|} = \sum_{k=1}^{(K_x-1)\wedge n} v^k = \ddot{Y}_{x:\overline{n+1}|} - 1$, $\quad Y_{\overline{x:\overline{n}|}} = \sum_{k=1}^{(K_x-1)\vee n} v^k = \ddot{Y}_{\overline{x:\overline{n+1}|}} - 1$

cts	present value=$\overline{a}_{\overline{n}\rvert} = \underline{\int_0^n v^t dt}$	APV
whole life	$\overline{Y}_x = \int_0^{T_x} v^t dt = \underline{\frac{1-\overline{Z}_x}{\delta}}$	$\overline{a}_x = \underline{\int_0^\infty v^t {}_tp_x dt}$
n–y. def.	${}_n\lvert\overline{Y}_x = \int_n^{T_x} v^t dt \underline{I(T_x > n)}$	${}_n\lvert\overline{a}_x = \underline{\int_n^\infty v^t {}_tp_x dt}$
n–y. tem.	$\overline{Y}_{x:\overline{n}\rvert} = \int_0^{T_x \wedge n} v^t dt = \underline{\frac{1-v^{T_x\wedge n}}{\delta}}$	$\overline{a}_{x:\overline{n}\rvert} = \underline{\int_0^n v^t {}_tp_x dt}$
n–y. cer.	$\overline{Y}_{\overline{x:\overline{n}\rvert}} = \int_0^{T_x \vee n} v^t dt = \underline{\overline{a}_{\overline{n}\rvert} + {}_n\lvert\overline{Y}_x}$	$\overline{a}_{\overline{x:\overline{n}\rvert}} = \underline{\overline{a}_{\overline{n}\rvert} + {}_n\lvert\overline{a}_x}$

18. $\underline{\ddot{Y}_x} = \ddot{Y}_{x:\overline{n}\rvert} + {}_n\lvert\ddot{Y}_x$, $\quad E((\ddot{Y}_x)^2)\underline{\neq}\ddot{a}_x(v^2)$, $\ddot{a}_x = \underline{1 + vp_x\ddot{a}_{x+1}}$. $\quad \overline{a}_x = \underline{\overline{a}_{x:\overline{n}\rvert} + v^n {}_np_x\overline{a}_{x+n}}$.
${}_n\lvert\ddot{a}_x = {}_nE_x\ddot{a}_{x+n} = \underline{vp_x \cdot {}_{n-1}\lvert\ddot{a}_{x+1}}$. $\quad {}_n\lvert\overline{a}_x = {}_nE_x \cdot \overline{a}_{x+n} = \underline{vp_x \cdot {}_{n-1}\lvert\overline{a}_{x+1}}$.
$\ddot{a}_{x:\overline{n+m}\rvert} = \ddot{a}_{x:\overline{n}\rvert}\underline{+{}_nE_x \cdot \ddot{a}_{x+n:\overline{m}\rvert}}$.

19.

Plan	Loss
Whole life insurance	$\underline{Z_x - P\ddot{Y}_x}$
t–year funded whole life insurance	$\underline{Z_x - P\ddot{Y}_{x:\overline{t}\rvert}}$
n–year term insurance	$\underline{Z^1_{x:\overline{n}\rvert} - P\ddot{Y}_{x:\overline{n}\rvert}}$
t–year funded n–year term insurance	$\underline{Z^1_{x:\overline{n}\rvert} - P\ddot{Y}_{x:\overline{t}\rvert}}$
n–year pure endowment insurance	$\underline{Z_{x:\overset{1}{\overline{n}\rvert}} - P\ddot{Y}_{x:\overline{n}\rvert}}$
t–year funded n–year pure endowment insurance	$\underline{Z_{x:\overset{1}{\overline{n}\rvert}} - P\ddot{Y}_{x:\overline{t}\rvert}}$
n–year endowment	$\underline{Z_{x:\overline{n}\rvert} - P\ddot{Y}_{x:\overline{n}\rvert}}$
t–year funded n–year endowment insurance	$\underline{Z_{x:\overline{n}\rvert} - P\ddot{Y}_{x:\overline{t}\rvert}}$
n–year deferred insurance	$\underline{{}_n\lvert Z_x - P\ddot{Y}_x}$
t–year funded n–year deferred insurance	$\underline{{}_n\lvert Z_x - P\ddot{Y}_{x:\overline{t}\rvert}}$

Loss in the fully discrete case

13. **Application of Woolhouse's formula:**

$$\overline{a}_x \approx \underline{\ddot{a}_x - \frac{1}{2}} - \frac{1}{12}(\delta + \mu_x) \approx \underline{\ddot{a}_x^{(m)} - \frac{1}{2m}} - \frac{1}{12m^2}(\delta + \mu_x)$$

Notations in 450: $k, i, n \geq 0$ and $0 \leq m \leq n$. $P(X \geq 0) = 1$ and $x, t > 0$. $S_X(x) = s(x) = \mathbb{P}\{X > x\}$,

1. If $H(0) = 0$ and _____ ≥ 0, then __________ $= \int_0^\infty s(t)H'(t)\,dt$, *e.g.*,

   __________ $= \int_0^\infty s(t)\,dt$, __________ $= \int_0^\infty s(t)pt^{p-1}\,dt$, $E[X \wedge a] = \int_0^{\_\_\_} s(t)\,dt$.

2. If $P(X \in \{0, 1, 2, ...\}) =$ _____ and _____, then

$E[\_\_\_\_\_] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(H(k) - H(k-1))$, $E[\_\_\_\_\_] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}$,

   $E[\_\_\_\_\_] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(2k-1)$, $E[\min(X, a)] = \sum_{\_\_\_\_\_}^{\_\_\_\_\_} \mathbb{P}\{X \geq k\}$.

3. $T(x) = T_x =$ _______________, ____________ $= S_{T(x)}(t) = \frac{s(x+t)}{s(x)}$, _____ $= F_{T(x)}(t) = \frac{s(x)-s(x+t)}{s(x)}$,

   __________ $= \mathbb{P}\{s < T(x) \leq s+t\} = {}_sp_x \cdot {}_tq_{x+s}$, _____ $= {}_s|{}_1q_x$, _____ $= {}_1p_x$, _____ $= {}_1q_x$,

4. __________ $= {}_mp_x \cdot {}_np_{x+m}$, __________ $= p_x p_{x+1} \cdots p_{x+n-1}$,

   __________ $= {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}$.

5. The force of mortality is $\mu_X(x) = \mu(x) = \mu_x =$ __________. $\mu_{T(x)}(t) =$ ________

   If X is cts, _____ $= -\frac{d}{dx}\ln S_X(x)$, _____ $= \exp\left(-\int_0^x \mu(t)\,dt\right)$, _____ $= {}_tp_x\mu(x+t)$. $\mu_x(t) =$ _____

6. __________ $= E[T(x)] = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$, __________ $= E[T(x) \wedge n] = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$.

7. The central rate of failure on $(x, x+n]$ is ________ $= \frac{\int_0^n {}_tp_x\mu_x(t)\,dt}{\int_0^n {}_tp_x\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$,

   ________ $= {}_1m_x$, ________ $= E(T(x)|T(x) \leq n) = \frac{\mathring{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$, ________ $= {}_1a(x)$.

8. _____ $= \lceil T(x) \rceil$, $\lceil t \rceil =$ _____ if $t \in (k-1, k]$, $K(x) = K_x$_____,

   __________ $= {}_{k-1}|q_x = {}_{k-1}p_x \cdot q_{x+k-1} = (\prod_{j\geq 0}^{k-2} p_{x+j})q_{x+k-1}$

9. ________ $= E[K(x)] = p_x(1 + e_{x+1}) = \sum_{k=1}^\infty {}_kp_x = e_{x:\overline{n}|} + {}_np_xe_{x+n}$,

$\underline{\hspace{2cm}} = E(K(x) \wedge n) = \sum_{k=1}^{n} {}_kp_x$, $\underline{\hspace{2.5cm}} = \sum_{k=1}^{\infty}(2k-1) \cdot {}_kp_x$.

10. The KME $\hat{S}_{pl}(t) = \prod_{t_k \le t}(\underline{\hspace{2cm}})$, and $\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \underline{\hspace{1.5cm}} \sum_{k:\ t_k \le t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}$.

The Nelson-Aalen estimator: $\tilde{S}_{NA}(t) = \underline{\hspace{3cm}}$, where $H(t) = \sum_{t_k \le t} \frac{d_k}{r_k}$.

$\hat{\sigma}^2_{\tilde{S}_{NA}(t)} = (\tilde{S}_{NA}(t))^2 \hat{\sigma}^2_{H(t)}$, where $\hat{\sigma}^2_{H(t)} = \underline{\hspace{3cm}}$.

11. $\underline{\hspace{3cm}}$ = **# of individuals alive** at age x, $\underline{\hspace{3cm}} = L_x$,

${}_td_x = \underline{\hspace{3cm}}$, $d_x = \underline{\hspace{3cm}}$, $s(x) = \underline{\hspace{2.5cm}}$, ${}_tp_x = \underline{\hspace{1.5cm}}$,

$\underline{\hspace{1.5cm}} = \prod_{x \le k < x+t}(1 - d_k/l_k)$. $\underline{\hspace{2cm}} = \ell_x \overset{\circ}{e}_x = \int_0^{\infty} \ell_{x+t}\, dt = \sum_{k \ge x} L_k$
= E(# of years lived beyong age x by the cohort group with l_0 members),

$\underline{\hspace{2.5cm}} = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = T_x - T_{x+n}$, ($T_x$ in #3 differs from T_x in #11)

12.

Interpolation	ℓ_{x+t}	${}_tp_x$
UDD		
exponential		
Balducci		

, where ____.

13. **Application of Woolhouse's formula:**

$$\bar{a}_x \approx \underline{\hspace{3cm}} - \frac{1}{12}(\delta + \mu(x)) \approx \underline{\hspace{3cm}} - \frac{1}{12m^2}(\delta + \mu(x))$$

14. Life Insurance $Z = \underline{\hspace{2cm}}$

Whole life ins: $Z_x = \underline{\hspace{2cm}}$, $\overline{Z}_x = \underline{\hspace{2cm}}$, $A_x = A_x(v) = \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$,

${}^2A_x = E[Z_x^2] = \underline{\hspace{2cm}}$,

n–year term : $Z^1_{x:\overline{n}|} = \underline{\hspace{2cm}}$, $\overline{Z}^1_{x:\overline{n}|} = \underline{\hspace{2cm}}$, $A^1_{x:\overline{n}|} = E[Z^1_{x:\overline{n}|}] = \underline{\hspace{3cm}} = \underline{\hspace{4cm}}$,

${}^2A^1_{x:\overline{n}|} = E((Z^1_{x:\overline{n}|})^2) = \underline{\hspace{2cm}}$,

n–year deferred : ${}_n|Z_x = \underline{\hspace{4cm}}$, ${}_n|\overline{Z}_x = \underline{\hspace{4cm}}$,

${}^2{}_n|A_x = {}_n|A_x \underline{\hspace{1.5cm}}$, ${}_n|A_x = E[{}_n|Z_x] = \underline{\hspace{2cm}}\ v^k f_{K_x}(k) = \underline{\hspace{5cm}} = vp_x \cdot {}_{n-1}|A_{x+1}$.

n–year pure endowment : $Z_{x:\overline{n}|}^{\ \ 1} = \underline{\hspace{2cm}}$, $\overline{Z}_{x:\overline{n}|}^{\ \ 1} = \underline{\hspace{2.5cm}}$,

$A_{x:\overline{n}|}^{\ \ 1} = {}_nE_x = \underline{\hspace{3cm}}$, ${}^2A_{x:\overline{n}|}^{\ \ 1} = E((Z_{x:\overline{n}|}^{\ \ 1})^2) = A_{x:\overline{n}|}^{\ \ 1}\underline{\hspace{2cm}}$,

n–year endowment : $Z_{x:\overline{n}|} = \underline{\hspace{2cm}}$, $\overline{Z}_{x:\overline{n}|} = \underline{\hspace{2cm}}$, $A_{x:\overline{n}|} = \underline{\hspace{5cm}}$,

${}^2A_{x:\overline{n}|} = E((Z_{x:\overline{n}|})^2) = A_{x:\overline{n}|}\underline{\hspace{1.5cm}}$,

m–year defer n-year term : ${}_{m}|_{n}Z_x = \underline{\hspace{6cm}}$, ${}_{m}|_{n}\overline{Z}_x = \underline{\hspace{5cm}}$,

${}_{m}|_{n}A_x = {}_mE_x\underline{\hspace{4cm}}$, ${}^2{}_{m}|_{n}A_{x:\overline{n}|} = E[({}_{m}|_{n}Z_x)^2] = \underline{\hspace{2.5cm}}\ v^{2k}f_{K_x}(k)$,

$\underline{\hspace{2cm}} = Z^1_{x:\overline{n}|} + {}_n|Z_x$, $Z^1_{x:\overline{n}|} \cdot {}_n|Z_x = \underline{\hspace{1.5cm}}$, $\underline{\hspace{3cm}} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, $Z^1_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ 1} = \underline{\hspace{1.5cm}}$,

$\underline{\hspace{3cm}} = {}_nE_xA_{x+n}$, $\underline{\hspace{2cm}} = A^1_{x:\overline{n}|} + {}_nE_xA_{x+n}$.

15. $(IZ)_x =$ _____. $(\overline{IZ})_x =$ _____. $(I\overline{Z})_x =$ _____. $(DZ)^1_{x:\overline{n}|} =$ __________________.

$(\overline{DZ})^1_{x:\overline{n}|} =$ __________________. $(D\overline{Z})^1_{x:\overline{n}|} =$ __________________.

16. _____ $= \frac{1}{i+1}$, _____ $= -\ln v$, _____ $= 1 - v$. $\sum_{k=1}^{n} kx^{k-1} =$ _________ $a_{\overline{n}|} = \sum_{k=1}^{n} v^k$ $=$ _____

$\sum_{k=0}^{n} v^k =$ _____.

17.

due	PV $\ddot{a}_{\overline{n}\rvert} =$ ______ $=$ ______	APV
whole life	$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k =$ ______	$\ddot{a}_x =$ ______
n–y. def.	${}_n\lvert\ddot{Y}_x =$ ______ $= \frac{v^n - v^{K_x}}{1-v}$ ______	${}_n\lvert\ddot{a}_x =$ ______
n–y. temp.	$\ddot{Y}_{x:\overline{n}\rvert} =$ ______ $= \frac{1-Z_{x:\overline{n}\rvert}}{d}$	$\ddot{a}_{x:\overline{n}\rvert} =$ ______
n–y. cer.	$\ddot{Y}_{\overline{x:\overline{n}\rvert}} =$ ______ $= \ddot{a}_{\overline{n}\rvert} + {}_n\lvert\ddot{Y}_x$	$\ddot{a}_{\overline{x:\overline{n}\rvert}} =$ ______

$i \neq 0.$

immediate: $Y_x = \sum_{k\geq 1}^{K_x-1} v^k =$ ______, ${}_n\lvert Y_x = \sum_{k>n}^{K_x-1} v^k =$ ______,

$Y_{x:\overline{n}\rvert} = \sum_{k=1}^{(K_x-1)\wedge n} v^k =$ ______, $Y_{\overline{x:\overline{n}\rvert}} = \sum_{k=1}^{(K_x-1)\vee n} v^k =$ ______

cts	present value$=\bar{a}_{\overline{n}\rvert} =$ ______	APV
whole life	$\overline{Y}_x = \int_0^{T_x} v^t dt =$ ______	$\bar{a}_x =$ ______ dt
n–y. def.	${}_n\lvert\overline{Y}_x = \int_n^{T_x} v^t dt$ ______	${}_n\lvert\bar{a}_x =$ ______ dt
n–y. temp.	$\overline{Y}_{x:\overline{n}\rvert} = \int_0^{T_x\wedge n} v^t dt =$ ______	$\bar{a}_{x:\overline{n}\rvert} =$ ______ dt
n–y. cer.	$\overline{Y}_{\overline{x:\overline{n}\rvert}} = \int_0^{n\vee T_x} v^t dt =$ ______	$\bar{a}_{\overline{x:\overline{n}\rvert}} =$ ______

18. ____ $= \ddot{Y}_{x:\overline{n}\rvert} + {}_n\lvert\ddot{Y}_x$. $E((\ddot{Y}_x)^2)$____$\ddot{a}_x(v^2)$. $\ddot{a}_x =$ ______.

$\bar{a}_x =$ ______. ${}_n\lvert\ddot{a}_x = {}_nE_x\ddot{a}_{x+n} =$ ______.

${}_n\lvert\bar{a}_x = {}_nE_x \cdot \bar{a}_{x+n} =$ ______. $\ddot{a}_{x:\overline{n+m}\rvert} = \ddot{a}_{x:\overline{n}\rvert}$______.

19. Loss in the fully discrete case

Plan	Loss	Loss
Whole life insurance	________	
t–year funded whole life insurance		________
n–year term insurance	________	
t–year funded n–year term insurance		________
n–year pure endowment insurance	________	
t–year funded n–year pure endowment insurance		________
n–year endowment	________	
t–year funded n–year endowment insurance		________
n–year deferred insurance	________	
t–year funded n–year deferred insurance		________

line-up

CHAPTER 1

MATH 450, Syllabus

MATH 450, Life Contingency Models I

The course is a preparation for Advanced Long-Term Actuarial Mathematics Exam.

MWF 2:20 - 3:50 FA 209

No class on 9/2 M. 10/2-4(MF), 10/11(F), 11/27-29(WF).

Class on 10/8 (Tu), 11/26(Tu)

Professor: Qiqing Yu qiyu@binghamton.edu

Office hours: M 4:05pm-5:05pm in my office WH132; T 7-8pm, through zoom.

https://binghamton.zoom.us/j/8265526594?pwd=d3l6OGx1cmZ4M3cxZEJwVGd1RGcrUT09

Meeting ID: 826 552 6594

Passcode: 031320

Textbook: Arcones' Manual For SOA Exam MLC (First Volumn).

(Chapters to be covered: 2-6)

It is in my website

http://www2.math.binghamton.edu/p/people/qyu/qyu_personal

Course materials for 450, lecture note 1. **It will dispear 9/1**!

A pdf file with some tables needed in the homework can be downloaded from **my website**.
http://www.math.binghamton.edu/qyu/qyu_personal
e.g, the Illustrative Life Table needed in some of the homework problems.

The lecture notes will be posted on my website.

Exams: (closed book) 3 tests + final, Sept. 23 (M), Oct. 21 (M) Nov 25 (M), Final Fri. Dec 13, 12:50 – 02:50 PM FA 209
You can bring a calculator **without the function of installing formulas**. I will check !!

Quizzes: once a week, on Friday;

this week on Friday, Formulas #1-10 for 447 in page 1 (keys are in my website).

Homework: Due Wednesday in class, late homework will be taken 3 points off (out of 10).
Homework due this Friday: Do the final exam of 447 in my website **including Part A !**

Grading Policy:

1. 10% hw +10% quiz +45% tests +35% final

2. Correction: If you make correction and hand in with the old exam, **the next class** after I return the test **in class**, you can get 40% of the missing grades back. No partial credit for correction. Cannot ask me to help you in correction.

3. A− = 85 + and C = 60 +.

10+10+45*(0.3+0.4*0.7)+35*0.3=56

Student Attendance in Class:

The Bulletin states, "Students are expected to attend all scheduled classes, laboratories and discussions. Instructors may establish their own attendance criteria for a course. They may establish both the number of absences permitted to receive credit ....

If you miss a test or quiz, and have a decent reason with a proof, then I will give the lowest grade of the class.

CHAPTER 2
Survival models

2.1 Survival models.

2.1.1 A short probability review.

Definition 2.1. *Given a set* Ω, *a* **probability** $\mathbb{P}$ *on* Ω *is a function defined on the collection of all events (subsets) of* Ω *such that*
(i) $\mathbb{P}(\emptyset) = 0$;
(ii) $\mathbb{P}(\Omega) = 1$;
(iii) If $\{A_n\}_{n=1}^{\infty}$ *are disjoint events, then* $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

Ω is called the **sample space**.

Definition 2.2. *A* **random variable** *(r.v.)* X *is a function from* Ω *into* $\mathbb{R}$.

Definition 2.3. *The* **cumulative distribution function** *(cdf) of the r.v.* X *is* $F_X(x) = P\{X \le x\}$, $x \in \mathbb{R}$.

If X is the age at the death (or failure) of a life, then $X > 0$.

Theorem 2.1. F_X *is a cdf iff*
(i) $F_X \uparrow$, *i.e., for each* $x_1 \le x_2$, $F_X(x_1) \le F_X(x_2)$.
(ii) F_X *is right continuous (cts)* $(\lim_{h \downarrow 0} F_X(x+h) = F_X(x)\ \forall\ x)$
(or $F(x+) = F(x)\ \forall\ x$*).*
(iii) $\lim_{x \to -\infty} F_X(x) = 0$ *and* $\lim_{x \to \infty} F_X(x) = 1$.

For the c.d.f. of an age–at–failure, we only need to define it for $x > 0$ **Why ??**

Theorem 2.2.

Definition 2.4. *A r.v.* X *is called* **discrete**
if there is a countable set $C \subset \mathbb{R}$ *such that* $\mathbb{P}\{X \in C\} = 1$.

Meaning of countable set C ?
Ans. C is either a finite set or $C = \{c_i : i = 1, ..., \infty\}$.

Definition 2.5. *The probability mass function (or frequency function) (pmf) of the discrete r.v.* X *is the function* $p : \mathbb{R} \to \mathbb{R}$ *defined by*
$p(x) = \mathbb{P}\{X = x\},\ x \in \mathbb{R}$.

If X is a discrete r.v. with pmf p and $A \subset \mathbb{R}$, then
$\mathbb{P}\{X \in A\} = \sum_{x \in A} \mathbb{P}\{X = x\} = \sum_{x \in A} p(x)$.

Theorem 2.3. $p(x) \ge 0\ \forall\ x$ *and* $\sum_x p(x) = 1$.

Definition 2.6. *A r.v. X is called* **cts** *if there exists a nonnegative function f called the probability density function (pdf) of X such that $\forall\ A \subset \mathbb{R}$,*
$\mathbb{P}\{X \in A\} = \int_A f(x)\,dx$ *($= \int_{x\in A} f(x)\,dx = \int I(x \in A)f(x)\,dx$), where $I(x \in A) = 1$ if $x \in A$.*

Theorem 2.4. *$f(x) \geq 0\ \forall\ x$ and $\int f(x)\,dx = 1$ ($\int f(x)\,dx = \int_{-\infty}^{\infty} f(x)dx$).*

Q: If a r.v. X is positive and cts, and $x < 0$, **then** $f_X(x) =$ **??**

Theorem 2.5.

Definition 2.7. *A r.v. X has a* **mixed distribution** *if there are functions $f(\cdot)$ and $p(\cdot)$, and a countable set D such that*
for each $A \subset \mathbb{R}$, $\mathbb{P}\{X \in A\} = \int_A f(x)\,dx + \sum_{x\in A\cap D} p(x)$.

Remark. For convenience, we call both the pmf and pdf the density function (df) hereafter.

A mixed distribution X has two parts: a cts part and a discrete part (together with D). The function f in the previous definition is the cts part of df and the function p is the discrete part of df. Note that
$\int f(x)\,dx + \sum_{x\in D} p(x) = 1$, $f(x) \geq 0$ and $p(x) \geq 0\ \forall\ x$.
Abusing notations, we may use $f(x)$ rather than $f(x)$ and $p(x)$.
For the mixed distribution, one can let $f(x) = p(x)$ if $x \in D$, then

$\mathbb{P}\{X \in A\} = \int_A f(x)\,dx + \sum_{x\in A\cap D} f(x)$.

$\int f(x)\,dx + \sum_{x\in D} f(x) = 1$.

Why ?

if $A = \{0, 1, 2\}$ then $\int_A xdx = ?$

$$\int_{(0,1)} xdx \neq \int_{[0,1]} xdx \ \ ???$$

Theorem 2.6. *Let X be a r.v. with a mixed distribution. Then*
$p(x) = F_X(x) - F_X(x-)$ and
$f(x) = F'_X(x)$ if $F'_X(x)$ exists. **What to do OW ?**

Example 2.1. **Examples of $F'(x)$ does not exit ?**

Example 2.2.

Example 2.3. $G(x) = |x|$

Example 2.4.

Example 2.5. *Find D, f and p if* $F(x) = \begin{cases} 0 & \text{if } \ x < 0, \\ \frac{x+2}{8} & \text{if } 0 \leq x < 1, \\ \frac{3x^2+4}{16} & \text{if } 1 \leq x < 2, \\ 1 & \text{if } \ 2 \leq x. \end{cases}$

Solution: How to find D ?

$F(x) - F(x-) = 0$ except, perhaps at $\{0, 1, 2\}$ **Why ?**

$p(0) = F(0) - F(0-) = \frac{2}{8} - 0 = \frac{1}{4}$,
$p(1) = F(1) - F(1-) = \frac{7}{16} - \frac{3}{8} = \frac{1}{16}$,
$p(2) = F(2) - F(2-) = 1 - \frac{16}{16} = 0$.

One solution: $p(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, \\ \frac{1}{16} & \text{if } x = 1. \end{cases}$ $\qquad D = \{0, 1, 2\}$ or $\{0, 1\}$?

$$f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2. \end{cases}$$ **What happens OW ?**

Why not $1 < x \leq 2$ **?**
Is p **a pmf ?**
Is f **a df ?**

Can we write $f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2, \\ 0 & otherwise \end{cases}$?

How about $f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2, \\ 0 & if\ x < 0\ or\ x > 2 \end{cases}$?

How about $f(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2, \\ 0 & otherwise \end{cases}$ **???**

Another solution: $f(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0 \\ \frac{1}{8} & \text{if } 0 < x < 1 \\ \frac{1}{16} & \text{if } x = 1 \\ \frac{3x}{8} & \text{if } 1 < x < 2 \end{cases}$ and $D = \{0, 1\}$.

$f(x) =$? otherwise.

Example 2.5 (continued). $P(0 \leq X \leq 1) =$?

Sol. Two ways:
(1) $P(a \leq X \leq b) = F(b) - F(a-)$, $\qquad P(a < X \leq b) = F(b) - F(a)$.
(2) $P(a \leq X \leq b) = \int_a^b f(x)dx + \sum_{x \in [a,b] \cap D} f(x)$.

Answer: (1) $P(0 \leq X \leq 1) = F(b) - F(a-) = \frac{1+2}{8} - 0$ or $\frac{3+4}{16} - 0$? or $\frac{3+4}{16} - \frac{2}{8}$?
(2) $P(0 \leq X \leq 1) = \int_0^1 \frac{1}{8} dx +$ **??**

2.1.2 Survival function.

Definition 2.8. *The* **survival function** *of a r.v.* X *is* $S_X = 1 - F_X$.

$S_X(x) = \mathbb{P}\{X > x\}$, $x \in \mathbb{R}$. Sometimes denote $S_X(x)$ by $s(x)$.

Most of time, we only consider S_X of an age-at-death X. Then $P(X > 0) = 1$,

$S_X(-1) = \mathbf{??}$
In this course we often suppress the phrase "$S_X(t) = 1$ for $t < 0$"
and only define S_X on $[0, \infty)$.

Theorem 2.7. *A function $S_X : (-\infty, \infty) \to \mathbb{R}$ is the survival function of a* <u>*positive*</u> *r.v. X*

$$\textit{iff} \begin{cases} (1)\ S_X(x)\ \mathbf{?} \\ (2)\ S_X\ \textbf{is ??} \\ (3)\ S_X(0) = \mathbf{??} \\ (4)\ \lim\limits_{x\to\infty} S_X(x) = \mathbf{??} \end{cases} \qquad F_X \textit{ is a cdf iff} \begin{cases} F_X(x) \uparrow, \\ F_X \textit{ is right cts,} \\ \lim_{x\to-\infty} F_X(x) = 0, \\ \lim\limits_{x\to\infty} F_X(x) = 1. \end{cases}$$

Example 2.6. *Determine which of the following functions is a survival function of a non-negative r.v.:*

(i) $s(x) = \frac{2}{x+2}$, for $x \geq 0$.
(ii) $s(x) = (1 - x)e^{-x}$, for $x \geq 0$.
(iii) $s(x) = \frac{1+\frac{2}{x+2}}{2}$, for $x \geq 0$.
(iv) $s(x) = (1 + x)e^{-x}$, for $x \geq 0$.
(v) $s(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \leq x < 90, \\ 0 & \text{for } x \geq 90. \end{cases}$

Solution: (i) s is a survival function **why ??**

(1) $(\frac{2}{x+2})' = 2((x + 2)^{-1})' = -2(x + 2)^{-2} < 0$ on $[0, \infty) =>$?

(2) $\frac{2}{x+2}$ is continuos except at $x = -2 =>$? $\frac{2}{x+2}$ is continuos on $[0, \infty) =>$?

(3) $s(0) =$?

(4) $s(\infty) =$?

(ii) $s(x) = (1 - x)e^{-x}$ is not a survival function because
(1) $s(2) = -e^{-2} < 0 = s(\infty) => s(t)$ is not $\downarrow$. **Can we say $s(t)$ instead of $s(x)$?**
or (1) $s'(x) = -e^{-x} - (1 - x)e^{-x} = e^{-x}(-2 + x) > 0$ if $x \geq 3$; $=> s(x)$ is not $\downarrow$.
Do we need to point out both ?
(iii) $s(x) = \frac{1+\frac{2}{x+2}}{2}$ is not a survival function because
(4) $\lim_{x\to\infty} \frac{1+\frac{2}{x+2}}{2} = \frac{1}{2} \neq 0$.
(iv) s is a survival function (**why ?)**
(v) s is a survival function (**why ?**)

Example 2.7. *Find the density function for the following survival functions:*
(i) $s(x) = (1 + x)e^{-x}$, for $x \geq 0$.
(ii) $s(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \leq x < 90, \\ 0 & \text{for } 90 \leq x. \end{cases}$
(iii) $s(x) = I(x < 1) + \frac{2I(x\geq1)}{x+2}$, $x < 8$.

Solution: The df is $\begin{cases} f(x) = -S'(x) & \text{if } S'(x) \text{ exists} \\ p(x) = S(x-) - S(x) & \textit{otherwise.} \end{cases}$ **Why ?**

(i) $f_X(x) = xe^{-x}$, for $x > 0$.

(ii) The df is $f(x) = \begin{cases} \frac{2x}{10{,}000} & \text{for } 0 < x < 90 \\ 1 - \frac{8100}{10^4} & \text{if } x = 90 \end{cases}$ with $D = \{90\}$.

(iii) $s(x) = I(x < 1) + \frac{2I(x \geq 1)}{x+2}$, $x < 8$, then $f_X(x) = \begin{cases} \frac{2I(x>1)}{(x+2)^2} & \text{if } x \in (1, 8) \\ 1 - \frac{2}{1+2} & \text{if } x = 1 \\ 0.2 & \text{if } x = 8 \end{cases}$, **Done ?**

with $D = \{1, 8\}$.

Does it matter to write $f(x)$ **v.s.** $f_X(x)$ **?**

Does it matter to write $f(x)$ **v.s.** $f(t)$ **?**

Example 2.8. *Let the survival function of a person be* $S_X(x) = \frac{90-x}{90}$, *for* $0 \leq x \leq 90$.

(i) Find the probability that a person dies before reaching 20 years old.

(ii) Find the probability that a person lives more than 60 years.

(iii) Find the probability that a 20-year-old lives more than 60 years.

Solution: $P(a < X \leq b) = F(b) - F(a) = S(a) - S(b) = \int_a^b f(x)dx + \sum_{x \in D \cap (a,b]} f(x)$.

(i) $\mathbb{P}\{X < 20\} = 1 - S_X(20-) = 1 - \frac{90-20}{90} = \frac{2}{9}$.

(ii) $\mathbb{P}\{X > 60\} = S_X(60) = \frac{90-60}{90} = \frac{1}{3}$.

(iii) $\mathbb{P}\{X > 60 | X = 20\}$ or $\mathbb{P}\{X > 60 | X \geq 20\}$?

$= P(A|B) = P(A \cap B)/P(B) = \frac{P(X>60)}{P(X \geq 20)} = \frac{1/3}{1-2/9} = \frac{3}{7}$.

2.1.3 Expectation.

Definition 2.9. *If* X *is discrete r.v. then* $E[X] = \sum_x x p_X(x)$ *(if the series converges).*

Definition 2.10. *If* X *is cts r.v. then* $E[X] = \int x f_X(x)\, dx$ *(if the integral exists).*

Definition 2.11. *If* X *is a mixed r.v. then*
$E[X] = \sum_x x p_X(x) + \int x f_X(x)\, dx$ *(if the series and the integral are finite).*

$E[X]$ is called the **expectation** of the r.v. X, or the **expected value**, or the **mean**.

Definition 2.12.

Given a r.v. X and a function $g : \mathbb{R} \to \mathbb{R}$, $Y = g(X)$ is another r.v., i.e. by composing the functions $X : \Omega \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we get the r.v. $g(X) : \Omega \to \mathbb{R}$. **Examples of $g(X)$?**

$$\textbf{Formula:}\quad E(Y) = E(g(X)) = \begin{cases} \sum_x g(x)p_X(x) & \text{if } X \text{ is discrete} \\ \int g(x)f_X(x)dx & \text{if } X \text{ is cts} \\ \sum_x g(x)p_X(x) + \int g(x)f_X(x)dx & \text{if } X \text{ is mixed} \\ (\sum_{x\in D} g(x)f_X(x) + \int g(x)f_X(x)dx & \text{if } X \text{ is mixed)} \end{cases} \quad (1)$$

$$= \begin{cases} \sum_x xp_Y(x) & \text{if } Y \text{ is discrete} \\ \int xf_Y(x)dx & \text{if } Y \text{ is cts} \\ \sum_x xp_Y(x) + \int xf_Y(x)dx & \text{if } Y \text{ is mixed} \\ (\sum_{x\in D} xf_Y(x) + \int xf_Y(x)dx & \text{if } Y \text{ is mixed)} \end{cases} \quad (2)$$

Q: What is the difference between (1) and (2) ? between black and red ?
Ans. (1) $g(x)$ vs x; (2) f_X vs f_Y. **Can we say $f(x)$ vs $f(y)$ instead ?**
Are the D the same in (1) and (2) ?

Let $X \sim U(0,2)$ and $Y = X \wedge 1$ ($= \min\{X, 1\}$). Then $D = \begin{cases} \emptyset & \text{for } p_X \text{ or } f_X \\ \{1\} & \text{for } p_Y \text{ or } f_Y. \end{cases}$

Often, to find expectations, instead of the density we will use the survival function. We will often use the following theorem:

Theorem 2.8. *Let (1) X be a nonnegative r.v. with survival function s;*
(2) $h : [0,\infty) \to [0,\infty)$ be a function which is integrable in bounded intervals;
(3)$H(x) = \int_0^x h(t)\,dt$, $x \geq 0$.
Then, $E[H(X)] = \int_0^\infty s(t)h(t)\,dt$.

$\frac{d}{dx}\int_0^x h(t)dt = h(x)$ **if ... ???** In some sense Theorem 2.8 says
$E[H(X)] = \int_0^\infty s(t)H'(t)\,dt$, <u>**where** $H' \geq 0$ **and** $H(0) = 0$</u>.

Actually, Th 2.8 applies to functions H where H' exists on $[0,\infty) \setminus D$ and D is countable.
4 corollaries to be proved later, assuming $P(X \geq 0) = 1$, letting $X \wedge a = \min\{X, a\}$:

where $H' \geq 0$ **and** $H(0) = 0$

1. $E(X^p) = \int_0^\infty pt^{p-1}S_X(t)dt$, where $p > 0$. $\quad H(x) = x^p,\ \ (x^p)' = px^{p-1}$ *and* $x^p\big|_{x=0} = 0$
2. $E(X) = \int_0^\infty S_X(t)dt$. $\quad p = 1$. $\quad H(x) = x,\ \ (x)' = 1$ *and* $x^1\big|_{x=0} = 0$
3. $E(X^2) = \int_0^\infty 2tS_X(t)dt$. $\quad p = 2$. $\quad H(x) = x^2,\ \ (x^2)' = 2x^1$ *and* $x^2\big|_{x=0} = 0$
4. $E(X \wedge a) = \int_0^a S_X(t)dt$, where $a > 0$. $\quad H(x) = x \wedge a,\ \ (x \wedge a)' = ?$ *and* $(x' \wedge a') = ???$

$(x \wedge a)' = x' \wedge a'??$

$$(x \wedge a)' = \begin{cases} x' & \text{if } x \in (0,a) \\ a' & \text{if } x \in (a,\infty) \end{cases} = \begin{cases} 1 & \text{if } x \in (0,a) \\ 0 & \text{if } x \in (a,\infty), \end{cases} \text{ and } (x' \wedge a') = 0.$$

Proof of Th 2.8. $E[H(X)] = \int_0^\infty s(t)H'(t)\,dt$.

Recall $Y \sim bin(n,p)$ => E(Y)= ?? $\quad$ $Y \sim bin(1,p)$ => E(Y)= ??

$E[H(X)] = E\left[\int_0^X h(t)\,dt\right]$
$= E\left[\int_0^\infty I(X>t)h(t)\,dt\right]$ $\quad$ $I(X>t) \sim ???$
$= \int_0^\infty E[I(X>t)]h(t)\,dt$
$= \int_0^\infty P(X>t)h(t)\,dt$ $\quad$ **Why**?
$= \int_0^\infty s(t)h(t)\,dt.$ □

Example 2.9. *Let $S_X(x) = s(x) = e^{-x}(x+1)$, $x \geq 0$. (a) $E(X)$ = ? (b) $E(X \wedge 10)$ = ?*

Solution: (a) Compute $E(X)$ $(= E(H(X)))$. 3 approaches for $E(H(X))$ for a cts r.v. $H(X)$:

$$E(H(X)) = \begin{cases} \int H(x)f_X(x)dx & method\ (i) \\ \int_0^\infty s(x)h(x)dx & method\ (ii) \\ \underbrace{\int y f_{H(X)}(y)dy}_{=E(Y),\quad Y=??} & method\ (iii) \end{cases} \quad H(x)=?\ \ h=? \quad H(x) = x = \int_0^x 1dt \tag{1}$$

The 2nd needs $s(x)$ and $h(x)$, the others need $f_X(x)$ or $f_{H(X)}$. Which is most convenient ?

How many approaches for $E(X)$ based on Eq. (1) ? $\quad$ 2 or 3 ?
(i) $f(x) = f_X(x) = -s'(x) = -e^{-x}(-1)(x+1) - e^{-x}(1) = e^{-x}x$, **is it done ?**

$E[X] = \int x f_X(x)\,dx = \int_0^\infty x^2 e^{-x}\,dx = 2\int_0^\infty \frac{x^{3-1}e^{-x}}{\Gamma(3)1^3}dx = 2.$

(ii) $E[X] = \int_0^\infty s(t)\,dt = \int_0^\infty e^{-t}(t+1)\,dt = \int_0^\infty te^{-t}\,dt + \int_0^\infty e^{-t}\,dt = \int_0^\infty \overbrace{\frac{t^{2-1}e^{-t}}{\Gamma(2)1^2}}^{why\ do\ it\ ?}\,dt + 1 = 2.$
(b) Compute $E(X \wedge 10)$ $(= E(H(X)))$, $H(x) = ?$ $h = ?$ $\quad$ $x \wedge 10 = \int_0^x I(t \in (0,10))dt$

Three approaches for $E(H(X))$ for a discontinuous $H(X)$:

$$E(H(X)) = \begin{cases} \int H(x)f_X(x)dx & method\ (i) \\ \int_0^\infty s(x)h(x)dx & method\ (ii) \\ \underbrace{\int y f_{H(X)}(y)dy + \sum_{y\in D} y f_{H(X)}(y)}_{=E(Y),\quad Y=??} & method\ (iii) \end{cases} \tag{2}$$

How many approaches for $E(X \wedge 10)$ based on Eq. (2) ? $h(x) = ?$ $f_{H(X)} = ?$ $f_X(x) = ?$

$$h(x) = I(x \in (0,10)), \quad f_{H(X)}(x) = \begin{cases} f_X(x) & if\ x \in (0,10) \\ s(10) & if\ x = 10 \end{cases} \text{ with } D = \{10\}.$$

$$
\begin{aligned}
(ii)\quad E[\min(X,10)] &= \int_0^\infty s(t)h(t)\,dt = \int_0^{10} e^{-t}(t+1)\,dt && \text{can we use } G(\alpha,\beta) \text{ trick ?}\\
&= -\int_0^{10} (t+1)\,de^{-t} && \text{integration by parts } \textstyle\int_a^b udv = ..\\
&= -\left[(t+1)e^{-t}\Big|_0^{10} - \int_0^{10} e^{-t}d(t+1)\right]\\
&= -\left[11e^{-10} - 1 + e^{-t}\Big|_0^{10}\right] = 2 - 12e^{-10}.\\
(i)\quad E[\min(X,10)] &= \int_0^\infty (t\wedge 10)f(t)\,dt\\
&= \int_0^{10} tf(t)dt + \int_{10}^\infty 10f(t)dt\\
&= \int_0^{10} te^{-t}tdt + \int_{10}^\infty 10e^{-t}tdt = \cdots\\
(iii)\quad E[\min(X,10)] &= \int_0^{10} tf_{H(X)}(t)\,dt + \sum_{t\in D} tf_{H(X)}(t) \qquad D=?\\
&= \int_0^{10} te^{-t}tdt + 10e^{-10} = \cdots
\end{aligned}
$$

Announcement:
1. Quiz on Friday: 447 6-22, 44. and 1–2 in 450. (See page 1, 2, 3.)

Corollary 2.1. *Let X be a nonnegative r.v. with survival function s. Let $\delta > 0$. Then, $E[e^{-\delta X}] = 1 - \int_0^\infty \delta e^{-\delta t}s(t)\,dt$.*

Proof. Q: Can we try $E[e^{-\delta X}] = \int_0^\infty (e^{-\delta t})'s(t)\,dt$ **(by** $E(H(X)) = \int_0^\infty s(t)H'(t)dt$**) ??**
$H'(x) = (e^{-\delta x})' = -\delta e^{-\delta x} < 0$ **!!**
We shall show $E[1 - e^{-\delta X}] = \int_0^\infty \delta e^{-\delta t}s(t)\,dt$ **Why ??**
Let $H(t) = 1 - e^{-\delta t}$, then **(1)** $\underline{H(0) = 0}$ and **(2)** $h(t) = H'(t) = \delta e^{-\delta t} > 0$. By Th 2.8,
$E[1 - e^{-\delta X}] = E[H(X)] = \int_0^\infty h(t)s(t)\,dt = \int_0^\infty \delta e^{-\delta t}s(t)\,dt$.
$=> E[e^{-\delta X}] = 1 - \int_0^\infty \delta e^{-\delta t}s(t)\,dt$. ▫

The special case of Th. 2.8 for discrete X **(**$E(H(X)) = \int_0^\infty s(t)H'(t)dt$**):**

Theorem 2.9. *Let X be a discrete r.v. whose possible values are $\underline{non-negative}$ integers. Let $h : [0,\infty) \to [0,\infty)$ be a function. Let $H(x) = \int_0^x h(t)\,dt$, $x \geq 0$. Then,*

$$E[H(X)] = \textstyle\sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

Corollaries:

(2.1) $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}[k-(k-1)] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$ by Th 2.9, $s(t) = P(X > t)$

$= \sum_{k=1}^{\infty} s(k)$**??** **or** $= \sum_{k=1}^{\infty} s(k-)$**??** **or** $= \sum_{k=0}^{\infty} s(k)$**??**

If $X \sim bin(1,p)$, then $E(X)=$?

$E(X) = \int_0^{\infty} s(t)dt=$? $(s(t) = pI(t \in [0,1)),\ t \geq 0)$.

$E(X) = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} =$?

If $X \sim U(0,1)$, then $E(X) = \int_0^{\infty} s(t)dt$? $E(X) = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} = P(X \geq 1)$ **???**

$$(2.2) \qquad E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(k^2 - (k-1)^2) = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k-1),$$

$$(2.3) \qquad E[\min(X,n)] = \sum_{k=1}^{n} \mathbb{P}\{X \geq k\}[k \wedge n - (k-1) \wedge n] = \sum_{k=1}^{n} \mathbb{P}\{X \geq k\},\ n \geq 1.$$

Proof of Th 2.9. $E[H(X)] = \int_0^{\infty} s(t)h(t)\,dt$ (by Theorem 2.8 in page 20)

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} s(t)h(t)\,dt$$

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} \mathbb{P}\{X \geq k\}h(t)\,dt \qquad (s(t) = \mathbb{P}\{X > t\} = \mathbb{P}\{X \geq k\}, \text{ for } k-1 \leq t < k)$$

($s(k) = P(X \geq k)$ **??** $s(k-) = P(X \geq k)$ **??)**

$$= \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} \int_{k-1}^{k} h(t)\,dt \qquad H(x) = \int_0^x h(t)dt$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

□

Example 2.10. *Find* $E[X]$ *and* $E[X^2]$ *if*

k	*0*	*1*	*2*
$\mathbb{P}\{X = k\}$	*0.2*	*0.3*	*0.5*

Solution: (a) $E(X)$: $H(x) = x$; (b) $E(X^2)$: $H(x) = x^2$.
Which two are convenient approaches among 4 below ?

(i) using that $E[H(X)] = \sum_{k=0}^{2} H(k)\mathbb{P}\{X = k\}$.

(a) $E(X) = \sum_x x f_X(x)=$ ...

(b) $E(X^2) = \sum_x x^2 f_X(x) = \cdots$

(ii) using (2.1) $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$, (2.2) $E[X^2]=\sum_{k=1}^{\infty}\mathbb{P}\{X\ge k\}(2k-1)$.

(iii) $E(H(X))=\int_0^\infty H'(x)s(x)dx$.

(a) $E(X)=\int_0^\infty s(x)dx=$ ...

(b) $E(X^2)=\int_0^\infty 2xs(x)dx=$ ...

(iv) $E(H(X))=\sum_x xf_{H(X)}(x)$.

(a) $E(X)=\sum_x xf_X(x)=$ ...

(b) $E(X^2)=\sum_x xf_{X^2}(x)=\cdots$

Answer to the question above: The first 2.

(i) $E[X]=(0)(0.2)+(1)(0.3)+(2)(0.5)=1.3$
$E[X^2]=(0)^2(0.2)+(1)^2(0.3)+(2)^2(0.5)=2.3$.

(ii) We have that $\mathbb{P}\{X\ge 1\}=0.8$, $\mathbb{P}\{X\ge 2\}=0.5$, and $\mathbb{P}\{X\ge k\}=0$, for each $k\ge 3$.

$$\begin{aligned}
=> \quad E[X] &= \mathbb{P}\{X\ge 1\}+\mathbb{P}\{X\ge 2\} && \textit{by } (2.1)\\
&= 0.8+0.5=1.3\\
E[X^2] &= \mathbb{P}\{X\ge 1\}((2)(1)-1)+\mathbb{P}\{X\ge 2\}((2)(2)-1) && \textit{by } (2.2)\\
&= 0.8+0.5(3)=2.3.
\end{aligned}$$

Skip the next page.

Corollary 2.2. *Let X be a nonnegative r.v. and $a\ge 0$. Then, $E[\min(X,a)]=\int_0^a S_X(t)\,dt$.*

Proof. Let $H(t)=min\{t,a\}$ for each $t\ge 0$.
$h(t)=H'=I(t\in(0,a))$ if $t\in(0,\infty)\setminus\{a\}$. H' does not exist at $\{0,a\}$. Notice that $H(x)=\min(x,a)=\begin{cases}x & \text{if } x<a\\ a & \text{if } a\le x\end{cases}$ is cts in $[0,\infty)$ and ctsly differentiable in $(0,a)\cup(a,\infty)$, but not at a and 0.

Check the condition in Th 2.8 directly. For $x\ge 0$,

(1) $h(x)=I(x\in(0,a))\ge 0$ and

(2) $H(x)=\int_0^x h(t)\,dt=\int_0^x I(t\in[0,a])\,dt=\int_0^{\min(x,a)}1\,dt=\min(x,a)$.

By Theorem 2.8,

$E[\min(X,a)]=E(H(X))=\int_0^\infty h(t)s(t)dt=\int_0^\infty I(t\in(0,a))s(t)dt=\int_0^a s(t)\,dt.$ ■

Corollary 2.3.

Corollary 2.4. *Let X be a nonnegative r.v.. Then, $E[X^p]=\int_0^\infty S_X(t)pt^{p-1}\,dt$ if $p>0$.*

Proof. Let $H(t)=t^p$, for each $t\ge 0$. Hence, $h(t)=H'=pt^{p-1}\ge 0$, and $H(0)=0$. By Theorem 2.8, $E[X^p]=\int_0^\infty s(t)pt^{p-1}\,dt$, and $\begin{cases}E(X^2)=?\\ E(X)=?\end{cases}$ ■

Theorem 2.10.

Theorem 2.11.

Theorem 2.12.

Theorem 2.13.

Theorem 2.14.

Theorem 2.15.

Theorem 2.16.

Theorem 2.17.

Theorem 2.18.

Theorem 2.19.

Theorem 2.20.

Theorem 2.21.

Theorem 2.22.

Example 2.11.

Example 2.12.

Example 2.13.

2.1.4 Quantiles

Definition 2.13. *Given* $0 < p < 1$, *the* $100p$**–th percentile** *(or* p**–th quantile***) of a r.v.* X *is a value* ξ_p *such that*

$$\mathbb{P}\{X < \xi_p\} \le p \le \mathbb{P}\{X \le \xi_p\}.$$

Definition 2.14. *median* $= 0.50$*–th quantile.*

Definition 2.15. *The* **first quartile** Q_1 *of a r.v.* X *is the* 25*–th percentile of the r.v.* X.
The **third quartile** Q_3 *of a r.v.* X *is the* 75*–th percentile of the r.v.* X.
The **second quartile** $Q_2 =$ *median.*

3 ways to find ξ_p.

1. If X has a cts strictly increasing ($\uparrow\uparrow$) cdf, then solve $F(\xi_p) = p$.
2. Theorem 2.17.
3. Definition. $\xi_p = \xi_p^{**} = \inf\{x : F_X(x) \ge p\} \qquad = \min\{x : F_X(x) \ge p\}$?
   or $\xi_p = \xi_p^{*} = \sup\{x : F_X(x-) \le p\} \qquad = \max\{x : F_X(x) \ge p\}$?

$\inf\{x : x \in (0,1)\} = 0$, but $\min\{x : x \in (0,1)\}$ does not exist.

$\sup\{x : x \in (0,1)\} = 1$, but $\max\{x : x \in (0,1)\}$ does not exist.
$\inf\{x : x \in [0,1]\} = 0 = \min\{x : x \in [0,1]\}$
$\sup\{x : x \in [0,1]\} = 1 = \max\{x : x \in [0,1]\}$
What is the difference between them ?

Theorem 2.17. Let X be a cts r.v. with range (a,b). Let $0 < p < 1$.
Let $h : (a,b) \to (c,d)$ be a one–to–one and onto function and $Y = h(X)$.
Let ξ_p be a p–th quantile of X.

A p–th quantile of Y is $\zeta_p = h(\xi_p)$ if $h \uparrow$.

A p–th quantile of Y is $\zeta_p = h(\xi_{1-p})$ if $h \downarrow$.

Theorem 2.18. The p–th quantile ξ_p of a normal r.v. with mean μ and variance σ^2 is $\mu + \Phi^{-1}(p)\sigma$. $(\mu + z_p\sigma, \quad \overline{X} \pm 1.64\hat{\sigma}, \quad \mu + 1.64\sigma, \quad \xi_{0.95})$.

Proof. (i) Let $Z \sim N(0,1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$.
The cdf Φ of $N(0,1)$ satisfies $\Phi(t) \uparrow\uparrow$ in t. So
$\Phi^{-1}(p)$ is p-th quantile of $N(0,1)$ (method 1).
$\xi_p = h(\Phi^{-1}(p))$, where $h(z) = \mu + \sigma z$ (method 3) **(where does h() come from ?)** ■

Theorem 2.23.

Theorem 2.24.

Example 2.14. *Let $Z \sim U(0,1)$, $X = 2Z + 1$ and $Y = X^2 + X$. Find the 70th percentile of Z, X and Y.*

Solution: Use Th2.17: h is 1-1 and onto function and
$X = h(Z)$. Let ξ_p be a p–th quantile of Z.
A p–th quantile of X is $\zeta_p = h(\xi_p)$ if $h \uparrow$.
A p–th quantile of X is $\zeta_p = h(\xi_{1-p})$ if $h \downarrow$.

$F_Z(t) = t \ \forall\, t \in (0,1)$. The 70th percentile of Z is 0.7.
Since $g(z) = 2z + 1 \uparrow\uparrow$ in z, the 70th percentile of X is $g(0.7) = 2 * 0.7 + 1 = 2.4$.
Since $h(x) = x^2 + x \uparrow\uparrow$, the 70th percentile ζ of Y is $h(2.4) = (2.4)^2 + 2.4 = 8.16$.

Example 2.15.

Often, we will assume that the individuals do not live more than a certain age. This age ω is called the **terminal age** or **limiting age** of the population. So, $S(t) = 0$, for each $t \geq \omega$.

Example 2.16. *Suppose that the age–at–failure r.v. X has density*
$f_X(x) = 5x^4k^{-5}I(0 < x < k)$ and the expected age–at–failure is 70 years.
Find the 4 intervals determined by the 3 quartiles and the terminal age.

Solution: Let ξ_p be p–th quantile of the age–at–failure.

$\xi_p = ?$ for $p = 0.25$, 0.5, and 0.75.

$$p = F(\xi_p) = \mathbb{P}\{X \leq \xi_p\} = \int_0^{\xi_p} \frac{5x^4}{k^5}\,dx = \frac{x^5}{k^5}\bigg|_0^{\xi_p} = \frac{\xi_p^5}{k^5}$$

$\xi_p = kp^{1/5} = ?$ k= ? **Need to solve k using $70 = E[X]$.**

$$70 = E[X] = \int_0^k x\frac{5x^4}{k^5}\,dx = \frac{5x^6}{6k^5}\Big|_0^k = \frac{5k}{6} \quad => \quad k = \frac{(70)(6)}{5} = 84. => \quad \xi_p = 84p^{\frac{1}{5}}$$

$$\xi_p = 84p^{\frac{1}{5}} = \begin{cases} 84p^{\frac{1}{5}}|_{p=1/4} & 1st\ quartile \\ 84p^{\frac{1}{5}}|_{p=2/4} & 2nd\ quartile \\ 84p^{\frac{1}{5}}|_{p=3/4} & 3nd\ quartile \end{cases} = \begin{cases} 63.66 & 1st\ quartile \\ 73.13 & 2nd\ quartile \\ 79.30 & 3nd\ quartile \end{cases}$$

The 4 intervals determined by the 3 quartiles and the terminal age are
[0, 63.66], (63.66, 73.13], (73.13, 79.30], (79.30, 84].

Example 2.17.

Definition 2.16.

Definition 2.17.

Definition 2.18.

Definition 2.19.

Definition 2.20.

Example 2.18. *Let X be a r.v such $\mathbb{P}\{X = 1\} = \frac{1}{2}$ and $\mathbb{P}\{X = 2\} = \frac{1}{2}$. Find the first quartile Q_1 and median of X, say m.*

Solution. 3 ways to find ξ_p $(F(\xi_p-) \le p \le F(\xi_p))$.

1. If X has a cts strictly increasing cdf, then solve $F(\xi_p) = p$.
2. Definition. $\xi_p = \xi_p^* = \sup\{x : F_X(x-) \le p\}$,
   or $\xi_p = \xi_p^{**} = \inf\{x : F_X(x) \ge p\}$.
3. Theorem 2.17. Relation of the quantiles of $g(X)$ and X.

Does Method 1 work here ?
Does Method 2 work here ?
Does Method 3 work here ?

	$F(x-)$ $(= \mathbb{P}\{X < x\})$	$F(x)$ $(= \mathbb{P}\{X \le x\})$
$x \in (-\infty, 1)$	0	0
$x = 1$	0	$\frac{1}{2}$
$x \in (1, 2)$	$\frac{1}{2}$	$\frac{1}{2}$
$x = 2$	$\frac{1}{2}$	1
$x \in (2, \infty)$	1	1
	$\xi_p^* = \sup\{x : F(x-) \le p\}$	$\xi_p^{**} = \inf\{x : F(x) \ge p\}$
$Q_1 =$	$\xi_{1/4}^* = 1$	$\xi_{1/4}^{**} =$ **??**
$m =$	$\xi_{1/2}^* =?$	$\xi_{1/2}^{**} =$ **??**

$Q_1 = 1$.
m**: The values of x that satisfy $\mathbb{P}\{X < m\} \le \frac{1}{2} \le \mathbb{P}\{X \le m\}$??**

Ans: $m \in [1,2]$.
Thus m is a median of X if and only if $m \in [1,2]$.

Remark. A quantile may not be unique.

2.2 Actuarial notation for survival analysis.

Def. In Actuary, denote (x) a life that survives to age x
(x) is called a life-age-x or a life aged x.
Let X be the lifetime of the person.
$T(x)$ or T_x $(= (X-x)|\{X>x\})$ – the future lifetime of (x).
Notice that $T(x)$ is a conditional random variable.
${}_tp_x = S_{T_x}(t)$ – probability that (x) survives t years.
${}_tq_x = F_{T_x}(t)$ – probability that (x) dies within t years.
$p_x = {}_1p_x = P(T(x) > 1)$.
${}_{s|t}q_x = P(s < T(x) \le s+t)$.
${}_{t|}q_x = {}_{t|1}q_x$.
$q_x = {}_{0|1}q_x = {}_1q_x = P(T(x) \le 1)$.

Q: ${}_tp_x + {}_tq_x =$ **??**
$p_x + q_x =$ **??**

Remark. ${}_tp_x = P(T(x) > t) = P(X - x > t|X > x) = \frac{P(X>x+t)}{P(X>x)} = S_X(x+t)/S_X(x)$.
$p_x = P(T(x) > 1) = S_X(x+1)/S_X(x)$.

Q: How about the other notations ?

$$
\begin{aligned}
{}_{s|t}q_x &= P(s < T(x) \le s+t) = P(s < X - x \le s+t|X > x) \\
&= \frac{P(x+s < X \le x+s+t)}{P(X>x)} = \frac{S_X(x+s) - S_X(x+s+t)}{S_X(x)} = {}_sp_x - {}_{s+t}\,p_x.
\end{aligned}
$$

$q_x = P(T(x) \le 1) = P(X \le x+1|X > x) = \frac{S_X(x) - S_X(x+1)}{S_X(x)}$.

Theorem 2.25. *For each* $t, s \ge 0$, ${}_{t+s}p_x = {}_tp_x \cdot {}_sp_{x+t}$.

Proof.
$$ {}_{t+s}p_x = \frac{S_X(x+t+s)}{S_X(x)} = \frac{S_X(x+t)}{S_X(x)} \frac{S_X(x+t+s)}{S_X(x+t)} = {}_tp_x \cdot {}_sp_{x+t} $$

Theorem 2.26. ${}_np_x = p_x p_{x+1} \cdots p_{x+n-1} = \prod_{j=0}^{n-1} p_{x+j}$.
For each $t_1, \ldots, t_m \ge 0$, ${}_{t_1+\cdots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot {}_{t_3}p_{x+t_1+t_2} \cdots {}_{t_m}p_{x+t_1+\cdots+t_{m-1}}$.

Q: Relation between Theorems 2.25 and 2.26 ??

Def. $X \simeq Y$ means X and Y have the same distribution.

Question. Let $X \sim N(0,1)$, then $X = -X$? or $X \simeq -X$??

Theorem 2.27. $(T(x) - t)|\{T(x) > t\} \simeq T(x+t)$. *(Denoted by* $U \simeq W$ $(\equiv T(x+t))$*).*

Proof. It suffices to show either their F's, or S's or f's are the same.

Use S's here. Recall that $\forall\, y > 0$, $P(T(x) > y) = P(X - x > y | X > y) = \frac{S_X(x+y)}{S_X(x)}$;

$$S_U(y) = P((T(x) - t) > y | T(x) > t) = \frac{P((T(x) - t) > y)}{P(T(x) > t)} = \frac{P(T(x) > y + t)}{P(T(x) > t)} = \frac{\frac{S_X(x+t+y)}{S_X(x)}}{\frac{S_X(x+t)}{S_X(x)}}$$

$= \frac{S_X(x+t+y)}{S_X(x+t)} = P(T(x+t) > y) = S_W(y)$.

Example 2.19. *Let* $s(t) = \frac{85-t}{85}$, $0 \le t \le 85$, *where* $s = S_X$. *What distribution is* X *?*
(i) Calculate ${}_tp_{40}$.
(ii) Calculate the density function of $T(40)$.

Solution: (i) ${}_tp_{40} = P(X > t + 40 | X > 40) = \frac{s(40+t)}{s(40)} = \frac{\frac{85-(40+t)}{85}}{\frac{85-40}{85}} = \frac{\frac{45-t}{85}}{\frac{85-40}{85}} = \frac{45-t}{45}$, ????

$t \in [0, 45]$.
${}_tp_{40} = 0$ **if** $t > 45$ **?** ${}_tp_{40} = 0$ **if** $t < 0$ **?**
(ii) The density function of $T(40)$ is

$$f_{T(40)}(t) = -\frac{d}{dt}{}_tp_{40} = -\frac{d}{dt}\left(\frac{45-t}{45}\right) = \frac{1}{45},\ t \in (0, 45).$$

$f_{T(40)}(t) = 0$ **if** $t \ge 45$ **?** $f_{T(40)}(t) = 0$ **if** $t \le 0$ **?**
Notice the difference between the domains of $S_X(t)$ and $f_X(t)$.

Example 2.20. *If* ${}_tp_x = 1 - \frac{t}{90-x}$, $0 \le t \le 90 - x$, *find the probability that a 25–year–old reaches age 80 and the density of* $T(x)$.

Solution: The probability that a 25–year-old reaches age 80 is ${}_{80}p_{25}$ or ${}_{80-25}p_{25}$?
${}_{80-25}p_{25} = {}_{55}p_{25} = 1 - \frac{55}{90-25} = \frac{2}{13}$.
The density of $T(x)$ is $-(1 - \frac{t}{90-x})'_x$ **or** $-(1 - \frac{t}{90-x})'_t$ **??**

$$f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x = \frac{1}{90-x}, 0 \le t \le 90 - x\textbf{??}\ \textbf{or}\ 0 < t < 90 - x?$$

Example 2.21.

Example 2.22. *Suppose that probability that a 30–year–old reaches age 40 is* 0.95, *the probability that a 40–year–old reaches age 50 is* 0.99, *and the probability that a 50–year–old reaches age 60 is* 0.95. *Find the probability that a 30–year–old reaches age 60.*

Solution: ${}_{60}p_{30} = ?$ or ${}_{60-30}p_{30} = ?$
Given conditions:

${}_{40-30}p_{30} = {}_{10}p_{30} = 0.95$ – probability that a 30–year–old reaches age 40,

${}_{50-40}p_{40} = {}_{10}p_{40} = 0.99$ – the probability that a 40–year–old reaches age 50,

${}_{60-50}p_{50} = {}_{10}p_{50} = 0.95$ – the probability that a 50–year–old reaches age 60.
Formula: ${}_{t_1+\cdots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot {}_{t_3}p_{x+t_1+t_2} \cdots {}_{t_m}p_{x+t_1+\cdots+t_{m-1}}$. (Formula 4).

The probability that a 30–year–old reaches age 60 is

$${}_{30}p_{30} = {}_{10}p_{30} \cdot {}_{10}p_{40} \cdot {}_{10}p_{50} = (0.95)(0.99)(0.95) = 0.893475.$$

Definition 2.21.

Example 2.23.

Example 2.24.

Example 2.25.

Theorem 2.28.

Quiz on Friday: 450: 1-5; 447: 10-22, 44

Example 2.26. *Suppose that the survival function of a person is given by* $S_X(x) = \frac{90-x}{90}$*, for* $0 \le x \le 90$*.*
Given a married couple with husband aged 40 and wife aged 35,
what is the probability that
the husband will die before age 60 and the wife will survive to age 75?
Here, we assume that their times of death are independent r.v.'s.

Solution: $\underbrace{{}_{60-40}q_{40} \times {}_{75-35}p_{35}}_{\textbf{Why??}} = ?$ Formula #3: ${}_tp_x = \frac{s(x+t)}{s(x)}$ and ${}_tq_x = \frac{s(x)-s(x+t)}{s(x)}$

$${}_{60-40}q_{40} = {}_{20}q_{40} = \frac{s(40)-s(60)}{s(40)} = \frac{\frac{90-40}{90} - \frac{90-60}{90}}{\frac{90-40}{90}} = \frac{20}{50} = \frac{2}{5}.$$

$${}_{75-35}p_{35} = \frac{s(75)}{s(35)} = \frac{\frac{90-75}{90}}{\frac{90-35}{90}} = \frac{15}{55} = \frac{3}{11}.$$

$$Answer: \ {}_{60-40}q_{40} \times {}_{75-35}\,p_{35} = \frac{2}{5} \times \frac{3}{11} = \frac{6}{55}.$$

Example 2.27.

Example 2.28. *Suppose that* $s(t) = \frac{85-t}{85}$*,* $0 \le t \le 85$*, find the probability that a 40–year–old will die in less than one year*

Solution: $q_{40} = ?$
Formula: ${}_tq_x = 1 - {}_tp_x$ and ${}_tp_x = \frac{s(x+t)}{s(x)}$ or #3 ${}_tq_x = \frac{s(x)-s(x+t)}{s(x)}$

$$q_{40} = 1 - \frac{s(41)}{s(40)} = 1 - \frac{\frac{85-41}{85}}{\frac{85-40}{85}} = \frac{1}{45}.$$

Example 2.29. *Suppose that:*
(i) The probability that a 30–year–old will die in less than one year is 0.012
(ii) The probability that a 31–year–old will die in less than one year is 0.013

(iii) The probability that a 32–year–old will die in less than one year is 0.014.
Find the probability that a 30–year–old will die in less than three years.

Solution: ${}_3q_{30} = ?$
Given conditions: (i) q_{30}, (ii) q_{31}, (iii) q_{32}. thus know: ${}_tp_x = 1 - {}_tq_x$, $t = ?$
Formula #4: ${}_{t_1+\cdots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot {}_{t_3}p_{x+t_1+t_2} \cdots {}_{t_m}p_{x+t_1+\cdots+t_{m-1}}$.

$$ {}_3p_{30} = p_{30}p_{31}p_{32} = (1-0.012)(1-0.013)(1-0.014) \approx 0.9615. $$

$$ {}_3q_{30} = 1 - {}_3p_{30} \approx 1 - 0.9615 = 0.038. $$

Theorem 2.29. ${}_{s|t}q_x = {}_sp_x - {}_{s+t}p_x = {}_{s+t}q_x - {}_sq_x = {}_sp_x \cdot {}_tq_{x+s}$. *Formula #3.*

Proof. We have that

$$
\begin{aligned}
{}_{s|t}q_x =& \mathbb{P}\{s < T(x) \le s+t\} && P(a < Y \le b) \\
=& S_{T(x)}(s) - S_{T(x)}(s+t) = {}_sp_x - {}_{s+t}p_x, && S_Y(a) - S_Y(b) \\
=& F_{T(x)}(s+t) - F_{T(x)}(s) = {}_{s+t}q_x - {}_sq_x, && F_Y(b) - F_Y(a) \\
{}_sp_x \cdot {}_tq_{x+s} =& {}_sp_x \cdot (1 - {}_tp_{x+s}) = {}_sp_x - {}_sp_x \cdot {}_tp_{x+s} \\
=& {}_sp_x - {}_{s+t}p_x = {}_{s|t}q_x. && \textit{last one}
\end{aligned}
$$

Example 2.30. *Let* $S_X(x) = (\frac{90-x}{90})^2$, $x \in (0,90)$.
(i) Find ${}_{s|t}q_x$, *where* $0 < x, s, t$ *and* $x+s+t \le 90$.
(ii) Find the probability that a 30–year–old dies between ages 55 and 60.

Solution: (i) ${}_{s|t}q_x = ?$ (ii) ${}_{55-30|5}q_{30} = ?$

$$
\begin{aligned}
{}_{s|t}q_x =& P(x+s < X \le x+s+t)|X > x) \quad P(a < X \le b) = F(b) - F(a) = S(a) - S(b) \text{ which ?} \\
=& \frac{(\frac{90-(x+s)}{90})^2}{(\frac{90-x}{90})^2} - \frac{(\frac{90-(x+s+t)}{90})^2}{(\frac{90-x}{90})^2} \qquad (\textbf{ Recall } a^2 - b^2 = (a+b)(a-b)) \\
=& \frac{(180-2x-2s-t)t}{(90-x)^2} \textbf{...... done ?}
\end{aligned}
$$

x, s, $t \ge 0$ and $s+t+x \le 90$.
(ii) ${}_{55-30|60-55}q_{30} = {}_{25|5}q_{30} = \frac{(180-2(30)-2(25)-5)5}{(90-30)^2} = \frac{65\times 5}{60^2} = \frac{13}{12^2} \approx 0.09.$

Theorem 2.30. *For* $x \ge 0$, *and each positive integer* n,

$$ {}_nq_x = \sum_{j=1}^{n} {}_{j-1|}q_x = \sum_{j=1}^{n} {}_{j-1}p_x q_{x+j-1}. $$

Proof. ${}_nq_x = \mathbb{P}\{T(x) \le n\} = \sum_{j=1}^n \mathbb{P}\{j-1 < T(x) \le j\} = \sum_{j=1}^n {}_{j-1|}q_x$
$= \sum_{j=1}^n {}_{j-1}p_x q_{x+j-1}$. ∎

Theorem 2.31. $\sum_{j=1}^{\infty} {}_{j-1|}q_x = \sum_{j=0}^{\infty} {}_{j-1}p_x q_{x+j-1} = 1.$

Theorem 2.32. ${}_{t+s|u}q_x = {}_tp_x \cdot {}_{s|u}q_{x+t}$.

Theorem 2.33. ${}_{n+m|}q_x = {}_np_x \cdot {}_{m|}q_{x+n}$.

When $n = 1$ and $m = j - 1 \geq 0$, we get that

$$ {}_{j|}q_x = p_x \cdot {}_{j-1|}q_{x+1}. $$

Example 2.31.

Definition 2.22.

Definition 2.23.

Example 2.32.

Example 2.33.

Example 2.34.

Definition 2.24.

Example 2.35.

Example 2.36.

2.3 Force of mortality

Definition. The **hazard function** of the survival function $S_X(x)$ or the **force of mortality** (denoted by $\mu_X(x)$, $\mu(x)$, μ_x and $\lambda_X(x)$), is defined as

$$ \lambda_X(x) = \frac{f_X(x)}{S_X(x-)}, $$

where f_X is the pdf or pmf of the r.v. X.

Denote $\mu_x(t) = \mu_{T_x}(t)$ ($\neq \mu_X(t)$ as $T_x = X - x|(X > x)$).

$\mu_x = \mu_0(x) = \mu_x(0)$ ($\neq \mu_x(\cdot)$) $= \mu_{T_x}(\cdot)$).

Theorem 2.28. If $X \geq 0$ is cts and has the force of mortality $\mu(\cdot)$, then,

(i) $-\frac{d}{dt}\ln S_X(t) = \frac{f_X(t)}{S_X(t)} = \mu(t)$ and $S_X(t) = \exp\left(-\int_0^t \mu(s)\,ds\right), t \geq 0$.

(ii) $f_X(t) = S_X(t)\mu(t) = \exp\left(-\int_0^t \mu(s)\,ds\right)\mu(t), t \geq 0$.

(iii) $S_{T(x)}(t) = {}_tp_x = \exp\left(-\int_0^t \mu_x(s)\,ds\right), t \geq 0$.

(iv) $\mu_x(t) = \mu(x + t)$

(v) $f_{T(x)}(t) = {}_tp_x\mu_x(t) = \exp(-\int_0^t \mu(x + s)ds)\mu(x + t), t \geq 0$.

Proof. (i), (ii), (iii) and (v) are obvious from the definition.

(iv) The survival function of $T(x)$ is $S_{T(x)}(t) = \frac{S_X(x+t)}{S_X(x)}$, $t \geq 0$. ,

$$\mu_x(t) = -\frac{d}{dt}\ln S_{T(x)}(t) = -\frac{d}{dt}\ln\left(\frac{S_X(x+t)}{S_X(x)}\right) = \frac{f_X(x+t)}{S_X(x+t)} = \mu(x+t). \square$$

Remark. The distribution can be specified by either of $F(\cdot)$, $S(\cdot)$, $f(\cdot)$ and $\mu(\cdot)$ if f exists. *e.g.*, if X is continuous, then $F(x)$ yields $S(x)$ yields $f(x)$ yields $\mu(x)$ yields $F(x)$.
$F => S = 1 - F$

$=> f(x) = -S'(x)$

$=> \mu(x) = \frac{f(x)}{\underbrace{\int_x^\infty f(t)dt}_{=S(x)}}$

$=> F(x) = \int_{-\infty}^x \underbrace{\mu(t) \underbrace{\exp(-\int_0^t \mu(y)dy)}_{=S(t)}}_{=f(t)} dt$ (see (i) of the next theorem),

as $\mu(x) = \frac{f(x)}{S(x)}$, $f(x) = \mu(x)S(x)$ and $S(x) = f(x)/\mu(x)$.

Hereafter, when we consider the force of mortality, we assume that X is cts.

The force of mortality of a life at time x, $x > 0$, satisfies that $\mu(x) = \lim_{t\to 0} \frac{{}_tq_x}{t}$, as

$$\lim_{t\to 0} \frac{{}_tq_x}{t} = \lim_{t\to 0} \frac{\frac{s(x)-s(x+t)}{s(x)}}{t} = -\lim_{t\to 0} \frac{s(x+t)-s(x)}{ts(x)} = -\frac{s'(x)}{s(x)} = \frac{f(x)}{s(x)} = \mu(x),\ x > 0. \tag{2.4}$$

If t is small, the proportion of people aged x who will die within t years is $\frac{s(x)-s(x+t)}{s(x)} \approx t\mu_x$. For example, if $\mu(x) = 0.06$ and t is $1/12$ (a month), we expect that from each 1,000 individuals with age x, $t\mu(x)10^3 = \frac{60}{12} = 5$ individuals will die within a month.

The force of mortality is the rate of death for lives aged x. For a life aged x, the force of mortality t years later is the force of mortality for a $(x+t)$–year old.

Theorem 2.34. *Let $\mu : [0,\infty] \to \mathbb{R}$ be a function which is cts everywhere except at finitely many points. Then, μ is the force of mortality of an age–at–death r.v.*
iff (1) $\mu(x) \geq 0\ \forall\ x$ and (2) $\int_0^\infty \mu(t)\,dt = \infty$.

Example 2.37. *Suppose that the survival function of a new born is*
$S_X(t) = \frac{85^4-t^4}{85^4}$, *for* $0 < t < 85$.
(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged 20.

Solution: (i) $\mu(t) = -\frac{d}{dt}\ln S_X(t) = -\frac{d}{dt}\ln\left(\frac{85^4-t^4}{85^4}\right) = -\frac{d}{dt}[\ln(85^4-t^4) - \ln(85^4)] = \frac{4t^3}{85^4-t^4}$, $0 < t < 85$

(ii) $\mu_{20}(t) = \mu(20+t) = \frac{4(20+t)^3}{85^4-(20+t)^4}$, $\quad 0 < t <$ **??**

Definition 2.25.

Example 2.38.

Example 2.39. *If* $\mu(x) = \frac{1}{x+1}$ *for* $x \geq 0$, *find* S_X, f_X, $\mu_{T(x)}$, ${}_tp_x$ *and* $f_{T(x)}$.

Solution: Note: S_X, f_X, $\mu_{T(x)}$, ${}_tp_x$ and $f_{T(x)}$ are functions, *e.g.*, we can write $S_X(x)$ or $S_X(t)$. **Do we write** $\mu_{T(x)}(x)$, $\mu_{T(x)}(t)$, $\mu_x(x)$ or $\mu_x(t)$?

$$s(x) = \exp\left(-\int_0^x \mu(t)\,dt\right) = \exp(-\int_0^x \frac{1}{t+1}\,dt) = \exp(-\ln(1+x)) = \frac{1}{x+1}, \quad x \geq 0,$$

as $\int \frac{1}{u}du = \ln u + c$, u=??? $\qquad S_X(x)$ =??

$$f_X(x) = \mu(x)s(x-) = \frac{1}{(x+1)^2}, \quad x \geq 0, \qquad (\mu(x) = f(x)/s(x-))$$

$${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\frac{1}{x+t+1}}{\frac{1}{x+1}} = \frac{x+1}{x+t+1}, \quad x, t \geq 0,$$

$$f_{T(x)}(t) = {}_tp_x\mu_x(t) = {}_tp_x\mu(x+t) = \frac{x+1}{x+t+1}\frac{1}{x+t+1} = \frac{x+1}{(x+t+1)^2}, \ t \geq 0,$$

2.4 Expectation of life

Definition 2.26. $\overset{\circ}{e}_x = E[T(x)]$ *is called the* **expected future lifetime at age** x *or the* **complete expectation of a life at age** x.
$\overset{\circ}{e}_0$ *is also called the* **complete expectation of life at birth**.

Definition 2.27. *The* n*–year* **temporary** *complete life expectancy is* $\overset{\circ}{e}_{x:\overline{n}|} = E(T(x) \wedge n)$, *the expected number of years lived between ages* x *and* $x+n$ *by a survivor aged* x. $0 \leq \overset{\circ}{e}_{x:\overline{n}|} \leq n$.

Example 2.40. *An actuary models the lifetime in years of a random selected person as a r.v.* X *with* $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find:*
(i) $\overset{\circ}{e}_0$ *and* $\mathrm{Var}(X)$; *(ii)* $\overset{\circ}{e}_{30}$; *(iii)* $\overset{\circ}{e}_{30:\overline{10}|}$. $\qquad E(X)$, $V(X)$, $E(X|X > x)$, $E(T(X) \wedge 10)$

Solution: Formulae: $\overset{\circ}{e}_0 = E(X)$ and $\mathrm{Var}(X) = E[X^2] - (E[X])^2$.
Two ways for taking expectation.

(1) $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt$, where ${}_tp_x = \frac{s(x+t)}{s(x)}$, $f_{T(x)}(t) = {}_tp_x\mu_x(t)$

$$(2)\ \overset{\circ}{e}_x = \begin{cases} \int_0^\infty t f_{T(x)}(t)dt & \text{if } X \text{ is cts} \quad (\ f_{T(x)}(t) = {}_tp_x\mu_x(t)), \quad \text{f=s'} \\ \sum_t t f_{T(x)}(t) & \text{if } X \text{ is discrete,} \\ \int_0^\infty t f_{T(x)}(t)dt + \sum_{t \in D} t f_{T(x)}(t) & \text{if } X \text{ is mixed.} \end{cases}$$

Which way is more convenient here, (1) or (2) ?

(i) $\overset{\circ}{e}_0$ and σ_X^2. Method 1: $\overset{\circ}{e}_0 = \int_0^\infty S_X(x)dx = \int_0^{90} \frac{90^6 - x^6}{90^6}dx = [x - \frac{x^7}{(7)90^6}]\Big|_0^{90} = 77.142857,$

$$E[X^2] = \int_0^\infty 2xS_X(x)dx = \int_0^{90} 2x\frac{90^6 - x^6}{90^6}dx = [x^2 - \frac{x^8}{(4)90^6}]\Big|_0^{90} = 6075,$$

Method 2 needs $f_{T(0)}(x) = f_X(x) = -\frac{d}{dx}S_X(x) = -\frac{d}{dx}\frac{90^6 - x^6}{90^6} = \frac{6x^5}{90^6}, 0 < x < 90.$

$$\overset{\circ}{e}_0 = \int_0^\infty xf_X(x)dx = \int_0^{90} x\frac{6x^5}{90^6}dx = \frac{6x^7}{(7)90^6}\Big|_0^{90} \approx 77.142857,$$

$$E[X^2] = \int_0^\infty x^2 f_X(x)dx = \int_0^{90} x^2\frac{6x^5}{90^6}dx = \frac{6x^8}{(8)90^6}\Big|_0^{90} = 6075,$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = 6075 - (77.143)^2 \approx 1123.980.$$

(ii) Method 1 for $\overset{\circ}{e}_{30}$: $S_{T(30)}(t) = {}_tp_{30} = \frac{S_X(30+t)}{S_X(30)} = \frac{90^6-(30+t)^6}{(90)^6-(30)^6}$ if $0 \le t <$?? (Check S_X)

$$\overset{\circ}{e}_{30} = \int_0^\infty {}_tp_{30}\,dt = \int_0^{60} \frac{90^6 - (30+t)^6}{90^6 - (30)^6}\,dt = \frac{90^6 t - (30+t)^7/7}{90^6 - (30)^6}\Big|_0^{60} = 47.21.$$

(ii) Method 2 for $\overset{\circ}{e}_{30}$: $f_{T(30)}(t) = -\frac{d}{dt}{}_tp_{30} = \frac{f_X(30+t)}{S_X(30)} = \frac{\frac{6(30+t)^5}{90^6}}{\frac{(90)^6-(30)^6}{90^6}} = \frac{6(30+t)^5}{(90)^6-(30)^6}, \quad 0 < t <$??.

$$\overset{\circ}{e}_{30} = \int_0^\infty tf_{T(30)}(t)\,dt = \int_0^{60} t\frac{6(30+t)^5}{(90)^6 - (30)^6}\,dt \qquad (S_X(x) = \tfrac{90^6-x^6}{90^6}, \text{ for } 0 < x < 90)$$

$$= \int_{30}^{90} (s-30)\frac{6s^5}{(90)^6 - (30)^6}\,ds \quad \text{(change of variables } s = 30+t\text{),} \quad \textbf{Why change ??}$$

$$= \frac{6s^7}{(7)(90^6 - (30)^6)} - \frac{(30)s^6}{(90)^6 - (30)^6}\Big|_{30}^{90} = 47.21.$$

(iii) The two methods for $\overset{\circ}{e}_{x:\overline{n}|}$ are stated as follows.

Theorem 2.35. *(1)* $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,dt$; *(2)* $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n tf_{T(x)}(t)\,dt + n \cdot {}_np_x$.

Method 1. ${}_tp_{30} = \frac{s(30+t)}{s(30)} = \frac{90^6-(30+t)^6}{90^6-30^6}$, for $0 \le t < 60$. So,

$$\overset{\circ}{e}_{30:\overline{10}|} = \int_0^{10} {}_tp_{30}\;dt = \int_0^{10} \frac{90^6 - (30+t)^6}{90^6 - 30^6}\,dt$$

$$= \frac{(90^6)t}{90^6 - 30^6} - \frac{(30+t)^7}{(7)(90^6 - 30^6)}\Big|_0^{10} = 9.975520756.$$

Method 2. $Y = T(x) \wedge n$ has a mixed distribution with df. $f_Y(t) = \begin{cases} f_{T(x)}(t) & \text{if } t \in (0,n) \\ {}_np_x & \text{if } t \in D = \{n\}. \end{cases}$

$\mathring{e}_{30:\overline{10}|} = \int_0^{10} t \frac{6(30+t)^5}{(90)^6-(30)^6}\, dt + 10\,{}_{10}p_{30} = \cdots.$

Theorem 2.36. *For* $0 < m < n$, $\mathring{e}_{x:\overline{n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x \mathring{e}_{x+m:\overline{n-m}|}$

Letting $n \to \infty$=> $\mathring{e}_x = \mathring{e}_{x:\overline{m}|} + {}_mp_x \mathring{e}_{x+m}$.

Example 2.41. *You are given that:*
The expected # of years lived between ages 40 *&* 50 *by a 40-year old is 9.7.* $\mathring{e}_{40:\overline{10}|}$
The probability that a 40–year old survives to age 50 is 0.98. ${}_{10}p_{40}$
The expected # of years lived between ages 50 *&* 70 *by a 50-year old is 19.5.* $\mathring{e}_{50:\overline{20}|}$
Find the expected # of years lived between ages 40 *and* 70 *by a 40-year old.* $\mathring{e}_{40:\overline{30}|}$

Solution: Given (i) $\mathring{e}_{40:\overline{10}|}$ =? (ii) ${}_{10}p_{40}$ =? (iii) $\mathring{e}_{50:\overline{20}|}$ =? Find $\mathring{e}_{40:\overline{30}|}$ = ? #6

$$\mathring{e}_{40:\overline{30}|} = \mathring{e}_{40:\overline{10}|} + {}_{10}p_{40}\mathring{e}_{50:\overline{20}|} = 9.7 + (0.98)(19.5) = 28.81.$$

Example 2.42. *Assume that*
(i) The expected future lifetime of a 40–year old is 45 years.
(ii) The expected future lifetime of a 50–year old is 36 years.
(iii) The probability that a 40–year old survives to age 50 is 0.98.
The expected number of years lived between ages 40 *and* 50 *by a 40-year old ?*

Solution: Given: $\mathring{e}_{40} = 45$; $\mathring{e}_{50} = 36$; ${}_{50-40}p_{40} = 0.98$. $\mathring{e}_{40:\overline{10}|}$=?
Formula 6: $\underbrace{\mathring{e}_x = \mathring{e}_{x:\overline{m}|} + {}_mp_x \mathring{e}_{x+m},}_{45=\mathring{e}_{40:\overline{10}|}+(0.98)(36)}$ => $\mathring{e}_{40:\overline{10}|} = 45 - (0.98)(36) = 9.72.$

Definition 2.28. ${}_nm_x = \frac{\int_x^{x+n} S_X(t)\mu_X(t)\,dt}{\int_x^{x+n} S_X(u)\,du}$ *(*$= \int_x^{x+n} \frac{S_X(t)}{\int_x^{x+n} S_X(u)\,du}\mu_X(t)\,dt$*) is called the* **central death rate** *or the* **central rate of failure** *over the age interval x and $x+n$.*

${}_nm_x$ is the weighted average of the force mortality on the interval $[x, x+n]$ using the survival function as a weight *i.e.*, $\frac{S_X(t)}{\int_x^{x+n} S_X(t)dt}$. Denote $m_x = {}_1m_x$. Recall $\mu(x) = \frac{f(x)}{s(x-)}$ and
$\mu(x)t$ is the proportion of people die within t year for people of age x $(t \approx 0)$.

Example 2.43.

Example 2.44.

Theorem 2.37. ${}_nm_x = \frac{S_X(x)-S_X(x+n)}{\int_x^{x+n} S_X(t)\,dt} = \frac{\int_0^n {}_tp_x\mu_{T_x}(t)\,dt}{\int_0^n {}_tp_x\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$.

Proof. ${}_nm_x = \frac{\int_x^{x+n} S_X(t)\mu_X(t)\,dt}{\int_x^{x+n} S_X(u)\,du} = \frac{\int_x^{x+n} f_X(t)\,dt}{\int_x^{x+n} S_X(u)\,du}$ as $S_X(t)\mu_X(t) = f_X(t)$, (1)

By Eq. (1), ${}_nm_x = \frac{\int_x^{x+n} f_X(t)dt}{\int_x^{x+n} S_X(t)dt} = \frac{\int_0^n f_X(x+u)du}{\int_0^n S_X(x+u)du} = \frac{\int_0^n {}_tp_x\mu_{T_x}(t)\,dt}{\int_0^n {}_tp_x\,dt}$

$${}_nm_x = \frac{S_X(x)-S_X(x+n)}{\int_x^{x+n} S_X(t)\,dt} = \frac{S_X(x)-S_X(x+n)}{\int_0^n S_X(x+u)\,du} = \frac{\frac{S_X(x)-S_X(x+n)}{S_X(x)}}{\frac{\int_0^n S_X(x+u)\,du}{S_X(x)}} = \frac{1-{}_np_x}{\int_0^n {}_up_x\,du} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}. \qquad \blacksquare$$

Definition 2.29. *The* **median future lifetime** *of* (x) *is* $m(x)$ *(=median of* $T(x)$*).* $\mathbb{P}\{T(x) < m(x)\} \le \frac{1}{2} \le \mathbb{P}\{T(x) \le m(x)\}$.

Definition 2.30. ${}_na(x) = E(T(x)|T(x) \le n)$, *the average future lifetime of those who survive to age* x*, but die within the next* n *years.* $a(x) = {}_1a(x)$.

$x + {}_na(x)$ is the mean age at death of those who survive to age x, but die in the next n years.

Theorem 2.38. ${}_na(x) = \frac{\mathring{e}_{x:\overline{n}|} - n\cdot {}_np_x}{{}_nq_x}$.

Formulas:

3. $T(x) = T_x = \underline{(X-x)|(X>x)}$,
   $\underline{{}_tp_x} = S_{T(x)}(t) = \frac{s(x+t)}{s(x)}$, $\underline{{}_tq_x} = F_{T(x)}(t) = \frac{s(x)-s(x+t)}{s(x)}$,
   $\underline{{}_{s|t}q_x} = \mathbb{P}\{s < T(x) \le s+t\} = {}_sp_x \cdot {}_tq_{x+s}$, $\underline{{}_{s|}q_x} = {}_{s|1}q_x$, $\underline{p_x} = {}_1p_x$, $\underline{q_x} = {}_1q_x$,
4. $\underline{{}_{m+n}p_x} = {}_mp_x \cdot {}_np_{x+m}$, $\underline{{}_np_x} = p_xp_{x+1}\cdots p_{x+n-1}$,
   $\underline{{}_{\sum_{j=1}^k n_j}p_x} = {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}$.
5. The force of mortality is $\mu_X(x) = \mu(x) = \mu_x = \underline{\frac{f_X(x)}{S_X(x-)}}$. $\mu_{T(x)}(t) = \underline{\mu_x(t)}$. If X is cts,
   $\underline{\mu(x)} = -\frac{d}{dx}\ln S_X(x)$, $\underline{S_X(x)} = \exp\left(-\int_0^x \mu(t)\,dt\right)$, $\underline{f_{T(x)}(t)} = {}_tp_x\mu(x+t)$. $\mu_x(t) = \underline{\mu(x+t)}$.
6. $\underline{\mathring{e}_x} = E[T(x)] = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$, $\underline{\mathring{e}_{x:\overline{n}|}} = E[T(x)\wedge n] = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$.
7. The central rate of failure on $(x, x+n]$ is $\underline{{}_nm_x} = \frac{\int_0^n {}_tp_x\mu_x(t)\,dt}{\int_0^n {}_tp_x\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$,
   $\underline{m_x} = {}_1m_x$, $\underline{{}_na(x)} = E(T(x)|T(x) \le n) = \frac{\mathring{e}_{x:\overline{n}|} - n\cdot {}_np_x}{{}_nq_x}$, $\underline{a(x)} = {}_1a(x)$.

Example 2.45. *(=Ex.2.40). For the survival function* $S_X(x) = \frac{90^6 - x^6}{(90)^6}$*, for* $0 < x < 90$*. Find (1) the median future lifetime of* (30)*, (2)* ${}_{10}m_{30}$*, (3)* ${}_{10}a(30)$*.*

Solution: (1) Formula: $p = F(\xi_p) = S(\xi_{1-p})$, as F is continuous and $\uparrow\uparrow$. We have that

$$0.5 = \mathbb{P}\{T(30) > \xi_{0.5}\} = \frac{S_X(30+\xi_{0.5})}{S_X(30)} = \frac{90^6 - (30+\xi_{0.5})^6}{90^6 - 30^6}$$

$=> \xi_{0.5} = \left(90^6 - (90^6 - 30^6) * 0.5\right)^{1/6} - 30 \approx 50.19920541$.

(2) Formulas [7]: ${}_nm_x \overset{def}{=} \frac{\int_0^n {}_tp_x\mu_x(t)\,dt}{\int_0^n {}_tp_x\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$. Try the 2nd ${}_{10}m_{30} = {}_{10}q_{30}/\mathring{e}_{30:\overline{10}|}$, where

(a) ${}_{10}q_{30} = 1 - {}_{10}p_{30} = 1 - \frac{S_X(40)}{S_X(30)} \approx 0.006344$,

(b) $\overset{\circ}{e}_{x:\overline{n}|} = \overset{\circ}{e}_{30:\overline{10}|} = \int_0^n {}_tp_x dt = \int_0^n \frac{s(x+t)}{s(x)} dt \approx 9.976$ (see Ex. 2.40).
Thus ${}_{10}m_{30} = \frac{{}_{10}q_{30}}{\overset{\circ}{e}_{30:\overline{10}|}} \approx \frac{0.00634}{9.976} \approx 0.000636$.

(3) Formula [7]: ${}_na(x) = E(T(x)|T(x) \le n)) = \frac{\overset{\circ}{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$.

$${}_{10}a(30) = \frac{\overset{\circ}{e}_{30:\overline{10}|} - 10 \cdot {}_{10}p_{30}}{{}_{10}q_{30}} \approx \frac{9.9755 - (10)(0.9937)}{0.00634430727} \approx 6.1415.$$

Example 2.46.

Example 2.47.

Quiz on Friday: 447: 19-22, 44. 450: 1-9.

2.5 Future curtate lifetime.

Definition 2.31. *Let $K_x = \lceil T(x) \rceil$, where $\lceil t \rceil$ is called the* **ceiling** *of t, that is,*

$$\lceil t \rceil = kI(k-1 < t \le k) = \begin{cases} . & \cdots \\ 0 & \text{if } -1 < t \le 0, \\ 1 & \text{if } 0 < t \le 1, \\ 2 & \text{if } 1 < t \le 2, \\ 3 & \text{if } 2 < t \le 3, \\ . & \cdots \end{cases} \qquad (\lfloor t \rfloor = (k-1)I(k-1 \le t < k)).$$

Definition 2.32. *$K(x) = \lceil T(x) \rceil - 1$ is the* **future curtate lifetime of a life aged** *x.*
$e_x = E[K(x)]$ is the **curtate life expectation of a life aged** *x.*
$e_{x:\overline{n}|} = E(K(x) \wedge n)$, the expected number of whole years lived in the interval $(x, x+n]$ by (x).

$$K(x) = K_x - 1 = \begin{cases} 0 & \text{if } 0 < T(x) \le 1, \\ 1 & \text{if } 1 < T(x) \le 2, \\ 2 & \text{if } 2 < T(x) \le 3, \\ . & \cdots \end{cases} \ne \lfloor x \rfloor \text{ and } K_x = \begin{cases} 1 & \text{if } 0 < T(x) \le 1, \\ 2 & \text{if } 1 < T(x) \le 2, \\ 3 & \text{if } 2 < T(x) \le 3, \\ . & \cdots \end{cases}$$

Q: Which of e_x, $\overset{\circ}{e}_x$, $E(K_x)$ is larger ? $K(x)$ $(= K_x - 1)$, $T(x)$, K_x $(= \lceil T(x) \rceil)$.

Class Exercise Q: $f_{K_x}(t) = P(K_x = t) = \begin{cases} f_{K(x)}(t-1) & ? \\ f_{K(x)}(t+1) & ? \end{cases}$ $E(K_x) = \begin{cases} E(K(x)) - 1 & ? \\ E(K(x)) + 1 & ? \end{cases}$

Remark. Recall $K_x = \lceil T(x) \rceil$ and $K(x) = \lceil T(x) \rceil - 1$.
What is the meaning of $T(30) = 0.5$?
What is the meaning of $T(30) > 0.5$?
Q: If $T(30) = 0.5$, $K_{30} = ?$ $K(30) = ?$

Theorem 2.39. $\mathbb{P}\{K_x = k\} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j=0}^{k-2} p_{x+j}\right) q_{x+k-1} = \frac{s(x+k-1) - s(x+k)}{s(x)}$
$= {}_{k-1|}q_x = \mathbb{P}\{k-1 < T(x) \le k\} = \mathbb{P}\{k-1 < X - x \le k | X > x\} = {}_{k-1}p_x - {}_kp_x = {}_kq_x - {}_{k-1}q_x$.

Example 2.48. *Suppose* $p_{90} = 0.05, \quad p_{91} = 0.01, \quad p_{92} = 0.001, \quad p_{93} = 0$ *(*$S_X(94) =$ *?) Calculate the probability mass function of* K_{90}.

Solution: Which formula to use in Th 2.39 ?? How about the formulas sheet ?

$$\begin{aligned}
Th.2.39\ \mathbb{P}\{K_{90} = 1\} &= q_{90} = 1 - p_{90} = (1 - 0.05) = 0.95, && sheet \\
2nd: \mathbb{P}\{K_{90} = 2\} &= p_{90}q_{91} = (0.05)(1 - 0.01) = 0.0495, && [8]3rd \\
3rd: \mathbb{P}\{K_{90} = 3\} &= p_{90}p_{91}q_{92} = (0.05)(0.01)(1 - 0.001) = 0.0004995, && [8]4th \\
\mathbb{P}\{K_{90} = 4\} &= p_{90}p_{91}p_{92}q_{93} = (0.05)(0.01)(0.001)(1 - 0) = 0.0000005, \\
\mathbb{P}\{K_{90} = k\} &= 0, \text{ for } k = 5, 6, 7, \ldots, \text{ as } p_{93} = P(X > 94|X > 93) = 0
\end{aligned}$$

Example 2.49. *Calculate the probability mass function of* K_{90} *for given*

$$p_{90} = 0.2, \quad {}_2p_{90} = 0.1, \quad {}_3p_{90} = 0.01, \quad {}_4p_{90} = 0.005, \quad {}_5p_{90} = 0.$$

Sol: ${}_0p_{90} =$? ${}_1p_{90} =$? $\mathbb{P}\{K_{90} = k\} = {}_{k-1}q_x$=? **Which formula in Th 2.39 to use ??**

$$\begin{aligned}
\mathbb{P}\{K_{90} = 1\} &= 1 - p_{90} = 1 - 0.2 = 0.8, \\
7th: \mathbb{P}\{K_{90} = 2\} &= p_{90} - {}_2p_{90} = 0.2 - 0.1 = 0.1, && [8]:\ K(x) = \lceil T(x) \rceil - 1 \\
\mathbb{P}\{K_{90} = 3\} &= {}_2p_{90} - {}_3p_{90} = 0.1 - 0.01 = 0.09, && K_x = \lceil T(x) \rceil \\
\mathbb{P}\{K_{90} = 4\} &= {}_3p_{90} - {}_4p_{90} = 0.01 - 0.005 = 0.005, && \mathbb{P}\{K_{90} = k\} = f_{K_x}(k) = {}_{k-1}|q_x \\
\mathbb{P}\{K_{90} = 5\} &= {}_4p_{90} - {}_5p_{90} = 0.005 - 0 = 0.005, \\
\mathbb{P}\{K_{90} = k\} &= 0, \text{ for } k = 6, 7, \ldots, \ \textbf{Why ??}
\end{aligned}$$

Definition 2.33.

Definition 2.34.

Definition 2.35.

Theorem 2.40.

Theorem 2.41.

Theorem 2.42. $e_x = E[K(x)] = \sum_{k=1}^{\infty} k \cdot {}_k|q_x = \sum_{k=1}^{\infty} {}_kp_x$ *and*

$$E[(K(x))^2] = \sum_{k=1}^{\infty} k^2 \cdot {}_k|q_x = \sum_{k=1}^{\infty} (2k - 1) \cdot {}_kp_x.$$

Example 2.50. *Suppose that*

x	*90*	*91*	*92*	*93*	*94*
p_x	*0.2*	*0.1*	*0.05*	*0.01*	*0*

. $e_{90} =$ *?*

Solution: Theorem 2.42: $e_x = E(K(x)) = \sum_{k=1}^{\infty} k \cdot {}_k|q_x = \sum_{k=1}^{\infty} {}_kp_x$ (formula [9]).

Formula [4]: ${}_np_x = p_x \cdots p_{x+n-1}$,

$$e_{90} = \sum_{k=1}^{\infty} {}_kp_{90} = p_{90} + p_{90}\cdot p_{91} + p_{90}\cdot p_{91}\cdot p_{92} + p_{90}\cdot p_{91}\cdot p_{92}\cdot p_{93} + \cdots$$
$$=(0.2) + (0.2)(0.1) + (0.2)(0.1)(0.05) + (0.2)(0.1)(0.05)(0.01) + 0 \approx 0.22.$$

Example 2.51. *Suppose* $s(t) = \frac{100-t}{100}$, $0 \le t \le 100$. *Find* $\mathring{e}_x$ *and* e_x, *where* x *is an integer.*

Solution: Formulas:
[1] $\mathring{e}_x = E(T(x)) = \int_0^\infty {}_tp_x\, dt$,
[3] ${}_tp_x = \frac{s(x+t)}{s(x)}$ and
[9] $e_x = E(K(x)) = \sum_{k=1}^\infty {}_kp_x$.
${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{100-x-t}{100-x}$, $0 < x+t \le 100$.

$$\mathring{e}_x = \int_0^{100-x} {}_tp_x\, dt = \int_0^{100-x} \frac{100-x-t}{100-x}\, dt = -\int_0^{100-x} \frac{\overbrace{100-x-t}^{=u}}{100-x}\, d\overbrace{(100-x-t)}^{=u}$$
$$= -\frac{(100-x-t)^2}{2(100-x)}\Bigg|_0^{100-x} = -0 + \frac{100-x}{2} ...?$$
$$e_x = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{100-x} \frac{100-x-k}{100-x} = \sum_{k=1}^{100-x} [1 - \frac{k}{100-x}]$$
$$=100 - x - \frac{1}{100-x}\frac{(100-x)(100-x+1)}{2} \qquad (\sum_{k=1}^{n} k = \frac{n(n+1)}{2})$$
$$=100 - x - \frac{100-x+1}{2} = \frac{99-x}{2},\ x \in [0,99] \text{ or } x \in [0,100) \text{ or } x \in \{0,1,...,99\}\ ?$$

Theorem 2.43. *(Iterative formula for* e_x*)* $e_x = p_x(1 + e_{x+1})$.

Theorem 2.44. *If* $p_{x+k} = p_x$, *for each integer* $k \ge 1$. *Then,* $e_x = p_x + {}_2p_x + \cdots = \frac{p_x}{1-p_x}$.

Remark:

$$(1-p)(1+p) = 1-p^2,\ (1-p)(1+p+\cdots+p^k) = 1-p^{k+1},\ k = 1,\ 2,\ 3,\ ... \tag{1}$$

Theorem 2.45. $e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k \cdot {}_{k|}q_x + n \cdot {}_np_x = \sum_{k=1}^{n} {}_kp_x$.

Theorem 2.46. $e_x = e_{x:\overline{n}|} + {}_np_x e_{x+n}$.

Theorem 2.47. *If* $p_{x+k} = p$, *for each integer* $k \ge 0$. *Then,* $e_{x:\overline{n}|} = \frac{p-p^{n+1}}{1-p}$.

Theorem 2.48. $e_{x:\overline{n}|} = p_x \left(1 + e_{x+1:\overline{n-1}|}\right)$.

Theorems 2.44-2.49 above are summarized as formulas as follows.

[9] $e_x = E[K(x)] = p_x(1+e_{x+1}) = e_{x:\overline{n}|} + {}_np_xe_{x+n} = \sum_{k=1}^{\infty} {}_kp_x$. $\quad K(x) = \lceil T(x) \rceil - 1$.

$e_{x:\overline{n}|} = E(K(x) \wedge n) = \sum_{k=1}^{n} {}_kp_x$.

[16] $a_{\overline{n}|} \stackrel{def}{=} \sum_{k=1}^{n} v^k = v\frac{1-v^n}{1-v}$, $\sum_{k=0}^{n} v^k = \frac{1-v^{n+1}}{1-v}$. [16] yields Th.2.44 and Eq.(1).

Example 2.52. *Suppose that $e_x = 30$, $p_x = 0.97$ and $p_{x+1} = 0.95$. $e_{x+2} =$?*

Solution: [9] $e_x = E[K(x)] = p_x(1+e_{x+1}) = e_{x:\overline{n}|} + {}_np_xe_{x+n} = \sum_{k=1}^{\infty} {}_kp_x$. **Which to use ?**

Method 1. $\overbrace{e_x}^{=?} = \overbrace{p_x}^{=?}(1+e_{x+1}) => e_{x+1} = \frac{e_x}{p_x} - 1$ and $e_{x+2} = \frac{e_{x+1}}{p_{x+1}} - 1$. Then

$e_{x+1} = \frac{30}{0.97} - 1 \approx 29.9 < 30 = e_x$ and

$e_{x+2} = \frac{e_{x+1}}{p_{x+1}} - 1 = \frac{29.9}{0.95} - 1 \approx 30.5 > 29.9 = e_{x+1}$.

Question: Should $e_{x+1} \begin{cases} > e_x? \\ < e_x? \end{cases}$

> (x=30/0.97-1) [1] 29.92784
> x/0.95-1
> x/0.99-1 [1] 29.23014 # if most people survival after 1 year, 29.2 < 29.9.
> x/0.40-1 [1] 73.81959 # if most people died before 1 year, 73.8 > 29.9.

In the other cases, it fluctuates around 29.9.

Method 2. $e_x = e_{x:\overline{2}|} + {}_2p_xe_{x+2} => \frac{e_x - e_{x:\overline{2}|}}{{}_2p_x} = e_{x+2}$.

by [9] $e_{x:\overline{2}|} = \sum_{k=1}^{2} {}_kp_x$,

by [4], ${}_2p_x = p_x \cdot p_{x+1}$,

$$e_{x:\overline{2}|} = p_x + {}_2p_x = 0.97 + (0.97)(0.95) = 1.8915.$$

$$e_{x+2} = \frac{e_x - e_{x:\overline{2}|}}{{}_2p_x} = \frac{30 - 1.8915}{(0.97)(0.95)} = 30.50298426.$$

Example 2.53. *Suppose that*

x	*90*	*91*	*92*	*93*	*94*
p_x	*0.2*	*0.1*	*0.05*	*0.01*	*0*

. $e_{90:\overline{2}|} =$? $\quad E(K(x) \wedge 2)$

Solution: By [9] or Th2.46, $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x$. **n= ?**

${}_1p_x =$? By [4] ${}_2p_x = {}_1p_x \cdot {}_1p_{x+1}$,

$$e_{90:\overline{2}|} = p_{90} + p_{90}p_{91} = (0.2) + (0.2)(0.1) = 0.22.$$

Definition 2.36. *Let $S_x = T(x) - K(x)$, the period of time lived through the death interval of an entity aged x.*

S_x is a r.v. taking values in the interval $(0, 1]$. Notice that $E[S_x] = \mathring{e}_x - e_x$. $E(S_X) =$?

2.6 Selected survival models.

A **select table** is a mortality table for a group of people subject to a special circumstance (disability, retirement, etc). The variable in common of this group of people is called the

concomitant variable. The probability of surviving from time x, to time $x+t$ for an entity selected at time x is ${}_tp_{[x]}$. Here, the age at selection is denoted by $[x]$. The select survival function is denoted by $S(x;t) = {}_tp_{[x]}$. The force of mortality is $\mu_{[x]+t} = -\frac{d}{dt}\ln S(x;t)$. The expected future life is $\mathring{e}_{[x]} = \int_0^\infty S(x;t)\,dt$. Here,

$[x]$, $S(x;t)$, ${}_tp_{[x]}$, $\mu_{[x]+t}$ are all **special** notations.

Example 2.54.

2.7 Common analytical survival models

2.7.1 De Moivre model. Ignore this section.

Definition 2.37. *The age–at–death X follows* **De Moivre mortality law** *with terminal age ω, if the distribution of $X \sim U(0,\omega)$.*

Definition 2.38.

Definition 2.39.

Example 2.55.

Example 2.56.

Example 2.57.

Example 2.58.

Example 2.59.

Example 2.60.

Example 2.61.

Example 2.62.

Theorem 2.49.

Theorem 2.50.

Theorem 2.51.

Theorem 2.52.

Theorem 2.53.

Theorem 2.54.

Theorem 2.55.

Theorem 2.56.

Theorem 2.57.

Theorem 2.58.

2.7.2 Generalized De Moivre model.

Definition 2.40. *The age–at–death X follows a* **generalized De Moivre mortality** *if $s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$, for $0 \le x \le \omega$, where $\alpha > 0$. If $\alpha = 1$, it is $U(0, \omega)$, called De Moivre law.*

Example 2.63. *The future lifetime of a new born has survival function $s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$, for $0 \le x \le \omega$, where $\alpha, \omega > 0$. Suppose that $\mathring{e}_{40} = 8$ and $\mathring{e}_{60} = 4$. Calculate α and ω.*

Solution: Given $\mathring{e}_x = \int t f_{T(x)}(t)dt = \int_0^{\infty} {}_tp_x\, dt = \begin{cases} 8 & x = 40 \\ 4 & x = 60 \end{cases}$,

$$ {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{(\omega - x - t)^{\alpha}}{(\omega - x)^{\alpha}},\ 0 \le t \le ? \qquad\qquad s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha},\ for\ \ 0 \le x \le \omega $$

$$ \mathring{e}_x = \int_0^{\infty} {}_tp_x\, dt = \int_0^{\omega - x} \frac{(\overbrace{\omega - x - t}^{=u})^{\alpha}}{(\omega - x)^{\alpha}}\, dt = \frac{-(\omega - x - t)^{\alpha+1}}{(\alpha+1)(\omega - x)^{\alpha}} \Bigg|_0^{\omega - x} = \frac{\omega - x}{\alpha + 1} = \begin{cases} 8 & x = 40 \\ 4 & x = 60 \end{cases} $$

$$ \begin{cases} 8 = \mathring{e}_{40} = \frac{\omega - 40}{\alpha+1} \\ 4 = \mathring{e}_{60} = \frac{\omega - 60}{\alpha+1} \end{cases} \Rightarrow\ 2 = \frac{\omega - 40}{\omega - 60} \Rightarrow\ (2)(\omega - 60) = \omega - 40 \Rightarrow \omega = 80. $$

Hence, $8 = \frac{80-40}{\alpha+1} \Rightarrow \alpha = \frac{80-40}{8} - 1 = 4$.

2.7.3 Exponential model.

Theorem 2.59. *An exponential r.v. X with $E(X) = \theta > 0$ (letting $\mu = 1/\theta$) satisfies, for $x \ge 0$, $f(x) = \frac{1}{\theta}e^{-x/\theta} = \mu e^{-\mu x}$, $s(x) = e^{-x/\theta} = e^{-\mu x}$, $h(x) = \mu$, $E[X] = \theta$ and $\mathrm{Var}(X) = \theta^2$.*

We write $X \sim Exp(1/\mu)$ or $Exp(\theta)$ and X has constant force of mortality $h(x) = \mu$. The exponential model is also called the **constant force model**. By 447, $Exp(\alpha) = \Gamma(1, \alpha)$. α=?

Theorem 2.60.

Theorem 2.61. (**Memoryless property of the exponential distribution**) *Let X have an exponential distribution. Then, for each $s, t > 0$,*

$$ \mathbb{P}\{T(s) > t\} = \mathbb{P}\{X > s + t \mid X > s\} = \mathbb{P}\{X > t\}. \qquad\qquad E(T(x)) = E(X)-? $$

Definition 2.41. *A r.v. X has a* **geometric distribution** *with parameter p, if*

$\mathbb{P}\{X=k\}=(1-p)^k p,\ k=0,1,2\ldots$ *where* $0<p\leq 1$
(X=# of failures until the 1st success).

Remark. The geometric distribution in Math 447 $Y \sim G(p)$ **([10]) is**
$Y=X+1$ (=# of trials until the 1st success).

Theorem 2.62. (Memoryless property of the geometric distribution) *Let X be a r.v. with a geometric distribution ($X \sim G(p)$). Then, for each integers $k, n \geq 1$,*

$$\mathbb{P}\{T(k)\geq n\}=\mathbb{P}\{X\geq k+n \mid X\geq k\}=P\{X\geq n\}.$$

Proof. $\vdash$: $\mathbb{P}\{X\geq n\}=(1-p)^n=\mathbb{P}\{X\geq k+n \mid X\geq k\},\ k=0,1,2\ldots$

$\mathbb{P}\{X\geq n\}$
$=\sum_{j=n}^{\infty}\mathbb{P}\{X=j\}=\sum_{j=n}^{\infty}(1-p)^j p \qquad (\sum_{j=0}^{n}x^j=\frac{1-x^{n+1}}{1-x}) \qquad (see\ formula[17]).$
$=(1-p)^n p\sum_{k=0}^{\infty}(1-p)^k \qquad (k=j?)$
$=(1-p)^n p\frac{1-(1-p)^\infty}{1-(1-p)}=(1-p)^n.$

$\mathbb{P}\{X\geq k+n \mid X\geq k\} \qquad P(A|B)=P(AB)/P(B)=P(A)/P(B)$??
$=\frac{\mathbb{P}\{X\geq k+n\}}{\mathbb{P}\{X\geq k\}}=\frac{(1-p)^{k+n}}{(1-p)^k}=(1-p)^n.$ □

Theorem 2.63. *Let $X \sim G(p)$. Then, $E[X]=\frac{q}{p}$ and $\mathrm{Var}(X)=\frac{q}{p^2}$.*

Proof. (Math 447) [10]: $E(Y)=1/p$ and $V(Y)=q/p^2$, where $Y=X+1$.

$Y=\#$ of trials to have a success.

$X=\#$ of trials before a success.

$E(X)=E(Y-1)=(1/p)-1=(1-p)/p=q/p. \quad V(X)=V(Y-1)=V(Y)=q/p^2.$

Theorem 2.64. *Suppose that for each $k=1,2,\ldots,$ $p_{x+k}=p_x$. Then, the curtate lifetime $K(x)$ follows a geometric distribution with parameter $p=1-p_x$.*

Proof: $\vdash$: $\mathbb{P}\{K(x)=k\}=p_x^k(1-p_x)$.

$\mathbb{P}\{K(x)\geq k\}={}_kp_x=p_xp_{x+1}\cdots p_{x+k-1}=p_x^k$ (by 450 [4]) and

$$\mathbb{P}\{K(x)=k\}=\mathbb{P}\{K(x)\geq k\}-\mathbb{P}\{K(x)\geq k+1\}=p_x^k-p_x^{k+1}=p_x^k(1-p_x).$$

Example 2.64. *(***is Example 2.67 in the textbook***). Suppose that:*
(a) the force of mortality is constant.
(b) the probability that a 30–year–old will survive to age 40 is 0.95.
(i) the expected future lifetime of a 40–year–old=? $\qquad \mathring{e}_{40}=E(T(40))=$?
(ii) the curtate life expectation of a 40–year–old=? $\qquad e_x$=E(K(x))= ?

Q: $X \sim Exp(\theta)$ with $E(X)=\theta$, $\mu=\theta$ or $\frac{1}{\theta}$?

$S(x)=e^{-x\mu}$; $f(x)=e^{-x\mu}$, $h(x)=\frac{f(x)}{S(x-)}=\mu$, $E(X)=\frac{1}{\mu}$.

Solution: (i) $\mathring{e}_{40}=E(T(40))=E(X)=?$

Condition (a) => $T(x)\sim Exp(1/\mu)$. => $\qquad \mathring{e}_{40}=E(T(40))=1/\mu=E(X)$, μ unkonwn.

Condition (b) => ${}_{10}p_{30} = e^{-10\mu} = 0.95$. => $\mu = \frac{-\ln(0.95)}{10}$, thus $\mathring{e}_{40} = \frac{1}{\mu} = \frac{10}{-\ln(0.95)} \approx 195$?
(ii) $e_x = \sum_{k=1}^{\infty} {}_kp_x$, formula [9]
$\quad {}_kp_x = e^{-\mu k}$,
$\quad \sum_{k=1}^{n} x^k = x\frac{1-x^n}{1-x}$ (see formula [16]).

$$e_x = \sum_{k=1}^{\infty}(e^{-\mu})^k = \sum_{k=1}^{\infty} x^k \Big|_{x=?} = e^{-\mu}\frac{1-e^{-\mu\infty}}{1-e^{-\mu}} \approx 194.5.$$

Example 2.65.

Example 2.66.

Example 2.67.

Theorem 2.65. *Suppose that for each $k = 1, 2, \ldots,$ $p_{x+k} = p_x$. Then,* $e_{x:\overline{n}|} = \frac{p_x(1-p_x^n)}{q_x}$.

Proof. [9]: $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x = \sum_{k=1}^{n} \underbrace{p_x \cdots p_x}_{k \text{ terms by } [4]} = \sum_{k=1}^{n} p_x^k = \frac{p_x(1-p_x^n)}{1-p_x}$ by # [16]: $\sum_{k=1}^{n} p^k = p\frac{1-p^n}{1-p}$.

2.7.4 Gompertz model. (1825). $\mu(x) = Bc^x$, where $B > 0$ and $c > 1$. Hence, $S_X(x) = exp(-\int_o^x Bc^t dt) = e^{-m(c^x-1)}$ for $x \geq 0$, where $m = \frac{B}{\ln c}$.

2.7.5 Makeham model. Makehan (1860) introduced the model $\mu(x) = A + Bc^x$, where $A \geq -B$, $B > 0$ and $c > 1$. **What model is it if $A = 0$?**

2.7.6 Weibull model. (1939). $\mu(x) = kx^n$, for $x \geq 0$, where $k > 0$ and $n > -1$. Then, $S_X(x) = e^{-\frac{kx^{n+1}}{n+1}}, x \geq 0$. **What distribution is it if $n = 0$?**

2.7.7 Pareto model with parameters α (> 0) and θ (> 0). $S_X(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$, $x > 0$.

Theorem 2.66.

2.8 Mixture distributions

Math 447: $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$ and $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$.

$$f_X(x) = \begin{cases} \int f_{X,Y}(x,y)dy & \text{if } Y \text{ is cts.} \\ \sum_y f_{X,Y}(x,y) & \text{if } Y \text{ is dis} \end{cases} = \begin{cases} \int f_{X|Y}(x|y)f_Y(y)dy & \text{if } Y \text{ is cts.} \\ \sum_y f_{X|Y}(x|y)f_Y(y) & \text{if } Y \text{ is dis} \end{cases}$$

F_X is called the marginal cdf of X in 447, and is called the mixture distribution here.

Question. Can we write $f(x|y) = \frac{f(x,y)}{f_Y(y)}$? $\quad f(x|y) = \frac{f(x,y)}{f(y)}$? $\quad \frac{f(2,2)}{f(2)} = $?
Can we define two distributions by that
$\quad X$ has df $f(x) = e^{-x}$, $x > 0$ and Y has df $f(y) = 1$, $y \in (0,1)$ **??**
How about : X has df $f_X(x) = e^{-x}$, $x > 0$ and Y has df $f_Y(x) = 1$, $x \in (0,1)$?

Theorem 2.67. *(Double expectation theorem for expectations)* $E[E[X|Y]] = E[X]$.

Theorem 2.68. *(Double expectation theorem for variances)* $\sigma^2_X = \sigma^2_{E(X|Y)} + E(\sigma^2_{X|Y})$.

$$\mathrm{Var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)].$$

Example 2.68. *You are given that:*
(a) Men follow a de Moivre model with terminal age 100.
(b) Women follow a de Moivre model with terminal age 110.
(c) 55% of births are male.
(i) Calculate the expectated life of a randomly chosen life.
(ii) Calculate the probability that a newborn survives 80 years.
(iii) Calculate the density of the future lifetime T of a randomly chosen life.

Solution: Let X be the lifetime of a newborn and $Y = I$(a birth is a male).
(a) $=> X|Y = 1$ follows $U(0, 100)$, So, $E[X|Y = 1] =?$
(b) $=> X|Y = 0$ follows a $U(0, 110)$. So, $E[X|Y = 0] =?$
(c) $P(Y = 1) =?$

$$
\begin{aligned}
(i) \qquad E[X] =& E[E[X|Y]] = E(\overbrace{g(Y)}^{=?}) = \int g(y) f_Y(y)dy? \ \ or \ \sum_i E(X|Y = i)f_Y(i)? \\
=& E[X|Y = 1]\mathbb{P}\{Y = 1\} + E[X|Y = 0]\mathbb{P}\{Y = 0\} \\
=& (50)(0.55) + (55)(0.45) = 52.25.
\end{aligned}
$$

(ii) $\mathbb{P}\{X > 80\} = ?$ $\qquad = E(Z)$, where $Z = I(X > 80) \sim bin(n, p)$?

$$
\begin{aligned}
&\mathbb{P}\{X > 80\} = E(Z) = E(E(Z|Y)) \\
=& E(Z|Y = 1)P(Y = 1) + E(Z|Y = 0)P(Y = 0) \\
=& \mathbb{P}\{X > 80|Y = 1\}\mathbb{P}\{Y = 1\} + \mathbb{P}\{X > 80|Y = 0\}\mathbb{P}\{Y = 0\} \qquad Z|Y \sim bin(1, p_Y) \\
=& \int_{80}^{100} \frac{1}{100} dt P(Y = 1) + \int_{80}^{110} \frac{1}{110} dt P(Y = 0) \\
=& \frac{100 - 80}{100}(0.55) + \frac{110 - 80}{110}(0.45) \approx 0.23.
\end{aligned}
$$

(iii) $f_X(x) = \sum_j f_{X,Y}(x, j) = \sum_j f_Y(j) f_{X|Y}(x|j) = (0.55)\underbrace{f_{X|Y}(x|1)}_{U(0,100)} + (0.45)\underbrace{f_{X|Y}(x|0)}_{U(0,110)}$

$$
= \begin{cases} 0.55\frac{1}{100} + 0.45\frac{1}{110} & \text{if } 0 < x < 100, \\ 0.55 * 0 + 0.45\frac{1}{110} & \text{if } 100 \le x < 110, \\ 0 & \text{OW.} \end{cases}
$$

Example 2.69. *The future lifetime $T(x)$ of (x) has constant force of mortality μ. $\mu \sim U(0.01, 0.05)$. (i) Calculate $\mathring{e}_x$. (ii) Calculate* $\mathrm{Var}(T(x))$.

Solution: (i) $\overset{\circ}{e}_x = E(T(x))$=?
Given conditions: $T(x)|\mu \sim Exp(\beta)$, $\beta = 1/\mu$, and $\mu \sim U(0.01, 0.05)$.

$E[E[X|Y]] = E[X]$. $\qquad$ $\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}(X|Y)]$.

$E[E[T(x)|\mu]] = E[T(x)]$. $\qquad$ $\text{Var}(T(x)) = \text{Var}(E[T(x)|\mu]) + E[\text{Var}(T(x)|\mu)]$.

Formula 447 [23] or [12]: $X \sim \mathcal{G}(\alpha, \beta)$, $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$, $x > 0$,

$E(X) = \alpha\beta$, $\sigma^2 = \alpha\beta^2$, $Exp(\beta) = \mathcal{G}(1, \beta)$.
Thus $f_{T(x)|\mu}(t) = \frac{1}{\beta}\exp(-t/\beta) = \mu e^{-\mu t}$ and $S_{T(x)|\mu}(t) = \exp(-t/\beta) = e^{-\mu t}$, $t > 0$.

$E[T(x)|\mu] = \alpha\beta = \frac{1}{\mu}$ and $\text{Var}(T(x)|\mu) = \alpha\beta^2 = \beta^2 = \frac{1}{\mu^2}$. So,

$$
\begin{aligned}
(i) \quad E[T(x)] =& E[E[T(x)|\mu]] = E\left[\frac{1}{\mu}\right] = \int \frac{1}{x} f_\mu(x)dx = \int_{0.01}^{0.05} \frac{1}{\mu}\frac{1}{0.05-0.01}\,d\mu \ ??\\
=& \frac{\ln(\mu)}{0.05-0.01}\Bigg|_{0.01}^{0.05} = \frac{\ln(0.05)-\ln(0.01)}{0.05-0.01} \approx 40.2.\\
(ii) \quad \text{Var}(T(x)) =& \text{Var}(E[T(x)|\mu]) + E[\text{Var}(T(x)|\mu)] = \text{Var}\left(\frac{1}{\mu}\right) + E\left[\frac{1}{\mu^2}\right]\\
=& E\left[\frac{1}{\mu^2}\right] - \left(E\left[\frac{1}{\mu}\right]\right)^2 + E\left[\frac{1}{\mu^2}\right]\\
=& 2E\left[\frac{1}{\mu^2}\right] - \left(E\left[\frac{1}{\mu}\right]\right)^2 = 2E\left[\frac{1}{\mu^2}\right] - (40.2)^2 = ??\\
E\left[\frac{1}{\mu^2}\right] =& \int_{0.01}^{0.05} \mu^{-2}\frac{1}{0.05-0.01}\,d\mu\\
=& \frac{1}{0.05-0.01}\frac{\mu^{-2+1}}{-2+1}\Bigg|_{0.01}^{0.05} = \frac{1}{0.05-0.01}\left(\frac{1}{0.01}-\frac{1}{0.05}\right) = 2000.\\
=> \quad \text{Var}(T(x)) =& 2*2000 - (40.2)^2 \approx 2381.1.
\end{aligned}
$$

Theorem 2.69.

Theorem 2.70.

2.9 Estimation of the survival function

Given a distribution of a r.v. X, we may either know the parametric form of the distribution, say $X \sim U(a, b)$, or $\text{Exp}(\theta)$, or $\mathcal{G}(\alpha, \beta)$ etc., or do not know the form of the distribution. In the last case, we say S_X is a non-parametric form, and the first case S_X is of a parametric form with parameter, say θ ($= (a, b)$, or θ, or (α, β)). To compute $P(X > 2)$ or $E(X)$, we need to know the value of S_X. *e.g.*,
Q: What is the life expectancy of American ? Ans: $E(X)$ or 79, which makes sense ?

79 is called an estimate. $\qquad$ $E(X) = 79$? $\qquad$ $E(X) \approx 79$?

There are two typical types of estimators of the distribution or survival functions:
(1) Parametric estimators such as
the maximum likelihood estimator (MLE),
the method of moment estimator (MME), *etc.*.
(2) Non-parametric estimators such as
the non-parametric maximum likelihood estimator (NPMLE) and
the Nelson-Aalen estimator *etc.*

Given a parametric distribution form, say $S_X(x;\theta)$, such as $U(a,b)$, or Exp(θ), or $\mathcal{G}(\alpha,\beta)$ etc., where $\theta=$?? is the parameter, the MLE of θ maximizes
$L(\theta)=\prod_{i=1}^n f_X(X_i;\theta)$ over $\theta\in\Theta$, the parameter space.
Let $\hat\theta$ be the MLE, then the MLE of S_X is $S_X(x;\hat\theta)$.

An MME $\tilde\theta$ satifies $g_i(\tilde\theta)=\overline{X^i}$, where $g_i(\theta)=E(X^i)$ for $i\in\{1,...,p\}$, and $\theta\in\mathcal{R}^p$.
The MME of S_X is $S_X(x;\tilde\theta)$.

Example 1. If X_1, ..., X_n are i.i.d. from Exp(θ) find the MLE and MME of θ and S_X.
Sol. $f(x;\theta)=\theta e^{-x\theta}$, $x,\theta>0$. $E(X)=$?
The MLE of $Exp(\theta)$: $L(\theta)=\prod_{i=1}^n\theta e^{-\theta X_i}=\theta^n\exp(-\theta\sum_{i=1}^n X_i)=\theta^n\exp(-n\theta\overline{X})$.
$\ln L(\theta)=n\ln\theta-n\theta\overline{X}$.
$(\ln L(\theta))'_\theta=n/\theta-n\overline{X}=0 => \theta=1/\overline{X}$.
$(\ln L(\theta))''_\theta=-n/\theta^2<0 => \theta=1/\overline{X}$ maximizes $L(\theta)$.
The MLE of θ is $1/\overline{X}$; the MLE of $S_X(t)$ is $\hat S_X(t)=exp(-tI(t>0)/\overline{X})$,
or $\hat S_X(t)=e^{-t/\overline{X}}$, $t>0$, as $S_X(t)=e^{-t\theta}$, $t>0$ **Which of S_X or $\hat S_X$ is better ?**
MME of θ: $E(X)=1/\theta=\overline{X} => \tilde\theta=1/\overline{X}$. So MME $\tilde S=\hat S$.

Example 2. If X_1, ..., X_n are i.i.d. from $U(0,\theta)$, the MLE and MME of $S_X(t)$?
Sol. MLE: $L(\theta)=\prod_{i=1}^n\frac{I(X_i\in[0,\theta])}{\theta}$? or $=\prod_{i=1}^n\frac{1}{\theta}$ **??** $\qquad=\frac{I(0\le X_{(n)}\le\theta)}{\theta^n}$???
$\frac{d\ln L(\theta)}{d\theta}=-n/\theta=0$? **Solution ?** **What to do next ?**

$$L(\theta)=\frac{I(0\le X_{(n)}\le\theta)}{\theta^n}\downarrow \text{ in } \theta\in[X_{(n)},\infty).$$

$=> \hat\theta=X_{(n)}$ maximizes $L(\theta)$.
Since $S_X(t)=I(t<0)+\frac{\theta-t}{\theta}I(t\in[0,\theta))$, the MLE of S_X is
$\hat S_X(t)=I(t<0)+\frac{X_{(n)}-t}{X_{(n)}}I(t\in[0,X_{(n)}))$. Or just write $\hat S_X(t)=\frac{X_{(n)}-t}{X_{(n)}}$, $t\in[0,X_{(n)}]$.
MME: Since $E(X)=\theta/2$, letting $\theta/2=\overline{X}$ yields $\tilde\theta=2\overline{X}$. Thus the MME of $S_X(t)$ is

$$\tilde S_X(t)=I(t<0)+\frac{2\overline{X}-t}{2\overline{X}}I(t\in[0,2\overline{X})).$$

If we know nothing about $F(t)$, we make use of the NPMLE of $F(t)$:

$$\hat F(t)=\frac{1}{n}\sum_{i=1}^n I(X_i\le t)=\sum_{j=1}^m\frac{n_j}{n}I(t_j\le t), \tag{1}$$

where $t_1<t_2<\cdots<t_m$ are all the distinct points of X_i's and $n_i=\sum_{j=1}^n I(X_j=t_i)$,
$\hat F$ is also called the empirical distribution function (EDF), and the estimator of S_X is

$\hat{S}(t) = 1 - \hat{F}(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i > t)$? ? or $\frac{1}{n}\sum_{i=1}^{n} I(X_i \geq t)$??

Notice that the EDF is a discrete cdf with the d.f.

$$\hat{f}(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i = t) = \sum_{j=1}^{m} \frac{n_j}{n} I(t = t_j). \qquad (\textit{see Eq. } (1))$$

Remark. It is interesting to notice the following facts:

$\sum_x x\hat{f}(x) = \overline{X}, \quad = \underbrace{E(X)}_{=\sum_x x f_X(x)}$? $\overline{X}$ is an estimator of $E(X)$.

$\hat{\sigma}^2 = \sum_x (x - \overline{X})^2 \hat{f}(x) = \overline{X^2} - (\overline{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2,$ $\hat{\sigma}^2$ is an estimator of $V(X)$.

$\underbrace{= \sum_x (x - E(X))^2 f_X(x)}_{V(X)}$?

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$ another estimator of $V(X)$.

A modification of $\hat{F}$ is to smooth it.
then $\check{F}(t)$ is a continuous piecewise linear function:

$$\check{F}(t) = \begin{cases} 0 & \text{if } t < 0 = t_0, \\ \hat{F}(t_i) & t \in \{t_0, ..., t_m\}, \\ s\hat{F}(t_i) + (1-s)\hat{F}(t_{i+1}) & \text{if } t = st_i + (1-s)t_{i+1},\ s \in (0,1),\ \ i \in \{0, ..., m-1\}, \\ 1 & \text{if } t > t_m. \end{cases}$$

with d.f. $\check{f}(t) = \frac{n_{i+1}}{n(t_{i+1} - t_i)}$ if $t \in (t_i, t_{i+1})$, $i \in \{0, ..., m-1\}$ and
the survival function $\check{S}(t) = 1 - \check{F}(t)$.

Example 2.68 Derive the MLE of F and $S(2.5)$ under $Exp(\theta)$ and $U(0,\theta)$, and $\hat{F}$ and $\check{F}$, with the data X_i's: 1, 1, 2, 2, 3, 4, 4, 5, 5, 8, 8, 8, 8, 11, 11, 12, 12, 15, 17, 22, 23 (n=21).

Sol. Under $Exp(\theta)$ the MLE of θ is $\hat{\theta} = 1/\overline{X} \approx 1/8.7$, or use the R codes as follows:

```
x=c(1, 1, 2, 2, 3, 4, 4, 5, 5, 8, 8, 8, 8, 11, 11, 12, 12, 15, 17, 22, 23)
mean(x)                                8.7
```

The MLE of $F_X(t)$ under $Exp(\theta)$ is $\tilde{F}(t) = 1 - e^{-(t/8.7)I(t>0)}$. $\tilde{S}(2.5) = e^{-2.5/8.7} \approx 0.7502$.

Under $U(0,\theta)$ the MLE of θ is $\check{\theta} = X_{(n)} = 23$.

The MLE of $F_X(t)$ under $U(0,\theta)$ is $U(0,23)$, $\check{F}(t) = \begin{cases} t/23 & t \in (0,23) \\ 1 & t \geq 23, \end{cases}$ $\check{S}(2.5) \approx 0.891$.

The MME of $F(t)$ under $U(0,\theta)$ is $\tilde{F}(t) = \begin{cases} \frac{I(t \in (0, 2\overline{X}])}{2\overline{X}} & t \in (0, 2\overline{X}) \\ 1 & t \geq 2\overline{X} \end{cases}$. $\tilde{S}(2.5) \approx 0.86$.

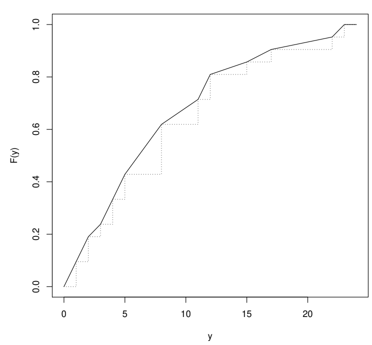
Use the EDF $\hat{F}(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t)$ $\hat{S}(2.5) \approx 0.81$.

t:	1, 2, 3, 4, 5, 8, 11, 12, 15, 17, 22, 23
$\sum_i I(X_i \leq t)$:	2, 4, 5, 7, 9, 13, 15, 17, 18, 19, 20, 21

$$\hat{F}(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{2}{21} & \text{if } t \in [1,2) \\ \frac{4}{21} & \text{if } t \in [2,3) \\ \frac{5}{21} & \text{if } t \in [3,4) \\ \frac{7}{21} & \text{if } t \in [4,5) \\ \frac{9}{21} & \text{if } t \in [5,8) \\ \frac{13}{21} & \text{if } t \in [8,11) \\ \frac{15}{21} & \text{if } t \in [11,12) \\ \frac{17}{21} & \text{if } t \in [12,15) \\ \frac{18}{21} & \text{if } t \in [15,17) \\ \frac{19}{21} & \text{if } t \in [17,22) \\ \frac{20}{21} & \text{if } t \in [22,23) \\ 1 & \text{if } t \geq 23 \end{cases} \quad \check{F}(t) = \begin{cases} 0 & \text{if } t < 0 \\ (1-s)2/21 & \text{if } t = 1-s \text{ and } s \in [0,1) \\ s\frac{2}{21} + (1-s)\frac{4}{21} & \text{if } t = s + 2(1-s) \text{ and } s \in [0,1) \\ s\frac{4}{21} + (1-s)\frac{5}{21} & \text{if } t = 2s + 3(1-s) \text{ and } s \in [0,1) \\ s\frac{5}{21} + (1-s)\frac{7}{21} & \text{if } t = 3s + 4(1-s) \text{ and } s \in [0,1) \\ s\frac{7}{21} + (1-s)\frac{9}{21} & \text{if } t = 4s + 5(1-s) \text{ and } s \in [0,1) \\ s\frac{9}{21} + (1-s)\frac{13}{21} & \text{if } t = 5s + 8(1-s) \text{ and } s \in [0,1) \\ s\frac{13}{21} + (1-s)\frac{15}{21} & \text{if } t = 8s + 11(1-s) \text{ and } s \in [0,1) \\ s\frac{15}{21} + (1-s)\frac{17}{21} & \text{if } t = 11s + 12(1-s) \text{ and } s \in [0,1) \\ s\frac{17}{21} + (1-s)\frac{18}{21} & \text{if } t = 12s + 15(1-s) \text{ and } s \in [0,1) \\ s\frac{18}{21} + (1-s)\frac{19}{21} & \text{if } t = 15s + 17(1-s) \text{ and } s \in [0,1) \\ s\frac{19}{21} + (1-s)\frac{20}{21} & \text{if } t = 17s + 22(1-s) \text{ and } s \in [0,1) \\ s\frac{20}{21} + (1-s)\frac{21}{21} & \text{if } t = 22s + 23(1-s) \text{ and } s \in [0,1) \\ 1 & \text{if } t \geq 23 \end{cases}$$

The curve of $\hat{F}(t)$ is a step function with jumps at $\{1, 2, 3, 4, 5, 8, 11, 12, 15, 17, 22, 23\}$.

The curve of $\check{F}(t)$ is a piecewise linear curve of $\hat{F}$.



$$\begin{cases} X = c(0,1,2,3,4,5,8,11,12,15,17,22,23,24) \\ y = c(0,2,4,5,7,9,13,15,17,18,19,20,21,21)/21 \\ plot(X, y, type = "s", lty = 3, xlab = "y", ylab = "F(y)") \\ lines(X, y, type = "l", lty = 1) \end{cases}$$

$\tilde{S}(2.5) \approx 0.8$, $\ddot{S}(2.5) \approx 0.78$, $\hat{S}(2.5) \approx 0.75$, $\check{S}(2.5) \approx 0.89$.

Right censored data.

If one observed X_1, ..., X_n, which are i.i.d. from F_X, it is called a **complete data** set.
Sometimes, one cannot observed each X_i, *e.g.*, in life expectancy survey for $E(T(0))$.

Recall $T(x)$. $T(0) = ?$

Ideally, we collect $T_1(0)$, ..., $T_n(0)$, then $E(T(0))$ can be estimated by $\overline{T(0)}$ (its meaning ?)
In a survey, we try to record $T_1(0)$, ..., $T_n(0)$, but end up with *e.g.*, 2+, 10+, 70+, 89, 95, 70.

What do they mean ?

Sometimes, one cannot observed each X_i. Instead, one observed data as above, or (Z_1, δ_1), ..., (Z_n, δ_n), where $Z_i = \min\{X_i, C_i\}$, (X_1, C_1), ..., (X_n, C_n) are i.i.d. from $F_{X,C}$

$(= F_X F_C)$, and $\delta_i = I(X_i \le C_i)$,

it is called a right censored (RC) data set.

The RC data are often recorded as Z_i (if exact) or Z_i+ (if right-censored).

What are (Z_i, δ_i)'s corresponding to 2+, 10+, 70+, 89, 95, 70 ?

For complete data, the common non-parametric estimator of S is the NPMLE, *i.e.*, $\hat{S} = \frac{1}{n}\sum_{i=1}^{n} I(X_i > t)$, as $S(t) = P(X > t)$. $\hat{E}(X) = \overline{X} = \sum_t t\hat{f}(t)$, where $\hat{f}(t) = \overline{I(X = t)}$. For RC data, can we use $\hat{S}$ and $E(T(0)) \approx \overline{T(0)}$? *e.g.*, can we say

$$E(T(0)) \approx \frac{2 + 10 + 70 + 89 + 95 + 70}{6} = 56?$$

Thus we need a new estimator of S as well. There are two common non-parametric estimators of a survival function with RC data. One is the NPMLE, which is also called

the **Kaplan-Meier estimator** (KME) or the **product-limit-estimator** (PLE):

$$\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \frac{d_k}{r_k}) \text{ and } \hat{f}_{pl}(t) = \hat{S}_{pl}(t-) - \hat{S}_{pl}(t),$$

where $t_1 < \cdots < t_m$ are distinct values of Z_i's with $\delta_i = 1$,

d_k is the number of person died (or event happened) at time t_k, and

r_k is the number of person at risk at time t_k $(= \sum_{i=1}^{n} I(Z_i \ge t_k))$.

How about given 2+, 10+, 70+, 89, 95, 70 ?

$k:$	1	2	3
$t_k:$	70	89	95
d_k	1	1	1
r_k	4	2	1
$\hat{S}(t_k)$	$1 - \frac{1}{4}$	$(1 - \frac{1}{4})(1 - \frac{1}{2})$	?
$\hat{f}(t_k):$	$\frac{1}{4}$	?	$\frac{3}{4}\frac{1}{2}$

$E(\hat{T}(0)) = 70/4 + (89 + 95)3/8 = 86.5$ v.s. 56 treating $T(0)+$ as $T(0)$.

An estimator of $\sigma^2_{\hat{S}_{pl}(t)}$ is

$$\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \frac{1}{n}(\hat{S}_{pl}(t))^2 \sum_{k:\ t_k \le t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}, \ \textit{where} \qquad \hat{S}_Z(t) = \frac{1}{n}\sum_{i=1}^{n} I(Z_i > t).$$

A 95% confidence interval (CI) of $S_X(t)$ is $\hat{S}_{pl}(t) \pm 1.96\hat{\sigma}_{\hat{S}_{pl}(t)}$. **What does CI mean ?**

$$P(S_X(t) \in [\hat{S}_{pl}(t) - 1.96\hat{\sigma}_{S_{pl}(t)}\hat{S}_{pl}(t) + 1.96\hat{\sigma}_{S_{pl}(t)}]) \approx 0.95.$$

Another estimator is the **Nelson-Aalen estimator**:

$$\tilde{S}_{NA}(t) = e^{-\sum_{t_k \le t} \frac{d_k}{r_k}}$$

Its variance can be estimated by $\hat{\sigma}^2_{\tilde{S}_{NA}(t)} = (\tilde{S}_{NA}(t))^2 \hat{\sigma}^2_{H(t)}$, where

$\hat{\sigma}^2_{H(t)} = \sum_{t_j \le t} \frac{(r_j - d_j)d_j}{(r_j - 1)r_j^2}$.

A 95% CI of $S_X(t)$ is $\tilde{S}_{NA}(t) \pm 1.96\hat{\sigma}_{\tilde{S}_{NA}(t)}$.

Questions: $\hat{S}_{pl}(0) = ?$ why ? $\tilde{S}_{NA}(0) = ?$ why ?

The PLE can be derived by the **Redistribution to the right method** too (in 3 steps):
(1) Order the observations $Z_{(1)} \le \cdots \le Z_{(n)}$.
(2) Initially, each observation has equal weight;
(3) Iteratively from $Z_{(1)}$ to $Z_{(n-1)}$, each RC observation assigns its up-dated weight equally to the Z_i's to its right.
See the next example.

Example 2.70. *PLE of $S(t)$ based on simple data: 60+, 70+, 70, 32, 62, 95.*

Ordered :	32	60+	62	70	70+	95	
$\hat{f}_{pl}(t)$:	$\frac{1}{6}$	0	$\frac{5}{6}\frac{1}{4}=\frac{5}{24}$	$\frac{5}{6}\frac{1}{4}=\frac{5}{24}$	0	$\frac{5}{6}\frac{2}{4}=\frac{5}{12}$	$E(\hat{X})=$?
$\hat{S}_{pl}(t)=\sum_{x>t}\hat{f}_{pl}(x)$	$\frac{5}{6}$	?	$\frac{15}{24}$	?	?	?	

How about 2+, 10+, 70+, 89, 95, 70 ?

Example 2.71. *A follow-up study on a five-year insurance policies is summarized in the next table:*

i	x_i	u_i	i	x_i	u_i
1	–	0.1	16	4.8	–
2	–	0.5	17	–	4.8
3	–	0.8	18	–	4.8
4	0.8	–	19 – 30	–	5
5	–	1.8	31	–	5
6	–	1.8	32	–	5
7	–	2.1	33	4.1	–
8	–	2.5	34	3.1	–
9	–	2.8	35	–	3.9
10	2.9	–	36	–	5
11	2.9	–	37	–	4.8
12	–	3.9	38	4.0	–
13	4.0	–	39	–	5
14	–	4.0	40	–	5
15	–	4.1			

, where

(1) i is the policy number, 1-40;

(2) x_i is the duration at which the insured was observed to die.
Those who didn't die has "-" in that column;

(3) u_i is the last duration at which those who did not die were observed.

Compute the KME and the Nelson-Aalen estimator of the survival function, as well as the estimators of their variances.

Solution. First try to understand the data, in terms of (Z_i, δ_i).

	i	1	2	3	4	5	6	...
	Z_i	0.1	0.5	0.8	0.8	1.8	1.8	...
In terms of (Z_i, δ_i):	δ_i	0	0	0	1	0	0	...
In terms of Z_i or Z_i+:	Z_i	0.1+	0.5?	0.8+	0.8?	1.8?	1.8?	...

Then rearrange the data according to time Z_i's (x_i or u_i):

old table

i	x_i	u_i	i	x_i	u_i
1	–	0.1	16	4.8	–
2	–	0.5	17	–	4.8
3	–	0.8	18	–	4.8
4	0.8	–	19 – 30	–	5
5	–	1.8	31	–	5
6	–	1.8	32	–	5
7	–	2.1	33	4.1	–
8	–	2.5	34	3.1	–
9	–	2.8	35	–	3.9
10	2.9	–	36	–	5
11	2.9	–	37	–	4.8
12	–	3.9	38	4.0	–
13	4.0	–	39	–	5
14	–	4.0	40	–	5
15	–	4.1			

is ordered as

i	x_i	u_i	i	x_i	u_i
1	–	0.1	37	–	4.8
2	–	0.5	17	–	4.8
4?	0.8	–	18	–	4.8
3?	–	0.8	19 – 30	–	5
5	–	1.8	31	–	5
6	–	1.8	32	–	5
7	–	2.1			
8	–	2.5			
9	–	2.8			
10	2.9	–	36	–	5
11	2.9	–			
34	3.1	–			
12	–	3.9			
35	–	3.9			
38	4.0	–			
13	4.0	–	39	–	5
14	–	4.0	40	–	5
33	4.1	–			
15	–	4.1			
16	4.8	–			

The two estimators of $S(t)$ are $\hat{S}_{pl}(t) = \prod_{t_k \le t} (1 - \frac{d_k}{r_k})$ and $\tilde{S}_{NA}(t) = \exp(-\sum_{t_k \le t} \frac{d_k}{r_k})$,

where $t_1 < \cdots < t_m$ are distinct values of Z_i's with $\delta_i = 1$,
d_k is the number of person died at time t_k, and
r_k is the number of person at risk at time t_k $(= \sum_{i=1}^n I(Z_i \ge t_k))$.

To find t_i, d_i and r_i, two steps:

1.

time	0.1	0.5	0.8	1.8	2.1	2.5	2.8	2.9	3.1	3.9	4.0	4.1	4.8	5.0	
# of events	1	1	2	2	1	1	1	2	1	2	3	2	4	17	
# of deaths	0	0	1	0	0	0	0	2	1	0	2	1	1	0	d_i?
# in risk	40	39	38	36	34	33	32	31	$\rightarrow$	$\leftarrow$	26	23	21	17	r_i?

2. (t_i, d_i, r_i) are given as folllows.

i	1	2	3	4	5	6
t_i	0.8	2.9	3.1	4.0	4.1	4.8
d_i	1	2	1	2	1	1
r_i	38	31	29	26	23	21

2. (t_i, d_i, r_i) are given as folllows.

i	1	2	3	4	5	6
t_i	0.8	2.9	3.1	4.0	4.1	4.8
d_i	1	2	1	2	1	1
r_i	38	31	29	26	23	21

The following table calculates the KME. $\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \frac{d_k}{r_k})$

Table 1. Calculation of PLE or KME		
Survival Time	$(1 - \frac{d_k}{r_k})$	$\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \frac{d_k}{r_k})$
0.8	37/38	37/38≈ 0.97
2.9	29/31	(37/38)(29/31)≈ 0.91
3.1	28/29	(37/38)(29/31)(28/29)
4	24/26	(37/38)(29/31)(28/29)(24/26)
4.1	22/23	(37/38)(29/31)(28/29)(24/26)(22/23)
4.8	20/21	(37/38)(29/31)(28/29)(24/26)(22/23)(20/21)

$$\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \tfrac{d_k}{r_k}) \approx \begin{cases} 1 & \text{if } t < 0.8, \\ 0.97 & \text{if } t = 0.8, \\ 0.91 & \text{if } t = 2.9, \\ 0.88 & \text{if } t = 3.1, \\ ... & ... \\ ... & \text{if } t \ge 4.8. \end{cases} \textbf{?? or } \approx \begin{cases} 1 & \text{if } t < 0.8, \\ 0.97 & \text{if } t \in [0.8, 2.9), \\ 0.91 & \text{if } t \in [2.9, 3.1), \\ 0.88 & \text{if } t \in [3.1, 4), \\ ... & ... \\ ... & \text{if } t \ge 4.8. \end{cases} ??$$

$$\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \tfrac{d_k}{r_k}) \approx \begin{cases} 1 & \text{if } t < 0.8, \\ 0.97 & \text{if } t \in [0.8, 2.9), \\ 0.91 & \text{if } t \in [2.9, 3.1), \\ 0.88 & \text{if } t \in [3.1, 4), \\ ... & ... \\ ... & \text{if } t \ge 4.8. \end{cases} \qquad \hat{\sigma}^2_{\hat{S}_{pl}(t)} = ??$$

$$\begin{aligned} \hat{\sigma}^2_{\hat{S}_{pl}(t)} =& \frac{1}{n}(\hat{S}_{pl}(t))^2 \sum_{k:\ t_k \le t} \frac{\hat{f}_{pl}(t_k) \quad (\textit{see (2) below})}{\underbrace{\hat{S}_Z(t_k-)}_{\textit{see (3)}} \hat{S}_{pl}(t_k)}, \\ =& \begin{cases} ? & \text{if } t < 0.8 \\ \frac{1}{40}(\hat{S}_{pl}(0.8))^2 \frac{\hat{f}_{pl}(0.8)}{\hat{S}_Z(0.8-)\hat{S}_{pl}(0.8)} & \text{if } t \in [0.8, 2.9) \\ \frac{1}{40}(\hat{S}_{pl}(2.9))^2 [\frac{\hat{f}_{pl}(0.8)}{\hat{S}_Z(0.8-)\hat{S}_{pl}(0.8)} + \frac{\hat{f}_{pl}(2.9)}{\hat{S}_Z(2.9-)\hat{S}_{pl}(2.9)}] & \text{if } t \in [2.9, 3.1) \\ \cdots & \cdots \end{cases} =? \end{aligned}$$

$$\text{need to derive } \hat{f}_{pl}(t) = \hat{S}_{pl}(t-) - \hat{S}_{pl}(t) \approx \begin{cases} 1 - 0.97 = 0.03 & \text{if } t = 0.8, \\ 0.97 - 0.91 = 0.06 & \text{if } t = 2.9, \\ 0.91 - 0.88 = 0.03 & \text{if } t = 3.1, \\ ... & ... \end{cases} \tag{2}$$

$$\hat{S}_Z(t_k) = \frac{1}{n}\sum_{i=1}^n I(Z_i > t_k) \text{ and } \hat{S}_Z(t_k-) = \frac{1}{n}\sum_{i=1}^n I(Z_i \ge t_k) = \frac{r_k}{n}. \tag{3}$$

$$\hat\sigma^2_{\hat S_{pl}(t)} = \begin{cases} \frac{1}{40}(0.97)^2\frac{0.03}{\frac{38}{40}\times 0.97} & \text{if } t\in[0.8,2.9) \\ \frac{1}{40}(0.91)^2[\frac{0.03}{\frac{38}{40}\times 0.97}+\frac{0.06}{\frac{31}{40}\times 0.91}] & \text{if } t\in[2.9,3.1) \\ \cdots & \cdots \end{cases} = \begin{cases} 0.0007657895 & \text{if } t\in[0.8,2.9) \\ 0.002435273 & \text{if } t\in[2.9,3.1) \\ \cdots & \cdots \end{cases}$$

$draft$

i	1	2	3	4	5	6
t_i	0.8	2.9	3.1	4.0	4.1	4.8
d_i	1	2	1	2	1	1
r_i	38	31	29	26	23	21

R codes:

```
> (1/40)*0.97**2*0.03*40/38/0.97
    [1] 0.0007657895
> (1/40)*0.91**2*(0.03*40/38/0.97+0.06*40/31/0.91)
    [1] 0.002435273
```

The Nelson-Aalen estimator.

$$\begin{aligned}\tilde S_{NA}(t) =& e^{-\sum_{t_k\le t}\frac{d_k}{r_k}} \\ =& \begin{cases} 1 & \text{if } t<0.8, \\ \exp(-1/38) & \text{if } t\in[0.8,2.9), \\ \exp(-(\frac{1}{38}+\frac{2}{31})) & \text{if } t\in[2.9,3.1), \\ \exp(-(\frac{1}{38}+\frac{2}{31}+\frac{1}{29})) & \text{if } t\in[3.1,4), \\ ... & ... \\ \exp(-(\frac{1}{38}+\cdots+\frac{1}{21})) & \text{if } t\in[4.8,\infty). \end{cases}\end{aligned}$$

i	1	2	3	4	5	6
t_i	0.8	2.9	3.1	4.0	4.1	4.8
d_i	1	2	1	2	1	1
r_i	38	31	29	26	23	21

$\hat S_{NA}(100) =$**0 ???**

$$\hat\sigma^2_{\tilde S_{NA}(t)} = (\tilde S_{NA}(t))^2\hat\sigma^2_{H(t)}, \text{ where } \hat\sigma^2_{H(t)} = \sum_{t_j\le t}\frac{(r_j-d_j)d_j}{(r_j-1)r_j^2}.$$

$$\hat\sigma^2_{\tilde S_{NA}(t)} = \begin{cases} 1 \textit{ or } 0? & \text{if } t<0.8 \\ \exp(-2/38)\frac{(38-1)\cdot 1}{(38-1)38^2} & \text{if } t\in[0.8,2.9) \\ \exp(-2(\frac{1}{38}+\frac{2}{31}))[\frac{(38-1)\cdot 1}{(38-1)38^2}+\frac{(31-2)2}{(31-1)31^2}] & \text{if } t\in[2.9,3.1) \\ ... & ... \\ ... & \text{if } t\in[4.8,\infty) \end{cases} = \cdots$$

Announcement.

Exam on coming Monday.

First midterm exam Formulae: 447: 6-20, and 44. 450: 1-10 and 16

The blank quiz pages for 447 and 450 are on my website: 450 lecture notes 2.

Class exercise. (1) Find D, f, p, $S(x)$ and $h(x)$ if $F(x) = \begin{cases} 0 & \text{if } \ x < 0, \\ \frac{x+2}{8} & \text{if } 0 \le x < 1, \\ \frac{3x^2+4}{16} & \text{if } 1 \le x < 2, \\ 1 & \text{if } \ 2 \le x. \end{cases}$

(2) Compute $E(X)$ using two formulae: $E(X) = \int_0^\infty S(t)dt = \int tf(t)dt + \sum_{t\in D} tp(t)$.

Solution: How to find D ?

$F(x) - F(x-) = 0$ except, perhaps at $\{0, 1, 2\}$ **Why ?**

$p(0) = F(0) - F(0-) = \frac{2}{8} - 0 = \frac{1}{4}$,
$p(1) = F(1) - F(1-) = \frac{7}{16} - \frac{3}{8} = \frac{1}{16}$,
$p(2) = F(2) - F(2-) = 1 - \frac{16}{16} = 0$.

One solution: $p(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, \\ \frac{1}{16} & \text{if } x = 1. \end{cases}$ $\qquad D = \{0, 1, 2\}$ or $\{0, 1\}$?

$$f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2. \end{cases}$$ **What happens OW ?**

Why not $1 < x \le 2$?
Is p a pmf ?
Is f a df ?

Can we write $f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2, \\ 0 & otherwise \end{cases}$?

How about $f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2, \\ 0 & if\ x < 0\ or\ x > 2 \end{cases}$?

How about $f(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2, \\ 0 & otherwise \end{cases}$ **???**

Another solution: $f(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0 \\ \frac{1}{8} & \text{if } 0 < x < 1 \\ \frac{1}{16} & \text{if } x = 1 \\ \frac{3x}{8} & \text{if } 1 < x < 2 \end{cases}$ and $D = \{0, 1\}$.

$$S(x) = 1 - F(x) = \begin{cases} 1 & \text{if } \ x < 0, \\ 1 - \frac{x+2}{8} = \frac{6-x}{8} & \text{if } 0 \le x < 1, \\ 1 - \frac{3x^2+4}{16} = \frac{12-3x^2}{16} & \text{if } 1 \le x < 2, \\ 0 & \text{if } \ 2 \le x. \end{cases}$$

$$h(x) = \frac{f(x)}{S(x-)} = \begin{cases} \frac{1}{4}/1 & \text{if } \ x = 0, \\ \frac{1}{8}/\frac{6-x}{8} = \frac{1}{6-x} & \text{if } 0 < x < 1, \\ \frac{1}{16}/\frac{6-1}{8} = \frac{1}{10} & \text{if } x = 1. \\ \frac{3x}{8}/\frac{12-3x^2}{16} = \frac{2x}{12-3x^2} & \text{if } 1 < x < 2. \end{cases}$$

$$E(X) = \int_0^\infty S(t)dt = \int tf(t)dt + \sum_{t \in D} tf(t)$$

$$E(X) = \int_0^\infty S(x)dx = \int_0^1 \frac{6-x}{8}dx + \int_1^2 \frac{12-3x^2}{16}dx = \frac{6x-x^2/2}{8}\Big|_0^1 + \frac{12x-x^3}{16}\Big|_1^2$$

$$E(X) = \int xf(x)dx + \sum_{x \in D} xf(x) = 0 \cdot \frac{1}{4} + \frac{1}{8}x^2/2\Big|_0^1 + \frac{1}{16} \cdot 1 + \frac{3x^2}{4}\Big|_1^2$$

CHAPTER 3

Life Tables

3.1 Life tables

Definition 3.1.
ℓ_x denotes the **number of individuals alive** *at age x, where $x \geq 0$.*
ℓ_x is also called the **number living** *or the* **number of lives** *at age x.*
ℓ_0 is called the **radix** *of a life table.*
A **life table** *(see Example 3.1) is a display of ℓ_k, for each $k = 0, 1, 2, \ldots$*
${}_td_x$ denotes the number of people which died between ages in $[x, x+t)$. $\quad d_x = {}_1d_x$.

Based on life table, one can estimate $F_X(t)$ by the EDF

$\hat{F}(t) = \frac{1}{n}\sum_{i=1}^n I(X_i \leq t) = 1 - \frac{\ell_t}{l_0}$ at $t = 0, 1, 2, \ldots$, **any problem ?**

main formula:
$$\hat{S}_X(x) = \frac{\ell_x}{\ell_0}, \text{ and } {}_tp_x = \frac{\ell_{x+t}}{\ell_x}, \qquad (in\ [11])$$

secondary:
$$ {}_tq_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} = \frac{{}_td_x}{\ell_x},\ {}_td_x = \ell_x - \ell_{x+t},$$
$$p_x = \frac{\ell_{x+1}}{\ell_x},\ q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x},\ {}_{n|m}q_x = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x} = \frac{{}_md_{x+n}}{\ell_x}.$$

Example 3.1. *Complete the entries in the table:*

Age	ℓ_x	d_x	p_x	q_x
0	100000	.	.	.
1	97523	.	.	.
2	94123	.	.	.
3	91174	.	.	.
4	87234	.	.	.
5	85938	–	–	–

Solution:

Age x	ℓ_x	$d_x = \ell_x - \ell_{x+1}$	$p_x = \ell_{x+1}/\ell_x$	$q_x = 1 - p_x = d_x/\ell_x$
0	100000	2477	0.97523	0.02477
1	97523	3400	0.96514	0.03486
2	94123	2949	0.96867	0.03133
3	91174	3940	0.95679	0.04321
4	87234	?	?	?
5	85938	–	–	–
	?????	1290	0.98514	0.01486

By Eq.(3.1) and Formula [1]-[9], we can compute $E(K_x)$ $(= E(\lceil T(x) \rceil))$, e_x, $e_{x:\overline{n}|}$, ..., *e.g.* $e_{1:\overline{2}|} = ?$ (see [9]).

$$e_{1:\overline{2}|} = \sum_{k=1}^n {}_kp_x = {}_1p_1 + {}_2p_1 = p_1 + p_1 \cdot p_2 = p_1 + \frac{\ell_2}{\ell_1}\frac{\ell_3}{\ell_2} = p_1 + \frac{\ell_3}{\ell_1} \qquad \neq p_1 + \frac{\ell_1 - \ell_3}{\ell_1}$$
$$= 0.96514 + 0.96514 * 0.96867 = 1.900042.$$

Example 3.2. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

(i) Calculate d_x *for* $x = 80, 81, \ldots, 86$.
(ii) Calculate the d.f. of the curtate life $K(80)$ *and the time interval of death* K_{80}. *(*f_{K_x}*)*
(iii) Calculate e_{80}, $\mathrm{Var}(K(80))$, *and* $e_{80:\overline{3}|}$

Solution: (i) $d_x = ?$ Try !

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
$d_x = \ell_x - \ell_{x+1}$	33	56	54	45	34	28	0

(ii) $f_{K(x)} = ?$ $K(x) = K_x - 1$, $K_x = \lceil T(x) \rceil$ (see [8]). which do you prefer ?
$f_{K(80)}(k) = P\{k < T(x) \le k+1\} = {}_kp_x - {}_{k+1}p_x = \frac{\ell_{x+k} - \ell_{x+k+1}}{\ell_x} = \frac{d_{x+k}}{\ell_x}$ (see [11]).

$f_{K(x)}(k) = \frac{d_{x+k}}{\ell_x}$

k	0	1	2	3	4	5	6
$f_{K(x)}(k)$	33/250	56/250	54/250	45/250	34/250	28/250	0

$f_{K_x}(k+1) = f_{K(x)}(k)$

k	1	2	3	4	5	6	7
$f_{K_x}(k)$	?	?	?	?	?	?	0

(iii) $e_{80} = E[K(80)] = ?$ $V(K(80)) = E[(K(80))^2] - e_{80}^2 = ?$
Two ways for $E(K(x))$ and $E((K(x))^2)$:
(1) $E(K(x)) = \sum_y y f_{K(x)}(y)$ and $E((K(x))^2) = \sum_y y^2 f_{K(x)}(y)$
(2) $E(K(x)) = \sum_{k=1}^{\infty} {}_kp_x$ [2] and $E((K(x))^2) = \sum_{k=1}^{\infty}(2k-1){}_kp_x$ $({}_kp_x = \frac{\ell_{x+k}}{\ell_x}$ (see [11])).

$$
\begin{aligned}
(1) \qquad e_{80} =& E[K(80)] = \sum_k k f_{K(x)}(k) \\
=& (0)\frac{33}{250} + (1)\frac{56}{250} + (2)\frac{54}{250} + (3)\frac{45}{250} + (4)\frac{34}{250} + (5)\frac{28}{250} = 2.3, \\
E[(K(80))^2] =& (0)^2\frac{33}{250} + (1)^2\frac{56}{250} + (2)^2\frac{54}{250} + (3)^2\frac{45}{250} + (4)^2\frac{34}{250} + (5)^2\frac{28}{250} = 7.684. \\
(2) \qquad e_{80} =& \sum_{k=1}^{\infty} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217}{250} + \frac{161}{250} + \frac{107}{250} + \frac{62}{250} + \frac{28}{250} = 2.3, \\
E[(K(80))^2] =& \sum_{k=1}^{\infty}(2k-1)\frac{\ell_{x+k}}{\ell_x} \\
=& (1)\frac{217}{250} + (3)\frac{161}{250} + (5)\frac{107}{250} + (7)\frac{62}{250} + (9)\frac{28}{250} = 7.684, \\
\mathrm{Var}(K(80)) =& 7.684 - (2.3)^2 = 2.394.
\end{aligned}
$$

Note: Without (ii), method (2) is faster.

Two ways for $e_{x:\overline{n}|} = E(K(x) \wedge n)$:
(1) $E(g(X)) = \sum_x g(x) f_X(x)$ **Q:** $X = ?$ $g(X)$ **??**
(2) $E(K(x) \wedge n) = \sum_{k=1}^{n} {}_kp_x = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x}$.

$$E(g(X)) = \sum_{k=0}^{n} k f_X(k) + \sum_{k>n} n f_X(k) = \sum_{k=0}^{n} k f_{K(x)}(k) + \sum_{k>n} n f_{K(x)}(k).$$

$$e_{80:\overline{3}|} = \sum_{k=1}^{3} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217}{250} + \frac{161}{250} + \frac{107}{250} \approx 1.94.$$

Example 3.3. *Using the life table (Table D.1) in the end of the textbook find:*

(i) ℓ_{10}.
(ii) d_{35}.
(iii) ${}_5d_{35}$.
(iv) P(a newborn will die before reaching 50 years).
(v) P(a newborn will live more than 60 years).
(vi) P(a newborn will die when his age is between 45 and 65 years old).
(vii) P(a 25–year old will die before reaching 50 years).
(viii) P(a 25–year old will live more than 60 years).
(ix) The probability that a 25–year old will die when his age is between 50 and 65 years old.

Solution: (i) $\ell_{10} = 99129$.

(ii) $d_{35} = \ell_{35} - \ell_{36} = 97250 - 97126 = 124$.

(iii) ${}_5d_{35} = \ell_{35} - \ell_{40} = 97250 - 96517 = 733$.

(iv) $P(T(0) \leq 50) = 1 - s(50) = 1 - \frac{\ell_{50}}{\ell_0} = 1 - \frac{93735}{100000} = 0.06265$.

(v) $P(T(0) > 60) = s(60) = \frac{\ell_{60}}{\ell_0} = \frac{88038}{100000} = 0.88038$.

(vi) $P(45 < T(0) \leq 65) = s(45) - s(65) = \frac{\ell_{45}-\ell_{65}}{\ell_0} = \frac{95406-83114}{100000} = 0.12292$.

(vii) ${}_{25}q_{25} = 1 - {}_{25}p_{25} = 1 - \frac{\ell_{50}}{\ell_{25}} = \frac{98246-93735}{98246} = 0.04591535533$.

(viii) ${}_{35}p_{25} = \frac{\ell_{60}}{\ell_{25}} = \frac{88038}{98246} = 0.896097551$.

(ix) ${}_{25|15}q_{25} = {}_{25}p_{25} - {}_{40}p_{25} = \frac{\ell_{50}-\ell_{65}}{\ell_{25}} = \frac{93735-83114}{98246} = 0.1081061824$.

One may skip to section 3.4.

Theorem 3.1.

Example 3.4.

Definition 3.2.

Definition 3.3.

Definition 3.4.

3.2 Mathematical models

3.3 Deterministic survivorship group and stochastic survivorship group

There are two interpretation about what a life table is.

deterministic survivorship group assumes

(i) The initial group consists of ℓ_0 lives at zero.

(ii) The group is closed. We are able to track all the initial lives and we do not add any individuals to the group.

(iii) ℓ_x denotes the number of individuals alive at time x.

According to the deterministic model, the proportion of people who die at a certain age is given by a life table.

Usually, it is very difficult to track an initial group of lives for a long time. We should expect life expectancies to change over time. A life table using data from people born 100 years ago is not very useful to determine the death rates of the current population.

Often life tables are constructed first estimating the survival function $s(\cdot)$ and assuming that the number of alive individuals follow the same survival function. If this happen, we have a **random survivorship group**, which assumes:

1. ℓ_0 individuals alive at time zero. Let $X_1, \dots, X_{\ell_0}$ be the age–at–death random variables of these individuals.

2. $X_1, \dots, X_{\ell_0}$ are i.i.d. r.v. with survival function $s(\cdot)$ $(s(t) = P(X_i > t))$.

3. The number of individuals alive at time x is the r.v. $\mathcal{L}(x)$.
$\mathcal{L}(x) = \sum_{j=1}^{\ell_0} I(X_j > x)$. $\mathcal{L}(x) \sim bin(\ell_0, s(x))$.
$E[\mathcal{L}(x)] = \ell_0 s(x)$ and $\mathrm{Var}(\mathcal{L}(x)) = \ell_0 s(x)(1 - s(x))$.
In a life table ℓ_x is $\ell_0 s(x)$ rounded up.

Both the deterministic survivorship group and the random survivorship group allow to use past data to predict future lifetimes of a group of individuals.

Some of the previous formulas have an intuitive interpretation using the group determinist approach to life tables.

Consider $e_x = \sum_{k=1}^{\infty} {}_k p_x$.

The number of survivors at time x is ℓ_x. The average complete years lived by the ℓ_x survivors at time x is e_x. So, the total number of complete future years lived by the ℓ_x survivors at time x is $\ell_x e_x$. ℓ_{x+k} is the number of the ℓ_x survivors at time x who live the k–th year, i.e. the period of time $(x + k - 1, x + k]$. Hence, $\sum_{k=1}^{\infty} \ell_{x+k}$ is the total number of complete future years lived by the ℓ_x survivors at time x. Hence,

$\ell_x e_x = \sum_{k=1}^{\infty} \ell_{x+k}$ and
$e_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x} = \sum_{k=1}^{\infty} {}_k p_x$.

Consider $e_x = p_x(1 + e_{x+1})$. The number of lives aged x is ℓ_x. The complete number of years lived by all lives aged x is $\ell_x e_x$. From these ℓ_x lives, during the first year $\ell_x - \ell_{x+1}$ lives die and do not live a complete year. From these ℓ_x lives, ℓ_{x+1} lives survive one year and live one year plus some complete of years after time $x + 1$. The complete number of years lived by all lives aged $x + 1$ is $\ell_{x+1} e_{x+1}$. Hence, $\ell_x e_x = \ell_{x+1} + \ell_{x+1} e_{x+1} = \ell_{x+1}(1 + e_{x+1})$, which implies that $e_x = p_x(1 + e_{x+1})$.

Consider $e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}$. The number of lives aged x is ℓ_x. We have that the complete number of years lived by all lives aged x is $\ell_x e_x$. $\ell_x e_x$ is the complete number of years lived by all lives aged x between times x and $x + n$ plus the complete number of years lived by all lives aged x after time $x + n$. The complete number of years lived by all lives aged x between times x and $x + n$ is $\ell_x e_{x:\overline{n}|}$. The lives aged x who live complete years after time $x + n$ are the ones that survive time $x + n$. The average complete years lived by each of the ℓ_{x+n} survivors at time $x + n$ is e_{x+n}. Hence, the complete number of years lived by all lives aged x after time $x + n$ is $\ell_{x+n} e_{x+n}$. Therefore, $\ell_x e_x = \ell_x e_{x:\overline{n}|} + \ell_{x+n} e_{x+n}$, which implies $e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}$.

The rest of the section can be ignored!!

Theorem 3.2.

Theorem 3.3.

Theorem 3.4.

Corollary 3.1.

Theorem 3.5.

Example 3.5.

Theorem 3.6.

Corollary 3.2.

Theorem 3.7.

Example 3.6.

Example 3.7.

3.4 Continuous computations.

Assuming X is cts, knowing ℓ_x for each real number $x \geq 0$, we can get the following:

$$\begin{array}{rl}
\textbf{main} & \textit{secondary} \\
s(x) = \dfrac{\ell_x}{\ell_0}, & f(x) = -\dfrac{d}{dx}\dfrac{\ell_x}{\ell_0},\ \mu(x) = -\dfrac{d}{dx}(\ln(\ell_x)) = -\dfrac{\ell'_x}{\ell_x}, \\
{}_tp_x = \dfrac{\ell_{x+t}}{\ell_x}, & {}_tq_x = \dfrac{\ell_x - \ell_{x+t}}{\ell_x},\ f_{T(x)}(t) = -\dfrac{d}{dt}\dfrac{\ell_{x+t}}{\ell_x},\ \mu_{x+t} = -\dfrac{d}{dt}\ln\dfrac{\ell_{x+t}}{\ell_x}, \\
 & \mathring{e}_0 = \displaystyle\int_0^\infty \frac{\ell_x}{\ell_0}dx,\ \mathring{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x}dt,\ \mathring{e}_{x:\overline{n}|} = \int_0^n \frac{\ell_{x+t}}{\ell_x}dt, \\
 & {}_nm_x = \dfrac{\int_0^n \ell_{x+t}\mu_x(t)\,dt}{\int_0^n \ell_{x+t}\,dt}.
\end{array}$$

However, from the life table, we only know ℓ_x at $x = 0, 1, 2, 3, ...$

Q: How to get ℓ_x for $x \in [0, \infty)$?

Ans: linear interpolation or non-linear interpolation.

Linear interpolation of $F(x)$ is just

$$\check{F}(t) = \begin{cases} 0 & \text{if } t < 0, \\ \hat{F}(i) & i = t \in \{0, ..., n\}, \\ s\hat{F}(i) + (1-s)\hat{F}(i+1) & \text{if } t = si + (1-s)(i+1),\ s \in (0,1),\ \ i \in \{0, ..., n-1\}, \\ 1 & \text{if } t > n. \end{cases}$$

Using linear interpolation, Figure 3.1-3.5 show the graphs of the survival function of the age–at–death using Table D.1 (see page 602 (textbook)) as well as the density, d_x and $\mu(x)$.

3.5 Interpolating life tables

Life tables only show the values of ℓ_x whenever x is a nonnegative integer. In many computations, we need to know ℓ_x for each $x \geq 0$.

Let $\underline{f(x) = \ell_x}$. Suppose that $x_1 < x_2 < \cdots < x_n$ and $f(x_i)$'s are known but not other $f(x)$. We can estimate the values of $f(x)$ for $x \in (x_1, x_n)$ by linear function or nonlinear function:
(1) **Linear interpolation.** Straight line equation: $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$ or $y = y_1 + \frac{y_2-y_1}{x_2-x_1}(x - x_1)$
$f(x) = f(x_j) + \frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j}(x - x_j)$
$= (1 - \frac{x-x_j}{x_{j+1}-x_j})f(x_j) + \frac{x-x_j}{x_{j+1}-x_j}f(x_{j+1}),\ x \in (x_j, x_{j+1})$.
If $x_{j+1} - x_j = 1$ and $x = x_j + t$, then
$f(x_j + t) = (1 - t)f(x_j) + tf(x_{j+1}),\ t \in (0, 1)$.
(2) **Nonlinear interpolation**. $f(x)$ is a curve passing through x_j's.

3.5.1 Uniform distribution of deaths is to assume

$$\ell_{j+t} = (1-t)\ell_j + t\ell_{j+1} = \ell_j - t \cdot d_j,\ \underbrace{0 \leq t \leq 1, \quad j \in \{0, 1, 2, ...\}}_{important}. \tag{3.1}$$

We say that X **is uniform on** $(j, j+1)$ or say a **uniform distribution of deaths (UDD)** or say **linear interpolation for the number of living**.

Q: How to compute the following quantities under UDD ?

$_tp_x = S_{T(x)}(t)$
$p_x = {}_1p_x = S_{T(x)}(1)$
$_{s|t}q_x = P(s < T(x) \leq s + t)$
$_{t|}q_x = {}_{t|1}q_x$,
$_tq_x = F_{T(x)}(t)$.

(3.2) Ans: Key formula: $_tp_x = \frac{l_{x+t}}{l_x}$ and Eq. (3.1).

Theorem 3.8. *Under form (3.1),* $\overbrace{\forall\ x = 0,\ 1,\ 2,\ \ldots\ \text{and}\ \forall\ t \in [0, 1]}^{important}$:
(i) $_tp_x = 1 - tq_x$. *(ii)* $_tq_x = tq_x$. **Notice the difference !**
(iii) $f_{T(x)}(t) = q_x$. *(iv)* $\mu_{x+t} = \frac{q_x}{1-tq_x}$

Remark. No need to memorize Th. 3.8, they can be derived easily by (3.2).

Proof. Notice the assumption: $t \in [0, 1]$. By (3.2) and (3.1),
(i) $S_{T(x)}(t) = {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_x - t \cdot d_x}{\ell_x} = 1 - t\frac{d_x}{\ell_x} = 1 - tq_x$, as $\frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = 1 - {}_1p_x = q_x$.
(ii) $F_{T(x)}(t) = {}_tq_x = 1 - {}_tp_x = tq_x$ (by (i)).
(iii) $f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x = q_x$ (by (1)).
(iv) $\mu_{x+t} = \frac{f_{T(x)}(t)}{S_{T(x)}(t)} = \frac{q_x}{1-tq_x}$. ∎

Q: $S_{T(x)}(t) = 1 - tq_x$ **for each** t **and each** x **?**

Example 3.8. *Using the life table D.1 p. 602-605 and assuming UDD, find: (i)* ${}_{0.1}p_{35}$ *and (ii)* ${}_{1.4}p_{35.3}$.

Solution: Formulas: ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\ell_{x+t}}{\ell_x}$, [12] $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$, $t \in [0,1]$, $x = 1, 2, ...$
(i) ${}_{0.1}p_{35} = \frac{\ell_{35.1}}{\ell_{35}} = \frac{\ell_{35.1}}{97250}$. $\ell_{35.1} = \ell_{x+t}$, $(x,t) = ?$
Ans: $\ell_{35.1} = \ell_{35+0.1} = (1-0.1)\ell_{35} + (0.1)\ell_{36} = (0.9)(97250) + (0.1)(97126) = 97237.6$
${}_{0.1}p_{35} = \frac{\ell_{35.1}}{\ell_{35}} = \frac{97237.6}{97250} \approx 0.9998725.$

(ii) ${}_{1.4}p_{35.3} = \frac{\ell_{36.7}}{\ell_{35.3}} = ??$ $\begin{cases} \ell_{36.7} = \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1} & (x,t) = ??? \\ \ell_{35.3} = \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1} & (x,t) = ??? \end{cases}$

Ans: $\ell_{36.7} = \ell_{36+0.7} = (1-0.7)\ell_{36} + (0.7)\ell_{37} = (0.3)(97126) + (0.7)(96993) = 97032.9,$
$\ell_{35.3} = \ell_{35+0.3} = (1-0.3)\ell_{35} + (0.3)\ell_{36} = (0.7)(97250) + (0.3)(97126) = 97212.8,$
${}_{1.4}p_{35.3} = \frac{\ell_{36.7}}{\ell_{35.3}} = \frac{97032.9}{97212.8} \approx 0.9981.$

Example 3.9. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assume linear interpolation.
(A) Calculate the complete expected life at 80.
(B) Calculate $\mathring{e}_{80:\overline{3}|}$.
(C) Calculate ${}_3m_{80}$ *(central death rate).*

Solution: (A) [1] $E(T(x)) = \int t f_{T(x)}(t)dt = \int_0^\infty {}_tp_x dt = ?$
(B) [6] $\mathring{e}_{80:\overline{3}|} = E(T(x) \wedge 3) = \int (t \wedge n) f_{T(x)}(t)dt = \int_0^n {}_tp_x dt = ?$
(C) [7] ${}_3m_{80} = \frac{\int_x^{x+n} S_X(t)\mu_X(t)\,dt}{\int_x^{x+n} S_X(t)\,dt} = \frac{\int_0^n S_{T(x)}(t)\mu_{T(x)}(t)\,dt}{\int_0^n S_{T(x)}(t)\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}} = \frac{{}_3q_{80}}{\mathring{e}_{80:\overline{3}|}} = ?$

Two usual ways for A, B: $\begin{cases} \text{(a) based on } {}_tp_x \ (= \frac{\ell_{x+t}}{\ell_x}); & \textit{why?} \\ \text{(b) based on } f_{T(x)} \ (f_{T(x)}(t) = (-{}_tp_x)'_t), & \textit{why?} \end{cases}$

which way is preferred here ?

Moreover, both (a) and (b) need formula [11] $\ell_{x+t} = \begin{cases} (1-t)\ell_x + t\ell_{x+1} \\ \ell_x - t \cdot d_x \end{cases} \quad 0 \le t \le 1.$

So approach (a) has 2 preliminary steps: step (i) ℓ_{x+t}=? step (ii) ${}_tp_x$=?
Approach (a): (i) $\ell_{80+t} = ?$ for all t
Try Formula: $\ell_{x+t} = \ell_x - t \cdot d_x$, $0 \le t \le 1$, as we had d_x in Ex. 3.2:

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

Can we say that the answer is $\ell_{80+t} = \ell_{80} - d_{80}t = \ell_{80} - 33t$?
Try $t \in (1,2)$ **first to avoid mistake !!** then try $t \ge 2$.

$$\ell_{80+t} = \ell_{80+k+(t-k)} = \ell_{80+k} - \underbrace{(t-k)}_{\in(0,1)} \cdot d_{80+k}, \qquad 1 = k \le t < k+1!!$$

Class exercise !

$$\ell_{80+t} = \begin{cases} 250-33t & \text{if } 0 \le t < 1 \\ 217-56(t-1) & \text{if } 1 \le t < 2 \\ 161-54(t-2) & \text{if } 2 \le t < 3 \\ 107-45(t-3) & \text{if } 3 \le t < 4 \\ 62-34(t-4) & \text{if } 4 \le t < 5 \\ 28-28(t-5) & \text{if } 5 \le t \le 6 \\ ? & \text{if } ?? \end{cases} \Rightarrow \text{(ii) } {}_tp_{80} = \frac{\ell_{80+t}}{\ell_{80}} = \begin{cases} \frac{250-33t}{250} & \text{if } 0 \le t < 1, \\ \frac{217-56(t-1)}{250} & \text{if } 1 \le t < 2, \\ \frac{161-54(t-2)}{250} & \text{if } 2 \le t < 3, \\ \frac{107-45(t-3)}{250} & \text{if } 3 \le t < 4, \\ \frac{62-34(t-4)}{250} & \text{if } 4 \le t < 5, \\ \frac{28-28(t-5)}{250} & \text{if } 5 \le t \le 6. \end{cases}$$

$$\begin{aligned}
(A) \quad \mathring{e}_{80} &= \int_0^\infty {}_tp_{80}dt = \int_0^1 {}_tp_{80}dt + \int_1^2 {}_tp_{80}dt + \cdots + \int_5^6 {}_tp_{80}dt + \cdots \quad ?? \\
&= \int_0^1 \frac{250-33t}{250}\,dt + \int_1^2 \frac{217-56(t-1)}{250}\,dt + \int_2^3 \frac{161-54(t-2)}{250}\,dt \\
&+ \int_3^4 \frac{107-45(t-3)}{250}\,dt + \int_4^5 \frac{62-34(t-4)}{250}\,dt + \int_5^6 \frac{28-28(t-5)}{250}\,dt \quad \text{use } u = t-k \\
&= \left(\frac{250t}{250} - \frac{33t^2}{(2)(250)}\right)\Bigg|_0^1 + \left(\frac{217t}{250} - \frac{56(t-1)^2}{(2)(250)}\right)\Bigg|_1^2 + \left(\frac{161t}{250} - \frac{54(t-2)^2}{(2)(250)}\right)\Bigg|_2^3 \\
&+ \left(\frac{107t}{250} - \frac{45(t-3)^2}{(2)(250)}\right)\Bigg|_3^4 + \left(\frac{62t}{250} - \frac{34(t-4)^2}{(2)(250)}\right)\Bigg|_4^5 + \left(\frac{28t}{250} - \frac{28(t-5)^2}{(2)(250)}\right)\Bigg|_5^6 = 2.8
\end{aligned}$$

$$(B) \quad \mathring{e}_{80:\overline{3}|} = \int_0^3 {}_tp_{80}dt = \int_0^1 \frac{250-33t}{250}\,dt + \int_1^2 \frac{217-56(t-1)}{250}\,dt + \int_2^3 \frac{161-54(t-2)}{250}\,dt$$
$$= 2.226.$$

$$(C) \quad {}_3m_{80} = \frac{{}_3q_{80}}{\mathring{e}_{80:\overline{3}|}} = \frac{1-{}_3p_{80}}{\mathring{e}_{80:\overline{3}|}} \quad or \quad = \frac{\frac{\ell_{80}-\ell_{83}}{\ell_{80}}}{\mathring{e}_{80:\overline{3}|}} = \frac{\frac{250-107}{250}}{2.226} = 0.257.$$

Another way for (A) and (B):

Using that $f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt}$ if the derivative exists,

$${}_tp_{80} = \begin{cases} \frac{250-33t}{250} & \text{if } 0 \le t < 1, \\ \frac{217-56(t-1)}{250} & \text{if } 1 \le t < 2, \\ \frac{161-54(t-2)}{250} & \text{if } 2 \le t < 3, \\ \frac{107-45(t-3)}{250} & \text{if } 3 \le t < 4, \\ \frac{62-34(t-4)}{250} & \text{if } 4 \le t < 5, \\ \frac{28-28(t-5)}{250} & \text{if } 5 \le t \le 6. \end{cases} \Rightarrow f_{T(80)}(t) = \begin{cases} \frac{33}{250} & \text{if } 0 \le t < 1 \\ \frac{56}{250} & \text{if } 1 \le t < 2 \\ \frac{54}{250} & \text{if } 2 \le t < 3 \\ \frac{45}{250} & \text{if } 3 \le t < 4 \\ \frac{34}{250} & \text{if } 4 \le t < 5 \\ \frac{28}{250} & \text{if } 5 \le t \le 6 \end{cases}$$

Can we write $f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt} = \begin{cases} \frac{33}{250} & \text{if } 0 \le t < 1 \\ \frac{56}{250} & \text{if } 1 \le t < 2 \\ \dots & \dots \end{cases}$ **??**

Notice that the derivative of ${}_tp_{80}$ does not exist at $1, 2, \dots, 5$. But, the density can be defined arbitrarily at finitely many points.

$$
\begin{aligned}
(b)\quad \mathring{e}_{80} =& \int_0^\infty t f_{T(80)}(t)dt \\
=&(\int_0^1 t\frac{33}{250} + \int_1^2 t\frac{56}{250} + \int_2^3 t\frac{54}{250} + \int_3^4 t\frac{45}{250} + \int_4^5 t\frac{34}{250} + \int_5^6 t\frac{28}{250})dt \\
=&\frac{1}{2}\frac{33}{250} + \frac{3}{2}\frac{56}{250} + \frac{5}{2}\frac{54}{250} + \frac{7}{2}\frac{45}{250} + \frac{9}{2}\frac{34}{250} + \frac{11}{2}\frac{28}{250} = 2.8. \\
\mathring{e}_{80:\overline{3}|} =& \int_0^\infty (t\wedge 3) f_{T(80)}(t)dt = [\int_0^3 + \int_3^\infty](t\wedge 3) f_{T(80)}(t)dt = \int_0^3 t f(t)dt + 3{}_3p_{80} \\
=& \int_0^1 t\frac{d_{80}}{\ell_{80}}dt + \int_1^2 t\frac{d_{81}}{\ell_{80}}dt + \int_2^3 t\frac{d_{82}}{\ell_{80}}dt + 3{}_3p_{80} \approx 2.2
\end{aligned}
$$

Example 3.10.

Announcement: 1. Hw for last week due tomorrow. Hw for this week due Wednesday.
2. Quiz tomorrow 450: # 1-11, 16

Theorem 3.9. *Given $t \in [k, k+1)$, where $k \geq 0$, under UDD,*
(i) $s(t) = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.
(ii) $f_X(t)(= {}_k|q_0) = \frac{d_k}{\ell_0}$.
(iii) $f_{T(x)}(t)\ (= {}_k|q_x) = \frac{d_{x+k}}{\ell_x}$.
(iv) $\mu(t) = \frac{d_k}{\ell_k-(t-k)d_k}$.

Notice that the theorem expresses the notations in terms of ℓ_x or d_x. Instead of using them directly, you should derive them yourself in doing the homework, based on [11] and [12]:

$s(t) = \frac{\ell_t}{\ell_0}$ and $\ell_t = \ell_{k+t-k} = \ell_k - (t-k)\cdot d_k$, $t-k \in (0,1)$.

Proof. (i) $s(t) = \frac{\ell_t}{\ell_0} = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.
(ii) $f_X(t) = -\frac{d}{dt}s(t) = \frac{d_k}{\ell_0}$.
(iii) $f_{T(x)}(t) = \frac{f_X(x+t)}{s(x)}$, $x+t \in [k+x, k+x+1)$; thus $f_{T(x)}(t) = \frac{f_X(x+t)}{s(x)} = \frac{d_{x+k}/l_0}{\ell_x/l_0} = \frac{d_{x+k}}{\ell_x}$.
(iv) $\mu(t) = \frac{f_X(t)}{s(t)} = \frac{d_k/\ell_0}{\ell_k/\ell_0-(t-k)d_k/\ell_0} = \ldots$ ∎

Example 3.11. *Under the assumption of uniform distribution of deaths, find the average number of years lived between x and $x+1$ by those who die between those ages.*

Solution: The average number of years lived between x and $x+1$ by those who die in $(x, x+1]$

$$E(X-x|X \in (x,x+1]) = E(T(x)|T(x) \in (0,1]). \qquad (X = T(0))$$

Is it 0.5 under UDD ?

$E(X-x|X \in (x,x+1])$
$= \int_x^{x+1}(t-x)\frac{f_X(t)}{\int_x^{x+1} f_X(u)du}dt$

$= \int_x^{x+1}(t-x)\frac{d_x/\ell_0}{\int_x^{x+1}(d_x/\ell_0)du}dt$ (by (ii) of Theorem 3.9)
$= \int_x^{x+1}(t-x)dt = (\frac{(t-x)^2}{2})|_x^{x+1} = \frac{1}{2}$.

Insurance companies need to know the total years for all their clients.

Definition 3.5. *Denote $T_x = \ell_x \overset{\circ}{e}_x$, the expected number of years lived beyond age x by* **the cohort group with ℓ_0 members** *(different from $T_x \overset{def}{=} T(x)$ in Chapter 2).*

Definition 3.6. *Denote ${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|}$, the expected number of years lived between age x and age $x+n$* **by the ℓ_x survivors at age** *x. Denote $L_x = {}_1L_x$. ($T_x = {}_\infty L_x$).*

Formula [11]: $\underline{\ell_x} = \#$ of individuals alive at age x, $\underline{{}_1L_x} = L_x$. ${}_td_x = \underline{\ell_x - \ell_{x+t}}$, $d_x = \underline{{}_1d_x}$,

$\underline{{}_tp_x} = \prod_{x \le k < x+t}(1 - d_k/l_k)$. $\underline{T_x} = \ell_x \overset{\circ}{e}_x = \int_0^\infty \ell_{x+t}\,dt = \sum_{k=x}^\infty L_k$
$=$ E(# of years lived beyong age x by the cohort group with l_0 members),
$\underline{{}_nL_x} = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = T_x - T_{x+n}$. $s(x) = \underline{\frac{\ell_x}{\ell_0}}$, ${}_tp_x = \underline{\frac{\ell_{x+t}}{\ell_x}}$, (not the T_x in #3).

Theorem 3.10. *Under a linear form for the number of living,*
(i) $L_x = \ell_x - \frac{d_x}{2} = \ell_{x+1} + \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2}$.
(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{1+p_x}{2}$.
(iii) $T_x = \frac{\ell_x}{2} + \sum_{k=x+1}^\infty \ell_k$.
(iv) $m_x = \frac{q_x}{1-\frac{q_x}{2}}$ *(central death rate over $(x, x+1]$).*
(v) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.
(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1} \frac{\frac{\ell_k + \ell_{k+1}}{2}}{\ell_x}$.

One needs to learn how to derive these formulas rather memorize the theorem.

Proof. (i) Formulas [11]: ${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|}$, $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt$, $\begin{cases} {}_tp_x = \ell_{x+t}/\ell_x, & t...? \\ [12]\ \ell_{x+t} = \ell_x - td_x, & t...? \end{cases}$

$L_x = \int_0^1 \ell_x \cdot {}_tp_x\,dt = \int_0^1 \ell_{x+t}\,dt = \int_0^1 (\ell_x - t \cdot d_x)\,dt = \ell_x - \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2} = \ell_{x+1} + \frac{d_x}{2}$.

(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{L_x}{\ell_x}$ **(Why ??)** $= \frac{\ell_x + \ell_{x+1}}{2\ell_x}$ *(check (i))?)* $= \frac{1+p_x}{2}$ **Why ?**

(iii) Formulas: $T_x = \ell_x \overset{\circ}{e}_x = \sum_{k=x}^\infty L_k$, $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt$, ${}_tp_x = \ell_{x+t}/\ell_x$.

$$T_x = \sum_{k=x}^\infty L_k = \sum_{k=x}^\infty \left(\underbrace{\frac{\ell_k + \ell_{k+1}}{2}}_{by\ (i)} \right) = \frac{\ell_x + \ell_{x+1}}{2} + \frac{\ell_{x+1} + \ell_{x+2}}{2} + \cdots = \frac{\ell_x}{2} + \sum_{k=x+1}^\infty \ell_k.$$

(iv) $\vdash$: $m_x = \frac{q_x}{1-\frac{q_x}{2}}$. Formula [7]: ${}_nm_x = \int_x^{x+n} \frac{S_X(t)}{\int_x^{x+n} S_X(u)du} \mu_X(t)dt = {}_nq_x/\overset{\circ}{e}_{x:\overline{n}|}$

$m_x = q_x/\overset{\circ}{e}_{x:\overline{1}|} = \frac{q_x}{(1+p_x)/2}$ *(see (ii))* $= \frac{q_x}{(1+1-q_x)/2} = \frac{q_x}{1-\frac{q_x}{2}}$.

(v) $\vdash$: $\overset{\circ}{e}_x = e_x + \frac{1}{2}$. Formula[11]: $T_x = \ell_x \overset{\circ}{e}_x$ and $T_x = \frac{\ell_x}{2} + \sum_{k=x+1}^\infty \ell_k$.

$\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{1}{2} + \sum_{j=x+1}^{\infty} \frac{\ell_j}{\ell_x} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x} = \frac{1}{2} + \sum_{k=1}^{\infty} {}_kp_x = \frac{1}{2} + e_x$ by [9].

(vi) $\vdash$: $\overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1} \frac{\frac{\ell_k+\ell_{k+1}}{2}}{\ell_x}$. Formula[11]: ${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1} L_k$

$\overset{\circ}{e}_{x:\overline{n}|} = \frac{{}_nL_x}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x} = \sum_{k=x}^{x+n-1} \frac{\frac{\ell_k+\ell_{k+1}}{2}}{\ell_x}$ by (i) in the Th. ∎

[11] $\underline{\ell_x}$ = # of individuals alive at age x, $\underline{{}_nL_x} = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = T_x - T_{x+n}$, $\underline{{}_1L_x} = L_x$,

$\underline{T_x} = \ell_x \overset{\circ}{e}_x = \int_0^{\infty} \ell_{x+t}\, dt = \sum_{k=x}^{\infty} L_k$ (not T_x in #3),

= E(# of years lived beyong age x by the cohort group with l_0 members),

$\underline{{}_tp_x} = \prod_{x \le k < x+t} (1 - d_k/l_k)$, $s(x) = \underline{\frac{\ell_x}{\ell_0}}$, ${}_tp_x = \underline{\frac{\ell_{x+t}}{\ell_x}}$, ${}_td_x = \underline{\ell_x - \ell_{x+t}}$, $d_x = \underline{{}_1d_x}$,

UDD: $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$.

Under UDD, we should know how to derive ${}_tp_x$, $f_{T(x)}$, $\mu_x(t)$, $\overset{\circ}{e}_x$, $\overset{\circ}{e}_{x:\overline{n}|}$, L_x $(= \ell_x \overset{\circ}{e}_{x:\overline{1}|})$,

T_x ($= E$(# of years lived beyong age x by the cohort group with l_0 members)),

m_x (the central death rate over (x,x+1]).

Theorem 3.10 is actually one of such exercise.

Recall $S_x = T(x) - K(x)$ (§2.5), where $K(x) = \lceil T(x) \rceil$ is the curtate duration. $S_x = S_X$?

Theorem 3.11. *Under UDD, for each x, $K(x)$ and S_x are independent r.v.'s and S_x has a distribution uniform on $(0,1)$ ($S_x \sim U(0,1)$).*

Corollary 3.3. *Under the assumption of uniform distribution of deaths:*

(i) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$. *(ii)* $\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \frac{1}{12}$.

Proof (i) Since $T(x) = K(x) + S_x$, $\overset{\circ}{e}_x = E[T(x)] = E[K(x)] + E[S_x] = e_x + \frac{1}{2}$.

(ii) Since $T(x) = K(x) + S_x$ and $K(x)$ and S_x are independent,

$\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \mathrm{Var}(S_x) = \mathrm{Var}(K(x)) + \frac{1}{12}$ (as $S_x \sim U(0,1)$ and $\frac{(b-a)^2}{12}$).

Theorem 3.12.

3.5.2 Exponential interpolation.

Exponential interpolation is a non-linear interpolation:

$$\ln\ell_{x+t} = (1-t)\ln\ell_x + t\ln\ell_{x+1} \qquad (\text{ v.s. } \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}.)$$

$$\ell_{x+t} = \ell_x p_x^t = \ell_x \left(\frac{\ell_{x+1}}{\ell_x}\right)^t = (\ell_x)^{1-t}(\ell_{x+1})^t \text{ for } t \in [0,1] \text{ and } x = 0, 1, \ldots \tag{3.2}$$

We should know how to derive ${}_tp_x$, $f_{T(x)}$, $\mu_x(t)$, $\overset{\circ}{e}_x$, $\overset{\circ}{e}_{x:\overline{n}|}$, L_x, T_x, m_x , etc.

Example 3.12. *Using the life table in page 602 and exponential interpolation, find:*

(i) ${}_{0.75}p_{80}$ *(ii)* ${}_{2.25}p_{80}$.

Solution: (i) Formulas: ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\ell_{x+t}}{\ell_x}$, (3.2) or [12] => $\ell_{x+t} = \ell_x(\frac{\ell_{x+1}}{\ell_x})^t$, $t \in (0,1]$. Thus ${}_tp_x = \frac{\ell_x(\frac{\ell_{x+1}}{\ell_x})^t}{\ell_x} = (\frac{\ell_{x+1}}{\ell_x})^t$, $t \in (0,1]$. ${}_{0.75}p_{80} = \underbrace{\frac{\ell_{x+t}}{\ell_x}}_{\mathbf{t=?}}$.

$${}_{0.75}p_{80} = \left(\frac{\ell_{81}}{\ell_{80}}\right)^{0.75} = \left(\frac{50987}{53925}\right)^{0.75} = 0.958852885.$$

(ii) Two ways for ${}_{2.25}p_{80}$: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, or ${}_{t+k}p_x = {}_kp_x \cdot {}_tp_{x+k}$.

[11] => ${}_{2.25}p_{80} = \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_{82.25}}{\ell_{80}} = \frac{\ell_x^{1-t}\ell_{x+1}^t}{\ell_{80}} = \frac{\ell_{82}^{0.75}\ell_{83}^{0.25}}{\ell_{80}} = \frac{(47940)^{0.75}(44803)^{0.25}}{53925} \approx 0.8450.$

[4] => ${}_{2.25}p_{80} = {}_2p_{80} \cdot {}_{0.25}p_{82} = \frac{\ell_{82}}{\ell_{80}} \cdot (\frac{\ell_{83}}{\ell_{82}})^{0.25} = \frac{\ell_{82}^{0.75}\ell_{83}^{0.25}}{\ell_{80}} \approx 0.8450$

Example 3.13. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Using exponential interpolation, calculate
(i) the complete expected life at 80;
(ii) $\overset{\circ}{e}_{80:\overline{3}|}$;
(iii) the density function of the future life $T(80)$;
(iv) $\mu(80+t)$, $0 \le t \le 6$.

Solution: Formulas: (i) $\overset{\circ}{e}_x = \int t f_{T(x)}(t)dt = \int_0^\infty {}_tp_x dt$, ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$.
(ii) $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt = \int_0^\infty (t \wedge n) f_{T(x)}(t)dt (= \int_0^n t f_{T(x)}(t)dt + n_np_x)$,
(iii) $f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt}$, (iv) $\mu(80+t) = \mu_{T(80)}(t) = -\frac{d(\ln({}_tp_{80}))}{dt} = \frac{f_{T(80)}(t)}{{}_tp_{80}}$,
The key is ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$ and $\ell_{x+t} = \ell_x\left(\frac{\ell_{x+1}}{\ell_x}\right)^t$, $0 \le t \le 1$.

Can we say $\ell_{80+t} = \ell_{80}\left(\frac{\ell_{80+1}}{\ell_{80}}\right)^t$**,** $t \ge 0$ **?**

For k = 0,1,2,... if $t \in [k, k+1)$, then $\ell_{80+t} = \ell_{80+k+(t-k)} = \ell_{80+k}\left(\frac{\ell_{80+k+1}}{\ell_{80+k}}\right)^{t-k}$, $t-k \in [0,1]$.
$\ell_{80+t} = \ell_{80+k}\left(\frac{\ell_{80+k+1}}{\ell_{80+k}}\right)^{t-k}$, $t-k \in [0,1]$.

$$\ell_{80+t} = \begin{cases} 250\left(\frac{217}{250}\right)^t & \text{if } 0 \le t < 1, \\ 217\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t < 2, \\ 161\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t < 3, \\ 107\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t < 4, \\ 62\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \\ . \end{cases} \quad {}_tp_{80} = \frac{\ell_{x+t}}{\ell_x} = \begin{cases} \frac{250}{250}\left(\frac{217}{250}\right)^t & \text{if } 0 \le t < 1, \\ \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t < 2, \\ \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t < 3, \\ \frac{107}{250}\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t < 4, \\ \frac{62}{250}\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \end{cases} \quad \begin{matrix} {}_6p_x = ? \\ {}_{-1}p_x = ? \end{matrix}$$

(i) Two ways: $\overset{\circ}{e}_x = \int t f_{T(x)}(t)dt = \int_0^\infty {}_tp_x dt$. Which is prefer here ?

$$(i)\ \overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt = \int_0^1 {}_tp_x dt + \int_1^2 {}_tp_x dt + \cdots + \int_4^5 {}_tp_x dt + \int_5^\infty {}_tp_x dt$$

$$= \int_0^1 \frac{250}{250}\left(\frac{217}{250}\right)^t dt + \int_1^2 \frac{217}{250}\left(\frac{161}{217}\right)^{-1}\left(\frac{161}{217}\right)^t dt \quad \textbf{why ??}$$

$$+ \int_2^3 \frac{161}{250}\left(\frac{107}{161}\right)^{-2}\left(\frac{107}{161}\right)^t dt + \int_3^4 \frac{107}{250}\left(\frac{62}{107}\right)^{-3}\left(\frac{62}{107}\right)^t dt + \int_4^5 \frac{62}{250}\left(\frac{28}{62}\right)^{-4}\left(\frac{28}{62}\right)^t dt$$

$$\text{(notice } \int ab^t dt = ab^t \frac{1}{\ln b} \text{ as } (b^t)' = b^t \ln b)$$

$$= \left(\frac{250}{250}\left(\frac{217}{250}\right)^t \frac{1}{\ln\left(\frac{217}{250}\right)}\right)\Bigg|_0^1 + \left(\frac{217}{250}\left(\frac{161}{217}\right)^{-1}\left(\frac{161}{217}\right)^t \frac{1}{\ln\left(\frac{161}{217}\right)}\right)\Bigg|_1^2$$

$$+ \left(\frac{161}{250}\left(\frac{107}{161}\right)^{t-2} \frac{1}{\ln\left(\frac{107}{161}\right)}\right)\Bigg|_2^3 + \left(\frac{107}{250}\left(\frac{62}{107}\right)^{t-3} \frac{1}{\ln\left(\frac{62}{107}\right)}\right)\Bigg|_3^4 + \left(\frac{62}{250}\left(\frac{28}{62}\right)^{t-4} \frac{1}{\ln\left(\frac{28}{62}\right)}\right)\Bigg|_4^5$$

$$= \frac{217-250}{250\ln\frac{217}{250}} + \frac{161-217}{250\ln\frac{161}{217}} + \frac{107-161}{250\ln\frac{107}{161}} + \frac{62-107}{250\ln\frac{62}{107}} + \frac{28-62}{250\ln\frac{28}{62}} \approx 2.71.$$

(ii) There are two ways: (1) $\overset{\circ}{e}_{80:\overline{3}|} = \int_0^n {}_tp_x dt$.
(2) $\overset{\circ}{e}_{80:\overline{3}|} = \int_0^\infty (x \wedge n) f_{T(80)}(x)dx = \int_0^3 x f_{T(80)}(x)dx + 3 \cdot {}_3p_{80}$ **Which way you like ?**

$$\overset{\circ}{e}_{80:\overline{3}|} = \int_0^3 {}_tp_x dt = \int_0^1 \frac{250}{250}\left(\frac{217}{250}\right)^t dt + \int_1^2 \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} dt + \int_2^3 \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} dt$$

$$= \frac{217-250}{250\ln\left(\frac{217}{250}\right)} + \frac{161-217}{250\ln\left(\frac{161}{217}\right)} + \frac{107-161}{250\ln\left(\frac{107}{161}\right)} \approx 2.21. \quad (\textit{see } (1))$$

(iii) $f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt}$, $\quad (a^x)' = a^x \ln a)$, $\quad -(a^{t-i})' = -a^{-i}(a^t)' = a^{t-i}(-\ln a)$.

$$f_{T(80)}(t) = \begin{cases} \frac{250}{250}\left(\frac{217}{250}\right)^t\left(-\ln\left(\frac{217}{250}\right)\right) & \text{if } 0 \le t < 1 \\ \frac{217}{250}\left(\frac{161}{217}\right)^{t-1}\left(-\ln\left(\frac{161}{217}\right)\right) & \text{if } 1 \le t < 2 \\ \frac{161}{250}\left(\frac{107}{161}\right)^{t-2}\left(-\ln\left(\frac{107}{161}\right)\right) & \text{if } 2 \le t < 3 \\ \frac{107}{250}\left(\frac{62}{107}\right)^{t-3}\left(-\ln\left(\frac{62}{107}\right)\right) & \text{if } 3 \le t < 4 \\ \frac{62}{250}\left(\frac{28}{62}\right)^{t-4}\left(-\ln\left(\frac{28}{62}\right)\right) & \text{if } 4 \le t \le 5. \end{cases}$$

(iv) $\mu(80+t) = \mu_{T(80)}(t) = -\frac{d(\ln({}_tp_{80}))}{dt} = \frac{f_{T(80)}(t)}{{}_tp_{80}}$,

$$\mu(80+t) = \begin{cases} \ln\left(\frac{250}{217}\right) & \text{if } 0 \le t \le 1 \\ \ln\left(\frac{217}{161}\right) & \text{if } 1 < t \le 2 \\ \ln\left(\frac{161}{107}\right) & \text{if } 2 < t \le 3 \\ \ln\left(\frac{107}{62}\right) & \text{if } 3 < t \le 4 \\ \ln\left(\frac{62}{28}\right) & \text{if } 4 < t \le 5, \end{cases} \text{ or } \mu(80+t) = \begin{cases} \ln\left(\frac{250}{217}\right) & \text{if } 0 < t < 1 \\ \ln\left(\frac{217}{161}\right) & \text{if } 1 < t < 2 \\ \ln\left(\frac{161}{107}\right) & \text{if } 2 < t < 3 \\ \ln\left(\frac{107}{62}\right) & \text{if } 3 < t < 4 \\ \ln\left(\frac{62}{28}\right) & \text{if } 4 < t < 5. \end{cases} \quad \textbf{Which one ?}$$

Notice that the derivative of $\ln({}_tp_{80})$ does not exist at $1, 2, \ldots, 5$. But, the density and force of mortality can be defined arbitrarily at finitely many points. So both are right.

Theorem 3.13.

Example 3.14.

Theorem 3.14. *Under an exponential form for ℓ_{x+t},*
(i) $L_x = \frac{d_x}{-\ln p_x}$. $\qquad$ *(*$L_x = \ell_x \overset{\circ}{e}_{x:\overline{1}|}$*)*
(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{q_x}{-\ln p_x}$.
(iii) $T_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ln p_k}$. *(= E(# of years lived beyong age x by the cohort group with* l_0 *members))*
(iv) $m_x = -\ln p_x$. $\qquad$ *(the central death rate over (x,x+1])*
(v) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \ln p_k}$.
(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1} \frac{d_k}{-\ell_x \ln p_k}$.

Proof. Need to learn how to derive them based on ℓ_x, p_x and d_x. **Formulas:**
[11] $\underline{\ell_x}$ = # of individuals alive at age x, $\underline{{}_1L_x} = L_x$, $\underline{{}_nL_x} = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = T_x - T_{x+n}$,

$\underline{T_x} = \ell_x \overset{\circ}{e}_x = \int_0^{\infty} \ell_{x+t}\, dt = \sum_{k=x}^{\infty} L_k$ (not T_x in #3),

= E(# of years lived beyong age x by the cohort group with l_0 members),

$\underline{{}_tp_x} = \prod_{x \le k < x+t}(1 - d_k/l_k)$, $s(x) = \underline{\frac{\ell_x}{\ell_0}}$, ${}_tp_x = \underline{\frac{\ell_{x+t}}{\ell_x}}$, ${}_td_x = \underline{\ell_x - \ell_{x+t}}$, $d_x = \underline{{}_1d_x}$,

$L_x = \ell_x \overset{\circ}{e}_{x:\overline{1}|}$, $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt$, ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, $\ell_{k+t} = \ell_k (\frac{\ell_{k+1}}{\ell_k})^t$, t ?

(i) $L_x = \ell_x \int_0^1 {}_tp_x \, dt = \ell_x \int_0^1 p_x^t \, dt = \frac{\ell_x p_x^t}{\ln p_x} \Big|_0^1 = \frac{\ell_x (p_x - 1)}{\ln p_x} = \frac{\ell_x q_x}{-\ln p_x} = \frac{d_x}{-\ln p_x}$.

(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{L_x}{\ell_x} = \frac{d_x}{-\ln p_x \ell_x} = \frac{\ell_x - \ell_{x+1}}{-\ln p_x \ell_x} = \frac{1-p_x}{-\ln p_x} = \frac{q_x}{-\ln p_x}$.

(iii) $T_x = \sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty} \frac{d_k}{-\ln p_k}$ by (i)

(iv) [7]: ${}_nm_x = \frac{\int_x^{x+n} s(t)\mu(t)dt}{\int_0^{x+n} s(u)du} = \frac{{}_nq_x}{\overset{\circ}{e}_{x:\overline{n}|}}$, $\qquad$ => $m_x = \frac{q_x}{\overset{\circ}{e}_{x:\overline{1}|}} = -\ln p_x$ by (ii).

(v) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \ln p_k}$ by (iii).

(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \frac{{}_nL_k}{\ell_x} = \frac{T_x - T_{x+n}}{\ell_x} = \frac{\sum_{k=x}^{\infty} L_k - \sum_{k=x+n}^{\infty} L_k}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x} = \sum_{k=x}^{x+n-1} \frac{d_k}{-\ell_x \ln p_k}$ by (i). ∎

Theorem 3.15. *Given* $t \ge 0$*, let* k *be the nonnegative integer such that* $k \le t < k+1$*. Under exponential interpolation:*
(i) $s(t) = {}_kp_0 \cdot p_k^{t-k}$. *(ii)* $f_X(t) = {}_kp_0 \cdot p_k^{t-k}(-\ln p_k)$. *(iii)* $f_{T(x)}(t) = {}_kp_x \cdot p_{x+k}^{t-k}(-\ln p_{x+k})$.

Proof. (i) By Formula [12] for each integer x and each $0 \le t \le 1$,

$$s(x+t) = \frac{\ell_{x+t}}{\ell_0} = \frac{\ell_x}{\ell_0} \left(\frac{\ell_{x+1}}{\ell_x} \right)^t = {}_xp_0 \cdot p_x^t.$$

Hence, for $t \ge 0$ and $k \le t < k+1$, $s(t) = s(k + \underbrace{(t-k)}_{\in(0,1)}) = {}_kp_0 \cdot p_k^{t-k}$.

The proofs for (ii) and (iii) can be skipped. ∎

3.5.3 Harmonic interpolation assumes

$$\frac{1}{\ell_{x+t}} = (1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}, \qquad t \in [0,1]$$

$$\text{(recall } linear: \quad \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}, \qquad t \in [0,1],$$

$$Exp: \quad \ln\ell_{x+t} = (1-t)\ln\ell_x + t\ln\ell_{x+1} \ \ or \ \ \ell_{x+t} = \ell_x\Big(\frac{\ell_{x+1}}{\ell_x}\Big)^t, \qquad t \in [0,1]).$$

[12]

Interpolation	ℓ_{x+t}	${}_tp_x$
UDD	$(1-t)\ell_x + t\ell_{x+1}$ or $\ell_x - td_x$	$1 - tq_x$
exponential	$(\ell_x)^{1-t}(\ell_{x+1})^t$ or $\ell_x p_x^t$	p_x^t
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$	$\frac{p_x}{t+(1-t)p_x}$

, $t \in [0,1]$.

A function of the form $\frac{1}{a+bx}$ is called a hyperbolic function.
Harmonic interpolation of the number of living is also called the **hyperbolic form of the number of living** or it satisfies the **Balducci assumption**.

Example 3.15. *Using the life table in page 602 and harmonic interpolation, find: (i)* ${}_{0.75}p_{80}$ *(ii)* ${}_{2.25}p_{80}$.

Solution: (i) Formulas: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, and $\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$ ([12])

$$=> \quad {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{1}{(1-t)+t\frac{\ell_x}{\ell_{x+1}}} \tag{3.5}$$

$${}_{0.75}p_{80} = \frac{1}{1-0.75+(0.75)\frac{\ell_{80}}{\ell_{81}}} = \frac{1}{1-0.75+(0.75)\frac{53925}{50987}} \approx 0.95857.$$

(ii) Formulas: ${}_{t+s}p_x = {}_tp_x \cdot {}_sp_{x+t}$ and ${}_tp_x = \frac{1}{1-t+t\ell_x/\ell_{x+1}} = \frac{1}{t/p_x+(1-t)}$ (see (3.5)),
or ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$ and [12]: ($\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$) **which do you prefer ?**

$${}_{2.25}p_{80} = \frac{\ell_{82.25}}{\ell_{80}} = \frac{\frac{1}{(1-0.25)\frac{1}{\ell_{82}}+0.25\frac{1}{\ell_{83}}}}{\ell_{80}} = \frac{1}{(1-0.25)\frac{1}{47940}+0.25\frac{1}{44803}}\Big/53925 \approx 0.8737.$$

Example 3.16.

Example 3.17. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming harmonic interpolation calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Solution: Two ways:
(1) $\overset{\circ}{e}_{80:\overline{3}|} = \int (t \wedge n) f_{T(x)}(t)dt$. (2) $\overset{\circ}{e}_{80:\overline{3}|} = \int_0^n {}_tp_x dt$ **Which is better here ?**

Formulas for $\overset{\circ}{e}_{80:\overline{3}|} = \int_0^n {}_tp_x dt$: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$ and $\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$, $0 \le t \le 1$, If $k \le t < k+1$, then $t-k \in [0,1]$ and $\ell_{x+t} = \ell_{x+k+(t-k)} = \frac{1}{(1-(t-k))\frac{1}{\ell_{x+k}}+(t-k)\frac{1}{\ell_{x+k+1}}}$.

$$\ell_{80+t} = \begin{cases} \frac{1}{(1-t)\frac{1}{250}+t\frac{1}{217}} & \text{if } 0 \le t < 1, \\ \frac{1}{(1-(t-1))\frac{1}{217}+(t-1)\frac{1}{161}} & \text{if } 1 \le t < 2, \\ \frac{1}{(1-(t-2))\frac{1}{161}+(t-2)\frac{1}{107}} & \text{if } 2 \le t < 3, \end{cases}$$

$${}_tp_{80} = \begin{cases} \frac{1}{(1-t)\frac{250}{250}+t\frac{250}{217}} & \text{if } 0 \le t < 1, \\ \frac{1}{(1-(t-1))\frac{250}{217}+(t-1)\frac{250}{161}} & \text{if } 1 \le t < 2, \\ \frac{1}{(1-(t-2))\frac{250}{161}+(t-2)\frac{250}{107}} & \text{if } 2 \le t < 3, \end{cases} = \begin{cases} \frac{1}{\frac{250}{250}+t(\frac{250}{217}-\frac{250}{250})} & \text{if } 0 \le t < 1, \\ \frac{1}{\frac{250}{217}+(t-1)(\frac{250}{161}-\frac{250}{217})} & \text{if } 1 \le t < 2, \\ \frac{1}{\frac{250}{161}+(t-2)(\frac{250}{107}-\frac{250}{161})} & \text{if } 2 \le t < 3, \end{cases}$$

$$\overset{\circ}{e}_{80:\overline{3}|} = \int_0^n {}_tp_x dt = \int_0^1 \frac{1}{\frac{250}{250}+t(\frac{250}{217}-\frac{250}{250})} dt$$
$$+ \int_1^2 \frac{1}{\frac{250}{217}+(t-1)(\frac{250}{161}-\frac{250}{217})} dt + \int_2^3 \frac{1}{\frac{250}{161}+(t-2)(\frac{250}{107}-\frac{250}{161})} dt,$$
$$(\int \frac{1}{a+(t-c)b} dt = \int \frac{1}{u} d\frac{u}{b} \qquad u = ???$$
$$= \frac{\ln(a+(t-c)b)}{b} \qquad as \quad \int \frac{1}{x} dx = \ln x \ or\ (ln x)' = 1/x).$$
$$\overset{\circ}{e}_{80:\overline{3}|} = \frac{\ln\left(\frac{250}{250}+t(\frac{250}{217}-\frac{250}{250})\right)}{\frac{250}{217}-\frac{250}{250}}\Bigg|_0^1$$
$$+ \frac{\ln\left(\frac{250}{217}+(t-1)(\frac{250}{161}-\frac{250}{217})\right)}{\frac{250}{161}-\frac{250}{217}}\Bigg|_1^2 + \frac{\ln\left(\frac{250}{161}+(t-2)(\frac{250}{107}-\frac{250}{161})\right)}{\frac{250}{107}-\frac{250}{161}}\Bigg|_2^3 \approx 2.197.$$

Theorem 3.16. *Under the Balducci assumption for ℓ_{x+t} with $0 \le t \le 1$,*
(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$ *(see [12]).* *(ii)* ${}_tq_x = \frac{tq_x}{1-(1-t)q_x}$.
(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$. *(iv)* $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$.

Proof. ${}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{1}{\ell_x[(1-t)/\ell_x+t/\ell_{x+1}]} = \frac{1}{(1-t)+t\ell_x/\ell_{x+1}} = \frac{1}{(1-t)+t/p_x}$

$$= \frac{p_x}{(1-t)p_x+t} = \frac{1-q_x}{t+(1-t)(1-q_x)} = \frac{1-q_x}{t+(1-t)-(1-t)q_x} = \frac{1-q_x}{1-(1-t)q_x}.$$

(ii) ${}_tq_x = 1 - {}_tp_x = 1 - \frac{1-q_x}{1-(1-t)q_x} = \frac{1-(1-t)q_x-(1-q_x)}{1-(1-t)q_x} = \frac{tq_x}{1-(1-t)q_x}$.

$$(iii) \quad \mu_{x+t} = -\frac{d}{dt}\ln {}_tp_x = -\frac{d}{dt}\ln\frac{p_x}{t+(1-t)p_x} = \frac{d}{dt}\ln(t+(1-t)p_x) = \frac{d}{dt}\ln(p_x+t(1-p_x))$$
$$= \frac{d}{dt}\ln(p_x+tq_x) = \frac{q_x}{p_x+tq_x} = \frac{q_x}{1-(1-t)q_x} = \frac{q_x}{t+(1-t)p_x}.$$

(iv) $f_{T(x)}(t) = {}_tp_x\mu_{x+t} = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$.

Skip the proofs of the rest theorems in this section.

Theorem 3.17. *Under the Balducci assumption,* ${}_{1-t}q_{x+t} = (1-t)q_x$*,* $t \in [0,1]$*.*

Remark. Under UDD ${}_tq_x = tq_x$, $t \in [0,1]$;
and under Exponentail interpolation, ${}_tp_x = p_x^t$, $t \in [0,1]$.

Theorem 3.18. *Given* $t \geq 0$*, let* k *be the nonnegative integer such that* $k \leq t < k+1$*. Under the Balducci assumption,*

(i) $s(t) = \frac{\ell_k}{\ell_0}\frac{p_k}{1-(1-(t-k))(1-p_k)} = s(k)\frac{1-q_k}{1-(1-\underbrace{(t-k)}_{=t_o\in(0,1)})q_k}$.

(ii) $f_X(t) = \frac{\ell_k}{\ell_0}\frac{p_k(1-p_k)}{(1-(1-(t-k))(1-p_k))^2}$.

(iii) $f_{T(x)}(t) = \frac{\ell_{x+k}}{\ell_x}\frac{p_{x+k}(1-p_{x+k})}{(1-(1-(t-k))(1-p_{x+k}))^2} = {}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-(t-k))(1-p_{x+k}))^2}$.

Theorem 3.19. *Under the Balducci assumption for* ℓ_{x+t}*,*

(i) $L_x = \frac{-\ell_{x+1}\ln p_x}{q_x}$. $(= \ell_x \overset{\circ}{e}_{x:\overline{1}|})$

(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{-p_x\ln p_x}{q_x}$.

(iii) $T_x = \sum_{k=x}^{\infty}\frac{-\ell_{k+1}\ln p_k}{q_k}$. $(= \ell_x\overset{\circ}{e}_x)$

(iv) $m_x = \frac{q_x^2}{-p_x\ln p_x}$. *(central death rate over* $(x, x+1]$*)*

(v) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty}\frac{-\ell_{k+1}\ln p_k}{\ell_x q_k}$.

(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1}\frac{-\ell_{k+1}\ln p_k}{\ell_x q_k}$.

Skip §3.5.4.

3.5.4 Review of interpolations.

For the previous interpolations, typically, we have

Interpolation	ℓ_{x+t}	${}_tp_x$	L_x	$\overset{\circ}{e}_{x:\overline{1}\vert}$
linear	$\ell_x - d_xt$ $(1-t)\ell_x + t\ell_{x+1}$	$1-tq_x$	$\frac{\ell_x+\ell_{x+1}}{2}$	$\frac{1+p_x}{2}$
exponential	$\ell_xp_x^t$ $\ln\ell_{x+t} = (1-t)\ln\ell_x + t\ln\ell_{x+1}$	p_x^t	$\frac{d_x}{-\ln p_x}$	$\frac{q_x}{-\log p_x}$
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$ $\frac{1}{\ell_{x+t}} = (1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}$	$\frac{p_x}{t+(1-t)p_x}$	$\frac{-\ell_{x+1}\ln p_x}{q_x}$	$\frac{-p_x\log p_x}{q_x}$

From ${}_tp_x$, we can get

$$ {}_tq_x = 1 - {}_tp_x,\ \ f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x,\ \ \mu_{x+t} = -\frac{d}{dt}\ln{}_tp_x, \qquad 0 \leq t \leq 1. $$

For the exponential and Balducci assumptions, it is more convenient to know how to derive L_x and $\overset{\circ}{e}_{x:\overline{1}|}$ than to trying remember the corresponding formulas.

3.6 Select and ultimate tables

A **select table** is a mortality table for a group of people subject to a special circumstance (disability, retirement, etc.). Usually, the cohort of people is given by a certain age.
Suppose that we start with $\ell_{[x]}$ lives of a certain cohort at time x.
The number of survivors at time t is denoted by $\ell_{[x]+t}$.
$\ell_{[x+t]}$ is # of lives at $x+t$ for another cohort.

Notations: ${}_n p_{[x]+t} = \frac{\ell_{[x]+t+n}}{\ell_{[x]+t}}$, ${}_n q_{[x]+t} = 1 - {}_n p_{[x]+t}$, $p_{[x]+t} = {}_1 p_{[x]+t}$, $q_{[x]+t} = {}_1 q_{[x]+t}$ and

$$p_{[x]}p_{[x]+1}\cdots p_{[x]+n-1} = \frac{\ell_{[x]+1}}{\ell_{[x]}}\frac{\ell_{[x]+2}}{\ell_{[x]+1}}\cdots\frac{\ell_{[x]+n}}{\ell_{[x]+n-1}} = \frac{\ell_{[x]+n}}{\ell_{[x]}} = {}_n p_{[x]}.$$

A select table of three cohorts:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$
43	958	823	768
44	854	738	701
45	723	687	667

$\ell_{[43]+2}$= ? $\ell_{[44]+1}$= ? $\ell_{[45]}$= ?
They all related to age 45.

A **select and ultimate table** displays the number of living using a select table for a certain number of years and a standard life table when the elapsed time is bigger than this number of years. The number of years such that the select table is used is called the **select period**. A life table which does not use the select period is called an **ultimate table**, *e.g.*,

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Suppose that the selection period is m.

$\ell_{[x]}$ —- # of living of a certain cohort selected at time x.
$\ell_{[x]+t}$ —- # of their survivors at time $x+t$.
ℓ_x —- # of living at time x for the ultimate table.

In a select and ultimate table, $\ell_{[x]+k} = \ell_{x+k}$, for each $k \geq m$.
Suppose that select period is three years. Then, a select and ultimate life table has the form

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
1	$\ell_{[1]}$	$\ell_{[1]+1}$	$\ell_{[1]+2}$	ℓ_4	4
2	$\ell_{[2]}$	$\ell_{[2]+1}$	$\ell_{[2]+2}$	ℓ_5	5
3	$\ell_{[3]}$	$\ell_{[3]+1}$	$\ell_{[3]+2}$	ℓ_6	6

$\ell_{[1]+4} = \ell_5$? $\ell_{[2]+3} = \ell_5$? $\ell_{[3]+2} = \ell_5$?

Example 3.18. *Complete the following 2 tables using the select table:*

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$
43	*958*	*823*	*768*
44	*854*	*738*	*701*
45	*723*	*687*	*667*

x	$p_{[x]}$	$p_{[x]+1}$
43		
44		
45		

and

x	$q_{[x]}$	$q_{[x]+1}$
43		
44		
45		

Try yourselves now !

Solution: Formulas: $p_{[x]+t} = \frac{\ell_{[x]+t+1}}{\ell_{[x]+t}}$ and $q_{[x]+t} = 1 - p_{[x]+t}$

x	$p_{[x]}$	$p_{[x]+1}$
43	$\frac{823}{958}$	$\frac{768}{823}$
44	$\frac{738}{854}$	$\frac{701}{738}$
45	$\frac{687}{723}$	$\frac{667}{687}$

and

x	$q_{[x]}$	$q_{[x]+1}$
43	$1-\frac{823}{958}$	$1-\frac{768}{823}$
44	$1-\frac{738}{854}$	$1-\frac{701}{738}$
45	$1-\frac{687}{723}$	$1-\frac{667}{687}$

Example 3.19. *Complete the table using the following select table:* **Skip Example 3.19!**

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$
35	*0.013*	*0.012*	*0.011*
36	*0.010*	*0.011*	*0.009*

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	$\ell_{[x]+3}$
35	*1000*			
36	*950*			

Solution: Formulas: $1 - q_{[x]+t} = p_{[x]+t} = \frac{\ell_{[x]+t+1}}{\ell_{[x]+t}}$. Hence $\ell_{[x]+t+1} = \ell_{[x]+t}p_{[x]+t} = \ell_{[x]+t}(1 - q_{[x]+t})$.

x	$p_{[x]}$	$p_{[x]+1}$	$p_{[x]+2}$
35	1- 0.013	1-0.012	1- 0.011
36	1-0.010	1- 0.011	1-0.009

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	$\ell_{[x]+3}$
		$\to$	$\to$	$\to$
35	1000	987	975.156	964.429284
36	950	940.5	930.1545	921.7831095

$$\ell_{[35]+1} = \ell_{[35]}p_{[35]} = (1000)(1 - 0.013) = 987,$$
$$\ell_{[35]+2} = \ell_{[35]+1}p_{[35]+1} = (987)(1 - 0.012) = 975.156,$$
$$\ell_{[35]+3} = \ell_{[35]+2}p_{[35]+2} = (975.156)(1 - 0.011) = 964.429284,$$
$$\ell_{[36]+1} = \ell_{[36]}p_{[36]} = (950)(1 - 0.01) = 940.5,$$
$$\ell_{[36]+2} = \ell_{[36]+1}p_{[36]+1} = (940.5)(1 - 0.011) = 930.1545,$$
$$\ell_{[36]+3} = \ell_{[36]+2}p_{[36]+2} = (930.1545)(1 - 0.009) = 921.7831095.$$

Example 3.20. *You are given the following entries extracted from a 2–year select–and–ultimate mortality table:*

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	1235	1124	1039	47
46	1135	1025	978	48
47	1012	996	965	49

(i) Complete the table

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45				47
46				48
47			–	49

(ii) Find ${}_2p_{[47]}$, ${}_2p_{[46]+1}$ *and* ${}_2p_{47}$.

Solution: (i) Formula: $q_x = 1 - \frac{\ell_{x+1}}{\ell_x}$.

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	$\to$	$\to$	$\downarrow$	47
46	$\to$	$\to$	$\downarrow$	48
47	$\to$	$\to$	–	49

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	$1 - \frac{1124}{1235}$	$1 - \frac{1039}{1124}$	$1 - \frac{978}{1039}$	47
46	$1 - \frac{1025}{1135}$	$1 - \frac{978}{1025}$	$1 - \frac{965}{978}$	48
47	$1 - \frac{996}{1012}$	$1 - \frac{965}{996}$	–	49

(ii) Find ${}_2p_{[47]}$, ${}_2p_{[46]+1}$ and ${}_2p_{47}$. Formula: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$. **Exercise, just write down the fractions.**

$$\begin{cases} {}_2p_{[47]} = \frac{\ell_{49}}{\ell_{[47]}} = \frac{965}{1012} \approx 0.954 \\ {}_2p_{[46]+1} = \frac{\ell_{49}}{\ell_{[46]+1}} = \frac{965}{1025} \approx 0.941 \text{ **Check yourself and hand in after class.**} \\ {}_2p_{47} = \frac{\ell_{49}}{\ell_{47}} = \frac{965}{1039} \approx 0.929 \end{cases}$$

Example 3.21. *You are given the following entries extracted from a 2–year select–and–ultimate mortality table:*

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	0.009	0.008	0.007	47
46	0.008	0.006	0.005	48
47	0.004	0.003	–	49

Complete the table

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	10000			47
46				48
47				49

Solution: Relation between ℓ_x and q_x ?
Formula: $1 - q_x = p_x = \ell_{x+1}/\ell_x =>$ $\ell_{x+1} = \ell_x(1-q_x)$ $(\to)$ or $\ell_x = \ell_{x+1}/(1-q_x)$ $(\leftarrow)$.

Why need 2 ? Flow:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	10000	$\to$	$\to$	47
46	$\leftarrow$	$\leftarrow$	$\downarrow$	48
47	$\leftarrow$	$\leftarrow$	$\downarrow$	49

$$\begin{aligned}
&\to\ell_{[45]+1} = \ell_{[45]}(1-q_{[45]}) = (10000)(1-0.009) = 9910, \\
&\to\ell_{47} = \ell_{[45]+1}(1-q_{[45]+1}) = 9910(1-0.008) = 9830.72, \\
&\downarrow\ell_{48} = \ell_{47}(1-q_{47}) = 9830.72(1-0.007) = 9761.90496, \\
&\leftarrow\ell_{[46]+1} = \frac{\ell_{48}}{(1-q_{[46]+1})} = \frac{9761.90496}{(1-0.006)} = 9820.82994, \\
&\leftarrow\ell_{[46]} = \frac{\ell_{[46]+1}}{(1-q_{[46]})} = \frac{9820.82994}{(1-0.008)} = 9900.030181, \\
&\downarrow\ell_{49} = \ell_{48}(1-q_{48}) = 9761.90496(1-0.005) = 9713.095435, \\
&\leftarrow\ell_{[47]+1} = \frac{\ell_{49}}{(1-q_{[47]+1})} = \frac{9713.095435}{(1-0.003)} = 9742.322402, \\
&\leftarrow\ell_{[47]} = \frac{\ell_{[47]+1}}{(1-q_{[47]})} = \frac{9742.322402}{(1-0.004)} = 9781.448195.
\end{aligned}$$

Hence,

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	10000	9910	9830.72	47
46	9900.030181	9820.82994	9761.90496	48
47	9781.448195	9742.322402	9713.095435	49

Example 3.22. *You are given the following entries extracted from a 3–year select mortality table:*

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
40	96489	96319	96084	95906	43
41	96312	96164	95998	95667	44
42	96157	95954	95265	95406	45
43	95895	95480	95243	95122	46
44	98743	96812	95012	94813	47
45	97239	95123	94753	94479	48

Compute (a) $e_{[44]:\overline{4}|}$ (b) $e_{[42]+2:\overline{4}|}$ (c) $e_{44:\overline{4}|}$.

Solution: Formulas: $e_{[x]:\overline{n}|} = \sum_{k=1}^{n} {}_kp_{[x]}$ and ${}_kp_{[x]} = \frac{\ell_{[x]+k}}{\ell_{[x]}}$.

(a) $e_{[44]:\overline{4}|} = p_{[44]} + {}_2p_{[44]} + {}_3p_{[44]} + {}_4p_{[44]} = \frac{\ell_{[44]+1}}{\ell_{[44]}} + \frac{\ell_{[44]+2}}{\ell_{[44]}} + \frac{\ell_{44+3}}{\ell_{[44]}} + \frac{\ell_{44+4}}{\ell_{[44]}}$.

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
44	98743	96812	95012	94813	47
45				94479	48

$$e_{[44]:\overline{4}|} = \frac{96812 + 95012 + 94813 + 94479}{98743} = 3.859676129.$$

(b) $e_{[42]+2:\overline{4}|} = p_{[42]+2} + {}_2p_{[42]+2} + {}_3p_{[42]+2} + {}_4p_{[42]+2} = \frac{\ell_{42+3}}{\ell_{[42]+2}} + \frac{\ell_{42+4}}{\ell_{[42]+2}} + \frac{\ell_{42+5}}{\ell_{[42]+2}} + \frac{\ell_{42+6}}{\ell_{[42]+2}}$.

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
42			95265	95406	45
43				95122	46
44				94813	47
45				94479	48

$$e_{[42]+2:\overline{4}|} = \frac{95406}{95265} + \frac{95122}{95265} + \frac{94813}{95265} + \frac{94479}{95265} = 3.986983677.$$

(c) $e_{44:\overline{4}|} = p_{44} + {}_2p_{44} + {}_3p_{44} + {}_4p_{44} = \frac{\ell_{44+1}}{\ell_{44}} + \frac{\ell_{44+2}}{\ell_{44}} + \frac{\ell_{44+3}}{\ell_{44}} + \frac{\ell_{44+4}}{\ell_{44}}$.

ℓ_{x+3}	$x+3$
95667	44
95406	45
95122	46
94813	47
94479	48

$$e_{44:\overline{4}|} = \frac{95406+95122+94813+94479}{95667} = 3.970230069.$$

Quiz on Friday of next week: 450: [1]-[12], [16]

Skip the rest ! $\ell_x =$ **# of individuals alive** at age x.
${}_t d_x = \ell_x - \ell_{x+t} =$ # of individuals which died in $(x, x+t]$.
$d_x = {}_1 d_x = \ell_x - \ell_{x+1}$.
$T_x = \ell_x \overset{\circ}{e}_x = \int_0^\infty \ell_{x+t}\, dt$ ($\neq T_x$ **in other sections)**
$= \text{E}$(# of years lived beyong age x by the cohort group with l_0 members).
${}_n L_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = T_x - T_{x+n}$.
$s(x) = \frac{\ell_x}{\ell_0}$,
Estimators based on life table:

$$F_X(x) = \frac{\ell_0 - \ell_x}{\ell_0},$$
$${}_t p_x = \frac{\ell_{x+t}}{\ell_x},\ {}_t q_x = \frac{{}_t d_x}{\ell_x},\ q_x = \frac{d_x}{\ell_x},\ {}_n|_m q_x = \frac{{}_m d_{x+n}}{\ell_x}.$$
$$\mu(x) = -\frac{d}{dx}\log(\ell_x),$$
$$\overset{\circ}{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x} dt,\ \overset{\circ}{e}_{x:\overline{n}|} = \int_0^n \frac{\ell_{x+t}}{\ell_x} dt,$$
$$e_x = \sum_{k=1}^\infty \frac{\ell_{x+k}}{\ell_x},\ e_{x:\overline{n}|} = \sum_{k=1}^n \frac{\ell_{x+k}}{\ell_x}$$
$$\overset{\circ}{e}_x = E[T(x)] = \frac{T_x}{\ell_x},\ E[(T(x))^2] = \frac{2\int_x^\infty T_y\, dy}{\ell_x}.$$
$${}_n L_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = L_x + L_{x+1} + \cdots + L_{x+n-1},\ L_x = {}_1 L_x$$
$$T_x = \sum_{k=x}^\infty L_k,\ {}_n m_x = \frac{{}_n d_x}{{}_n L_x},\ m_x = \frac{d_x}{L_x},$$
$$\overset{\circ}{e}_x = \frac{\sum_{k=x}^\infty L_k}{\ell_x},\ \overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}.$$

Interpolation	ℓ_{x+t}	${}_t p_x$	L_x
UDD	$\ell_x + t(\ell_{x+1} - \ell_x)$	$1 - tq_x$	$\frac{\ell_x + \ell_{x+1}}{2}$
exponential	$\ell_x p_x^t = (\ell_x)^{1-t}(\ell_{x+1})^t$	p_x^t	$\frac{d_x}{-\log p_x}$
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}}$	$\frac{p_x}{t+(1-t)p_x}$	$\frac{-\ell_{x+1}\log p_x}{q_x}$

where $t \in [0, 1]$

UDD : $\mu_{x+t} = \frac{q_x}{1-tq_x},$, $m_x = \frac{q_x}{1-\frac{q_x}{2}}$, $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.
exponential : $\mu_{x+t} = -\log p_x$, $m_x = -\log p_x$.
Balducci : $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2}$, $m_x = \frac{q_x^2}{-p_x \log p_x}$.

#5 (#35, Exam M, Fall 2005) An actuary for a medical device manufacturer initially models the failure time for a particular device with an exponential distribution with mean 4 years. This distribution is replaced with a spliced model whose density function:
(i) is uniform over $[0, 3]$
(ii) is proportional to the initial modeled density function after 3 years
(iii) is continuous
Calculate the probability of failure in the first 3 years under the revised distribution.
(A) 0.43 (B) 0.45 (C) 0.47 (D) 0.49 (E) 0.51

Solution. (A) Since the density of the exponential with mean four is $4e^{-x/4}$, the density has the form

$$f(x) = \begin{cases} ae^{-x/4} & \text{if } 3 \leq x \ \ (from\ (ii)), \\ ae^{-3/4} & \text{if } 0 \leq x < 3 \ \ (from\ (i)\ and\ (iii)), \end{cases}$$

where we have used that f is continuous. Since

$$1 = \int_0^\infty f(x)\,dx = \int_0^3 ae^{-3/4}\,dx + \int_3^\infty ae^{-x/4}\,dx = 3ae^{-3/4} + a4e^{-3/4} = 7ae^{-3/4},$$

$a = \frac{e^{3/4}}{7}$. Hence,

$$f(x) = \begin{cases} \frac{1}{7} & \text{if } 0 \leq x < 3, \\ \frac{e^{3/4}}{7}e^{-x/4} & \text{if } 3 \leq x, \end{cases}$$

and $P\{X \leq 3\} = \frac{3}{7} = 0.4285714286$.

CHAPTER 4

Life Insurance

4.1 Introduction to life insurance.

We will consider a cashflow of **contingent payments**, i.e. the payments depend on uncertain events modeled as a random variable. We call such cashflow the **contingent cashflow**.

Definition 4.1. *The mean of the present value at the time of purchase of a cashflow is called its* **actuarial present value (APV)** *of the cashflow of payments,*
its **expected present value**, *or*
its **net single premium**.

Recall that under compound interest: $v = (1+i)^{-1} = 1 - d = e^{-\delta}$.
i is the annual effective rate of interest,
v is the annual discount factor,
d is the annual discount rate,
δ is the force of interest.
In general, for $t > 0$, let v_t be the t–year discount factor,
the force of interest is $\delta_t = -\frac{d}{dt}\ln v_t$ (similar to $\mu(t) = -\frac{d}{dt}\ln S_X(t)$)
$v_t = e^{-\int_0^t \delta_s\,ds}$ (similar to $S_X(t) = e^{-\int_0^t \mu(x)dx}$).
Under compound interest, $v_t = v^t = (1+i)^{-t}$, and $\delta_t = \delta = \ln(1+i) = -\ln v$.
Otherwise, v_t can be different.

Example 4.1. *On January 1, 2000, John entered a whole life insurance contract.*
This contract pays a death benefit of \$50,000 at the end of the year of death.
On June 13, 2009, John died. The annual effective rate of interest is 6%.
Calculate the **present value** *of the benefit payment at the time of the issue of this contract.*

Solution: Time of death: 6/13/2009. Time of payment: 12/21/2009, treated as 1/1/2010.
Present value= bv^t. Time of present value: 1/1/2000.
The present value is $bv^t = (50,000)(\frac{1}{1+0.06})^{10} \approx 27,919.79$.

Example 4.2. *John pays for its electric bill at the end of each month. John estimates that its electric bill* $X_j \sim U[100, 300]$. *Assume that John is going to pay his bill precisely at the end of each month. Find the APV of the total amount which John will pay in electricity in the next 12 months if* $i = 6\%$.

Solution: $X_1, \ldots, X_{12}$ are the amounts in John's electric bill for the next 12 months. So the total amount is $\sum_{j=1}^{12} X_j$,
but their present values is $Z = \sum_{j=1}^{12} X_j v^{t_j} = \sum_{j=1}^{12} X_j v^{j/12}$, where $v = 1/(1+i) = 1/1.06$.
The APV of Z is $E(Z) = E\left[\sum_{j=1}^{12} X_j v^{j/12}\right] = \left[\sum_{j=1}^{12} E(X_j) v^{j/12}\right] = E[X_1]\sum_{j=1}^{12} p^j$, $p =$??
Thus $E(Z) = \frac{100+300}{2} p \frac{1-p^n}{1-p}\Big|_{n=12, p=v^{1/12}} \approx 2325.76$ as $\sum_{j=1}^{n} p^j = p\frac{1-p^n}{1-p}$.

Example 4.3.

In this chapter, we consider an insurance policy on a certain entity.
Most of the times, the considered entity is (x), a live aged x.
Let T be the age–at–death of this entity.
The policyholder receives a payment at a certain time in the future.
Both the amount of the payment and the payment date depend on T.

Let b_t be the benefit payment made when failure happens at time t.

Let v_t be the discount factor when failure happens at time t.

The present value of the benefit payment is $b_T v_T$ and is denoted by $\begin{cases} \overline{Z} = b_T v_T & \text{if } T \text{ is cts,} \\ Z = b_T v_T & \text{if } T \text{ is discrete.} \end{cases}$
$v_t = v^t$ if the benefit payment is made at the time of death and compound interest is assumed.

In this section, we will see different insurance policies. Each policy has a different (b_t, v_t), $t \geq 0$. The theory in this section applies to life insurance as well as insurance related with the time at failure of inanimate objects.

Example 4.4. *An insurance guarantees a payment at the time of failure of a machine.*
(i) The age–at–failure T of this machine satisfies $T \sim U(0, 40)$.
(ii) $i = 7\%$.
(iii) The payment is $b_t = (20000)(1.04)^t$.
Find the mean and the SD of the present value random variable for this insurance.

Solution: The present value random variable of the payment benefit is

$$\overline{Z} = b_T v_T = (20000)(1.04)^T (1.07)^{-T} = (20000)\left(\frac{1.04}{1.07}\right)^T \stackrel{def}{=} g(T).$$

Possible formulas: $E(\overline{Z}) = \int t \underbrace{f_{\overline{Z}}(t)}_{=?} dt = \int \underbrace{S_{\overline{Z}}(t)}_{=?} dt = \int g(t) \underbrace{f_T(t)}_{=?} dt$ **Which to choose ?**

$E[\overline{Z}] = E(g(T)) = \int g(t) f_T(t) dt$, where $f_T(t) = \frac{1}{40}$, $0 \leq t \leq 40$.

$$E[\overline{Z}] = \int_0^{40} (20000)\left(\frac{1.04}{1.07}\right)^t \frac{1}{40}\, dt = \frac{20000}{40} \int_0^{40} \left(\frac{1.04}{1.07}\right)^t dt \qquad (\int a^t dt = \frac{a^t}{\ln a} + c), \quad a = ?$$

$$= \frac{(20000)a^t}{40\ln(a)} \Bigg|_0^{40} = \frac{(20000)\left(\frac{1.04}{1.07}\right)^t}{40\ln(1.04/1.07)} \Bigg|_0^{40} \approx 11945.07,$$

$$E[\overline{Z}^2] = \int_0^{40} (20000)^2 \left(\frac{1.04}{1.07}\right)^{2t} \frac{1}{40}\, dt = \int_0^{40} \frac{20000^2}{40} (\left(\frac{1.04}{1.07}\right)^2)^t\, dt \qquad (\int a^t dt = \frac{a^t}{\ln a} + c), \quad a = ?$$

$$= \frac{20000^2\left((\frac{1.04}{1.07})^2\right)^t}{40\ln((1.04/1.07)^2)} \Bigg|_0^{40} \approx 157748208.7,$$

$$\sigma_{\overline{Z}} = \sqrt{\mathrm{Var}(\overline{Z})} = \sqrt{157748208.7 - (11945.07)^2} \approx 3881.19.$$

Example 4.5. *A four–year warranty on a digital television will pay \$$400(5 - k)$ if the television breaks during the k–th year, $k = 1, \ldots, 4$.*
The payment will be paid at the end of the year.

The effective annual discount rate is $d = 4\%$.
The survival function $s(x) = \frac{1000}{(x+10)^3}$, $x \geq 0$.
Find the actuarial present value of this warranty benefit.

Solution: T is the time to break of the TV. The present value of the benefit payment is

$$Z = b_T v_T = 400(5 - \lceil T \rceil) v^{\lceil T \rceil} = 400(5-K)v^K, \text{ where } K = \lceil T \rceil, \textit{ i.e.,} \tag{4.1}$$

$(b_t, v_t) = (400(5-k), v^k) = (400(5-k), (1-d)^k)$ if $k = \lceil t \rceil$ (*i.e.*, $t \in (k-1, k]$), $k \in \{1, 2, 3, 4\}$.

$$APV = E(Z) = \sum_k k \underbrace{f_Z(k)}_{=?\ (1)} = \int \underbrace{S_Z(t)}_{=?\ (2)} dt = \int g(x) \underbrace{f_T(x)}_{=?\ (3)} dx = \sum_i g(i) \underbrace{f_K(i)}_{=?\ (4)}.$$

Given $S_T(x) = s(x) = \frac{1000}{(x+10)^3}$, $x \geq 0$, **which among the four to choose?**

Methods (1) and (2): not convenient.

(3) $E(Z) = E(400(5 - \lceil T \rceil) v^{\lceil T \rceil}) = \int 400(5 - \lceil t \rceil) v^{\lceil t \rceil} f_T(t) dt = \sum_{k=1}^{4} \int_{k-1}^{k} 400(5-k) v^k f_T(t) dt$

(4) $E(Z) = \sum_{k=1}^{4} g(k) f_K(k) = \sum_{k=1}^{4} b_k v^k \mathbb{P}\{k - 1 < T \leq k\}$

$$= \sum_{k=1}^{4} 400(5-k) v^k (s(k-1) - s(k)) \qquad s(x) = \frac{10^3}{(x+10)^3}$$

$$= \sum_{k=1}^{4} 400(5-k) v^k 10^3 \left[\frac{1}{(k-1+10)^3} - \frac{1}{(k+10)^3}\right] \qquad (v = 1-d)$$

$$= 400(5-1)(0.96)\left(\frac{1000}{(10)^3} - \frac{1000}{(1+10)^3}\right) + 400(5-2)(0.96)^2\left(\frac{1000}{(1+10)^3} - \frac{1000}{(2+10)^3}\right)$$
$$+ 400(5-3)(0.96)^3\left(\frac{1000}{(2+10)^3} - \frac{1000}{(3+10)^3}\right) + 400(5-4)(0.96)^4\left(\frac{1000}{(3+10)^3} - \frac{1000}{(4+10)^3}\right)$$
$$\approx 712.14.$$

4.2 Payments paid at the end of the year of death.

4.2.1 Whole life insurance.

Definition 4.2. *A policy is called a* **whole life policy** *if it pays a fixed amount, called the* **face value** *or* **death benefit**, *after the death of the policyholder.*

Theorem 4.1.

The payment in a whole life insurance can be paid at different times. In this section, we consider the situation when the face value is paid at the end of the year of death.

An insurer offering life insurance takes a liability. It is of interest to know the amount of this liability.

Definition 4.3. *The present value at time of issue of the death benefit payment of a unit whole life insurance payable at the end of the year of the death is denoted by Z_x. Its APV is denoted by A_x, also called the* **premium**. *The APV of a contingent contract is called the* **net single premium** *($\neq A_x$).*

[14] $Z_x = v^{K_x}$, where $K_x = \lceil T(x) \rceil$. $A_x = E[Z_x] = E[v^{K_x}]$
and $v^{\omega - x} \le A_x \le v$, where ω is the terminal age of the population.

The insurer would like that a policy holder will die as late as possible. In this way, the present value of the death benefit is low. The whole life insurance $Z = b_T v_T$ is a r.v..
An insurer may estimate Z_x using its APV ($= E(Z)$).

If x and ω are integers, Z_x is a discrete random variable taking the values $v, v^2, \ldots, v^{\omega - x}$.
The model $Z = b_T v_T$ applies with $b_t = b$ and $v_t = v^{\lceil t \rceil}$.
Or use the model $Z = b_K v_K = b_{\lceil T \rceil} v_{\lceil T \rceil}$ with $b_k = b$ and $v_k = v^k$, where $k = 1, 2, \ldots$

Example 4.6. *Let $i = 5\%$ and*

k	1	2	3
$\mathbb{P}\{K_x = k\}$	0.2	0.3	0.5

(1) Find f_{Z_x} and S_{Z_x}, (2) Find A_x (i.e., $E(Z_x)$) and (3) $\mathrm{Var}(Z_x)$.

Solution: (1) $f_{Z_x} = ?$ $Z_x = v^{K_x}$.
If $Z_x = g(K_x)$, $P(Z_x = z) = \sum_{k: g(k) = z} P(K_x = k)$. **Q:** g **= ??**

k	1	2	3
$\mathbb{P}\{K_x = k\}$	0.2	0.3	0.5
$Z_x = v^{K_x} = t$	v^1	v^2	v^3
t	1.05^{-1}	1.05^{-2}	1.05^{-3}
$f_{Z_x}(t)$	?	?	?
$S_{Z_x}(t)$	?	?	?

t	1.05^{-1}	1.05^{-2}	1.05^{-3}	$< 1.05^{-3}$	
$f_{Z_x}(t)$	0.2	0.3	0.5	0	
$S_{Z_x}(t)$	0.8	0.5	0.0	0	?
$S_{Z_x}(t)$	0.0	0.2	0.5	1	?

(2) Formula: 3 methods: $A_x = \sum_k v^k f_{K_x}(k) = E(Z_x) = \sum_z z f_{Z_x}(z) = \int_0^\infty S_{Z_x}(t)dt$.

Which method is better to compute both $E(Z_x)$ and $V(Z_x)$?

$A_x = (1.05)^{-1}(0.2) + (1.05)^{-2}(0.3) + (1.05)^{-3}(0.5) = 0.8945038333$ **is it Method 1 or 2 ?**
$= \int_0^{1.05^{-3}} 1dt + \int_{1.05^{-3}}^{1.05^{-2}} 0.5dt + \int_{1.05^{-2}}^{1.05^{-1}} 0.2dt$ (Method 3).
$= 1 \cdot (1.05^{-3} - 0) + 0.5(1.05^{-2} - 1.05^{-3}) + 0.2(1.05^{-1} - 1.05^{-2})$.
(3) $E[Z_x^2] = \sum_t t^2 f_{Z_x}(t) = (1.05)^{-2}(0.2) + (1.05)^{-4}(0.3) + (1.05)^{-6}(0.5) = 0.8013243364$,
$\mathrm{Var}(Z_x) = E[Z_x^2] - (E[Z_x])^2 = 0.8013243364 - (0.8945038333)^2 = 0.001187228612$.

Example 4.7.

Example 4.8.

Quiz on Friday : 447: [6]-20], 450: [1]-[12], [14](first 2 lines), [16]
Notations: $Z_x = v^{K_x}$, $A_x = A_x(v) = E(v^{K_x})$.
${}^m A_x = E(Z_x^m)$ ($= E(v^{mK_x}) = \sum_k v^{mk} f_{K_x}(k)$).

Formula: ${}^2A_x = E(Z_x^2)$ $(= E(v^{2K_x}) = \sum_k v^{2k} f_{K_x}(k) = A_x(v^2))$.
Recall under compound interest:
i is the annual effective rate of interest,
$v = (1+i)^{-1} = 1 - d = e^{-\delta}$ is the annual discount factor and
d is the annual discount rate.
$\delta = \ln(1+i) = -\ln v$, the force of interest.

Remark. If ${}^2A_x$ $(= E(Z_x^2))$ is written as a function of notations δ, i or d, then
(i) $A_x = A_x(\delta)$ and ${}^2A_x = A_x(2\delta)$.
(ii) $A_x = A_x(i)$ and ${}^2A_x = A_x(i(2+i))$.
(iii) $A_x = A_x(d)$ and ${}^2A_x = A_x(d(2-d))$.
(iv) $A_x = A_x(v)$ and ${}^2A_x = A(v^2)$ (**which of them is easier to remember ?**)

Example 4.9. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.
(i) Find the actuarial present value of this life insurance.
(ii) **Skip (ii).**
(iii) Find the probability that the APV of the life insurance is adequate to cover this insurance.
(iv) An insurance company offers this life insurance to 250 80–year old individuals. How much should each policyholder pay so that the insurer has a probability of 1% that the present value of these 250 policies exceed the total premiums received?

Solution: (i) Letting $Z = bZ_x$ $(b = ?)$, find $A = E(Z)$.
Formula: $Z_x = v^{K_x}$, $A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$.

$${}_{k-1|}q_x = \frac{s(x+k-1) - s(x+k)}{s(x)} = \frac{\ell_{x+k-1} - \ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1} \quad [8]$$

$$= {}_{k-1}p_x - {}_kp_x = {}_kq_x - {}_{k-1}q_x. \ [3]$$ **Which to choose ?**

${}_{k-1|}q_x = \frac{\ell_{x+k-1} - \ell_{x+k}}{\ell_x} = \frac{d_{x+k-1}}{\ell_x}$ =>

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
$\ell_x - \ell_{x+1} = d_x$	33	56	54	45	34	28	0

$$A_{80} = \sum_{k=1}^{\infty} v^k \frac{d_{80+k-1}}{\ell_{80}} = v^1 \frac{d_{80}}{\ell_{80}} + \cdots + v^6 \frac{d_{85}}{\ell_{80}} + v^7 \frac{d_{86}}{\ell_{80}} + 0$$
$$= (1.065)^{-1} \frac{33}{250} + (1.065)^{-2} \frac{56}{250} + (1.065)^{-3} \frac{54}{250} + (1.065)^{-4} \frac{45}{250} + (1.065)^{-5} \frac{34}{250} + (1.065)^{-6} \frac{28}{250}$$
$$\approx 0.8162.$$

Hence, the APV for this insurance is $A = bA_{80} \approx (50000)(0.8162) \approx 40810$.
(iii) P(A is adequate to cover the insurance)=?
$\mathbb{P}(A > 50000Z_x)$ $\quad$ $A \approx 40810$ in (i) is adequate if
$A > 50000Z_x$,

<=> $40810 > (50000)(1.065)^{-K_x}$,
<=> $\ln(40810/50000) > -K_x \ln(1.065)$
<=> $K_x > \frac{\ln(40810/50000)}{-\ln(1.065)} \approx 3.225$? or $K_x < 3.225$?
The probability that the APV is adequate is

$$P(A > 50000Z_x) = \mathbb{P}\{K_x > 3.225\} = \begin{pmatrix} \mathbb{P}\{K_x \geq 4\} & \mathbb{P}\{K_x > 3\} & \mathbb{P}\{T(x) > 3\} & \mathbb{P}\{T(x) \geq 3\} \\ & ? & ? & ? \end{pmatrix}$$

$= {}_3p_x = \frac{\ell_{x+3}}{\ell_x} = \frac{107}{250} = 0.428$. **What is its implication ?**
$\mathbb{P}(Z < E(Z)) < 0.5$, mean<median, A is far from enough.
(iv) The insurer offers a whole life insurance to n lives aged x $(= 80)$ with a benefit payment of b paid at the end of the year of death and a price P (for purchasing the insurance).
Let $Z_{x,1}, \ldots, Z_{x,n}$ be the present values per unit of their benefit payments.
Let $W = b\sum_{j=1}^{n} Z_{x,j}/n$. By the CLT or formulae [22] in 447 ($F_{\overline{Y}}(t) \approx \Phi(\frac{t-\mu_{\overline{Y}}}{\sigma_{\overline{Y}}})$)

$$\mathbb{P}(W \leq P) \approx \Phi(\frac{P - E(W)}{\sigma_W}) = 0.99 = \Phi(z_{0.01}) \text{ where } \Phi \text{ is the cdf of } N(0,1),$$

=> $\frac{P-E(W)}{\sigma_W} = z_{0.01}$
=> $P = E(W) + z_{0.01}\sigma_W$, where $E(W) = E\left[b\sum_{j=1}^{n} Z_{x,j}/n\right] = bA_x \quad (E(\overline{Y}) = E(Y))$, and

$$\sigma_W^2 = \mathrm{Var}(W) = \mathrm{Var}\left(b\sum_{j=1}^{n} Z_{x,j}/n\right) = b^2 V(Z_x)/n \qquad \sigma_{\overline{Y}}^2 = \sigma^2/n$$

$$= b^2({}^2A_x - A_x^2)/n$$

$$^2A_x = E(Z_x^2) = A_x(v^2) = \sum_{k=1}^{\infty} v^{2k}\frac{d_{80+k-1}}{\ell_{80}} = (1.065)^{-2}\frac{33}{250} + (1.065)^{-4}\frac{56}{250}$$

$$+ (1.065)^{-6}\frac{54}{250} + (1.065)^{-8}\frac{45}{250} + (1.065)^{-10}\frac{34}{250} + (1.065)^{-12}\frac{28}{250} \approx 0.672$$

Hence, $\sigma_W^2 = 50000^2(0.6723 - 0.8162^2)/250$. Each policyholder should pay
$P = E(W) + z_{0.01}\sigma_W = 40809.50 + 2.326\sqrt{50000^2(0.6723 - 0.8162^2)/250} \approx 41388$ v.s. 50000.

Common critical values:

$z_{0.05}$	$z_{0.025}$	$z_{0.01}$	$z_{0.005}$
1.64	1.96	2.33	2.58

If $i \geq 0$, $0 < A_x \leq 1$. The following is the table of A_{40}, using the life table in page 604.

i	2%	3%	4%	5%	6%	7%	8%
A_{40}	0.4658	0.3286	0.2373	0.1754	0.1326	0.1026	0.0812

$A_x \downarrow$ in i.

Table D.2 (see page 605) shows A_x and ${}^2A_x$ using the life table for the total population of United States in 2004 and $i = 6\%$.

Example 4.10. *An insurer issues a whole life insurance to 100 lives age* 40 *which pays \$20000 at the end of the year of their death.* $i = 0.06$. *Mortality follows the life table for the USA population in 2004 (see pages 602-605). The insurer has a fund with an amount of \$300,000 of dollars to paid for these 100 life insurances. Calculate the probability that this fund is not enough to cover the payments of these 100 life insurances.*

Solution: Let $Z = \sum_{j=1}^{100}(20000)Z_{40,j}$. $Z_{40,j}$'s are i.i.d. from $Z_{40} = v^{K_{40}}$. $E(Z_{40}) = A_{40}$

$\mathbb{P}\{Z > 300000\} = ?$

Formula: $P(\overline{Y} \le t) \approx \Phi(\frac{t-\mu_{\overline{Y}}}{\sigma_{\overline{Y}}}) => P(n\overline{Y} \le nt) \approx \Phi(\frac{nt-n\mu_{\overline{Y}}}{n\sigma_{\overline{Y}}}) => P(Z \le x) \approx \Phi(\frac{x-E(Z)}{\sigma_Z})$.

$$E[Z] = \underbrace{n}_{=?}\underbrace{b}_{=?}\underbrace{A_{40}}_{=?} \text{ and } \sigma_Z^2 = b^2 n({}^2A_{40} - (A_{40})^2).$$

From the life table D.2 (p.603), $A_{40} = 0.13264232$ and ${}^2A_{40} = E(Z_{40}^2) = 0.03648695$. Then

$$E[Z] = nbE(Z_{40}) = (100)(20000)(0.13264232) = 265284.64,$$
$$\text{Var}(Z_{40}) = 0.03648695 - (0.13264232)^2 = 0.01889296495,$$
$$\text{Var}(Z) = nb^2V(Z_{40}) = 100(20000)^2(0.01889296495) = 755718598.$$

The probability that the fund is not enough to cover the payments is

$$\mathbb{P}\{Z > 300000\} \approx 1 - \Phi(\frac{300000 - 265284.64}{\sqrt{755718598}}) \approx 1 - \Phi(1.26) \approx 0.1038.$$

Example 4.11. *If the mortality of* (x) *is given by*

k	*0*	*1*	*2*	*3*	*4*
p_{x+k}	0.05	0.01	0.005	0.001	0

calculate A_x *if* $i = 7.5\%$.

Solution: $A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k)$, where $v = \frac{1}{1+i}$. $f_{K_x}(k) = {}_{k-1|}q_x = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which to choose?**

$$A_x = \sum_{k=1}^{\infty} v^k \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1} \textbf{ (what happens to } k = 1 \textbf{ ?)}$$

$$\begin{aligned}=&vq_x + v^2 p_x q_{x+1} + v^3 p_x p_{x+1} q_{x+2} + v^4 p_x p_{x+1} p_{x+2} q_{x+3} + v^5 p_x p_{x+1} p_{x+2} p_{x+3} q_{x+4} \\ &+ v^6 p_x p_{x+1} p_{x+2} p_{x+3} \underbrace{p_{x+4}}_{=0} q_{x+5} + \cdots \qquad p_k s \ are \ different \\ =&(1.075)^{-1}(0.95) + (1.075)^{-2}(0.05)(0.99) + (1.075)^{-3}(0.05)(0.01)(0.995) \\ &+ (1.075)^{-4}(0.05)(0.01)(0.005)(0.999) + (1.075)^{-5}(0.05)(0.01)(0.005)(0.001)(1) \approx 0.9270.\end{aligned}$$

Theorem 4.2.

Example 4.12. *Rose is 40 years old. She buys a whole life policy insurance which will pay \$200000 at the end of the year of her death. Suppose that the de Moivre model holds with terminal age* 120*. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* 10%.

Solution: $Z = bZ_x$, $Z_x = v^{K_x}$, $b = 200,000$. $E(Z)$=?? $\sigma_Z = ?$
$\sigma_Z = b\sigma_{Z_x}$. $\quad \sigma_{Z_x}^2 = {}^2A_x - A_x^2$. $\quad A_x(v) = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$.

$$f_{K_x}(k) = \frac{s(x+k-1) - s(x+k)}{s(x)} = \frac{\ell_{x+k-1} - \ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$$

Which to choose for $f_{K_x}(k)$?

$$A_x(v)=\sum_{k=1}^{\omega-x} v^k \frac{s(x+k-1)-s(x+k)}{\underbrace{s(x)}_{=(w-x)/w}}=\sum_{k=1}^{\omega-x} v^k \frac{1}{\omega-x}=v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}. \quad (x,w,v)=?$$

$$^2A_x=(v^2)\frac{1-(v^2)^{w-x}}{1-(v^2)}\frac{1}{w-x}. \quad \textbf{Why}??$$

$A_{40}=v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}\Big|_{v=\frac{1}{1.1},\ w=120,\ x=40}=0.12$ and
$^2A_{40}=v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}\Big|_{v=(\frac{1}{1.1})^2,\ w=120,\ x=40}=0.06.$
The actuarial present value of this life insurance is $E(Z)=bA_{40}\approx 24987.80$ ($b=200000$).
$\sigma_Z=b(^2A_x-(A_x)^2)^{1/2}\approx 41911.36.$

Theorem 4.3. *(*Iterative formula for the APV of a discrete whole life insurance*) For each $x>0$, $A_x=vq_x+vp_xA_{x+1}$.*

Proof. (Skip the proof). $A_x=\sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$
$=vq_x+\sum_{k=2}^{\infty} v^k {}_{k-1|}q_x$
$=vq_x+\sum_{k=2}^{\infty} v^k p_x\cdot {}_{k-2|}q_{x+1}$ Formula [8]: ${}_{k|}q_x=p_x\cdot {}_{k-1|}q_{x+1}$
$=vq_x+vp_x\sum_{k-1=1}^{\infty} v^{k-1} {}_{(k-1)-1|}q_{x+1}$
$=vq_x+vp_x\sum_{j=1}^{\infty} v^j {}_{j-1|}q_{x+1}$ $j=k-1$
$=vq_x+vp_xA_{x+1}$ (which is Theorem 4.3 or formula [14]).

In particular, $^2A_x=\frac{(1-p_x)}{\frac{1}{v^2}-p_x}=\frac{q_x}{q_x+i(2+i)}$. It is better to derive Eq.(1) rather than memorizing it, as it is valid only for $p_{x+k}=p$.

Theorem 4.4. Skip! *Suppose that for each $k=1,2,\dots,$ $p_{x+k}=p_x$. Then,*

$$A_x=\frac{1-p_x}{\frac{1}{v}-p_x}=\frac{q_x}{q_x+i} \text{ and } {}^mA_x=\frac{1-p_x}{\frac{1}{v^m}-p_x}=\frac{q_x}{q_x+(1+i)^m-1} \tag{1}$$

Example 4.13. *Jess and Jane buy a whole life policy insurance on the day of their birthdays. Both policies will pay \$50000 at the end of the year of death. Jess is 45 years old and the net single premium of her insurance is \$25000. Jane is 44 years old and the net single premium of her insurance is \$23702. Suppose that $i=0.06$. Find the probability that a 44–year old will die within one year.*

Solution: Given bA_{44} and bA_{45}, $q_{44}=?$
$A_x=vq_x+vp_xA_{x+1}$ (which is formula [14] or Theorem 4.3).
$bA_{45}=25000$, $bA_{44}=23702$, $b=50000$. $A_{44}=\frac{23702}{50000}$, $A_{45}=\frac{25000}{50000}$, $v=\frac{1}{1+i}$ and $i=0.06$.
$A_x=vq_x+v(1-q_x)A_{x+1}=q_xv(1-A_{x+1})+vA_{x+1}$ =>

$$q_{44}=\frac{A_{44}-vA_{45}}{v(1-A_{45})}\Bigg|_{v=\frac{1}{1.06},A_{44}\approx 0.47,A_{45}=0.5}\approx 0.0049648$$

Example 4.14. *Jane is 30 years old. She buys a whole life policy insurance which will pay*

\$20000 at the end of the year of her death. Suppose that $\underline{p_x = 0.9, \text{ for each } x \geq 0}$*, and* $i = 5\%$*. Find the APV of this life insurance and its variance.*

Solution: $Z = bZ_x$. The APV $A = 20000A_x = ?$ $\quad V(Z) = V(bZ_x) = b^2V(Z_x)$?
Key is $A_x(v) = ?$ $\quad$ Formula: $A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$, and
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$

Which to choose ?

$$A_x = \sum_{k=1}^{\infty} v^k \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1} = \sum_{k=1}^{\infty} v^k \left(\prod_{j\geq 0}^{k-2} p_x\right) q_x \textbf{ why ?}$$

$$= \sum_{k=1}^{\infty} v^k p_x^{k-2+1}(1-p_x)? \text{ } or \text{ } = \sum_{k=1}^{\infty} v^k p_x^{k-2}(1-p_x)?$$

$$= \frac{1-p_x}{p_x}\sum_{k=1}^{\infty}(vp_x)^k \frac{1-p_x}{p_x}\sum_{k=1}^{\infty}(t)^k = \frac{1-p_x}{p_x} t \frac{1-(t)^{\infty}}{1-t}\Big|_{t=vp_x} = \frac{1-p_x}{p_x}\frac{vp_x(1-(vp_x)^{\infty})}{1-vp_x}$$

$$A_x = \frac{(1-p_x)v}{1-vp_x} = \frac{1-p_x}{\frac{1}{v}-p_x} = \frac{1-p_x}{1+i-p_x} = \frac{q_x}{q_x+i} = A_x \qquad \text{(which is Th 4.4)}$$

Q: Which of $A_x = \frac{(1-p_x)v}{1-vp_x} = \frac{1-p_x}{\frac{1}{v}-p_x} = \frac{q_x}{q_x+i}$ **do you prefer ?**

$${}^2A_x = A_x(v^2) = \frac{1-p_x}{\frac{1}{v^2}-p_x}$$

Since $b = 20000$, the actuarial present value is

$$\text{APV} = bA_{30} = (20000)\frac{1-p_x}{\frac{1}{v}-p_x}\Big|_{p_x=0.9, v=1/1.05} \approx 13333.$$

$${}^2A_{30} = A_x(v^2) = \frac{1-p_x}{\frac{1}{v}-p_x}\Big|_{p_x=0.9, v=1/1.05^2} \approx 0.493827.$$

$$\sigma_Z^2 = b^2({}^2A_x - (A_x)^2) = 20000^2({}^2A_x) - (bA_x)^2 = 19761991.$$

Example 4.15.

Example 4.16.

Definition 4.4.

Example 4.17. *An actuary models the future lifetime of* (30) *as follows. The actuary classifies lives according with health into 3 groups: good, average and poor health. The probabilities of belonging to a given group are given by the following table. Individuals for the same group have the same constant force of mortality. The force of mortality for each group is given in the following table The annual effective rate of interest is* $i = 7.5\%$*. Find* A_x *and* $\text{Var}(Z_x)$*.*

Group in health	*good*	*average*	*poor*
Probability	*0.1*	*0.3*	*0.6*
Force of mortality	*0.01*	*0.05*	*0.1*

Solution: [5]: For constant force of mortality $\mu_x = c$, $p_x = e^{-\int_0^1 \mu_x dx} = e^{-c}$. $\mu_x = ?$
[14] $A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$, $A_x = A_{x+1}$ **Why ?** ${}_tp_{x+1} = {}_tp_x$.
[14] $A_x = vq_x + vp_x A_{x+1} = vq_x + vp_x A_x$ yields $A_x = \frac{vq_x}{1+vp_x} = \frac{1-p_x}{1/v+p_x}$, or

$$ {}^mA_x = A_x(v^m) = \frac{1-p_x}{\frac{1}{v^m} - p_x} = \frac{1-e^{-\mu}}{\frac{1}{v^m} - e^{-\mu}}. \qquad \textbf{(Can we use it directly here ?)}(3)$$

We introduce a new r.v. Y to denote the health status of an insuree.

$$Y = \begin{cases} 2 & \text{if an insuree is in good health,} \\ 1 & \text{if an insuree is in average health,} \\ 0 & \text{if an insuree is in poor health,} \end{cases}$$

j	2	1	0
$f_Y(j)$	0.1	0.3	0.6
μ_j	0.01	0.05	0.1

Using the double expectation theorem,
${}^mA_x = E[Z_x^m] = E[E[Z_x^m|Y]] = \sum_{j=0}^{2} f_Y(j) E[Z_x^m|Y=j]$, $m = ?$

$$E[Z_x|Y=2] = A_x^{\text{good}} = \frac{1-p_{x,2}}{\frac{1}{v^1} - p_{x,2}} = \frac{1-e^{-0.01}}{1.075 - e^{-0.01}} = 0.11712945 \quad \textit{by } (3),$$

$$E[Z_x^2|Y=2] = {}^2A_x^{\text{good}} = \frac{1-p_{x,2}}{\frac{1}{v^2} - p_{x,2}} = \frac{1-e^{-0.01}}{(1.075)^2 - e^{-0.01}} = 0.06009455691.$$

$$E[Z_x|Y=1] = A_x^{\text{average}} = \frac{1-p_{x,1}}{\frac{1}{v^1} - p_{x,1}} = \frac{1-e^{-0.05}}{1.075 - e^{-0.05}} = 0.3940401449,$$

$$E[Z_x^2|Y=1] = {}^2A_x^{\text{average}} = \frac{1-e^{-0.05}}{(1.075)^2 - e^{-0.05}} = 0.2386087633.$$

$$E[Z_x|Y=0] = A_x^{\text{poor}} = \frac{1-e^{-0.1}}{1.075 - e^{-0.1}} = 0.5592450518,$$

$$E[Z_x^2|Y=0] = {}^2A_x^{\text{poor}} = \frac{1-e^{-0.1}}{(1.075)^2 - e^{-0.1}} = 0.3794549205.$$

$$\begin{aligned} A_x = E[Z_x] =& E[E[Z_x|Y]] \\ =& E[Z_x|Y=2]\mathbb{P}\{Y=2\} + E[Z_x|Y=1]\mathbb{P}\{Y=1\} + E[Z_x|Y=0]\mathbb{P}\{Y=0\} \\ \approx& (0.1)(0.1171) + (0.3)(0.3940) + (0.6)(0.5592) \approx 0.3053, \\ {}^2A_x = E[Z_x^2] =& E[E[Z_x^2|Y]] \\ =& E[Z_x^2|Y=2]\mathbb{P}\{Y=2\} + E[Z_x^2|Y=1]\mathbb{P}\{Y=1\} + E[Z_x^2|Y=0]\mathbb{P}\{Y=0\} \\ \approx& (0.1)(0.0601) + (0.3)(0.2386) + (0.6)(0.3795) \approx 0.4480, \\ \mathrm{Var}(Z_x) =& {}^2A_x - (A_x)^2 \approx 0.448 - (0.3053)^2 \approx 0.0886. \end{aligned}$$

4.2.2 n–year term life insurance.

Definition 4.5. *The n–**th term life insurance policy** (or n–year term life insurance): It pays a face value b if $T(x) \le n$ (the insured dies within n years of the issue of the policy).*

Definition 4.6. *The present value and the APV of an n–year term life insurance policy which pays a **unit** face value at the end of the year of the death is denoted by $Z^1_{x:\overline{n}|}$ and $A^1_{x:\overline{n}|}$, respectively ($A^1_{x:\overline{n}|} = A^1_{x:\overline{n}|}(v) = E[Z^1_{x:\overline{n}|}]$).*

Definition 4.7. $Z^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n) = \begin{cases} v^{K_x} & \text{if } K_x \le n, \\ 0 & \text{if } K_x > n. \end{cases}$

The model $Z = b_k v_k$ applies with $b_k = bI(K_x \le n)$ and $v_k = v^k$, $k \ge 1$.
3 types of problems:
(1) $A = bA^1_{x:\overline{n}|} = ?$
(2) $\sigma^2 = b^2 V(Z^1_{x:\overline{n}|}) = ?$
(3) p, or n, or $P(Z < p)$? (such that $P(Z < p) = \Phi(\frac{p-A}{\sigma_Z}) > 0.99$ or 0.95.)

Theorem 4.5. ${}^m A^1_{x:\overline{n}|} = A^1_{x:\overline{n}|}(v^m) = E[(Z^1_{x:\overline{n}|})^m] = \sum_{k=1}^n (v^m)^k \cdot {}_{k-1}|q_x$.

Example 4.18. *Let $i = 0.05$, $q_x = 0.05$ and $q_{x+1} = 0.02$. Find $A^1_{x:\overline{2}|}$ and $\mathrm{Var}(Z^1_{x:\overline{2}|})$.*

Solution: Formulas: $A^1_{x:\overline{n}|} = E(v^{K_x} I(K_x \le n)) = \sum_{k=1}^n v^k \cdot f_{K_x}(k) = v f_{K_x}(1) + v^2 f_{K_x}(2)$.
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which one?**

$$\begin{aligned} A^1_{x:\overline{2}|}(v) =& v f_{K_x}(1) + v^2 f_{K_x}(2) = v q_x + v^2 p_x q_{x+1} \quad \textbf{why } ? \\ =&(1.05)^{-1}(0.05) + (1.05)^{-2}(1-0.05)(0.02) = 0.06485260771, \\ {}^2A^1_{x:\overline{2}|} =& A^1_{x:\overline{2}|}(v^2) = v^2 q_x + v^4 p_x q_{x+1} \\ =&(1.05)^{-2}(0.05) + (1.05)^{-4}(1-0.05)(0.02) = 0.06098282094, \\ \mathrm{Var}(Z^1_{x:\overline{2}|}) =& {}^2A^1_{x:\overline{2}|} - \left(A^1_{x:\overline{2}|}\right)^2 \approx 0.06098 - (0.06485)^2 \approx 0.0568. \end{aligned}$$

Example 4.19. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year term life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the APV and the standard deviation of the present value of this life insurance.

(ii) Find the probability that the APV of this life insurance is adequate to cover it.

(iii) Find the probability that the present value of this life insurance exceeds one standard deviation to its APV.

Solution: (i) The APV of this life insurance $Z = bv^{K_x} I(K_x \le n)$ is $A = (50000)A^1_{80:\overline{3}|}$
Formula: $A^1_{x:\overline{n}|}(v) = \sum_{k=1}^n v^k \cdot f_{K_x}(k)$, $\sigma_Z = 50000\sqrt{{}^2A_{80:\overline{3}|} - A^2_{80:\overline{3}|}}$ and
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which one?**

$$\begin{aligned}
A^1_{80:\overline{3}|} =& \sum_{k=1}^{3} v^k \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}} \\
=& (1.065)^{-1}\frac{250-217}{250} + (1.065)^{-2}\frac{217-161}{250} + (1.065)^{-3}\frac{161-107}{250} = 0.5002507 \\
{}^2A^1_{80:\overline{3}|} =& A^1_{80:\overline{3}|}(v^2) = (1.065)^{-2}\frac{250-217}{250} + (1.065)^{-4}\frac{217-161}{250} + (1.065)^{-6}\frac{161-107}{250} = 0.4385316, \\
A =& (50000)A^1_{80:\overline{3}|} = 25012.53726. \\
\sigma_Z =& 50000\sqrt{0.4385316 - 0.5002507^2} = 21695.66542.
\end{aligned}$$

(ii) The probability that 25012.53726 is adequate is

$$\begin{aligned}
P(B) =& \mathbb{P}\{(50000)Z^1_{80:\overline{3}|} \le 25012.53726\} \\
=& \mathbb{P}\{Z^1_{80:\overline{3}|} \le 25012.53726/50000\} \\
=& \mathbb{P}\{(1.065)^{-K_x} I(K_x \le 3) \le 25012.53726/50000\} \qquad = P(B) \\
=& \mathbb{P}\{B, K_x \le 3\} + \mathbb{P}\{B, K_x > 3\} \qquad B = \{(1.065)^{-K_x} I(K_x \le 3) \le 25012.5/50000\} \\
=& \mathbb{P}\{(1.065)^{-K_x} \le 25012.53726/50000, K_x \le 3\} + \mathbb{P}\{0 \le 25012.53726/50000, K_x > 3\}\mathbf{??} \\
=& \mathbb{P}\{-K_x \ln(1.065) \le \ln(25012.53726/50000), K_x \le 3\} + \mathbb{P}\{K_x > 3\} \\
=& \mathbb{P}\{K_x \ge -\frac{\ln(25012.53726/50000)}{\ln(1.065)}, K_x \le 3\} + \mathbb{P}\{K_x > 3\} \\
=& \mathbb{P}\{K_x \ge 10.999, K_x \le 3\} + \mathbb{P}\{K_x > 3\} \\
=& \mathbb{P}\{T_x > 3\} = {}_3p_x = \frac{\ell_{x+3}}{\ell_x}\Big|_{x=80} = \frac{107}{250} = 0.428.
\end{aligned}$$

(iii) $\mathbb{P}(U) = \mathbb{P}(Z > A + \sigma_Z) = \mathbb{P}\{(50000)Z^1_{80:\overline{3}|} > 25012.53726 + 21695.66542\}$

$$\begin{aligned}
=& \mathbb{P}\{(50000)(1.065)^{-K_x} I(K_x \le 3) > 46708.20268\} \\
=& \mathbb{P}\{U, K_x \le 3) + \mathbb{P}\{(U, K_x > 3) \\
=& \mathbb{P}\{(50000)(1.065)^{-K_x} > 46708.20268, K_x \le 3\} + \mathbb{P}\{0 > 46708.20268, K_x > 3\} \\
=& \mathbb{P}\{K_x < -\frac{\ln(46708.20268/50000)}{\ln(1.065)}, K_x \le 3\} \\
=& \mathbb{P}\{K_x < 1.081435921, K_x \le 3\} \\
=& \mathbb{P}\{K_x \le 1\} \mathbf{= ??} \\
=& P(T(x) \le 1) = q_x = 1 - p_x \\
=& 1 - \frac{\ell_{x+1}}{\ell_x}|_{x=80} = 1 - \frac{217}{250} = 0.132.
\end{aligned}$$

Theorem 4.6.

Corollary 4.1.

Example 4.20. *If* $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$, $(A^1_{x:\overline{10}|}, \mathrm{Var}(Z^1_{x:\overline{10}|})) =?$

Solution: Formulae: $Z^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n)$, $A^1_{x:\overline{n}|}(v) = \sum_{k=1}^n v^k \cdot f_{K_x}(k)$, and $f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which one?**

$$s(t) = {}_tp_x = e^{-\int_0^t \mu(u)du} = e^{-\mu t} => p_x = e^{-\mu},$$ **which one?**

$$\begin{aligned}
A^1_{x:\overline{10}|}(v) =& \sum_{k=1}^{10} v^k f_{K_x}(k) = \sum_{k=1}^{10} v^k (p_x)^{k-1} q_x \ \textbf{(Why ??)}\\
=& p_x^{-1}(1-p_x)\sum_{k=1}^{10}(vp_x)^k \qquad [16]: \quad \sum_{k=1}^n v^k = v\frac{1-v^n}{1-v}, \quad v=?\\
=& \frac{1-p_x}{p_x}\cdot\frac{vp_x(1-(vp_x)^{10})}{1-vp_x} \qquad => Th.4.7\\
=& (1-p_x)\frac{v(1-(vp_x)^{10})}{1-vp_x}\Bigg|_{p_x = e^{-0.03} \text{ and } v = e^{-\delta} = e^{-0.04}}\\
=& 0.2114417945,\\
{}^2A^1_{x:\overline{10}|}(v) =& A^1_{x:\overline{10}|}(v^2) = (1-p_x)\frac{v^2(1-(v^2p_x)^{10})}{1-v^2p_x}\Bigg|_{p_x = e^{-0.03} \text{ and } v = e^{-0.04}}\\
=& 0.1747285636,\\
\mathrm{Var}(Z^1_{x:\overline{10}|}) =& {}^2A^1_{x:\overline{10}|} - {A^1_{x:\overline{10}|}}^2 = 0.1747285636 - (0.2114417945)^2 = 0.1300209311.
\end{aligned}$$

Midterm formulae : 447: [6]-[22], 450: [1]-[12], [14](first 2 lines), [16]

Theorem 4.7. *Under the de Moivre model, if* $n \le \omega - x$, $A^1_{x:\overline{n}|} = \frac{a_{\overline{n}|}}{\omega - x}$.

Theorem 4.8. $A^1_{x:\overline{n}|} = \frac{vq_x(1-p_x^n v^n)}{1-vp_x} = \frac{q_x(1-p_x^n v^n)}{q_x+i}$ *if* $f_{K_x}(k) = p_x^{k-1}(1-p_x), k = 1, 2, \ldots$

Theorem 4.9. *For* $n \ge 1$, $A^1_{x:\overline{n}|} = vq_x + vp_x A^1_{x+1:\overline{n-1}|}$. *[14]*

Proof. Formula: $A^1_{x:\overline{n}|} = \sum_{k=1}^n v^k {}_{k-1|}q_x$
$= vq_x + \sum_{k=2}^n v^k {}_{k-1|}q_x$ (notice
${}_{k-1|}q_x = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{s(x+1)}{s(x)}\frac{s(x+k-1)-s(x+k)}{s(x+1)} = p_x\frac{s\left((x+1)+(k-2)\right)-s\left((x+1)+(k-2)+1\right)}{s(x+1)}$
$= p_x \cdot {}_{k-2|}q_{x+1}$)

$A^1_{x:\overline{n}|}$
$= vq_x + \sum_{k=2}^n v^k p_x \cdot {}_{k-2|}q_{x+1}$
$= vq_x + \sum_{k-1=1}^n v^{k-1+1} p_x \cdot {}_{k-1-1|}q_{x+1}$
$= vq_x + vp_x \sum_{j=1}^{n-1} v^j {}_{j-1|}q_{x+1}$
$= vq_x + vp_x A^1_{x+1:\overline{n-1}|}$.

Example 4.21.

Example 4.22. *An insurer offers a 20–year term life insurance of \$$10^5$ to independent lives age 45. $i = 7.5\%$. Mortality follows de Moivre model with terminal age 110. The insurer has a fund with \$$10^6$ to pay for these insurances. Using the normal approximation, calculate the maximum number of policies the insurer can cover so that the probability that the aggregate present value for the issued policies exceeds the amount in the fund is less than 0.01.*

Solution: Let n be the number of policies that the insurer can cover. The present value for the aggregate n insurances is $Z = \sum_{j=1}^{n} 10^5 Y_j$, where Y_1, ..., Y_n are i.i.d. fron $Z^1_{45:\overline{20}|}$.

We need to determine n so that $\quad P(Z > 10^6) \approx 1 - \Phi(\frac{10^6 - E(Z)}{\sigma_Z}) = 1 - \Phi(z_{0.01}) = 0.01$,

$=> \frac{10^6 - E(Z)}{\sigma_Z} = z_{0.01} = 2.33 => 0 = E(Z) + 2.33\sigma_Z - 10^6.$

$0 = \underbrace{E(Z)}_{n10^5 A^1_{x:\overline{20}|}(v)} +2.33 \times \underbrace{\sigma_Z}_{\sqrt{n(10^5)^2\left(A^1_{x:\overline{20}|}(v^2) - (A^1_{x:\overline{20}|}(v))^2\right)}} -10^6$, or $\underbrace{an + b\sqrt{n} + c = 0}_{\sqrt{n} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}} => n = ?$

$A^1_{x:\overline{20}|} = \sum_{k=1}^{20} v^k f_{K_x}(k)$.

$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which one?**

$s(x) = 1 - x/w = \frac{w-x}{w}$ and $f_{K_x}(k) = \frac{w-x-k-(w-x-k-1)}{w-x} = \frac{1}{w-x}$, $0 < x < w$.

$$A^1_{x:\overline{20}|} = \sum_{k=1}^{20} v^k {}_{k-1|}q_x = \sum_{k=1}^{20} v^k \frac{1}{\omega - x} = \frac{1}{w-x}\frac{v(1-v^n)}{1-v}. \qquad (1)$$

$$E(Z) = n10^5 A^1_{45:\overline{20}|} = n10^5 \frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075} = 15683.83286n,$$

$$\mathrm{Var}(Z) = n(10^5)^2 (\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075^2} - (\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075})^2) = 687801161.6n.$$

Now solve n from equation $E(Z) + z_{0.01}\sigma_Z - 10^6 = 0$

$$15683.83286n + (2.3263479)\sqrt{687801161.6n} - 10^6 = 0 \qquad (\uparrow \text{ in } n) \qquad (2!)$$

$$=> 15683.83286n + 61010.71512\sqrt{n} - 10^6 = 0.$$

$\sqrt{n} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. $=> \sqrt{n} = \frac{-b-\sqrt{b^2-4ac}}{2a} < 0$, or $\sqrt{n} \approx 6.27$. So, $n \approx 6.3^2 \approx 39.4$.
The maximum number of policies that the insurer can cover is 39 or 40 ? Why ? (see (2!)).

Example 4.23. *Using $i = 0.05$ and a certain life table $A^1_{37:\overline{10}|} = 0.52$.*
Suppose that an actuary revises this life table and
changes p_{37} from 0.95 to 0.96.
Other values in the life table are unchanged, except $A^1_{37:\overline{10}|}$.
Find $A^1_{37:\overline{10}|}$ using the revised life table.

Solution: Using recursive formula [14]: $A^1_{x:\overline{n}|} = vq_x + vp_x A^1_{x+1:\overline{n-1}|}$.
Old: $A^1_{x:\overline{n}|} = 0.52 = \frac{1}{1.05}(0.05 + 0.95A^1_{x+1:\overline{n-1}|})$. $=> A^1_{x+1:\overline{n-1}|} = \frac{0.52*1.05-0.05}{0.95}$.

New: $A^1_{x:\overline{n}|} = \frac{1}{1.05}(0.04 + 0.96A^1_{x+1:\overline{n-1}|}) = \frac{1}{1.05}(0.04 + 0.96\frac{0.52*1.05-0.05}{0.95}) \approx 0.515.$

4.2.3 n–year deferred life insurance.

Definition 4.8. *The n–**year deferred life insurance**: It pays a face value b if $T(x) > n$ (the insured dies at least (after) n years after the issue of the policy).*

Definition 4.9. *The present value and APV of an n–year deferred life insurance with unit payment paid at the end of the year of death are denoted by ${}_n|Z_x$ and ${}_n|A_x$, respectively (${}_n|A_x = E[{}_n|Z_x]$).*

Definition 4.10. ${}_n|Z_x = v^{K_x} I(K_x > n) = \begin{cases} 0 & \text{if } K_x \le n \\ v^{K_x} & \text{if } n < K_x. \end{cases}$

The model $Z = b_T v_T$ applies with $b_k = bI(K_x > n)$ and $v_k = v^k$, $k = 1, 2, \ldots$

${}_n|A_x = {}_n|A_x(v) = E[{}_n|Z_x] = \sum_{k=n+1}^{\infty} v^k \mathbb{P}\{K_x = k\}$.

${}^m{}_n|A_x = {}^m{}_n|A_x(v) = {}_n|A_x(v^m) = \sum_{k=n+1}^{\infty} v^{mk} \mathbb{P}\{K_x = k\} = E\left[({}_n|Z_x)^m\right]$

and $\mathrm{Var}({}_n|Z_x) = {}^2{}_n|A_x - {}_n|A_x{}^2$.

Example 4.24. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

An 80–year old buys a three–year deferred policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

Find the probability that APV of this life insurance is adequate to cover this insurance.

Solution: $P(A \ge Z) = \,?$ $Z = b({}_n|Z_x))$, $b = \,?$ and $A = E(Z) = b\sum_{k>n}^{\infty} v^k f_{K_x}(k) = \,?$

$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which ?**

$$A = (50000)\sum_{k=4}^{\infty} v^k f_{K_x}(k) = (50000)\sum_{k=4}^{\infty} v^k \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}} \text{Annual discount factor } v = \,?$$

$$= (50000)[(1.065)^{-4}\frac{107-62}{250} + (1.065)^{-5}\frac{62-28}{250} + (1.065)^{-6}\frac{28-0}{250}] = 15796.96857.$$

$P(A \ge Z)$
$= P(A \ge Z,\ 3 < K_x) + P(A \ge Z,\ K_x \le 3)$
$= P(A \ge b(1.065)^{-K_x} I(3 < K_x),\ 3 < K_x) + P(A \ge b(1.065)^{-K_x} I(3 < K_x),\ K_x \le 3)$
$= P(A \ge (50000)(1.065)^{-K_x},\ 3 < K_x) + P(A \ge 0,\ K_x \le 3)$
$= P(K_x \ge \frac{-\ln(A/50000)}{\ln(1.065)},\ 3 < K_x) + P(K_x \le 3)$
$= P(K_x \ge 18.3,\ 3 < K_x) + P(K_x \le 3)$
$= P(K_{80} \ge 18.3) + P(K_{80} \le 3)$
$= P(T_{80} > 18) + 1 - P(T_{80} > 3)$
$= \frac{\ell_{98}}{\ell_{80}} + 1 - \frac{\ell_{83}}{\ell_{80}} = 0 + 1 - \frac{107}{250} = 0.572.$

Theorem 4.10. ${}_n|Z_x + Z^1_{x:\overline{n}|} = Z_x$ *and* ${}_n|A_x + A^1_{x:\overline{n}|} = A_x$.

Theorem 4.11. *Under constant force of mortality* μ, ${}_n|A_x = e^{-n(\mu+\delta)}\frac{q_x}{q_x+i} = \frac{(vp_x)^n q_x}{\frac{1}{v}-p_x}$.

Example 4.25. *Suppose that mortality of* (x) *is given by the table*

k	0	1	2	3	4
p_{x+k}	0.05	0.01	0.005	0.001	0

Calculate ${}_2|A_x$ *if* $i = 7.5\%$.

Solution: Formula: ${}_n|A_x = \sum_{k=n+1}^{\infty} v^k f_{K_x}(k)$ and
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which ?**

$$\begin{aligned}
{}_2|A_x &= \sum_{k=3}^{5} v^k \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1} \\
&= v^3 p_x p_{x+1} q_{x+2} + v^4 p_x p_{x+1} p_{x+2} q_{x+3} + v^5 p_x p_{x+1} p_{x+2} p_{x+3} q_{x+4} \\
&= (1.075)^{-3}(0.05)(0.01)(0.995) \\
&+ (1.075)^{-4}(0.05)(0.01)(0.005)(0.999) \\
&+ (1.075)^{-5}(0.05)(0.01)(0.005)(0.001)(1) \approx 0.0004.
\end{aligned}$$

[14] n–year deferred : ${}_n|Z_x = \underline{v^{K_x} I(n < K_x)}$, ${}_n|\overline{Z}_x = \underline{v^{T_x} I(n < T_x)}$, ${}^2{}_n|A_x = {}_n|A_x\ \underline{(v^2)}$,
${}_n|A_x = E[{}_n|Z_x] = \underline{\sum_{k=n+1}^{\infty}}\ \overline{v^k f_{K_x}(k)} = \underline{v^{n+1} f_{K_x}(n+1) + {}_{n+1}|A_x} = vp_x \cdot {}_{n-1}|A_{x+1}$.

Example 4.26. *Rose is 40 years old. She buys a 25-year deferred life policy insurance which will pay \$200,000 at the end of the year of her death. Suppose that the de Moivre model holds with terminal age* 120. *Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* $i = 10\%$.

Solution: $E(Z) = ?$ and $\sigma_Z = ?$ where $Z = b \cdot {}_n|Z_x$.
Formula: ${}_n|A_x = \sum_{k=n+1}^{\infty} v^k f_{K_x}(k)$, $s(t) = 1 - \frac{t}{w}$ and
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$. **which ?**

Some results: $f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \begin{cases} \frac{(w-x+k)-(w-x+k+1)}{w-x} = \frac{1}{w-x} & \text{if U(0,w)} \\ e^{-(k-1)\mu} - e^{-k\mu} = e^{-(k-1)\mu}(1-e^{-\mu}) & \text{if } Exp(1/\mu) \end{cases}$

$$\begin{aligned}
{}_n|A_x &= \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1}|q_x = \sum_{k=n+1}^{\omega-x} v^k \frac{1}{\omega-x} = v^n \sum_{k=1}^{\omega-x-n} v^k \frac{1}{\omega-x} = v^n \cdot v\frac{1-v^{w-x-n}}{(1-v)}\frac{1}{(w-x)} \\
&= v^{n+1}\frac{1-v^{w-x-n}}{(1-v)(w-x)}
\end{aligned}$$

(x,n,w)= ?

$$E(Z) = b \times {}_{25}|A_{40}(v) = b\sum_{k=n+1}^{\infty} v^k \frac{1}{80} = 200000 v^{26} \frac{1-v^{55}}{(1-v)80}\Bigg|_{v=1/1.1} \approx 2295.20.$$

$$E(Z^2) = b^2 \cdot {}_{25}|A_{40}(v^2) = (200000^2) v^{26} \frac{1-v^{55}}{(1-v)(80)}\Bigg|_{v=1/(1.1)^2} \approx 20281697.51.$$

$$\sigma_Z \approx \sqrt{20281697.51 - (2295.20)^2} \approx 3874.76.$$

Theorem 4.12.

Example 4.27. *Let* ${}_{14}|A_{35} = 0.24$, $i = 8\%$ *and* $p_{35} = 0.96$. ${}_{13}|A_{36} =$ *?*

Solution: Formula [14] ${}_n|A_x = vp_x \cdot {}_{n-1}|A_{x+1}$. $(n, x) = ?$
Thus $0.24 = {}_{14}|A_{35} = (1.08)^{-1}(0.96) \cdot {}_{13}|A_{36}$.

=> $${}_{13|}A_{36} = \frac{(0.24)(1.08)}{0.96} = 0.27.$$

Example 4.28. *An insurance company offers a 10–year deferred life insurance for individuals aged 25, which will pay \$250000 at the end of the year of his death. Suppose that $p_x = 0.95$, for each $x \geq 0$, and $\delta = 0.065$. 50 lives enter this insurance contract. Calculate the amount Q such that the probability that the aggregate present value of these 50 lives is less than this amount Q is 0.95.*

Solution: $Z = 250000{}_{m|}Z_x$, $m = ?$ annual discount factor $v = e^{-\delta}$ and $n = 50$.
Let Z_1, ..., Z_n be i.i.d. from Z. CLT => $\frac{\overline{Z}-E(\overline{Z})}{\sigma_{\overline{Z}}} = \frac{\sum_i Z_i - E(\sum_i Z_i)}{\sigma_{\sum_i Z_i}} \sim \Phi(0,1)$. $\sigma^2_{\sum_i Z_i} = n\sigma^2_{Z_1}$?
Find Q such that $P(\sum_{i=1}^n Z_i \leq Q) \approx \Phi(\frac{Q-nE(Z)}{\sqrt{n}\sigma_Z}) = \Phi(z_{0.05}) = 0.95.$ => $\frac{Q-nE(Z)}{\sqrt{n}\sigma_Z} = z_{0.05}$,
=> $Q = nE(Z) + z_{0.05}\sqrt{n}\sigma_Z = ?$
$E(Z) = b \cdot {}_{m|}A_x$ and $\sigma^2_Z = b^2\sigma^2_{{}_{m|}Z_x}$. So,

$$Q = b(n \cdot {}_{m|}A_x(v) + 1.645\sqrt{n}\sqrt{{}_{m|}A_x(v^2) - ({}_{m|}A_x(v))^2}).$$

${}_{m|}A_x(v) = \sum_{k=m+1}^{\infty} v^k f_{K_x}(k)$.
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$

$$\begin{aligned}{}_{m|}A_x(v) = &\sum_{k=m+1}^{\infty} v^k p_x^{k-1} q_x = \sum_{k=m+1}^{\infty} (vp_x)^k p_x^{-1} q_x \\ =&(vp_x)^m p_x^{-1} q_x \sum_{j=1}^{\infty-m} (vp_x)^j \quad (j = k-m \text{ or } k = j+m) \\ =&(vp_x)^m p_x^{-1} q_x \frac{vp_x}{1-vp_x} = \frac{(vp_x)^m q_x}{\frac{1}{v} - p_x}\end{aligned}$$

$Q = 250000(n \cdot {}_{m|}A_x(v) + 1.645\sqrt{n}\sqrt{{}_{m|}A_x(v^2) - ({}_{m|}A_x(v))^2})\Big|_{n=50, v=e^{-0.065}, p_x=0.95, q_x=0.05}$
≈ 2130968.

Example 4.29. *Suppose that ${}_{14|}A_{35} = 0.24$, $i = 8\%$, ${}_{14}p_{35} = 0.7$, $q_{49} = 0.03$. Find ${}_{15|}A_{35}$.*

Solution: Formula [3]: ${}_{s|t}q_x = {}_s p_x \cdot {}_t q_{x+s}$ => ${}_{14|}q_{35} = {}_{14}p_{35} \cdot q_{49}$.
[14] n–year deferred : ${}_{n|}Z_x = \underline{v^{K_x} I(n < K_x)}$, ${}_{n|}\overline{Z}_x = \underline{v^{T_x} I(n < T_x)}$, ${}^2{}_{n|}A_x = {}_{n|}A_x\ \underline{(v^2)}$,
${}_{n|}A_x = E[{}_{n|}Z_x] = \underline{\sum_{k=n+1}^{\infty}}\ \overline{v^k f_{K_x}(k)} = \underline{v^{n+1} f_{K_x}(n+1) + {}_{n+1|}A_x}$
$(n, x) = ?$
${}_{14|}A_{35} = v^{14+1} f_{K_{35}}(14+1) + {}_{14+1|}A_{35} = v^{14+1}{}_{14|}q_{35} + {}_{14+1|}A_{35}$.
Thus $0.24 = v^{15}{}_{14}p_{35} \cdot q_{49} + {}_{15|}A_{35} = (1.08)^{-15}(0.7)(0.03) + {}_{15|}A_{35}$.
=> ${}_{15|}A_{35} = 0.24 - (1.08)^{-15}(0.7)(0.03) \approx 0.23$.

Theorem 4.13. *For each $x > 0$, ${}_{n|}A_x = vp_x \cdot {}_{n-1|}A_{x+1}$.*

One may try to derive formula [14] directly as follows. Skip ————————-

Conditions:
$0.24 = {}_{14}|A_{35} = {}_{n}|A_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1}|q_x$
$= v^{n+1} \cdot {}_{n+1-1}|q_x + \sum_{k=n+2}^{\infty} v^k \cdot {}_{k-1}|q_x$
$= v^{n+1} \cdot {}_{n}|q_x + \sum_{k=n+1+1}^{\infty} v^k \cdot {}_{k-1}|q_x$
$= v^{n+1} \cdot {}_{n}|q_x + {}_{n+1}|A_x$
$= v^{15} \cdot {}_{14}|q_{35} + {}_{15}|A_{35}$. ——————-

Theorem 4.14. *For each $x > 0$, ${}_{n}|A_x = v^{n+1}{}_{n}|q_x + {}_{n+1}|A_x$. (see [14])*

4.2.4 n–year pure endowment life insurance.

Definition 4.11. *The n–***year pure endowment life insurance***: It pays a face value in n years when $T(x) > n$ (the insured dies at least (after) n years from the issue of the policy). Its present value and APV with unit payment are denoted by $Z_{x:\overline{n}|}^{\ \ 1}$ and $A_{x:\overline{n}|}^{\ \ 1}$, respectively.*

$Z_{x:\overline{n}|}^{\ \ 1} = v^n I(K_x > n) = v^n I(T(x) > n) = \overline{Z}_{x:\overline{n}|}^{\ \ 1}$, and $A_{x:\overline{n}|}^{\ \ 1} = \overline{A}_{x:\overline{n}|}^{\ \ 1} = E[Z_{x:\overline{n}|}^{\ \ 1}] = {}_{n}E_x$.
$Z_{x:\overline{n}|}^{1} = v^{K_x} I(K_x \le n)$,
$Z_{x:\overline{n}|}^{\ \ 1} = v^n I(K_x > n)$,
${}_{n}|Z_x = v^{K_x} I(K_x > n)$.

Withdraw deadline is approaching. 60- is an F !

The model in (4.1) applies with $b_k = I(K_x > n)$ and $v_k = v^n$, $k = 1, 2, \ldots$
Recall $Y = I(K_x > n) \sim bin(1, p)$ with $p = P(Y = 1) = P(K_x > n) = {}_{n}p_x$.

$E(Y) = p = {}_{n}p_x$ and $\sigma_Y^2 = pq = {}_{n}p_x \cdot {}_{n}q_x$.
Formulas [14]:

$Z_{x:\overline{n}|}^{\ \ 1} = v^n I(K_x > n)$, $A_{x:\overline{n}|}^{\ \ 1} = v^n \cdot {}_{n}p_x$, ${}^2A_{x:\overline{n}|}^{\ \ 1}(v) = A_{x:\overline{n}|}^{\ \ 1}(v^2)$ and $\mathrm{Var}(Z_{x:\overline{n}|}^{\ \ 1}) = v^{2n}{}_{n}p_x \cdot {}_{n}q_x$.
Is it right $\mathrm{Var}(Z_{x:\overline{n}|}^{\ \ 1}) = A_{x:\overline{n}|}^{\ \ 1}(v^2) - (A_{x:\overline{n}|}^{\ \ 1}(v))^2$?

Example 4.30. *An insurance company has 100 clients age 30 which will receive a payment of 50,000 at the end of 10 years if they are alive. Suppose that the probability that a life age 30 will die within 10 years is 0.02. The current annual effective rate of interest is 9%.*
The insurance company sets an account to meet these payments. Calculate the deposit made at time zero such that the probability that the insurance will have enough funds to the benefits is approximately 0.95.

Solution: Let $W = \sum_{i=1}^{100} Z_i$ be the total payments made to the 100 clients,
(Z_i's are i.i.d. from $Z = bZ_{x:\overline{n}|}^{\ \ 1}$). CLT: $\frac{\overline{Z}-E(\overline{Z})}{\sigma_{\overline{Z}}} = \frac{W-E(W)}{\sigma_W} \sim N(0, 1)$, $\quad (E(W), \sigma_W) = ?$
Let Q be the deposit which the insurance company needs to make. By the CLT,

$P(W \le Q) \approx \Phi(\frac{Q-E(W)}{\sigma_W}) = \Phi(1.645) = 0.95$. Hence, $\frac{Q-E(W)}{\sigma_W} = 1.645$.
$Q = E(W) + 1.645\sigma_W = ?$
$E(W) = 100b * A_{x:\overline{n}|}^{\ \ 1}(v)$, b=?
$\sigma_W^2 = 100b^2 V(Z_{x:\overline{n}|}^{\ \ 1})$? or $= 100^2 b^2 V(Z_{x:\overline{n}|}^{\ \ 1})$?

$A_{x:\overset{1}{\overline{n}|}}(v) = v^n \cdot {}_np_x$ $\qquad {}_np_x = ? \qquad v = ? \qquad n = ? \qquad b = ?$

$V(Z_{x:\overset{1}{\overline{n}|}}) = v^{2n}{}_np_x - (v^n{}_np_x)^2 = v^{2n}{}_np_x \cdot {}_nq_x$

Given ${}_{10}q_{30} = 0.02$ and $v = 1/(1+0.09)$,

$A_{x:\overset{1}{\overline{n}|}} = v^n{}_np_x = (1.09)^{-10}(0.98) = 0.4139626,$

$\text{Var}(Z_{x:\overset{1}{\overline{n}|}}) = v^{2n}{}_np_x \cdot {}_nq_x = (1.09)^{-20}(0.98)(1-0.98) = 0.003497266.$

Then $E[W] = 100bA_{x:\overset{1}{\overline{n}|}} = (100)(50000)(0.4139626) = 2069813,$

$\sigma_W^2 = 100b^2V(Z_{x:\overset{1}{\overline{n}|}}) = (100)(50000)^2(0.003497266) = 874316500.$

$Q = E(W) + 1.645\sigma = 2069813 + (1.645)\sqrt{874316500} = 2118454.$

Example 4.31. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three year pure endowment with an amount of \$50000. Suppose that $i = 6.5\%$. Find the APV and SD of this life insurance.

Solution: Let $Z = bZ_{x:\overset{1}{\overline{n}|}} = bv^nI(K_x > n)$. $\quad v = ? \quad b = ? \quad n = ?$

$E(Z) = bv^np$ and $\sigma_Z^2 = b^2v^{2n}pq$, *where* p= ${}_3p_{80}$ $\qquad {}_3p_{80} = ?$

$E(Z) = bv^n{}_3p_{80} = (50000)v^3\frac{\ell_{80+3}}{\ell_{80}} = (50000)(1.065)^{-3}\frac{107}{250} \approx 25850.12.$

$\sigma_Z = bv^n\sqrt{{}_3p_{80}(1-{}_3p_{80})} = 50000(1.065)^{-3}\sqrt{\frac{107}{250}(1-\frac{107}{250})} = 20480.52$

Example 4.32. *An actuary models the future lifetime of* (30) *as follows.* $T(30)$ *has force of mortality μ, where μ has pdf $f_\mu(u) = 400ue^{-20u}$, $u \ge 0$. The force of interest is $\delta = 0.1$. Calculate* $\text{Var}(Z_{x:\overset{1}{\overline{10}|}})$.

Solution: Let $Y = Z_{x:\overset{1}{\overline{10}|}}$. The exercise assumes that

conditional on μ, $f_{Y|\mu}(t|u) = ue^{-ut}\mathbf{1}(t>0)$, where

$f_\mu(u) = \frac{u^{\alpha-1}e^{-u/\beta}}{\Gamma(\alpha)\beta^\alpha}$, $u > 0$, $\qquad \alpha =? \qquad$ and $\beta =?$

Formulas: $E(Y) = \int yf_Y(y)dy = E(E(Y|\mu))$ from $E(X) = E(E(X|Y))$ (36).

Given conditions: annual discount factor $v = e^{-\delta} = e^{-0.1}$ and ${}_np_x = e^{-n\mu}$ conditional on μ.

$$\begin{aligned}
A_{x:\overset{1}{\overline{10}|}} =& E[Z_{x:\overset{1}{\overline{10}|}}] = E[E[Z_{x:\overset{1}{\overline{10}|}}|\mu]] = E\left[v^{10}e^{-10\mu}\right] \\
=& v^{10}\int_0^\infty e^{-10\mu}(400)\mu e^{-20\mu}\,d\mu \qquad = v^{10}\int_0^\infty e^{-10\mu}(400)te^{-20t}\,dt \ ? \\
=& v^{10}(400)\int_0^\infty \mu e^{-30\mu}\,d\mu = v^{10}\frac{400\Gamma(2)}{30^2}\int_0^\infty \frac{\mu^{2-1}e^{\frac{-\mu}{1/30}}}{\Gamma(2)/30^2}\,d\mu\bigg|_{v=e^{-0.1}} = 0.1635019739,
\end{aligned}$$

$${}^2A_{x:\overset{1}{10}} = A_{x:\overset{1}{10}}(v^2) = v^{10}\frac{400}{30^2}\bigg|_{v=(e^{-0.1})^2} \ \textbf{why ?} = 0.06014901477,$$

$$\text{Var}(\overline{Z}_{x:\overset{1}{10}}) = 0.06014901477 - (0.1635019739)^2 = 0.0334161193.$$

Example 4.33. *Assume that mortality follows the life table in page 606. $i = 6\%$. Calculate: $\overline{A}_{40:\overset{1}{10}}$ and $\overline{A}_{20:\overset{1}{15}}$ (i.e. ${}_{10}E_{40}$. ${}_{15}E_{20}$).* **Must learn how to solve it.**

Solution: (i) From the table in page 606, ${}_{10}E_{40} = 0.542299641$.
(ii) The life table only presents ${}_{5}E_x$, ${}_{10}E_x$ and ${}_{20}E_x$. No ${}_{15}E_{20}$.

But ${}_{15}E_{20} = {}_{10+5}E_{20}$.
Formula: ${}_{m+n}p_x = {}_{m}p_x \cdot {}_{n}p_{x+m}$, and ${}_{m}E_x = v^m \cdot {}_{m}p_x$.
${}_{m+n}E_x = v^{n+m} \cdot {}_{m+n}p_x = v^m v^n \cdot {}_{m+n}p_x = v^m \cdot {}_{m}p_x \cdot v^n \cdot {}_{n}p_{x+m} = \underline{{}_{m}E_x \cdot {}_{n}E_{x+m}}$.
${}_{15}E_{20} = {}_{10}E_{20} \cdot {}_{5}E_{30} = {}_{5}E_{20} \cdot {}_{10}E_{25}$. **Which to choose ?**
From the life table in page 606 ${}_{10}E_{20} = 0.553116815$, ${}_{10}E_{25} = 0.552733873$.
From the life table in page 606 ${}_{5}E_{30} = 0.74323819$, ${}_{5}E_{20} = 0.743753117$.
${}_{15}E_{20} = {}_{10}E_{20} \cdot {}_{5}E_{30} = (0.553116815)(0.74323819) = 0.4110975404$.

Theorem 4.15. $A_{x:\overline{n}|}^{\;\;1} = {}_{n}E_x = v^n \cdot {}_{n}p_x$,

Theorem 4.16.

Theorem 4.17.

Theorem 4.18.

Theorem 4.19.

Theorem 4.20. ${}_{m+n}E_x = {}_{m}E_x \cdot {}_{n}E_{x+m}$

Definition 4.12.

Definition 4.13.

4.2.5 n–year endowment life insurance.

Definition 4.14. *The n–**year endowment life insurance**: It makes a payment b at $K_x \wedge n$. Its present value and APV with unit payment paid at end of year of death is denoted by $Z_{x:\overline{n}|}$ and $A_{x:\overline{n}|} = E(Z_{x:\overline{n}|})$, respectively.*

Definition 4.15. $Z_{x:\overline{n}|} = v^{K_x \wedge n} = \begin{cases} v^{K_x} & \text{if } K_x \le n \\ v^n & \text{if } n < K_x \end{cases}$ $(= Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\;\;1} = \begin{cases} v^{K_x} & \text{if } K_x < n \\ v^n & \text{if } n \le K_x. \end{cases})$

The model $Z = b_t v_t$ applies with $b_k = b$ and $v_k = v^{\min(k,n)}$, $k = 1, 2, \ldots$.

$$ {}^{m}A_{x:\overline{n}|} = A_{x:\overline{n}|}(v^m) = E[Z^m_{x:\overline{n}|}] = \sum_{k=1}^{n} v^{mk}\mathbb{P}\{K_x = k\} + v^{mn}\mathbb{P}\{K_x > n\}. $$

Theorem 4.21. $A_{x:\overline{n}|}(v) = \sum_{k=1}^{n} v^k {}_{k-1}|q_x + v^n \cdot {}_{n}p_x = \sum_{k=1}^{n-1} v^k {}_{k-1}|q_x + v^n \cdot {}_{n-1}p_x$.

The proof makes use of $E(g(X)) = \sum_k g(k) f_X(k) \quad g(k) = v^{k \wedge n}$?? $f_X(k) = ?$
$(= \sum_{k<n} g(k) f_X(k) + g(n) f_X(n) + \sum_{k>n} g(n) f_X(k)$??
$= \sum_{k\le n} g(k) f_X(k) + \sum_{k>n} g(n) f_X(k) = \sum_{k<n} v^k f_X(k) + \sum_{k\ge n} g(n) f_X(k)$

Example 4.34. *Suppose that* $i = 0.05$ *and* $q_x = 0.05$. $A_{x:\overline{2}|}$ *and* $\mathrm{Var}(Z_{x:\overline{2}|})$*?*

Solution: $A_{x:\overline{2}|} = v_0|q_x + v^2{}_1|q_x + v^2{}_2p_x = vq_x + v^2p_x$ **why** q_x **not** ${}_0|q_x$ **?**
Are both equations applicable here ?

$$A_{x:\overline{2}|}(v) = vq_x + v^2p_x = (1.05)^{-1}(0.05) + (1.05)^{-2}(1-0.05) \approx 0.9,$$
$${}^2A_{x:\overline{2}|}(v) = A_{x:\overline{2}|}(v^2) = v^2q_x + v^4p_x = (1.05)^{-2}(0.05) + (1.05)^{-4}(1-0.05) \approx 0.8269188,$$
$$\mathrm{Var}(Z_{x:\overline{2}|}) = {}^2A_{x:\overline{2}|}(v) - (A_{x:\overline{2}|}(v))^2 = 0.000097695860391.$$

Example 4.35. *A 10–year endowment insurance pays \$20,000 at the end of the year of failure, or \$20,000 for survival to time 10, whichever occurs first. Find the actuarial present value and the variance of this endowment insurance for a 40–year old if* $s(x) = \frac{100-x}{100}$, $0 \le x \le 100$, *and* $i = 7.5\%$.

Solution: Let $Z = bZ_{x:\overline{n}|}$. $n=$? $b=$? Find $E(Z)$ and $V(Z)$.
$A_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_{k-1}|q_x + v^n \cdot {}_np_x = \sum_{k=1}^{n-1} v^k \cdot {}_{k-1}|q_x + v^n \cdot {}_{n-1}p_x$ **Which is better ?**
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which ?**
$A_{x:\overline{n}|}(v) = \sum_{k=1}^{n} v^k \frac{1}{\omega-x} + v^n(1-\frac{n}{\omega-x}) = \frac{v(1-v^n)}{1-v}\frac{1}{\omega-x} + v^n(1-\frac{n}{\omega-x})$. $\omega =$? $x =$?

$$A = (20000)A_{40:\overline{10}|}(v) = (20000)[v\frac{1-v^{10}}{1-v}\frac{1}{60} + v^{10}(1-\frac{10}{60})]\Big|_{v=1/1.075} \approx 10374.5.$$
$$V(Z) = b^2V(Z_{40:\overline{10}|}) = (20000)^2(A_{40:\overline{10}|}(v^2) - (A_{40:\overline{10}|}(v))^2)\Big|_{v=1/1.075} = \cdots$$

Definition 4.16.

Theorem 4.22. *(*$q_x = {}_0|_1q_x$*)*

Theorem 4.23. *For each* $x > 0$, $A_{x:\overline{n}|} = vq_x + vp_xA_{x+1:\overline{n-1}|}$. *(*$= \sum_{k=1}^{n} v^k \cdot {}_{k-1}|q_x + v^n \cdot {}_np_x$*)*

Theorem 4.24. $A_{x:\overline{n}|} = A_{x:\overline{n+1}|} + dA_{x:\overline{n}|}^{\;\;1}$. *(*$A_{x:\overline{n}|}^{\;\;1} = v^n{}_np_x$ *and* $d = 1-v$*).*

Example 4.36. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Let $i = 6.5\%$. *An 80–year old buys a 3-year endowment policy insurance which will pay \$50000.*

Find the probability that APV of this life insurance is adequate to cover this insurance.

Sol: Let $Z = bv^{K_x \wedge 3} = bZ_{x:\overline{3}|}$ ($b =$?) and $A = E(bZ_{x:\overline{3}|}) = bA_{x:\overline{3}|} =$? $P(Z \le A) =$?
3 steps. (1) $A_{x:\overline{3}|} =$? (2) Simplify $A \ge Z$, (3) Ans.
(1) $A_{x:\overline{3}|} = \sum_{k=1}^{2} v^k{}_{k-1}|q_x + v^3 \cdot {}_2p_x = \sum_{k=1}^{3} v^k{}_{k-1}|q_x + v^3 \cdot {}_3p_x$. **Q: Which ?**
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **which ?**

$A = (50000)A_{80:\overline{3}|} = (50000)[\sum_{k=1}^{2} v^k \frac{\ell_{80+k-1}-\ell_{80+k}}{\ell_{80}} + v^3\frac{\ell_{80+2}}{\ell_{80}}]$
$= (50000)[(1.065)^{-1}\frac{250-217}{250} + (1.065)^{-2}\frac{217-161}{250} + (1.065)^{-3}\frac{161}{250}] \approx 42728.5 = A.$

(2) Simplify $A \geq Z = bZ_{x:\overline{n}|}$: $A = 42728.50782$ is adequate if $A \geq Z = bZ_{x:\overline{n}|}$
$<=> A \geq (50000)(1.065)^{-K_x \wedge 3}$,
$<=> \min(K_x, 3) \geq \frac{-\ln(42728.50782/50000)}{\ln(1.065)} \approx 2.45$.
(3) $\mathbb{P}\{A \geq bv^{K_x \wedge 3}\} = P\{\min(K_x, 3) \geq 2.45\}$ $=?classexercise.$
$= \mathbb{P}\{\min(K_x, 3) \geq 2.45, K_x \geq 3\} + \mathbb{P}\{\min(K_x, 3) \geq 2.45, K_x < 3\}$
$= \mathbb{P}\{3 \geq 2.45, K_x \geq 3\} + \mathbb{P}\{K_x \geq 2.45, K_x < 3\}$
$= \mathbb{P}\{K_x \geq 3\} + \underbrace{\mathbb{P}\{2.45 \leq K_x < 3\}}_{=??}$
$= \mathbb{P}\{T_x > 2\}$ or $\mathbb{P}\{T_x > 3\}$ **???**

$= \frac{\ell_{82}}{\ell_{80}} = \frac{161}{250} = 0.644.$

4.2.6 m–year deferred n–year term life insurance.

Definition 4.17. *The m–**year deferred** n–**year term life insurance**: It makes a payment b if $T(x) \in (m, m+n]$ (death happens during the period of n years that starts m years from now). Its present value and APV with unit payment paid at the end of the year of death is denoted by ${}_{m|n}Z_x$ and ${}_{m|n}A_x$.*

Definition 4.18. ${}_{m|n}Z_x = v^{K_x} I(m < K_x \leq m+n)$ *and* ${}^k{}_{m|n}A_x(v) = {}_{m|n}A_x(v^k)$.

The model $Z = b_t v_t$ applies with $b_k = bI(m < k \leq m+n)$ and $v_k = v^k$.

Theorem 4.25. ${}_{m|n}A_x = E[v^{K_x} I(m < K_x \leq m+n)] = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x$.

Theorem 4.26. ${}_{m|n}Z_x = {}_{m|}Z_x - {}_{m+n|}Z_x$ *and* ${}_{m|n}A_x = {}_{m|}A_x - {}_{m+n|}A_x$.

Theorem 4.27. $Z_x = Z^1_{x:\overline{m}|} + {}_{m|}Z_x = Z^1_{x:\overline{m}|} + {}_{m|n}Z_x + {}_{m+n|}Z_x$.

The proofs of the last two theorems follows from the definitions:

$Z_x = v^{K_x}$,
$Z^1_{x:\overline{m}|} = v^{K_x} I(K_x \leq m)$,
${}_{m|}Z_x = v^{K_x} I(K_x > m)$.
${}_{m|n}Z_x = v^{K_x} I(K_x \in (m, m+n])$.
${}_{m+n|}Z_x = v^{K_x} I(K_x > m+n)$.
For instance,
${}_{m|n}Z_x = v^{K_x} I(K_x \in (m, m+n])$.
+) ${}_{m+n|}Z_x = v^{K_x} I(K_x > m+n)$.

${}_{m|}Z_x = v^{K_x} I(K_x > m)$.
That is, ${}_{m|n}Z_x + {}_{m+n|}Z_x = {}_{m|}Z_x$.
Thus the first theorem holds:

Theorem 4.28.

Theorem 4.29.

Theorem 4.30.

Theorem 4.31.

Definition 4.19.

4.3 Properties of the APV for discrete insurance.

type of life insurance	present value of unit payment
whole	$Z_x = v^{K_x}$
n–year term	$Z^1_{x:\overline{n}\vert} = v^{K_x} I(K_x \le n)$
n–year deferred	${}_n\vert Z_x = v^{K_x} I(n < K_x)$
n–year pure endowment	$Z_{x:\overline{n}\vert}^{\;\;1} = v^n I(n < K_x)$
n–year endowment	$Z_{x:\overline{n}\vert} = v^{\min(K_x,n)}$
m–year deferred n–year term	${}_m\vert{}_n Z_x = v^{K_x} I(m < K_x \le m+n)$

Formula[14]: (note $I(A)I(A^c) = 0$, $I(A) + I(A^c) = 1$, and $I(A) \times I(A) = I(A)$)
$\underline{Z_x} = Z^1_{x:\overline{n}|} + {}_n|Z_x$, $\quad Z^1_{x:\overline{n}|} \cdot {}_n|Z_x = \underline{0}$,
$\underline{Z_{x:\overline{n}|}} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\;\;1}$, $Z^1_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\;\;1} = \underline{0}$,
$\underline{{}_n|A_x} = {}_nE_x A_{x+n}$, $\quad \underline{A_x} = A^1_{x:\overline{n}|} + {}_nE_x A_{x+n}$.

Example 4.37. *Find* $E[{}_n|Z_x]$ *and* $\mathrm{Var}({}_n|Z_x)$ *if*

$$A_x = 0.75,\ \mathrm{Var}(Z_x) = 0.45,\ A^1_{x:\overline{n}|} = 0.5,\ \mathrm{Var}(Z^1_{x:\overline{n}|}) = 0.2.$$

Solution: By the given conditions, we know $(A_x(v), A_x(v^2), A^1_{x:\overline{n}|}(v), A^1_{x:\overline{n}|}(v^2))$,
${}_n|A_x(v)$= ? $\qquad Var({}_n|Z_x) = {}_n|A_x(v^2) - ({}_n|A_x)^2$=? $\qquad$ and ${}_n|A_x(v^2)$=?
Which formula in [14] above ?

$A_x(v) = A^1_{x:\overline{n}|}(v) + {}_n|A_x(v) => {}_n|A_x(v) = 0.75 - 0.5 = 0.25.$
Formulae: $\sigma^2_X = E(X^2) - (E(X))^2$ or $\sigma^2_X + (E(X))^2 = E(X^2)$.

$$[14] \qquad Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x \text{ and } 0 = Z^1_{x:\overline{n}|} \times {}_n|Z_x \tag{1}$$

$$(Z_x)^2 = (Z^1_{x:\overline{n}|})^2 + ({}_n|Z_x)^2 \text{ and } 0 = (Z^1_{x:\overline{n}|})^2 \times ({}_n|Z_x)^2 \tag{2}$$

$$(A_x(v), A_x(v^2), A^1_{x:\overline{n}|}(v), A^1_{x:\overline{n}|}(v^2)) = (0.75, 0.45 + 0.75^2, 0.5, 0.2 + 0.5^2) \tag{3}$$

$$\begin{aligned} \text{By Eq. (3), } 0.45 + (0.75)^2 &= {}^2A_x = {}^2A^1_{x:\overline{n}|} + {}^2{}_n|A_x && \text{(by Eq. (2))} \\ &= ((0.2 + (0.5)^2) + {}^2{}_n|A_x && \text{(by Eq. (3))} \end{aligned}$$

Hence,

$$^2{}_n|A_x = 0.45 + (0.75)^2 - 0.2 - (0.5)^2 = 0.5625.$$

and

$$\mathrm{Var}({}_n|Z_x) = 0.5625 - (0.25)^2 = 0.5.$$

Proof of Eq. (1) and (2):
$Z^1_{x:\overline{n}|} \times {}_n|Z_x = v^{K_x} I(K_x \le n) \times v^{K_x} I(n < K_x) = v^{2K_x} I(K_x \le n) I(K_x > n) = 0.$
$Z^1_{x:\overline{n}|} + {}_n|Z_x = v^{K_x} I(K_x \le n) + v^{K_x} I(n < K_x) = v^{K_x} = Z_x.$
${Z^1_{x:\overline{n}|}}^2 + {}_n|{Z_x}^2 = v^{2K_x} I(K_x \le n) + v^{2K_x} I(n < K_x) = v^{2K_x} = Z_x^2.$

Example 4.38. *Suppose that* $A^1_{x:\overline{n}|} = 0.5, \mathrm{Var}(Z^1_{x:\overline{n}|}) = 0.35, v^n = 0.4, {}_np_x = 0.6$. *Find* $E[Z_{x:\overline{n}|}]$ *and* $\mathrm{Var}(Z_{x:\overline{n}|})$.

Solution: Need to know $A_{x:\overline{n}|}(v) = ?$ $A_{x:\overline{n}|}(v^2) = ?$ **Why ?**
Formula[14]: $\underline{Z_{x:\overline{n}|}} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, $Z^1_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ 1} = \underline{0}$. In fact,

$Z_{x:\overline{n}|} = v^{\min(K_x,n)} = v^{K_x} I(K_x \le n) + v^n I(n < K_x) = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1} => A_{x:\overline{n}|} = \overbrace{A^1_{x:\overline{n}|}}^{=?} + A_{x:\overline{n}|}^{\ \ 1}.$
$(Z_{x:\overline{n}|})^2 = v^{2\min(K_x,n)} = v^{2K_x} I(K_x \le n) + v^{2n} I(n < K_x) = {Z^1_{x:\overline{n}|}}^2 + {Z_{x:\overline{n}|}^{\ \ 1}}^2.$
Given $v^n = 0.4$ and ${}_np_x = 0.6 =>$
$A_{x:\overline{n}|}^{\ \ 1} = v^n{}_np_x = 0.4 \times 0.6.$
$A_{x:\overline{n}|}^{\ \ 1}(v^2) = v^{2n}{}_np_x = 0.4^2 \times 0.6.$
$(A^1_{x:\overline{n}|}, {}^2A^1_{x:\overline{n}|}, A_{x:\overline{n}|}^{\ \ 1}, {}^2A_{x:\overline{n}|}^{\ \ 1}) = (0.5, 0.35 + 0.5^2, 0.24, 0.096).$

Thus
$$\begin{aligned} A_{x:\overline{n}|} &= A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} = 0.5 + 0.24 = 0.74 \\ {}^2A_{x:\overline{n}|} &= {}^2A^1_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1} = 0.35 + (0.5)^2 + 0.096 = 0.696. \\ \mathrm{Var}(Z^1_{x:\overline{n}|}) &= 0.696 - (0.74)^2 = 0.1484. \end{aligned}$$

Theorem 4.32. $Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x$ *and* $0 = Z^1_{x:\overline{n}|} \times {}_n|Z_x$.

Theorem 4.33.

Theorem 4.34. $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$ *and* $0 = Z^1_{x:\overline{n}|} \times Z_{x:\overline{n}|}^{\ \ 1}$.

Corollary 4.2.

Theorem 4.35.

Theorem 4.36.

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Example 4.39.

Example 4.40.

Example 4.41.

4.4 Non–level payments paid at the end of the year

Suppose that a life insurance provides a benefit of b_k paid at the end of the k–th year if death happens in this year.
The present value of this benefit is $Z = b_{K_x}\nu^{K_x}$. The net single premium is $P = E(b_{K_x}\nu^{K_x})$.

Example 4.42. *A whole life insurance on* (50) *pays a death benefit at the end of the year of death. The death benefit is \$50000 for the first year and it increases at annual rate of 3% per year. The annual effective rate of interest is 6.5%. If* $A^*_{50} = 0.47$ *when the annual effective rate of interest* i^* *is* $\frac{0.035}{1.03}$*. Calculate the net single premium* P *for this insurance.*

Solution: $P = E(b_{K_x}v^{K_x}) = \sum_{k=1}^{\infty} b_k v^k f_{K_x}(k)$=? but f_{K_x} is not given.
However, $A^*_{50} = \sum_{k=1}^{\infty} v^k f_{K_x}(k)\Big|_{v=\frac{1.03}{1.065}} = 0.47$ is given.

$$P = \sum_{k=1}^{\infty} \overbrace{[50000(1.03)^{k-1}]}^{b_k} \cdot \overbrace{(1.065)^{-k}}^{v^k} \cdot f_{K_x}(k) = (1.03)^{-1}50000 \overbrace{\sum_{k=1}^{\infty}(1.065/1.03)^{-k} \cdot f_{K_x}(k)}^{A^*_{50}}$$
$$=(1.03)^{-1}(50000)(0.47) = 22815.53398 \text{ (the net single premium).}$$

Example 4.43. *A whole life insurance on* (50) *pays \$50000 plus the return of the net single premium with interest at* $\delta^* = 0.03$ *at the end of the year of death. The survival function for (50) follows the de Moivre's law with* $\omega = 110$*. Calculate the net single premium for* $\delta = 0.07$*.*

Solution: Let $P = E(Z)$, $Z = b_{K_x}v^{K_x}$, $v = e^{-\delta}$, $b_k = 50000 + Pe^{\delta^* k}$.
$P = \sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1|}q_x$=?

${}_{k-1}|q_x = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ which ?

$$
\begin{aligned}
P =& \sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1}|q_x \\
=& \sum_{k=1}^{60} (50000 + Pe^{(0.03)k}) e^{-(0.07)k} \cdot \frac{1}{w-x} \qquad\qquad \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{1}{w-x} \\
=& \sum_{k=1}^{60} (50000) e^{-(0.07)k} \cdot \frac{1}{w-x} + \sum_{k=1}^{60} Pe^{-(0.04)k} \cdot \frac{1}{w-x} \\
=& 50000 \sum_{k=1}^{60} v^k \cdot \frac{1}{w-x} + P \sum_{k=1}^{60} r^k \frac{1}{w-x} \\
=& 50000 \frac{v(1-v^n)}{1-v}\bigg|_{v=e^{-0.07}} \frac{1}{w-x} + P \frac{r(1-r^n)}{1-r}\bigg|_{r=e^{-0.04}} \frac{1}{w-x} \qquad\qquad n = ?? \\
P =& 11320.61245 + 0.3713406834P.
\end{aligned}
$$

$=> \quad P = \frac{11320.61245}{1-0.3713406834} \approx 18007.55.$

Theorem 4.37.

Theorem 4.38.

Definition 4.20. *The* **increasing by one whole life insurance** *or* **annually increasing whole life insurance** *pays k (units) at time k, for each $k \geq 1$, if the failure happens in the k–th year. Its present value and APV are denoted by $(IZ)_x$ and $(IA)_x$.*

Under this policy a payment of K_x is made at time K_x.

$(IZ)_x = K_x v^{K_x}$.

$(IA)_x(v) = E[K_x v^{K_x}] = \sum_{k=1}^{\infty} k v^k \cdot {}_{k-1}|q_x$.

$E[(IZ)_x^2] = E[K_x^2 v^{2K_x}] = \sum_{k=1}^{\infty} k^2 v^{2k} \cdot {}_{k-1}|q_x \qquad = (IA)_x(v^2)$ **???**

Example 4.44. *A special whole life insurance on (40) makes non–level death benefits at the end of the year of death. The first year death benefit is \$10000. Each subsequent year death benefit is \$200 more than the previous year death benefit. $i = 0.06$. Mortality follows de Moivre model with terminal age 100. Calculate the net single premium of this life insurance P.*

Solution: $P = E(Z), \quad Z = 10,000v^{K_x} + 200K_x v^{K_x}$? or $10,000v^{K_x} + 200(K_x - 1)v^{K_x}$?

$Z = (10000 - 200)v^{K_x} + 200K_x v^{K_x} = (10000 - 200)Z_x + 200(IZ)_x.$

$P = E(Z) = (10000 - 200)A_{40} + (200)\,(IA)_{40}, \qquad A_{40} = ??\ (IA)_{40}{=}??$

$$
\begin{aligned}
A_x =&\sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{w-x} v^k \frac{1}{w-x} \qquad = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}, \quad w-x=?\\
(IA)_x =&\sum_{k=1}^{\infty} kv^k \cdot {}_{k-1|}q_x = \sum_{k=1}^{\omega-x} kv^k \frac{1}{\omega-x} = v\sum_{k=1}^{n} kv^{k-1}\frac{1}{\omega-x} \qquad n=??\\
=&(\sum_{k=0}^{n} v^k)'_v \frac{v}{\omega-x} = (\frac{1-v^{n+1}}{1-v})'_v\frac{v}{\omega-x} \qquad [16]\\
=&((1-v^{n+1})(1-v)^{-1})'_v\frac{v}{\omega-x}\\
=&[(-(n+1)v^n)(1-v)^{-1} + (1-v^{n+1})(1-v)^{-2}]\frac{v}{\omega-x}\\
=&\frac{\frac{v}{(1-v)^2}(1-(n+1)v^n + nv^{n+1})}{\omega-x}\\
P =&(10000-200)A_{40} + (200)\,(IA)_{40}\\
=&\big[9800\frac{v}{\omega-x}\frac{1-v^{60}}{1-v} + 200\frac{\frac{v}{(1-v)^2}(1-(n+1)v^n+nv^{n+1})}{\omega-x}\big]_{\omega-x=n=60, v=\frac{1}{1.06}}\\
=&3490.380588.
\end{aligned}
$$

Definition 4.21. Quiz on Friday : *447: [20]- [22], 450: [1]-[12], [14], [16]*

Definition 4.22.

Definition 4.23.

Example 4.45. *Let* $A_{30} = 0.13$*,* $(IA)_{30} = 0.45$*,* $v = 0.94$ *and* $p_{30} = 0.99$*.* $(IA)_{31} =$*?*

Solution: Need the relation between $(IA)_x$, A_x and $(IA)_{x+1}$.

$$
\begin{aligned}
(IA)_x =& \sum_{k=1}^{\infty} kv^k \cdot {}_{k-1}|q_x \\
=& \sum_{k=1}^{\infty}(1+k-1)v^k \cdot {}_{k-1}|q_x \quad \text{(try to get } A_x) \\
=& \sum_{k=1}^{\infty} v^k \cdot {}_{k-1}|q_x + \sum_{k=2}^{\infty}(k-1)v^k \cdot {}_{k-1}|q_x \qquad \textit{why } k=2\ ? \\
=& A_x + \sum_{k=2}^{\infty}(k-1)v^{k-1+1} \cdot {}_{k-1}|q_x \quad \text{(try to get } (IA)_{x+1} = \sum_{j=1}^{\infty} jv^j \cdot {}_{j-1}|q_{x+1}) \\
=& A_x + \sum_{j=1}^{\infty}(j)v^{j+1} \cdot {}_{j}|q_x \quad (j=k-1) \\
=& A_x + v\sum_{j=1}^{\infty} jv^j \cdot \underbrace{{}_{j}|q_x}_{need\ {}_{j-1}|q_{x+1}} \qquad ([3]:{}_s|_tq_x = P(s<T(x)\le s+t) \qquad t \qquad\qquad =?? \\
& ({}_j|q_x = \frac{S(x+j)-S(x+j+1)}{S(x)} \\
& = \frac{S((x+1)+(j-1))-S((x+1)+(j-1)+1)}{S(x+1)} \cdot \frac{S(x+1)}{S(x)} = {}_{j-1}|q_{x+1} \cdot p_x) \\
=& A_x + v\sum_{j=1}^{\infty} jv^j \cdot {}_{j-1}|q_{x+1} \cdot p_x \\
(IA)_x =& A_x + vp_x\,(IA)_{x+1} \qquad (1)
\end{aligned}
$$

$=> 0.45 = 0.13 + (0.94)(0.99)\,(IA)_{31}$,
$=> (IA)_{31} = \frac{0.45-0.13}{(0.94)(0.99)} = 0.3438641737.$

Typical cases of $f_{K_x}(k)$ in computing $(IA)_x$:
(1) Uniform $\frac{1}{w-x}$, (2) Exponential $(e^{-\mu(k-1)} - e^{-\mu k})$, (3) life table, (4) probability table.
Basic method $(IA)_x = \sum_k kv^k f_{K_x}(k)$ due to $E(g(Y)) = \sum_k g(k)f_Y(k)$.

Theorem 4.39. *Under constant force of mortality μ,*

$$(IA)_x = \frac{q_x(1+i)}{(q_x+i)^2} = \frac{vq_x}{(1-vp_x)^2}. \qquad (A_x = \tfrac{q_x}{q_x+i})$$

Proof 2. Since the mortality force is constant, $(IA)_x = (IA)_{x+1}$. By Eq.(1) above $(IA)_x = A_x + {}_1E_x\,(IA)_{x+1} = \frac{q_x}{q_x+i} + vp_x\,(IA)_x$ and $(IA)_x = \frac{\frac{q_x}{q_x+i}}{1-vp_x} = \frac{\frac{q_x}{q_x+i}}{1-\frac{1}{1+i}p_x} = \frac{q_x(1+i)}{(q_x+i)^2}$.

Definition 4.24.

Definition 4.25.

Example 4.46. *Suppose that* $\mu_x(t) = 0.03$, $t \geq 0$, *and* $\delta = 0.06$. $(IA)_x$=?

Solution: Given conditions: $v = e^{-\delta} = e^{-0.06}$ and $s(t) = e^{-\mu t} = e^{-0.03t}$.
$(IA)_x = E(K_x v^{K_x}) = \sum_{k=1}^{\infty} ke^{-\delta k} \cdot {}_{k-1|}q_x$.
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$

$$
\begin{aligned}
(IA)_x =& \sum_{k=1}^{\infty} ke^{-\delta k} \cdot {}_{k-1|}q_x \\
=& \sum_{k=1}^{\infty} ke^{-\delta k} \frac{e^{-\mu(x+k-1)} - e^{-\mu(x+k)}}{e^{-\mu x}} \\
=& \sum_{k=1}^{\infty} ke^{-\delta k} e^{-\mu(k-1)}(1 - e^{-\mu}) \\
=& (1 - e^{-\mu})e^{\mu} \sum_{k=1}^{\infty} ke^{-(\delta+\mu)k} \\
=& (1 - e^{-\mu})e^{\mu} \sum_{k=1}^{\infty} kt^k \Big|_{t=e^{-(\delta+\mu)}} \\
=& (1 - e^{-\mu})e^{\mu} t \times \sum_{k=1}^{\infty} kt^{k-1} \qquad formular[16]: \quad \sum_{k=1}^{\infty} kt^{k-1} = (\frac{1-t^{\infty+1}}{1-t})'_t = (\frac{1}{1-t})'_t \\
=& (1 - e^{-\mu})e^{\mu} t \times (\frac{1}{1-t})'_t \\
=& (1 - e^{-\mu})e^{\mu} t \frac{1}{(1-t)^2} \\
=& \frac{(1 - e^{-\mu})e^{\mu-\delta-\mu}}{(1 - e^{-(\delta+\mu)})^2} \\
=& \frac{e^{-\delta}(1 - e^{-\mu})}{(1 - e^{-\delta}e^{-\mu})^2} \qquad = \frac{vq_x}{(1 - vp_x)^2} \text{ (This actually proves in the last theorem)} \\
(IA)_x =& \frac{e^{-0.06}(1 - e^{-0.03})}{(1 - e^{-(0.03+0.06)})^2} = 3.757282156.
\end{aligned}
$$

Example 4.47.

Example 4.48.

Definition 4.26.

Definition 4.27.

Definition 4.28.

Definition 4.29. *A* **decreasing by one** $n-$ **year term life insurance** *pays* $n+1-k$ *(units) at time* k *if the failure happens in the* k*–th interval, where* $1 \le k \le n$*, that is,* $(DZ)^{1}_{x:\overline{n}|} = (n+1-K_x)v^{K_x}I(K_x \le n)$*. Its APV is denoted by* $(DA)^{1}_{x:\overline{n}|}$*.*

Example 4.49. *Suppose that* $\mu_x(t) = 0.03$*,* $t \ge 0$*, and* $\delta = 0.06$*.* $(DA)_{x:\overline{10}|} =$*?*

Solution: $v = e^{-\delta}$, and $S_{T_x}(t) = {}_tp_x = {}_tp_0 = e^{-\mu t}$, $t > 0$.

$$
\begin{aligned}
(DA)^{1}_{x:\overline{10}|} =& E((n+1-K_x)v^{K_x}I(K_x \le n)) = \sum_{k=1}^{n}(n+1-k)v^k f_{K_x}(k) \\
=& \sum_{k=1}^{10}(11-k)e^{-0.06k}(e^{-0.03(k-1)} - e^{-0.03k}) \qquad f_{K_x}(k) = {}_{k-1}|q_x = {}_{k-1}p_x - {}_kp_x \\
=& \sum_{k=1}^{10}(11-k)e^{-0.06k}e^{-0.03(k-1)}(1-e^{-0.03}) \\
=& e^{-0.06}(1-e^{-0.03})\sum_{k=1}^{10}(11-k)e^{-0.09(k-1)} \\
=& e^{-0.06}(1-e^{-0.03})\sum_{j=0}^{9}(10-j)e^{-0.09j} \quad (j = k-1) \\
=& e^{-0.06}(1-e^{-0.03})\sum_{j=0}^{9}(10-j)v^j \qquad v = ? \approx 1.196.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=0}^{9}(10-j)v^j =& \sum_{j=0}^{9}10v^j - \sum_{j=1}^{9}jv^j \\
=& 10\sum_{j=0}^{9}v^j - v\sum_{j=1}^{9}jv^{j-1} \\
=& 10\sum_{j=0}^{9}v^j - v(\sum_{j=0}^{9}v^j)'_v \\
=& 10\frac{1-v^{9+1}}{1-v} - v(\frac{1-v^{9+1}}{1-v})'_v \quad (see\ formula\ [16]) \\
=& 10\frac{1-v^{10}}{1-v} - \frac{v}{(1-v)^2}(1-10v^9+(9)v^{10}) \\
(DA)^{1}_{x:\overline{10}|} =& e^{-0.06}(1-e^{-0.03})[10\frac{1-v^{10}}{1-v} - \frac{v}{(1-v)^2}(1-10v^9+(9)v^{10})]|_{v=exp(-0.06)} \approx 1.196.
\end{aligned}
$$

4.5 Life insurance paid m–thly

It is unusual that claims are paid at the end of the year. A better model uses that claims are paid at the end of each month, or other period. In this section, we consider the case when payments can be made at m different equally spaced times a year. Previous insurance quantities are defined as before. To indicate that payments are made m–thly, a superindex $^{(m)}$ is added to the actuarial notation of insurance variables.

Suppose that a whole life insurance is paid at the end of the m–thly time interval in which failure occurs. Let $J_x^{(m)}$ be the m–thly time interval of death. We have that $J_x^{(m)} = j$ if $T_x \in \left(\frac{j-1}{m}, \frac{j}{m}\right]$, for some positive integer $j \geq 1$. In other words, $J_x^{(m)} = \lceil mT_x \rceil$. The present value of a whole life insurance paid at the end of the m–thly time interval in which failure occurs is $Z_x^{(m)} = v^{J_x^{(m)}/m}$. Thus

$$
\begin{aligned}
A_x^{(m)} =& A_x^{(m)}(v) = E[Z_x^{(m)}] = \sum_{j=1}^{\infty} v^{j/m}\mathbb{P}\{J_x^{(m)} = j\} = \sum_{j=1}^{\infty} v^{j/m} \cdot {}_{\frac{j-1}{m}|\frac{1}{m}}q_x, \qquad (1)\\
{}^2A_x^{(m)} =& E[(Z_x^{(m)})^2] = \sum_{j=1}^{\infty} v^{2j/m}\mathbb{P}\{J_x^{(m)} = j\} = A_x^{(m)}(v^2)
\end{aligned}
$$

Example 4.50. *Suppose that $\mu_x(t) = 0.03$, $t \geq 0$, and $\delta = 0.06$. Calculate $A_x^{(3)}$ and* $\mathrm{Var}(Z_x^{(3)})$.

Solution: $A_x^{(3)} = \sum_{j=1}^{\infty} v^{j/m}\mathbb{P}\{J_x^{(m)} = j\} = \sum_{j=1}^{\infty} v^{j/m} \cdot {}_{\frac{j-1}{m}|\frac{1}{m}}q_x$.

$m = 3$, $v = e^{-0.06}$, ${}_tp_x = e^{-0.03t}$, ${}_{\frac{j-1}{m}|\frac{1}{3}}q_x = S_{T_x}(\frac{j-1}{m}) - S_{T_x}(\frac{j}{m}) = e^{-\mu\frac{j-1}{m}} - e^{-\mu\frac{j}{m}}$, $\mu = ?$

$$
\begin{aligned}
A_x^{(3)} =& \sum_{j=1}^{\infty} v^{j/m}\mathbb{P}\{J_x^{(m)} = j\} = \sum_{j=1}^{\infty} v^{j/m} \cdot {}_{\frac{j-1}{m}|\frac{1}{m}}q_x, \qquad (1)\\
=& \sum_{j=1}^{\infty} v^{j/m}[e^{-\mu\frac{j-1}{m}} - e^{-\mu\frac{j}{m}}] \qquad {}_tp_x = S_{T_x}(t) = e^{-\mu t}\\
=& \sum_{j=1}^{\infty} v^{j/m}[e^{-\mu\frac{j}{m}}e^{\mu\frac{1}{m}} - e^{-\mu\frac{j}{m}}]\\
=&(e^{\mu\frac{1}{m}} - 1)\sum_{j=1}^{\infty} v^{j/m}e^{-\mu\frac{j}{m}}\\
=&(e^{\mu\frac{1}{m}} - 1)\sum_{j=1}^{\infty} t^j \qquad t = v^{1/m}e^{-\frac{\mu}{m}}\\
=&(e^{\mu\frac{1}{m}} - 1)t\frac{1-t^{\infty}}{1-t}\\
=&(e^{\mu\frac{1}{m}} - 1)t\frac{1}{1-t}\\
=&(e^{\mu\frac{1}{m}} - 1)\frac{v^{1/m}e^{-\frac{\mu}{m}}}{1-(v^{1/m}e^{-\frac{\mu}{m}})}\Bigg|_{v=e^{-0.06},\mu=0.03,m=3}\\
=&0.330005611.\\
{}^2A_x^{(3)} =& A_x^{(3)}(v^2) = 0.1960201321.\\
\mathrm{Var}(Z_x^{(3)}) =&0.1960201321 - (0.330005611)^2 \approx 0.087.
\end{aligned}
$$

The second way to find $A_x^{(m)}$ is to use the formulas for A_x, by changing the parameters to take in account that payments are m times a year, that is,

$$v \to v^{1/m} \text{ and } {}_{j-1|}q_x \to {}_{\frac{j-1}{m}|\frac{1}{m}}q_x \quad ({}_{0|}q_x = q_x \to {}_{\frac{1}{m}}q_x).$$

e.g., under constant force of mortality μ,

$$A_x = \frac{1-p_x}{\frac{1}{v}-p_x} = \frac{q_x}{\frac{1}{v}-1+q_x} \text{ becomes } A_x^{(m)} = \frac{{}_{\frac{1}{m}}q_x}{\frac{1}{v^{1/m}}-1+{}_{\frac{1}{m}}q_x}.$$

However, if A_x needs to be derived, then it is better to use Eq. (1) above.

Definition 4.30.

Example 4.51.

Example 4.52.

Example 4.53.

Example 4.54.

Example 4.55.

4.6 Level benefit insurance in the continuous case.

In this section, we consider the case of **benefits paid at the moment of death**. This is also called **immediate payment of a claim**.

4.6.1 Whole life insurance.

Definition 4.31. *The present value and the APV of a unit payment whole life insurance paid at the time of death are denoted by $\overline{Z}_x$ and $\overline{A}_x$, respectively.*

$\overline{Z}_x = v^{T_x}$.
$\overline{A}_x = E[\overline{Z}_x] = E[v^{T_x}] = \int_0^\infty v^t f_{T_x}(t)\,dt$ and ${}^m\overline{A}_x = {}^m\overline{A}_x(v) = E[\overline{Z}_x^m] = \overline{A}_x(v^m)$

Example 4.56. *The force of interest is* 0.06*.* (x) *has a constant force of mortality of* 0.05*. Consider the benefit of the whole life insurance to* (x) *with unity payment paid at the time of the death. (i) Find its APV and the variance.*
(ii) Find the density of its present value.
(iii) Find the first and third quartile of the present value of the benefit of a life insurance to (x) *with unity payment paid at the time of the death.*

Solution: (i) $\overline{A}_x = \int_0^\infty v^t f_{T_x}(t)dt = ?$
$v = e^{-\delta} = e^{-0.06}$. Constant force of mortality => $f_{T(x)}(t) = \mu e^{-\mu t}$, $t > 0$.

$$\overline{A}_x = \int_0^\infty v^t f_{T_x}(t)dt = \int_0^\infty e^{-\delta t}\mu e^{-\mu t}dt = \int_0^\infty \mu e^{-(\delta+\mu)t}dt$$
$$= \frac{\mu}{\mu+\delta}\int_0^\infty (\mu+\delta)e^{-(\delta+\mu)t}dt = \frac{\mu}{\mu+\delta}\bigg|_{\mu=0.05,\delta=0.06} \approx 0.45 \quad = \frac{\mu}{\mu - \ln v} \text{ Why do it ?}$$
$${}^2\overline{A}_x = \overline{A}_x(v^2) = \frac{\mu}{\mu - \ln v^2}\bigg|_{\mu=0.05,v=e^{-0.06}} \approx 0.29$$
$$\mathrm{Var}(\overline{Z}_x) = \overline{A}_x(v^2) - (\overline{A}_x(v))^2 \approx 0.0875.$$

(ii) $f_{\overline{Z}_x}(t) = f_{v^{T_x}}(t) = ?$ with $f_{T_x}(t) = \mu e^{-\mu t}$, $t > 0$.
Formula [20] for df of $U = h(Y)$: $f_U(u) = f_Y(h^{-1}(u))|\frac{dh^{-1}(u)}{du}|$. $(U, Y) = ?$ $h = ?$ $h^{-1} = ?$
$U = \overline{Z}_x = v^{T_x}$, $Y = T_x$ and $u = h(t) = v^t$ or $u = e^{-\delta t}$, $t > 0$.
$t = h^{-1}(u) = \frac{\ln u}{\ln v}$, $u \in (0, 1)$?

$f_Y(t) = f_{T_x}(t) = \mu e^{-\mu t}$, $t > 0$.

$$f_U(u) = f_Y(h^{-1}(u))|\frac{dh^{-1}(u)}{du}| = \mu \exp(-\mu \frac{\ln u}{\ln v})|\frac{d\frac{\ln u}{\ln v}}{du}| = \mu \exp(-\mu \frac{\ln u}{-\delta})|\frac{1}{u \ln v}| \quad (-\ln v = \delta)$$
$$= \mu \exp(\ln u^{\frac{\mu}{\delta}})|\frac{1}{u\delta}| = \frac{\mu}{\delta u} u^{\frac{\mu}{\delta}} = \frac{\mu}{\delta} u^{\frac{\mu}{\delta}-1}, \quad u \in (0,1). \quad \mu = ? \quad \delta = ??$$

$$f_{\overline{Z}_x}(u) = f_U(u) = \tfrac{5}{6} u^{-\frac{1}{6}}, \quad 0 \le u \le 1. \tag{1}$$

(iii) Quartiles of $\overline{Z}_x$? Two ways for $\overline{Z}_x = v^{T_x}$ or $U = h(Y)$: $(U, Y) = ?$

(1) $\xi_p = F_U^{-1}(p)$; $(F_U(t) = F_{\overline{Z}_x}(t) = \int_0^t f_{\overline{Z}_x}(z)dz = \int_0^t ??dz)$. (see (1))

(2) $\xi_p = \begin{cases} h(\xi_p^*) & \text{if } h(t) \uparrow \\ h(\xi_{1-p}^*) & \text{if } h(t) \downarrow \end{cases}$, where $U = h(Y)$, $\xi_p^* = F_Y^{-1}(p)$ and $F_Y(t) = F_{T_x}(t) = 1 - e^{-\mu t}$.

Method (1): Let ξ_p be the p–th quantile of $\overline{Z}_x$.
Solve $p = F_{\overline{Z}_x}(\xi_p)$ for ξ_p

$p = F_{\overline{Z}_x}(t) = \int_0^t f_{\overline{Z}_x}(z)dz = \int_0^t \frac{5}{6} z^{\frac{-1}{6}} dz = t^{5/6}$, $t \in (0,1)$.

$p = F_{\overline{Z}_x}(\xi_p) = \xi_p^{5/6}$. $=> \xi_p = p^{6/5}$.

The first quartile of $\overline{Z}_x$ is $(0.25)^{6/5} \approx 0.189$.
The third quartile of $\overline{Z}_x$ is $(0.75)^{6/5} \approx 0.708$.

Method (2): (2) $\xi_p = \begin{cases} h(\xi_p^*) & \text{if } h(t) \uparrow \\ h(\xi_{1-p}^*) & \text{if } h(t) \downarrow \end{cases}$, where $u = h(t) = v^t \downarrow {}_0^1$.

So $\xi_p = h(\xi_{1-p}^*)$. Need to solve ξ_p^*:

$F_Y(t) = F_{T_x}(t) = 1 - e^{-\mu t} = p$,

$=> 1 - p = e^{-\mu t}$

$=> \ln(1-p) = -\mu t$

$=> -\frac{\ln(1-p)}{\mu} = t = \xi_p^*$

$\xi_p = h(\xi_{1-p}^*)$ $\qquad h(t) = v^t$

$= v^{\xi_{1-p}^*} = v^{-\frac{\ln p}{\mu}}$ Why ??

$= e^{-\delta(-\frac{\ln p}{\mu})} = e^{\delta \frac{\ln p}{\mu})} = \exp(\ln(p^{\delta/\mu})) = p^{\delta/\mu} = p^{6/5}$.

Example 4.57. *An actuary models the future lifetime of* (15) *as follows.* $T(15)$ *has force of mortality* μ*, where* μ *has pdf* $f_\mu(t) = 25$*,* $0.01 \le t \le 0.05$*. The force of interest is* $\delta = 0.1$*. Calculate* $\overline{A}_{15}$*.*

Solution: Which ways: (1) $\overline{A}_{15} = E[\overline{Z}_x] = \int v^z f_{T_x}(z)dz$; (2) $\overline{A}_{15} = E[\overline{Z}_x] = E[E[\overline{Z}_x|\mu]]$?
Which of them is correct ? (1) $f_{T(15)}(t) = \mu e^{-\mu t}$, $t > 0$ (2) $f_{T(15)|\mu}(t|u) = ue^{-ut}$, $t > 0$?
Given conditions: (1)$S_{T(15)|\mu}(t|u) = e^{-ut}$, $t > 0$. (2) $f_\mu(u) = 25$, $u \in [0.01, 0.05]$.
A result from Ex. 4.56: $E(\overline{Z}_x) = \frac{\mu}{\mu+\delta} = \frac{\mu}{\mu - \ln v}$ under constant forces if μ is given.

$$\begin{aligned}
\overline{A}_{15} = E[\overline{Z}_x] =& E[E[\overline{Z}_{15}|\mu]] \\
=& E\left[\frac{\mu}{\mu+\delta}\right] = \int_{0.01}^{0.05} \frac{\mu}{\mu+0.1}(25)\,d\mu = (25)\int_{0.01}^{0.05} \frac{\mu+0.1-0.1}{\mu+0.1}\,d\mu \textbf{ Why do this ?} \\
=& (25)\int_{0.01}^{0.05}\left(1-\frac{0.1}{\mu+0.1}\right)d\mu \quad (\ln(x+c))' = \frac{1}{x+c} \\
=& (25)\left(\mu-(0.1)\ln(\mu+0.1)\right)\Big|_{0.01}^{0.05} \\
\approx& 0.22.
\end{aligned}$$

Example 4.58. *A cohort of lives age x consists of 10% of smokers and 90% of non–smokers. The force of mortality for smokers is $\mu_x(t) = 0.08$, $t \geq 0$. The force of mortality for non–smokers is $\mu_x(t) = 0.02$, $t \geq 0$. The force of interest is 0.04. Calculate $\overline{A}_x$ and $\mathrm{Var}(\overline{Z}_x)$.*

Solution: Given condition: $\mu = \begin{cases} 0.08 & \text{for 10\% of smokers} \\ 0.02 & \text{for 90\% of non-smokers} \end{cases}$ and $\delta = 0.04$

From Ex. 4.56, $\overline{A}_x = \frac{\mu}{\mu+\delta} = \frac{\mu}{\mu-\ln v}$. **Can we use it directly ?**

Let $Y = I(\text{(x) is a smoker})$, that is, $Y = \begin{cases} 1 & \text{if } (x) \text{ is a smoker,} \\ 0 & \text{if } (x) \text{ is a non} - \text{smoker.} \end{cases}$

Then the given assumption is
$f_{T_x|Y}(t|1) = f_1(t) = 0.08e^{-0.08t}$, $t > 0$;
$f_{T_x|Y}(t|0) = f_0(t) = 0.02e^{-0.02t}$, $t > 0$.
$Y \sim bin(1, 0.1)$.
Formula [19] $E(X) = E(E(X|Y))$ yields

$$\begin{aligned}
\overline{A}_x = E[v^{T_x}] =& E[E[\overline{Z}_x|Y]] \\
=& E[v^{T_x}|Y=1]\mathbb{P}\{Y=1\} + E[v^{T_x}|Y=0]\mathbb{P}\{Y=0\} \\
& (E[v^{T_x}|Y=1] = \overline{A}_x^{\text{smoker}} (= \frac{\mu}{\mu-\ln v}) = \frac{\mu}{\mu+\delta}\Big|_{\mu=\mu_{smoker}=0.08} = \frac{0.08}{0.08+0.04} = 2/3, \\
& E[v^{T_x}|Y=0] = \overline{A}_x^{\text{non-smoker}} = \frac{\mu}{\mu+\delta}\Big|_{\mu=0.02} = \frac{0.02}{0.02+0.04} = 1/3) \\
\overline{A}_x =& (2/3)(0.1) + (1/3)(0.9) = 1.1/3 \approx 0.37
\end{aligned}$$

$$\begin{aligned}
E[(v^{T_x})^2] =& E[E[(v^{T_x})^2|Y]] \\
=& E[(v^{T_x})^2|Y=1]\mathbb{P}\{Y=1\} + E[(v^{T_x})^2|Y=0]\mathbb{P}\{Y=0\} \\
& (E[(v^{T_x})^2|Y=1] = \overline{A}_x^{\text{smoker}}(v^2) = \frac{\mu}{\mu-\ln v^2} = \frac{0.08}{0.08+(2)0.04} = 0.5 \qquad (\text{as } -\ln v^2 = 2\delta) \\
& E[(v^{T_x})^2|Y=0] = \overline{A}_x^{\text{non-smoker}}(v^2) = \frac{0.02}{0.02+(2)0.04} = 0.2,) \\
E[(v^{T_x})^2] =& (0.5)(0.1) + (0.2)(0.9) = 0.23, \\
\mathrm{Var}(\overline{Z}_x) =& 0.23 - (1.1/3)^2 \approx 0.096.
\end{aligned}$$

Example 4.59. *A benefit of \$500 is paid at the failure time T of a home electronic product. The pdf of the time of failure of the product is $f_T(t) = \frac{t}{50}$ if $0 \le t \le 10$.*
(i) Calculate the actuarial present value of this benefit if $i = 0.075$.
(ii) Find the density of present value of this benefit.
(iii) Find the 25, 50 and 75th percentiles of the present value random variable of this benefit.

Solution: (i) $b\overline{A}_x = b\int v^t f_{T_x}(t)dt = (500)\int_0^{10}(1.075)^{-t}\frac{t}{50}\,dt = (500/50)\int_0^{10} t\cdot(1.075)^{-t}\,dt$=?

$\int ta^t dt = \int t da^t/\ln a$ $\qquad a =?$
$= \frac{1}{\ln a}[ta^t - \int a^t dt]$
$= \frac{1}{\ln a}[ta^t - a^t/\ln a]$
$= \frac{1}{\ln a}a^t[t - 1/\ln a]$, $(a = 1/1.075)$.

$$b\overline{A}_x = (500/50)\int_0^{10} t\cdot(1.075)^{-t}\,dt = \frac{10}{(-\ln(1.075))}[1.075^{-t}(t - \frac{1}{-\ln 1.075})]\Big|_0^{10} \approx 313.39.$$

(ii) Let $T = T(x)$ and $Z = bv^T = (500)(1.075)^{-T} = h(T)$ be the present value of the insurance benefit. Then $Z/500 = 1.075^{-T}$ and

$$T = h^{-1}(Z) = -\frac{\ln(Z/500)}{\ln(1.075)} \in [0,10] \quad \textbf{Why ?} \qquad (f_T(t) = t/50,\ t\in(0,10)) \tag{1}$$

and $f_Z(z) = f_T(h^{-1}(z))|\frac{dt}{dz}|$ (see 447 [20]).

$$\begin{aligned} f_Z(z) =& f_T\left(-\frac{\ln(z/500)}{\ln(1.075)}\right)\left|\frac{d}{dz}\left(-\frac{\ln(z)-\ln(500)}{\ln(1.075)}\right)\right| \quad \textbf{why ?} \\ =&(\frac{-\ln(z/500)}{\ln(1.075)}/50)\frac{1}{z\ln(1.075)} \qquad f_T(t) = t/50,\ t\in(0,10) \\ =&\frac{-\ln(z/500)}{50z(\ln(1.075))^2}, \quad \textbf{Done}? \\ &\text{if } 0 \le \underbrace{\frac{-\ln(z/500)}{\ln 1.075}}_{t\ (see\ Eq.(1))} \le 10, \text{ or simplify it as } 500(1.075)^{-10} \le z \le 500. \end{aligned}$$

(iii) Two ways: Let ζ_p be the p–th quantile of Z.
(1) Solve $p = \int_0^{\zeta_p} f_{\overline{Z}}(z)dz$ for ζ_p. $\qquad F_{\overline{Z}_x}(\zeta_p) = \int_0^{\zeta_p}\frac{-\ln(z/500)}{50z(\ln(1.075))^2}dz.$
(2) Since $z = h(t) = (500)(1.075)^{-t}$ is decreasing, we have that
$\zeta_p = h(\xi_{1-p})$, where ξ_{1-p} is a $(1-p)$–th quantile of T. $\qquad F_T(t) = \int_0^t \frac{x}{50}dx.$
Use method (2). **Why ??**

$$1 - p = \mathbb{P}\{T \le \xi_{1-p}\} = \frac{\xi_{1-p}^2}{100}$$

=> $\xi_{1-p} = 10\sqrt{1-p}$. Hence,

$$\zeta_p = h(\xi_{1-p}) = h(10\sqrt{1-p}) = 500(1.075)^{-10\sqrt{1-p}}.$$

The 25th percentile of the benefit is $\zeta_{0.25} = 500(1.075)^{-10\sqrt{0.75}} = 290.6742245$.
The 50th percentile of the benefit is $\zeta_{0.5} = 500(1.075)^{-10\sqrt{0.5}} = 348.2793162$.
The 75th percentile of the benefit is $\zeta_{0.75} = 500(1.075)^{-10\sqrt{0.25}} = 417.300441$.

Definition 4.32.

Example 4.60.

Theorem 4.40.

Theorem 4.41.

Theorem 4.42.

Theorem 4.43.

Theorem 4.44.

Theorem 4.45.

Theorem 4.46.

Theorem 4.47.

Theorem 4.48.

Theorem 4.49.

Corollary 4.3.

Theorem 4.50.

Theorem 4.51.

Theorem 4.52.

Theorem 4.53.

Theorem 4.54.

Theorem 4.55.

Definition 4.33.

4.6.2 n–year term life insurance.

Definition 4.34. *n–year term life insurance: a payment is made if the failure happens within the n–year term of an insurance commencing at issue. So, a payment is made only if the failure happens before n years. Its present value and APV of a unit payment n–year term life insurance paid at the moment of death is denoted by $\overline{Z}^{1}_{x:\overline{n}|}$ and $\overline{A}^{1}_{x:\overline{n}|}$, respectively.*

$\overline{Z}^{1}_{x:\overline{n}|} = v^{T_x} I(T_x \le n)$ $\qquad (Z^{1}_{x:\overline{n}|} = v^{K_x} I(K_x \le n)))$.

${}^m\overline{A}^{1}_{x:\overline{n}|} = E[(\overline{Z}^{1}_{x:\overline{n}|})^m] = \overline{A}^{1}_{x:\overline{n}|}(v^m)$ $\qquad = E(v^{mT_x} I(T_x \le n)) = \int_0^n v^{mt} f_{T_x}(t)\, dt$.

Example 4.61. *Julia is 40 year old. She buys a 15–year term life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. Suppose that the continuously compounded force of interest is $\delta = 0.05$. Find the APV and SD of the benefit of this life insurance.*

Solution: Let $Z = bv^{T_{40}} I(T_{40} \le 15)$. $T(40) \sim U(0,60)$ and $v = e^{-\delta} = e^{-0.05}$. $b = ?$
$E(Z) = b\overline{A}^{1}_{40:\overline{15}|}(v)$=? and $\sigma_Z = b\sqrt{\overline{A}^{1}_{40:\overline{15}|}(v^2) - \overline{A}^{1}_{40:\overline{15}|}(v))^2}$?

$$\overline{A}^{1}_{40:\overline{15}|} = E(v^{T_x} I(T_x \le n)) = \int_0^n v^t f_{T_x}(t) dt = \int_0^n v^t \frac{1}{60} dt = \frac{v^t}{60 \ln v}\Bigg|_0^{15} = \frac{v^{15}-1}{60 \ln v}\Bigg|_{v=e^{-0.05}} \approx 0.1759.$$

$${}^2\overline{A}^{1}_{40:\overline{15}|} = \overline{A}^{1}_{40:\overline{15}|}(v^2) = \frac{v^{15}-1}{60 \ln v}\Bigg|_{v=(e^{-0.05})^2} \approx 0.129.$$

The actuarial present value is $b\overline{A}^{1}_{40:\overline{15}|} = (50000)(0.1759) \approx 8793.9$.

$$\sigma_Z = b\sigma_{\overline{Z}^{1}_{40:\overline{15}|}} = (50000)\sqrt{{}^2\overline{A}^{1}_{40:\overline{15}|} - \left(\overline{A}^{1}_{40:\overline{15}|}\right)^2} \approx 15695.96.$$

Example 4.62. *Consider a 15-year term life insurance to (x) with unity payment. Assume that $\delta = 0.06$ and (x) has a constant force of mortality $\mu = 0.05$. Calculate the probability that the present value random variable is smaller than or equal to twice the APV.*

Solution: $P(\overline{Z}^{1}_{x:\overline{n}|} \le 2\overline{A}^{1}_{x:\overline{n}|})$=?
$\overline{A}^{1}_{x:\overline{n}|} = \int_0^n v^t f_{T_x}(t)\, dt = \int_0^n e^{-t\delta} \mu e^{-t\mu}\, dt = \int_0^n \mu e^{-t(\delta+\mu)}\, dt = \frac{\mu}{\delta+\mu}\int_0^n e^{-t(\delta+\mu)}\, dt(\delta+\mu)$ **why do this ?**
$= -\frac{\mu}{\delta+\mu} e^{-t(\delta+\mu)}\Big|_0^n$
$= \frac{\mu}{\mu+\delta}(-e^{-n(\mu+\delta)} + 1) = 0.36725$.

$\mathbb{P}\{\overline{Z}^{1}_{x:\overline{15}|} \le (2)(0.36725)\}$
$= \mathbb{P}\{e^{-(0.06)T_x} I(T_x \le 15) \le 0.73450\}$
$= \mathbb{P}\{e^{-(0.06)T_x} \le 0.73450, T_x \le 15\} + \mathbb{P}\{0 \le 0.7345, T_x > 15\}$ $\qquad -(0.06)T_x \le \ln 0.73450$
$= \mathbb{P}\{-\frac{\ln(0.73450)}{0.06} \le T_x, T_x \le 15\} + \mathbb{P}\{T_x > 15\}$
$= \mathbb{P}\{5.14275 \le T_x\} + \mathbb{P}\{T_x \le 15\} + \mathbb{P}\{15 < T_x\}$? *or* $\mathbb{P}\{5.14275 \le T_x \le 15\} + \mathbb{P}\{15 < T_x\}$**?**

$= \mathbb{P}\{5.14275 \le T_x\} = e^{-(5.14275)(0.05)} = 0.77326$.

4.6.3 n–year deferred life insurance.

In the case of n–year deferred life insurance, a payment is made only if the failure happens at least (after) n years following policy issue.

Definition 4.35. *The present value and APV of a unit payment n–year deferred life insurance paid at the moment of death are denoted by ${}_{n|}\overline{Z}_x$ and ${}_{n|}\overline{A}_x$, respectively.*

Definition 4.36. ${}_{n|}\overline{Z}_x = v^{T_x} I(n < T_x)$.

${}_{n|}\overline{A}_x = E[{}_{n|}\overline{Z}_x] = \int_n^\infty e^{-\delta t} f_{T_x}(t)\, dt$ and ${}^m{}_{n|}\overline{A}_x = E[({}_{n|}\overline{Z}_x)^m] = {}_{n|}\overline{A}_x(v^m)$.

Example 4.63. *Suppose* $p_k = \begin{cases} 0.9 & 0 \le k \le 50 \\ 0.6 & k > 50. \end{cases}$ *The force of interest is* 0.05. *Assume uniform distribution death (UDD) within each year (or linear interpolation).* ${}_{40|}\overline{A}_{30} =$ *?*

Sol. ${}_{40|}\overline{A}_{30} = ?$ ${}_{40|}\overline{A}_{30} = \int_{40}^\infty v^t \overbrace{f_{T_x}(t)}^{=?}\, dt$

Due to UUD, $f_{T_x}(t) = {}_{k-1|}q_x$ if $t \in (k-1, k]$, $k = 1, \ldots$

$${}_{40|}\overline{A}_{30} = \int_{40}^\infty v^t f_{T_x}(t) dt = \sum_{k>40}^\infty \int_{k-1}^k v^t\, {}_{k-1|}q_x dt = \sum_{k>40}^\infty {}_{k-1|}q_x \frac{v^t}{\ln v}\Big|_{k-1}^{k} = \sum_{k>40}^\infty {}_{k-1|}q_x v^k \frac{1-1/v}{\ln v}$$

$= {}_{40|}A_{30} \frac{1-v^{-1}}{\ln v}$ $\qquad {}_{40|}A_{30} = \sum_{k>40} v^k\, {}_{k-1|}q_{30} = ?$

$${}_{k-1|}q_{30} = {}_{k-1}p_{30} q_{30+k-1} = \overbrace{p_{30} \cdots p_{30+k-2}}^{={}_np_x \text{ by [4]},\ n=k-1} (q_{30+k-1}) \qquad {}_2p_x = p_x p_{x+1}, \ldots$$

$$= \begin{cases} (0.9)^{k-1}(0.1) & k = 1, \ldots, 21, \quad (30+k-2 \le 49 \textit{ or } k \le 21) \\ (0.9)^{22-1}(0.4) & k = 22, \quad (30+k-2 = 50) \\ (0.9)^{22-1} 0.6^{k-22}(0.4) & k \ge 23 \quad (30+k-2 \ge 51), \quad \textit{or } k \ge 22 \end{cases}$$

$${}_{40|}A_{30} = \sum_{k>40} v^k\, {}_{k-1|}q_{30} = \sum_{k>40} v^k 0.9^{21} 0.6^{k-22} 0.4$$

$$= 0.9^{21} \cdot 0.6^{-22} \cdot 0.4 \sum_{k>40} v^k 0.6^k \qquad = (3/2)^{21}(0.4/0.6) \sum_{k=41}^\infty (0.6v)^k$$

$$= (3/2)^{21}(0.4/0.6)(0.6v)^{41} \sum_{j=0}^\infty (0.6v)^j \qquad j = k - 41$$

$$= (3/2)^{21}(0.4/0.6)(0.6v)^{41} \frac{1}{1-0.6v} \qquad \textit{by } [16]$$

$$= 7.998342 \times 10^{-7}$$

$${}_{40|}\overline{A}_{30} = {}_{40|}A_{30} \frac{1-1/v}{\ln v}\Big|_{v=e^{-0.05}}$$

$$= 7.998342 \times 10^{-7} \frac{1-1/v}{\ln v}\Big|_{v=e^{-0.05}} \approx 8.201632$$

Remark. Quiz on Friday : 447: [20]- [22], 450: [1]-[12], [14], [16]

Theorem 4.56. $F_{{}_{n|}\overline{Z}_x}(z) = \begin{cases} {}_nq_x + S_{T_x}\left(\frac{-\ln z}{\delta}\right) & \textit{if } 0 \le z \le e^{-n\delta} \quad (v^n) \\ 1 & \textit{otherwise.} \end{cases}$

Proof. Since ${}_{n|}\overline{Z}_x = v^{T_x} I(n < T_x) = e^{-\delta T_x} I(n < T_x) \in [0,1]$, $F_{{}_{n|}\overline{Z}_x}(z) = \begin{cases} 0 & \text{if } z < 0, \\ 1 & \text{if } z > 1. \end{cases}$

Do we have $F_{{}_{n|}\overline{Z}_x}(1) = 1$? Why ?

If $z = 0$, $F_{{}_{n|}\overline{Z}_x}(z) = \mathbb{P}\{{}_{n|}\overline{Z}_x \le z\} = \mathbb{P}\{e^{-\delta T_x} I(n < T_x) = 0\} = P(T_x \le n)$ **or** $= P(n < T_x)$ **?**

$F_{{}_{n|}\overline{Z}_x}(0) = P(T_x \le n) > 0$.

$$\begin{aligned}
\text{If } 0 < z \le 1,\ F_{{}_{n|}\overline{Z}_x}(z) =& \mathbb{P}\{{}_{n|}\overline{Z}_x \le z\} \\
=& \mathbb{P}\{{}_{n|}\overline{Z}_x \le z, T_x \le n\} + \mathbb{P}\{{}_{n|}\overline{Z}_x \le z, T_x > n\} \qquad {}_{n|}\overline{Z} = v^{T_x} I(T_x > n) \\
=& \mathbb{P}\{0 \le z, T_x \le n\} + \mathbb{P}\{e^{-\delta T_x} \le z, T_x > n\} \\
=& \mathbb{P}\{T_x \le n\} + \mathbb{P}\{-\ln(z)/\delta \le T_x, n < T_x\} \quad (\max\{-\ln(z)/\delta, n\} < T_x) \\
=& \begin{cases} \mathbb{P}\{T_x \le n\} + \mathbb{P}\{-\ln(z)/\delta \le T_x\} & \text{if } n < \frac{-\ln z}{\delta}, \\ \mathbb{P}\{T_x \le n\} + \mathbb{P}\{n < T_x\} & \text{if } n \ge \frac{-\ln z}{\delta}, \end{cases} \\
=& \begin{cases} {}_nq_x + S_{T_x}\left(\frac{-\ln z}{\delta}\right) & \text{if } 0 < z \le e^{-n\delta}, \\ 1 & \text{if } z > e^{-n\delta}. \end{cases}
\end{aligned}$$

Question: $P({}_{n|}\overline{Z}_x = 0) = ?$

Example 4.64.

Example 4.65. *Under De Moivre's model with* $w = 100$, $f_{{}_{n|}\overline{Z}_x}(t) = ?$

Sol. Let $F(t) = F_{{}_{n|}\overline{Z}_x}(t)$. If $t \in (0, e^{-n\delta}]$, then $F(t) = {}_nq_x + S_{T_x}(-\ln t/\delta)$.

(1) If $t \notin [0, e^{-n\delta}]$, then $F'(t) = 0$.

(2) If $t \in (0, e^{-n\delta})$, then $F'(t) = S'_{T_x}(-\frac{\ln t}{\delta})\frac{-1}{t\delta} = (1 - \frac{-\frac{\ln t}{\delta}}{w-x})'_t \frac{-1}{t\delta}$ (> 0 **at** $t = 0+$ **or** ∞ **?**)

(3) If $t = 0$, $F(0) - F(0-) = F(0) = {}_nq_x = \frac{n}{w-x}$.

(4) If $t = e^{-n\delta}$, $F(t+) - F(t-) = 1 - {}_nq_x - S_{T_x}(\frac{-\ln e^{-n\delta}}{\delta}) = 1 - {}_nq_x - {}_np_x = 0$.

Thus the df $f(t) = \begin{cases} (1 - \frac{-\frac{\ln t}{\delta}}{w-x})'_t \frac{-1}{t\delta} & \text{if } t \in (0, e^{-n\delta}) \\ {}_nq_x & \text{if } t \in D = \{0\}. \end{cases}$ Is it cts ? discrete ?

4.6.4 n–year pure endowment life insurance. In the case of an n–**year pure endowment life insurance**, a payment at the end of n years is made if and only if the failure happens at least n years after issuing the policy. The present value of the benefit payment is

$$\overline{Z}_{x:\overline{n|}}^{\ \ 1} = v^n I(n < T_x) \ (= v^n I(n < K_x) = Z_{x:\overline{n|}}^{\ \ 1})$$

$$\overline{A}_{x:\overline{n|}}^{\ \ 1} = E[\overline{Z}_{x:\overline{n|}}^{\ \ 1}] = v^n {}_np_x \text{ and } \mathrm{Var}(\overline{Z}_{x:\overline{n|}}^{\ \ 1}) = v^{2n} \cdot {}_np_x \cdot {}_nq_x.$$

4.6.5 n–year endowment life insurance. In the case of an n–year endowment life insurance, a payment is made at either the time of death or in n years, which ever comes first.

Definition 4.37. *The present value and APV of a unit payment n–year endowment life insurance paid at the moment of death are denoted by* $\overline{Z}_{x:\overline{n}|}$ *and* $\overline{A}_{x:\overline{n}|}$*, respectively.*

Definition 4.38. $\overline{Z}_{x:\overline{n}|} = v^{T_x \wedge n} = g(T_x) = \begin{cases} v^{T_x} & \text{if } T_x \le n, \\ v^n & \text{if } n < T_x. \end{cases}$

$$
\begin{aligned}
{}^m\overline{A}_{x:\overline{n}|} =& E[(\overline{Z}_{x:\overline{n}|})^m] = \int_0^\infty v^{(t\wedge n)m} f_{T_x}(t)\,dt = \int_0^n v^{(t)m} f_{T_x}(t)\,dt + \int_n^\infty v^{(n)m} f_{T_x}(t)\,dt \\
=& \int_0^n v^{mt} f_{T_x}(t)\,dt + v^{mn}\mathbb{P}\{T_x > n\} = \overline{A}_{x:\overline{n}|}(v^m)
\end{aligned}
$$

Example 4.66. *Find the APV and variance of a* 15*–year endowment life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$*.*

Solution: $\overline{A}_{x:\overline{n}|} = E[v^{n\wedge T_x}] = \int_0^\infty g(t) f_{T_x}(t)\,dt = \int_0^n v^t f_{T_x}(t)\,dt + v^n \mathbb{P}\{T_x > n\}$ and $v = e^{-\delta}$.

$$
\begin{aligned}
\overline{A}_{x:\overline{15}|} =& \int_0^{15} v^t f_{T_x}(t)\,dt + v^{15}{}_{15}p_x \\
=& \int_0^{15} e^{-\delta t} e^{-\mu t}\mu\,dt + v^{15} e^{-(\mu)15} \\
=& \mu\int_0^{15} e^{-(\delta+\mu)t}\,dt + v^{15}e^{-(\mu)15} \qquad \textit{set } u = -(\delta+\mu)t \\
=& \mu\frac{-e^{-(\delta+\mu)t}}{\delta+\mu}\Big|_0^{15} + e^{-(\delta)(15)}e^{-(\mu)(15)} \\
=& \mu\frac{1-e^{-(\delta+\mu)15}}{\delta+\mu} + e^{-(\delta)(15)}e^{-(\mu)(15)} \qquad \delta = -\ln v \;\; (\textit{why put here?}) \qquad (1) \\
=& \frac{(0.05)(1-e^{-(0.11)(15)})}{(0.11)} + e^{-(0.06)(15)}e^{-(0.05)(15)} \approx 0.279. \\
V(\overline{A}_{x:\overline{15}|}) =& {}^2\overline{A}_{x:\overline{15}|}(v) - (\overline{A}_{x:\overline{15}|}(v))^2 \approx 0.3492341 - 0.279^2 \approx 0.2713931 \\
\textit{Note that } \; & {}^2\overline{A}_{x:\overline{15}|}(v) = \overline{A}_{x:\overline{15}|}(v^2) \\
=& \mu\frac{1-e^{-(\delta^*+\mu)15}}{\delta^*+\mu} + e^{-(\delta^*)(15)}e^{-(\mu)(15)}\Bigg|_{\delta^* = -\ln v^2} \qquad - \ln v^2 = (2)(-\ln v) = 2\delta \\
=& \frac{(\mu)(1-e^{-(2\delta+\mu)(15)})}{2\delta+\mu} + e^{-\delta(2)(15)}e^{-(\mu)(15)} = 0.3492341
\end{aligned}
$$

Theorem 4.57.

Theorem 4.58.

Theorem 4.59.

Example 4.67.

Theorem 4.60.

Theorem 4.61.

4.6.6 m–year deferred n–year term life insurance.

Definition 4.39. *In the case of m–year deferred n–year term life insurance, a payment is made if death happens during the period of n years that starts m years from now.*

Definition 4.40. *The present value and APV of an m–year deferred n–year term life insurance with unit payment paid at time of death are denoted by ${}_{m|n}\overline{Z}_x$ and ${}_{m|n}\overline{A}_x$, respectively.*

Definition 4.41. ${}_{m|n}\overline{Z}_x = v^{T_x} I(m < T_x \le m+n) = g(T_x) = \begin{cases} v^{T_x} & \text{if } m < T_x \le m+n, \\ 0 & \text{else.} \end{cases}$

$$ {}_{m|n}\overline{A}_x = E[{}_{m|n}\overline{Z}_x] = (\int_0^m + \int_m^{m+n} + \int_{m+n}^{\infty}) g(t) f_{T_x}(t)\,dt = \int_m^{m+n} v^t f_{T_x}(t)\,dt $$

$$ {}^2{}_{m|n}\overline{A}_x(v) = E[({}_{m|n}\overline{Z}_x)^2] = \int_m^{m+n} v^{2t} f_{T_x}(t)\,dt = {}_{m|n}\overline{A}_x(v^2) $$

4.7 Properties of the APV for continuous insurance

The following table shows the definition of all the variables in the previous section: **Level payment paid at the time of death**

type of life insurance	payment
whole	$\overline{Z}_x = v^{T_x}$
n–year term	$\overline{Z}^{1}_{x:\overline{n}\vert} = v^{T_x} I(T_x \le n)$
n–year deferred	${}_{n\vert}\overline{Z}_x = v^{T_x} I(n < T_x)$
n–year pure endowment	$\overline{Z}_{x:\overline{n}\vert}^{\ \ \ 1} = v^{n} I(n < T_x)$
n–year endowment	$\overline{Z}_{x:\overline{n}\vert} = v^{\min(T_x,n)}$
m–year deferred n–year term	${}_{m\vert n}\overline{Z}_x = v^{T_x} I(m < T_x \le m+n)$

[14]: $Z^{1}_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ \ 1} = \underline{0}$, $\quad \underline{Z_{x:\overline{n}|}} = Z^{1}_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ \ 1}$, $\quad \overline{Z}^{1}_{x:\overline{n}|} \times \overline{Z}_{x:\overline{n}|}^{\ \ \ 1} = 0$? $\overline{Z}_{x:\overline{n}|} = \overline{Z}^{1}_{x:\overline{n}|} + \overline{Z}_{x:\overline{n}|}^{\ \ \ 1}$?

$Z^{1}_{x:\overline{n}|} \cdot {}_{n|}Z_x = \underline{0}$, $\ \underline{Z_x} = Z^{1}_{x:\overline{n}|} + {}_{n|}Z_x$, $\quad \overline{Z}^{1}_{x:\overline{n}|} \cdot {}_{n|}\overline{Z}_x = \underline{0}$? $\ \underline{\overline{Z}_x} = \overline{Z}^{1}_{x:\overline{n}|} + {}_{n|}Z_x$?

$\underline{{}_{n|}A_x} = {}_nE_x A_{x+n}$, $\ \underline{A_x} = A^{1}_{x:\overline{n}|} + {}_nE_x A_{x+n}$. $\quad \underline{{}_{n|}\overline{A}_x} = {}_nE_x \overline{A}_{x+n}$? $\ \underline{\overline{A}_x} = \overline{A}^{1}_{x:\overline{n}|} + {}_nE_x \overline{A}_{x+n}$?

$A^{1}_{x:\overline{n}|} \cdot A_{x:\overline{n}|}^{\ \ \ 1} = \underline{0}$? $\quad \underline{A_{x:\overline{n}|}} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1}$? $\quad A^{1}_{x:\overline{n}|} \cdot {}_{n|}A_x = \underline{0}$? $\ \underline{A_x} = A^{1}_{x:\overline{n}|} + {}_{n|}A_x$?

${}_{m|}\overline{Z}_x = {}_{m|n}\overline{Z}_x + {}_{m+n|}\overline{Z}_x$, **which line above does it similar to ?** ${}^{m}\overline{A}_{x:\overline{n}|} = {}^{m}\overline{A}^{1}_{x:\overline{n}|} + {}^{m}\overline{A}_{x:\overline{n}|}^{\ \ \ 1}$.

Example 4.68. *If* $E[\overline{Z}^{1}_{x:\overline{n}|}] = 0.5, \mathrm{Var}(\overline{Z}^{1}_{x:\overline{n}|}) = 0.35, v^n = 0.4, {}_np_x = 0.6,$ *Find* $\overline{A}_{x:\overline{n}|}$ *and* $\mathrm{Var}(\overline{Z}_{x:\overline{n}|})$.

Solution: $\overline{A}_{x:\overline{n}|} = E(v^{T_x \wedge n}) = E(v^{T_x} I(T_x \le n) + v^n I(T_x > n))$
$= E(v^{T_x} I(T_x \le n)) + v^n P(T_x > n)) = 0.5 + (0.4)(0.6) = 0.74.$

$$\mathrm{Var}(\overline{Z}_{x:\overline{n}|}) = {}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2$$
$${}^2\overline{A}_{x:\overline{n}|} = {}^2\overline{A}^{1}_{x:\overline{n}|} + {}^2\overline{A}_{x:\overline{\overset{1}{n}}|} = ?$$
$${}^2\overline{A}^{1}_{x:\overline{n}|} = \mathrm{Var}(\overline{Z}^{1}_{x:\overline{n}|}) + \left(\overline{A}^{1}_{x:\overline{n}|}\right)^2 = 0.35 + (0.5)^2 = 0.6,$$
$${}^2\overline{A}_{x:\overline{\overset{1}{n}}|} = \overline{A}_{x:\overline{\overset{1}{n}}|}(v^2) = v^{2n}{}_np_x = (0.4)^2(0.6) = 0.096,$$
$${}^2\overline{A}_{x:\overline{n}|} = {}^2\overline{A}^{1}_{x:\overline{n}|} + {}^2\overline{A}_{x:\overline{\overset{1}{n}}|} = 0.6 + 0.096 = 0.696,$$
$$\mathrm{Var}(\overline{Z}_{x:\overline{n}|}) = 0.696 - (0.74)^2 = 0.148.$$

Theorem 4.62.

Theorem 4.63. ${}_{m|n}\overline{Z}_x = \overline{Z}^{1}_{x:\overline{m+n}|} - \overline{Z}^{1}_{x:\overline{m}|}$.

Theorem 4.64.

Skip the rest theorems.

Theorem 4.65. ${}_{m|n}\overline{A}_x = {}_mE_x \cdot \overline{A}^{1}_{x+m:\overline{n}|}$.

Theorem 4.66.

Theorem 4.67. ${}_{m|n}\overline{Z}_x = {}_{m|}\overline{Z}_x - {}_{m+n|}\overline{Z}_x$.

Theorem 4.68. $\overline{Z}_x = \overline{Z}^{1}_{x:\overline{m}|} + {}_{m|n}\overline{Z}_x + {}_{m+n|}\overline{Z}_x$.

Theorem 4.69.

Theorem 4.70.

Theorem 4.71.

Corollary 4.4.

Theorem 4.72.

Example 4.69.

Example 4.70.

Example 4.71.

4.8 Non–level payments paid at the time of death

In this section, we consider a life insurance with a general payment at the time of the death. Suppose that if failure happens at time t, then the benefit payment is b_t. The present value of the benefit payment is denoted by $\overline{B}_x = b_{T_x} v^{T_x}$. The actuarial present value of this benefit is

$$E[\overline{B}_x] = \int_0^\infty b_t v^t f_{T_x}(t)\, dt = \int_0^\infty b_t v^t \cdot {}_tp_x \mu_{x+t}\, dt.$$

Example 4.72.

Skip the next two examples.

Example 4.73. *For a whole life insurance on* (60)*, you are given:*
(i) Death benefits are paid at the moment of death.
(ii) Mortality follows a de Moivre model with terminal age 100.
(iii) $i = 7\%$.
(iv) $b_t = (20000)(1.04)^t$, $t \geq 0$.
Calculate the mean and the standard deviation of the present value random variable for this insurance.

Solution: The present value random variable is

$$Z = b_{T_{60}} v^{T_{60}} = (20000)(1.04)^{T_{60}} v^{T_{60}} = (20000)(1.04v)^{T_{60}}.$$

$f_{T_{60}}(t) = ?$

$$\begin{aligned}
E[Z] =& \int_0^{40} (20000)(1.04v)^t \frac{1}{40}\, dt = \frac{(20000)((1.04v)^{40} - 1)}{40 \ln(1.04v)} \\
=& 11945.06573, \\
E[Z^2] =& \qquad\qquad (= \frac{(20000)((1.04v^2)^{40} - 1)}{40 \ln(1.04v^2)} ??) \\
=& \int_0^{40} (20000)^2 (1.04v)^{2t} \frac{1}{40}\, dt \\
=& \frac{(20000)^2((1.04v)^{80} - 1)}{80 \ln(1.04v)} = 157748208.7, \\
\mathrm{Var}(Z) =& 157748208.7 - (11945.06573)^2 = 15063613.41, \\
\sqrt{\mathrm{Var}(Z)} =& \sqrt{15063613.41} = 3881.187114.
\end{aligned}$$

Example 4.74.

Definition 4.42. *An* **increasing life insurance in the continuous case** *(or a* **continuously increasing life insurance***) makes a payment of* T_x *at the time of death. Its present value and APV are denoted by* $\left(\overline{I}\ \overline{Z}\right)_x$ *(or by* $\left(\overline{IZ}\right)_x$*) and* $\left(\overline{I}\ \overline{A}\right)_x$ *(or by* $\left(\overline{IA}\right)_x$*, respectively.*
$\left(\overline{IZ}\right)_x = T_x v^{T_x}$

In this case $b_t = t$, $t \geq 0$ and $\left(\overline{I}\ \overline{A}\right)_x = \int_0^\infty tv^t \cdot f_{T_x}(t)\, dt$.

Definition 4.43. *Other* **increasing life insurances in the continuous case** *(or* **continuously increasing life insurances***) are defined as follows.*

type of life insurance	*payment*
whole	$(\overline{IZ})_x = T_x v^{T_x}$
n–year term	$(\overline{IZ})^1_{x:\overline{n}\vert} = T_x v^{T_x} I(T_x \leq n)$
n–year deferred	${}_{n}\vert(\overline{IZ})_x = T_x v^{T_x} I(n < T_x)$
n–year pure endowment	$(\overline{IZ})_{x:\overset{1}{\overline{n}\vert}} = T_x v^n I(n < T_x)$
n–year endowment	$(\overline{IZ})_{x:\overline{n}\vert} = \min\{T_x, n\} v^{\min(T_x,n)}$
m–year deferred n–year term	${}_{m\vert n}(\overline{IZ})_x = T_x v^{T_x} I(m < T_x \leq m+n)$

Theorem 4.73. *Under de Moivre's model,* $\left(\overline{I}\ \overline{A}\right)_x = \frac{1}{(\omega-x)\ln v}\{[(\omega-x)v^{\omega-x} - \frac{v^{\omega-x}}{\ln v}] + \frac{1}{\ln v}\}$.

Proof.

$$\begin{aligned}
\left(\overline{I}\ \overline{A}\right)_x =& E(T_x v^{T_x}) = \int_0^{\omega-x} tv^t \frac{1}{\omega-x}\, dt \\
=& \frac{1}{(\omega-x)\ln v}\int_0^{\omega-x} t\, dv^t \qquad (v^t)' = v^t \ln v \qquad \text{(integration by parts)} \\
=& \frac{1}{(\omega-x)\ln v}[tv^t - \int v^t\, dt]\Big|_0^{\omega-x} \\
=& \frac{1}{(\omega-x)\ln v}[tv^t - \frac{v^t}{\ln v}]\Big|_0^{\omega-x} \\
=& \frac{1}{(\omega-x)\ln v}\{[(\omega-x)v^{\omega-x} - \frac{v^{\omega-x}}{\ln v}] + \frac{1}{\ln v}\}.
\end{aligned}$$

Example 4.75. *Suppose that* $\mu_x(t) = 0.03$, $t \geq 0$, *and* $\delta = 0.06$.
Compute (1) ${}_{10}\vert\left(\overline{I}\ \overline{A}\right)_x$, *(2)* $\left(\overline{I}\ \overline{A}\right)_x$, *(3)* $(\overline{IA})^1_{x:\overline{10}\vert}$.

Solution: $\left(\overline{I}\ \overline{A}\right)_x = E(T_x v^{T_x}) = E(T_x v^{T_x} I(T_x \leq n)) + E(T_x v^{T_x} I(T_x > n))$
$= (\overline{IA})^1_{x:\overline{10}\vert} + {}_{10}\vert\left(\overline{I}\ \overline{A}\right)_x \qquad = {}_{0}\vert\left(\overline{I}\ \overline{A}\right)_x$?

Q1. How many integrations to do ?

Q2. Can we simplify to 2 questions ?

${}_{n}\vert\left(\overline{I}\ \overline{A}\right)_x$ with $n =$?

$$
\begin{aligned}
{}_{n|}\left(\overline{I}\ \overline{A}\right)_x =& E(T_x v^{T_x} I(T_x > n))\\
=& \int_n^\infty t v^t \mu e^{-\mu t} dt\\
=& \mu \int_n^\infty t e^{-t(\delta+\mu)} dt \qquad = \mu \int_n^\infty (t-n+n) e^{-(t-n+n)(\delta+\mu)} dt \qquad \textit{a trick}\\
=& \mu \int_0^\infty (y+n) e^{-(y+n)(\delta+\mu)} dy \qquad (y = t-n \textit{ or } t = y+n)\\
=& \mu \int_0^\infty (y+n) e^{-y(\delta+\mu)} e^{-n(\delta+\mu)} dy\\
=& \mu e^{-n(\delta+\mu)} \int_0^\infty y e^{-y(\delta+\mu)} + n e^{-y(\delta+\mu)} dy\\
=& \mu e^{-n(\delta+\mu)} [\int_0^\infty y e^{-y(\delta+\mu)} dy + \int_0^\infty n e^{-y(\delta+\mu)} dy] \qquad \int_0^\infty \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx = ? \quad \beta = ?\\
=& \mu e^{-n(\delta+\mu)} [\frac{\Gamma(2)}{(\delta+\mu)^2} \int_0^\infty \frac{(\delta+\mu)^2}{\Gamma(2)} y^{2-1} e^{-y(\delta+\mu)} dy + \frac{n}{\delta+\mu} \int_0^\infty (\delta+\mu) e^{-y(\delta+\mu)} dy]\\
=& \mu e^{-n(\delta+\mu)} [\frac{1}{(\delta+\mu)^2} + \frac{n}{\delta+\mu}]\\
{}_{10|}\left(\overline{I}\ \overline{A}\right)_x =& \mu e^{-n(\delta+\mu)} [\frac{1}{(\delta+\mu)^2} + \frac{n}{\mu+\delta}]\Big|_{n=10,\mu=0.03,\delta=0.06} \approx 2.86\\
\left(\overline{I}\ \overline{A}\right)_x =& {}_{0|}\left(\overline{I}\ \overline{A}\right)_x = \mu e^{-n(\delta+\mu)} [\frac{1}{(\delta+\mu)^2} + \frac{n}{\mu+\delta}]\Big|_{n=0,\mu=0.03,\delta=0.06} \approx 3.70.
\end{aligned}
$$

$(\overline{IA})^1_{x:\overline{10}|} \approx 0.84$ **why ?**

Remark. (1) No need to convert to v as $E(\left(\overline{I}\ \overline{Z}\right)_x^2) \neq \left(\overline{I}\ \overline{A}\right)_x (v^2)$.
(2) Avoid integration by parts and making use of $G(\alpha, \beta)$.

Definition 4.44. *An* **increasing whole life insurance in the piecewise–continuous case** *(or an* **annually increasing life insurance***) makes a payment of $\lceil T_x \rceil$ at the time of death. Its present value and APV are denoted by $\left(I\ \overline{Z}\right)_x$ and $\left(I\ \overline{A}\right)_x$, respectively.*
$\left(I\ \overline{Z}\right)_x = \lceil T_x \rceil v^{T_x} = K_x v^{T_x}$. *Other* **increasing life insurances in the piecewise–continuous case** *(or* **annually increasing life insurances***) can be defined similarly.*

Remember that $\lceil t \rceil$ is the least integer greater than or equal to t.

Theorem 4.74.

Example 4.76. *Suppose that $\mu_x(t) = 0.03$, $t \geq 0$, and $\delta = 0.06$. $\left(I\ \overline{A}\right)_x$=?.*

Solution: $\left(I\ \overline{A}\right)_x = E(\lceil T_x\rceil v^{T_x}) = \int_0^\infty \lceil t\rceil v^t \cdot f_{T_x}(t)\,dt$ and $\lceil T_x\rceil \in \{1, 2, \cdots\}$.

$$\begin{aligned}
\left(I\ \overline{A}\right)_x =& \sum_{k=1}^{\infty}\int_{k-1}^{k} \lceil t\rceil e^{-\delta t}\mu e^{-\mu t}\,dt \qquad\qquad = \sum_{k=1}^{\infty}\int_{k-1}^{k} k e^{-\delta t}\mu e^{-\mu t}\,dt \\
=& \sum_{k=1}^{\infty}\int_{k-1}^{k} k\mu e^{-(\delta+\mu)t}\,dt \qquad\qquad \int e^{ax}dx = e^{ax}/a \\
=& \sum_{k=1}^{\infty}\frac{k\mu}{\mu+\delta}[e^{-(\delta+\mu)(k-1)} - e^{-(\delta+\mu)k}] \qquad e^{-(\delta+\mu)k} = e^{-(\delta+\mu)(k-1+1)} = e^{-(\delta+\mu)(k-1)}e^{-(\delta+\mu)} \\
=& \frac{\mu}{\mu+\delta}[1-e^{-(\delta+\mu)}]\sum_{k=1}^{\infty} k e^{-(\delta+\mu)(k-1)} \qquad = c\sum_{k=1}^{\infty} kx^{k-1} \quad \mathbf{(x = ??)} \\
=& \frac{\mu}{\mu+\delta}[1-e^{-(\delta+\mu)}](\sum_{k=0}^{\infty} x^k)' \qquad \mathbf{(x = ??)} \\
=& \frac{\mu}{\mu+\delta}[1-e^{-(\delta+\mu)}](\frac{1-x^{\infty+1}}{1-x})'_x \quad (by\ [16]) \\
=& \frac{\mu}{\mu+\delta}[1-e^{-(\delta+\mu)}]\frac{1}{(1-x)^2}\Bigg|_{x=e^{-(\delta+\mu)}} \qquad ((1-x)^{-1})' \\
=& \frac{\mu}{(\mu+\delta)(1-e^{-(\delta+\mu)})}\Big|_{\mu=0.03,\delta=0.06} \approx 3.43.
\end{aligned}$$

Example 4.77. *Suppose that $\mu_x(t) = 0.03$, $t \ge 0$, and $\delta = 0.06$. $\left(\overline{D}\ \overline{A}\right)^{1}_{x:\overline{10}|} =?$*

Solution: $\left(\overline{D}\ \overline{A}\right)^{1}_{x:\overline{n}|} = E((n-T_x)v^{T_x}I(T_x \le n)) = \int_0^n (n-t)v^t \cdot f_{T_x}(t)\,dt$.
$= \int_0^n (n-t)e^{-\delta t}\mu e^{-t\mu}\,dt = \int_0^n (n-t)\mu e^{-t(\mu+\delta)}\,dt$ **2 ways: integration by parts or a trick :**

$$\begin{aligned}
\left(\overline{D}\ \overline{A}\right)^{1}_{x:\overline{10}|} =& [\int_0^\infty - \int_n^\infty]\mu e^{-t(\mu+\delta)}\,dt = \int_0^\infty (n-t)\mu e^{-t(\mu+\delta)}\,dt - \int_n^\infty (n-t)\mu e^{-t(\mu+\delta)}\,dt \\
=& \int_0^\infty (n-t)\mu e^{-t(\mu+\delta)}\,dt - \int_0^\infty (n-y-n)\mu e^{-(y+n)(\mu+\delta)}\,dy \quad (y = t-n \text{ or } t = y+n) \\
=& \int_0^\infty (n-t)\mu e^{-t(\mu+\delta)}\,dt + \int_0^\infty y\mu e^{-(y+n)(\mu+\delta)}\,dy \\
=& n\mu\int_0^\infty e^{-t(\mu+\delta)}\,dt - \mu\int_0^\infty t^{2-1}e^{-t(\mu+\delta)}\,dt + \mu e^{-n(\delta+\mu)}\int_0^\infty y^{2-1}e^{-y(\mu+\delta)}\,dy \\
=& n\mu\int_0^\infty e^{-t(\mu+\delta)}\,dt + \mu[-1+e^{-n(\delta+\mu)}]\int_0^\infty y^{2-1}e^{-y(\mu+\delta)}\,dy \qquad \int_0^\infty \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx =? \\
=& \{\frac{n\mu}{\delta+\mu} + \mu[-1+e^{-n(\delta+\mu)}]\frac{\Gamma(2)}{(\delta+\mu)^2}\}\Big|_{\mu=0.03,\delta=0.06,n=10} \approx 1.135.
\end{aligned}$$

Orders of Def. 4.45-4.51 are different from the textbook

Definition 4.45.

Theorem 4.75.

Theorem 4.76.

Definition 4.46.

Definition 4.47.

Theorem 4.77.

Definition 4.48.

Definition 4.49.

Theorem 4.78.

Example 4.78.

Example 4.79.

Theorem 4.79.

Theorem 4.80.

Definition 4.50.

Definition 4.51.

Definition 4.52. *An n–year term decreasing life insurance in the continuous case pays $n - T_x$ at the time of death, if $0 \le T_x \le n$. Its present value and APV are denoted by $\left(\overline{D}\ \overline{Z}\right)^{1}_{x:\overline{n}|}$ and $\left(\overline{D}\ \overline{A}\right)^{1}_{x:\overline{n}|}$. $\left(\overline{D}\ \overline{Z}\right)^{1}_{x:\overline{n}|} = (n - T_x)v^{T_x} I(T_x \le n)$.*

Definition 4.53. *An n–year term decreasing life insurance in the piecewise–continuous case pays $\lceil n - T_x \rceil$ at the time of death, if $T_x \le n$. Its present value and APV are denoted by $\left(D\ \overline{Z}\right)^{1}_{x:\overline{n}|}$ and $\left(D\ \overline{A}\right)^{1}_{x:\overline{n}|}$, respectively. $\left(D\ \overline{Z}\right)^{1}_{x:\overline{n}|} = \lceil n - T_x \rceil v^{T_x} I(T_x \le n)$.*

Example 4.80. *Suppose that $\mu_x(t) = 0.03$, $t \ge 0$, and $\delta = 0.06$. $\left(D\ \overline{A}\right)^{1}_{x:\overline{10}|}$=?*

Solution: $\left(D\ \overline{A}\right)^{1}_{x:\overline{n}|} = \int_0^n \lceil n - t \rceil v^t \cdot f_{T_x}(t)\, dt = \sum_{k=1}^n \int_{k-1}^k \lceil n - t \rceil v^t \cdot f_{T_x}(t)\, dt$.

$k =$	1	2	3	$\cdots$	
$t \in$	$(0,1)$	$(1,2)$	$(2,3)$	$\cdots$	$(\lceil n-t \rceil\big\|_{t=0.5} = \lceil 9.5 \rceil = 10-k+1,\ \ k=1)$
$\lceil 10 - t \rceil =$	10	9	8	$n-k+1$	

$$
\begin{aligned}
\left(D\,\overline{A}\right)^{1}_{x:\overline{10}|} =&\sum_{k=1}^{n}\int_{k-1}^{k}\lceil n-t\rceil v^t\cdot f_{T_x}(t)\,dt\\
=&\sum_{k=1}^{10}\int_{k-1}^{k}(n-k+1)e^{-t(0.06)}(0.03)e^{-t(0.03)}\,dt\\
=&\sum_{k=1}^{10}(0.03)\int_{k-1}^{k}(11-k)e^{-t(0.09)}\,dt \qquad (e^{ax})'=ae^{ax}\\
=&\frac{0.03}{0.09}\sum_{k=1}^{n}(11-k)[e^{-0.09(k-1)}-e^{-0.09k}] \qquad (formula[16])\\
=&\frac{0.03}{0.09}\sum_{k=1}^{n}(11-k)e^{-0.09(k-1)}[1-e^{-0.09}]\\
=&\frac{0.03}{0.09}\sum_{k=1}^{n}(11-k)p^{k-1}[1-e^{-0.09}] \quad (p=?)\\
=&\frac{0.03}{0.09}[1-e^{-0.09}](\sum_{k=1}^{n}11p^{k-1}-\sum_{k=1}^{n}kp^{k-1})|_{p=e^{-0.09}}\\
=&\frac{0.03}{0.09}[1-e^{-0.09}](11p^{-1}\sum_{k=1}^{n}p^{k}-(\sum_{k=0}^{n}p^{k})'_{p})|_{p=e^{-0.09}}\\
=&\frac{0.03}{0.09}[1-e^{-0.09}](11p^{-1}[p\frac{1-p^n}{1-p}]-(\frac{1-p^{n+1}}{1-p})'_{p})|_{p=e^{-0.09}}\\
\approx&1.23. \qquad\qquad (\frac{1-p^{n+1}}{1-p})'_{p}=\frac{1}{1-p^2}(1-(n+1)p^n+np^{n+1})
\end{aligned}
$$

Level payment paid at the end of the year of death

type of life insurance	present value of unit payment
whole	$Z_x=v^{K_x}$
n–year term	$Z^{1}_{x:\overline{n}\vert}=v^{K_x}I(K_x\le n)$
n–year deferred	${}_{n}\vert Z_x=v^{K_x}I(n<K_x)$
n–year pure endowment	$Z_{x:\overline{n}\vert}^{\ \ 1}=v^{n}I(n<K_x)$
n–year endowment	$Z_{x:\overline{n}\vert}=v^{\min(K_x,n)}$
m–year deferred n–year term	${}_{m}\vert_{n}Z_x=v^{K_x}I(m<K_x\le m+n)$

Level payment paid at the time of death

type of life insurance	payment
whole	$\overline{Z}_x = v^{T_x}$
n–year term	$\overline{Z}^1_{x:\overline{n}\vert} = v^{T_x} I(T_x \le n)$
n–year deferred	${}_{n\vert}\overline{Z}_x = v^{T_x} I(n < T_x)$
n–year pure endowment	$\overline{Z}_{x:\overline{n}\vert}^{\;\;1} = v^n I(n < T_x)$
n–year endowment	$\overline{Z}_{x:\overline{n}\vert} = v^{\min(T_x,n)}$
m–year deferred n–year term	${}_{m\vert n}\overline{Z}_x = v^{T_x} I(m < T_x \le m+n)$

Increasing by one life insurance in discrete case

type of life insurance	payment
whole	$(IZ)_x = K_x v^{K_x}$
n–year term	$(IZ)^1_{x:\overline{n}\vert} = K_x v^{K_x} I(K_x \le n)$
n–year deferred	${}_{n\vert}(IZ)_x = K_x v^{K_x} I(n < K_x)$
n–year pure endowment	$(IZ)_{x:\overline{n}\vert}^{\;\;1} = K_x v^n I(n < K_x)$
n–year endowment	$(IZ)_{x:\overline{n}\vert} = \min\{K_x, n\} v^{\min(K_x,n)}$
m–year deferred n–year term	${}_{m\vert n}(IZ)_x = K_x v^{K_x} I(m < K_x \le m+n)$

Increasing life insurance in continuous case

type of life insurance	payment
whole	$(\overline{IZ})_x = (\overline{I}\ \overline{Z})_x = T_x v^{T_x}$
n–year term	$(\overline{IZ})^1_{x:\overline{n}\vert} = T_x v^{T_x} I(T_x \le n)$
n–year deferred	${}_{n\vert}(\overline{IZ})_x = T_x v^{T_x} I(n < T_x)$
n–year pure endowment	$(\overline{IZ})_{x:\overline{n}\vert}^{\;\;1} = T_x v^n I(n < T_x)$
n–year endowment	$(\overline{IZ})_{x:\overline{n}\vert} = \min\{T_x, n\} v^{\min(T_x,n)}$
m–year deferred n–year term	${}_{m\vert n}(\overline{IZ})_x = T_x v^{T_x} I(m < T_x \le m+n)$

Increasing life insurance in piece-wise continuous case

type of life insurance	payment
whole	$(I\overline{Z})_x = \lceil T_x \rceil v^{T_x}$
n–year term	$(I\overline{Z})^1_{x:\overline{n}\vert} = \lceil T_x \rceil v^{T_x} I(T_x \le n)$
n–year deferred	${}_{n\vert}(I\overline{Z})_x = \lceil T_x \rceil v^{T_x} I(n < T_x)$
n–year pure endowment	$(I\overline{Z})_{x:\overline{n}\vert}^{\;\;1} = \lceil T_x \rceil v^n I(n < T_x)$
n–year endowment	$(I\overline{Z})_{x:\overline{n}\vert} = \min\{\lceil T_x \rceil, n\} v^{\min(T_x,n)}$
m–year deferred n–year term	${}_{m\vert n}(I\overline{Z})_x = \lceil T_x \rceil v^{T_x} I(m < T_x \le m+n)$

Skip the rest of this section.

4.9 Computing APV's from a life table

To calculate the actuarial present values of some life insurance products we need to know $f_{T_x}(t)$ for each $t \ge 0$, as

$$E(b_{T_x} v_{T_x}) = \int b_t v_t f_{T_x}(t) dt.$$

Usually, survival functions do not have an analytical form and mortality is given by a life table. Then f_{T_x} can be estimated by UDD or exponential interpolation in Chapter 3.

This section, we discuss some tricks under UDD.

Theorem 4.81. *Assume UDD, and b_t, $t \geq 0$, is constant in each interval $(k-1, k]$, $k = 1, 2, \ldots$. Then,*

$$E[b_{T_x} v^{T_x}] = \frac{i}{\delta} E[b_{K_x} v^{K_x}].$$

Proof: Under UDD, $f_{T(x)}(t) = {}_{k-1}|q_x$ for $k-1 \leq t < k$ (can be verified).

$$\begin{aligned}
=> \quad E[b_{T_x} v^{T_x}] &= \int_0^\infty b_t v^t f_{T(x)}(t)\, dt \\
&= \sum_{k=1}^\infty \int_{k-1}^k b_k v^t \cdot {}_{k-1}|q_x \, dt \\
&= \sum_{k=1}^\infty b_k \cdot {}_{k-1}|q_x \frac{v^t}{\ln v} \Big|_{k-1}^k = \sum_{k=1}^\infty b_k \cdot {}_{k-1}|q_x \frac{v^k - v^{k-1}}{\ln v} \\
&= \sum_{k=1}^\infty b_k v^k \cdot {}_{k-1}|q_x \frac{1 - v^{-1}}{\ln v} = \sum_{k=1}^\infty b_k e^{-\delta k} \cdot {}_{k-1}|q_x \frac{1-(i+1)}{-\delta} \\
&= E[b_{K_x} v^{K_x}] \frac{i}{\delta}
\end{aligned}$$

Theorem 4.82. *Assuming a uniform distribution of deaths, we have that:*
(i) $\overline{A}_x = \frac{i}{\delta} A_x$. *(Note $b_t = 1$).*
(ii) $\overline{A}^1_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|}$.
(iii) ${}_n|\overline{A}_x = \frac{i}{\delta} \cdot {}_n|A_x$.
(iv) $\overline{A}_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|i} + A_{x:\overset{1}{\overline{n}|}}$.

Question: (1) $\overline{A}_{x:\overline{n}|} = \frac{i}{\delta} A_{x:\overline{n}|}$?

(2) $\overline{A}_{x:\overset{1}{\overline{n}|}} = \frac{i}{\delta} A_{x:\overset{1}{\overline{n}|}}$?

Theorem 4.81 does not apply to $\overline{A}_{x:\overline{n}|}$ as $\overline{Z}_{x:\overline{n}|} = v^{\min(T_x, n)}$ $(v_t \neq v^t)$.
Theorem 4.81 does not apply to $\overline{A}_{x:\overset{1}{\overline{n}|}}$ as $\overline{Z}_{x:\overset{1}{\overline{n}|}} = v^n I(T_x > n)$ and $v_t = v^n \neq v^t$.
$\overline{A}_{x:\overset{1}{\overline{n}|}} = A_{x:\overset{1}{\overline{n}|}}$ **??**.

Example 4.81. *Assuming a uniform distribution of deaths, $i = 6\%$ and based on Tables 7.1 (for ℓ_x (see page **??**)), **??** (for A_x), 7.3 (for ${}_5E_x$) and 7.4 (for ${}_{10}E_x$ and ${}_{20}E_x$), find: $\overline{A}_{50}$, $\overline{A}_{50:\overset{1}{\overline{15}|}}$ (i.e., ${}_{15}E_{50}$), ${}_{15}|\overline{A}_{50}$, $\overline{A}^1_{50:\overline{15}|}$ and $\overline{A}_{50:\overline{15}|}$.*

Solution: $\overline{A}_{50} = \frac{i}{\delta} A_{50} = \frac{0.06}{\ln(1.06)}(0.20695786) = 0.2131063032$,
$\overline{A}_{50:\overset{1}{\overline{15}|}} = v^{15} {}_{15}p_{50} = (1.06)^{-15} \frac{\ell_{65}}{\ell_{50}} = (1.06)^{-15} \frac{83114}{93735} \approx 0.37$

(or $= {}_{15}E_{50} = {}_5E_{50} \cdot {}_{10}E_{50+5} \approx 0.728 * 0.508 \approx 0.37$).
${}_{15|}\overline{A}_{50} = \frac{i}{\delta}{}_{15|}A_{50}$ (not in the table), but ${}_{n|}A_x = {}_nE_x A_{x+n}$ (see [14])

$$
\begin{aligned}
{}_{15|}\overline{A}_{50} &= \frac{i}{\delta}{}_{15}E_{50}A_{65} \approx \frac{0.06}{\ln(1.06)}(0.37)(0.38) \approx 0.14,\\
\overline{A}^{1}_{50:\overline{15}|} &= \overline{A}_{50} - {}_{15|}\overline{A}_{50} = 0.2131063032 - 0.1432839903 = 0.0698223129,\\
\overline{A}_{50:\overline{15}|} &= \overline{A}^{1}_{50:\overline{15}|} + {}_{15}E_{50} = 0.0698223129 + 0.3699852591 = 0.439807572.
\end{aligned}
$$

Example 4.82.

Example 4.83. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Suppose that $i = 6.5\%$. Assume a uniform distribution of deaths. ${}_{3|}\overline{A}_{80}$=?

Solution: Two ways: (1) direct, (2) Th4.82.

$$
\begin{aligned}
(1)\quad {}_{3|}\overline{A}_{80} =& E(v^{T_x}I(T_x>3)) = \int_3^\infty v^t f_{T_x}(t)dt = \int_0^\infty v^t(-(\frac{\ell_{x+t}}{\ell_x})'_t)dt\\
=&\sum_{k=4}^\infty \int_{k-1}^k v^t(-(\frac{\ell_{x+k-1+(t-k+1)}}{\ell_x})'_t)dt \qquad (so\ that\ t-k+1\in(0,1)\ formula[12])\\
=&\sum_{k=4}^\infty \int_{k-1}^k v^t(-\frac{(\ell_{x+k-1+s})'_s}{\ell_x})dt \qquad (s=t-k+1)\\
=&\sum_{k=4}^\infty \int_{k-1}^k v^t\frac{\ell_{80+k-1}-\ell_{80+k}}{\ell_{80}}dt \quad (by\ [12])\\
=&\sum_{k=4}^\infty \frac{\ell_{80+k-1}-\ell_{80+k}}{\ell_{80}}\frac{v^t}{\ln v}\Big|_{k-1}^k\\
=&\sum_{k=4}^\infty \frac{\ell_{80+k-1}-\ell_{80+k}}{\ell_{80}}\frac{v^k-v^{k-1}}{\ln v}\\
=&\frac{1}{\ln v}\{[v^4-v^3]\frac{107-62}{250}+[v^5-v^4]\frac{62-28}{250}+[v^6-v^5]\frac{28-0}{250}\}\Big|_{v=1/1.065}\\
\approx& 0.326.
\end{aligned}
$$

$$
(2)\qquad {}_{3|}\overline{A}_{80} = \frac{i}{\delta}\cdot{}_{3|}A_{80} = \frac{i}{\delta}\sum_{k=4}^\infty v^k\frac{\ell_{80+k-1}-\ell_{80+k}}{\ell_{80}} \approx 0.326.
$$

Theorem 4.83. *Assuming a uniform distribution of deaths, we have that:*
(i) $(I\overline{A})_x = \frac{i}{\delta}(IA)_x$.
(ii) $(D\overline{A})^1_{x:\overline{n}|} = \frac{i}{\delta}(DA)^1_{x:\overline{n}|}$.
(iii) $(\overline{IA})_x = \frac{i}{\delta}(IA)_x + [e^\delta\left(\frac{1}{\delta}-\frac{1}{\delta^2}\right)+\frac{1}{\delta^2}]A_x$.

(iv) $(\overline{DA})^1_{x:\overline{n}|} = \frac{i}{\delta}(DA)^1_{x:\overline{n}|} + \frac{i}{\delta}\left(\frac{1}{\delta} - \frac{1}{d}\right) A^1_{x:\overline{n}|}$.

Proof (i) $(I\overline{A})_x = E[\lceil T_x \rceil v^{T_x}] = E[b_{K_x} v^{T_x}]$ and $(IA)_x = E[\lceil T_x \rceil v^{K_x}] = E[b_{K_x} v^{K_x}]$.
By Theorem 4.81 $E[b_{K_x} v^{T_x}] = \frac{i}{\delta} E[b_{K_x} v^{K_x}]$ with $b_t = \lceil t \rceil$, $t \ge 0$.
=> (i).
(ii) It follows similarly from Theorem 4.81 with $b_t = \lceil n - t \rceil$, $t \ge 0$,
(iii) Recall S_x $(= T_x - K(x) = T_x - K_x + 1)$.
$S_x \perp K_x$ and $S_x \sim U(0,1)$ by Theorem 3.12, and so is $1 - S_x$. Thus,

$$\begin{aligned}
(\overline{IA})_x =& E[T_x e^{-\delta T_x}] \\
=& E[(\underbrace{T_x - K_x}_{=S_x - 1} + K_x)e^{-\delta T_x}] \\
(\overline{IA})_x =& E[(S_x - 1 + K_x)e^{-\delta T_x}] = E[K_x e^{-\delta T_x}] - E[(1 - S_x)e^{-\delta T_x}] \\
=& (I\overline{A}_x)_x - E[(1 - S_x)e^{-\delta(K_x + S_x - 1)}] \\
=& (I\overline{A}_x)_x - E[(1 - S_x)e^{-\delta(S_x - 1)}e^{-\delta K_x}] \\
=& (I\overline{A}_x)_x - E[(1 - S_x)e^{\delta(1 - S_x)}]E[e^{-\delta K_x}] \quad (\text{as} \quad S_x \perp K_x) \\
=& \frac{i}{\delta}(IA)_x + \frac{i}{\delta}\left(\frac{1}{\delta} - \frac{1}{d}\right) A_x, \qquad \text{as } E(e^{-\delta K_x}) = E(v^{K_x}) = A_x \text{ and} \\
& E[(1 - S_x)e^{\delta(1 - S_x)}] = \int_0^1 t e^{\delta t}\, dt \qquad Y = 1 - S_x \sim U(0,1) \\
=& \frac{1}{\delta}\int_0^1 t de^{\delta t} = \frac{1}{\delta}[te^{\delta t} \Big|_0^1 - \int_0^1 e^{\delta t} dt] \\
=& \left(\frac{te^{\delta t}}{\delta} - \frac{e^{\delta t}}{\delta^2}\right)\Bigg|_0^1 = e^{\delta}\left(\frac{1}{\delta} - \frac{1}{\delta^2}\right) + \frac{1}{\delta^2}
\end{aligned}$$

$$\begin{aligned}
(iv) \qquad & (\overline{DA})^1_{x:\overline{n}|} = E[(n - T_x)e^{-\delta T_x} I(T_x \le n)] \\
=& E[(n - K_x - S_x + 1)e^{-\delta T_x} I(K_x \le n)] \\
=& E[(n - K_x)e^{-\delta(K_x + S_x - 1)} I(T_x \le n)] + E[(S_x - 1)e^{-\delta(K_x + S_x - 1)} I(T_x \le n)] \\
=& E[(n - K_x)e^{-\delta K_x} I(T_x \le n)]E[e^{-\delta(S_x - 1)}] + E[e^{-\delta K_x} I(T_x \le n)]E[(S_x - 1)e^{-\delta(S_x - 1)}] \\
=& (DA)^1_{x:\overline{n}|}\int_0^1 e^{\delta t} dt + E[e^{-\delta K_x} I(T_x \le n)]E[(S_x - 1)e^{-\delta(S_x - 1)}] \\
=& (DA)^1_{x:\overline{n}|}\frac{e^{\delta} - 1}{\delta} + A^1_{x:\overline{n}|}\frac{i}{\delta}\left(\frac{1}{\delta} - \frac{1}{d}\right) \\
=& \frac{i}{\delta}(DA)^1_{x:\overline{n}|} + \frac{i}{\delta}\left(\frac{1}{\delta} - \frac{1}{d}\right) A^1_{x:\overline{n}|}.
\end{aligned}$$

An insurer offers a 20–year term life insurance of $\$10^5$ to independent lives age 45. $i = 7.5\%$. Mortality follows de Moivre model with terminal age 110. The insurer has a fund with $\$10^6$ to pay for these insurances. Using the normal approximation, calculate the maximum number of policies the insurer can cover so that the probability that the aggregate present value for the issued policies exceeds the amount in the fund is less than 0.01.

Solution: Let n be the number of policies that the insurer can cover. The present value for the aggregate n insurances is $Z=\sum_{j=1}^n 10^5 Y_j$, where Y_1, ..., Y_n are i.i.d. fron $Z^1_{45:\overline{20}|}$.

We need to determine n so that $\quad P(Z>10^6)\approx 1-\Phi(\frac{10^6-E(Z)}{\sigma_Z})=1-\Phi(z_{0.01})=0.01,$

$=> \frac{10^6-E(Z)}{\sigma_Z}=z_{0.01}=2.33 => 0=E(Z)+2.33\sigma_Z-10^6.$

$0=\underbrace{E(Z)}_{n10^5A^1_{x:\overline{20}|}(v)}+2.33\times\underbrace{\sigma_Z}_{\sqrt{n(10^5)^2\left(A^1_{x:\overline{20}|}(v^2)-(A^1_{x:\overline{20}|}(v))^2\right)}}-10^6$, or $\quad\underbrace{an+b\sqrt{n}+c=0}_{\sqrt{n}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}} => n=?$

$A^1_{x:\overline{20}|}=\sum_{k=1}^{20}v^k f_{K_x}(k).$

$f_{K_x}(k)=\frac{s(x+k-1)-s(x+k)}{s(x)}=\frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x}={}_{k-1}p_x\cdot q_{x+k-1}=\left(\prod_{j\geq 0}^{k-2}p_{x+j}\right)q_{x+k-1}$ **which one?**

$s(x)=1-x/w=\frac{w-x}{w}$ and $f_{K_x}(k)=\frac{w-x-k-(w-x-k-1)}{w-x}=\frac{1}{w-x}$, $0<x<w$.

$$A^1_{x:\overline{20}|}=\sum_{k=1}^{20}v^k{}_{k-1}|q_x=\sum_{k=1}^{20}v^k\frac{1}{\omega-x}=\frac{1}{w-x}\frac{v(1-v^n)}{1-v}. \tag{1}$$

$$E(Z)=n10^5A^1_{45:\overline{20}|}=n10^5\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075}=15683.83286n,$$

$$\text{Var}(Z)=n(10^5)^2\Big(\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075^2}-\Big(\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075}\Big)^2\Big)=687801161.6n.$$

Now solve n from equation $E(Z)+z_{0.01}\sigma_Z-10^6=0$

$$15683.83286n+(2.3263479)\sqrt{687801161.6n}-10^6=0 \qquad (\uparrow \text{ in } n) \tag{2!}$$

$=> \quad 15683.83286n+61010.71512\sqrt{n}-10^6=0.$

$\sqrt{n}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. $=> \sqrt{n}=\frac{-b-\sqrt{b^2-4ac}}{2a}<0$, or $\sqrt{n}\approx 6.27$. So, $n\approx 6.3^2\approx 39.4$.
The maximum number of policies that the insurer can cover is 39 or 40 ? Why ? (see (2!)).

CHAPTER 5

Life Annuities (Exam on Nov 25 (M))

A **life annuity** is a financial contract according to which a seller (issuer) makes periodic payments in the future to the buyer (annuitant). Life annuities are one of the most often used plans to fund retirement. The payment for a life annuity can be made at the time of issue. But, in the case of retirement, contributions are made to the retirement fund while the annuitant works. Common retirement plans are 401(k) plans and (individual retirement accounts) IRA's. At the time of retirement, the insurance company uses the accumulated deposit to issue a life annuity. Contributions to this retirement fund can be made by either the employer and/or the employee. Contributions made by the employee can be tax free. Another way to get retirement funds is done by the Social Security. So, Social Security is some how similar to an insurance company issuing life annuities being funded while an individual works.

5.1 Whole life annuities

A **whole life annuity** is a series of payments made while (x) is alive.

5.1.1 Whole life due annuity

Definition 5.1. *A* **whole life due annuity** *is a series payments made at the beginning of each year while an individual is alive. Its present value and APV with* **unit** *payment are denoted by $\ddot{Y}_x$ and $\ddot{a}_x$.*

Definition 5.2. $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k$ *(*$= \ddot{a}_{\overline{K_x}|}$*, where* $\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n}|i} = \sum_{k=0}^{n-1} v^k$*).*

Definition 5.3. $\ddot{a}_x = E(\sum_{k=0}^{K_x-1} v^k) \quad (= E(\ddot{a}_{\overline{K_x}|}))$*.*

Theorem 5.1. *(i) If* $v = 1$, $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = K_x$*.*
[17] (ii) if $v \neq 1$, $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = \frac{1-v^{K_x}}{1-v} = \frac{1-Z_x}{d}$ *and* $\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \frac{1-A_x}{1-v}$*.*

Example 5.1. *Suppose that* $i = 5\%$ *and*

k	1	2	3
$\mathbb{P}\{K_x = k\}$	0.2	0.3	0.5

Find $\ddot{a}_x$ *and* $\mathrm{Var}(\ddot{Y}_x)$*.*

Solution: 4 ways for $\ddot{a}_x$:

(1) $E(\ddot{Y}_x) = \sum_y y f_{\ddot{Y}_x}(y)$ (from 447[6], needs $f_{\ddot{Y}_x}$? (

k	1	2	3
$\mathbb{P}\{K_x = k\}$	0.2	0.3	0.5
$\ddot{Y}_x$	1	$1+v$	$1+v+v^2$

)

(2) $E(\ddot{Y}_x) = E(\frac{1-v^{K_x}}{1-v}) = \sum_{k=1}^{\infty}(\frac{1-v^k}{1-v}) f_{K_x}(k)$ (from 447[6]) $v = 1/1.05$

(3) $E(\ddot{Y}_x) = \frac{1-A_x}{1-v}$ ($A_x = ?$),

(4) $E(\ddot{Y}_x) = \sum_{k=0}^{\infty} v^k {}_kp_x$ (${}_kp_x$= ? ${}_1p_x$= ?). Which you like the best here ?

$$\ddot{a}_x = \sum_y y f_{\ddot{Y}_x}(y) = 1 * 0.2 + (1+v)0.3 + (1+v+v^2)0.5 \quad often\ not\ convenient \tag{1}$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = 1 + 0.8v + 0.5v^2 = \cdots. \tag{4}$$

$$\ddot{a}_x = \frac{1-A_x}{1-v} \text{... need to first derive } A_x \ (= E(v^{K_x})) \tag{3}$$

$$\ddot{a}_x = E(\frac{1-v^{K_x}}{1-v}) = \sum_{k=1}^{3} \frac{1-1.05^{-k}}{1-1.05^{-1}} f_{K_x}(k) = 1 * 0.2 + 1.9524 * 0.3 + 2.8594 * 0.5 = 2.2154, \tag{2}$$

$$E[\ddot{Y}_x^2] = \sum_{k=1}^{3} (\frac{1-1.05^{-k}}{1-1.05^{-1}})^2 f_{K_x}(k) = (1)^2(0.2) + (1.9524)^2(0.3) + (2.8594)^2(0.5) = 5.4317,$$

$\text{Var}(\ddot{Y}_x) = 5.4317 - (2.2154)^2 = 0.5236$ or $= \text{Var}(\frac{1-Z_x}{1-v}) = \frac{1}{(1-v)^2}(A(v^2) - (A(v))^2)) = \cdots$

Example 5.2. *Suppose that $p_{x+k} = 0.97$, for $k \geq 0$ and $i = 6.5\%$. Find $\ddot{a}_x$ and $\text{Var}(\ddot{Y}_x)$.*

Solution: Which of the methods (2), (3) and (4) is better here ?
Most of the time, method (3) is more convenient.
Formula: $\ddot{Y}_x = \frac{1-Z_x}{1-v}$. $\quad \ddot{a}_x = \frac{1-A_x}{1-v}$=? $\quad A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k)$=?
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **Which ?**

$$A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k p_x^{k-1} q_x = q_x/p_x \sum_{k=1}^{\infty} (vp_x)^k = \frac{q_x}{p_x} vp_x \frac{1-(vp_x)^{\infty}}{1-vp_x} = \frac{q_x}{(1/v - p_x)}.$$

$$A_x = \frac{q_x}{\frac{1}{v} - p_x} = \frac{0.03}{\frac{1}{1.065^{-1}} + 0.97} = 0.315789, \qquad \ddot{a}_x = \frac{1-A_x}{1-v} = \frac{1-0.3157894737}{1-(1.065)^{-1}} = 11.2105.$$

Formulas: $\ddot{Y}_x = \frac{1-Z_x}{1-v}$. $\quad V(aX+b) = a^2V(X)$, and $\text{V}(\ddot{Y}_x) = \frac{Var(Z_x)}{(1-v)^2} = \frac{{}^2A_x - A_x^2}{(1-v)^2}$.

$$^2A_x = A_x(v^2) = \frac{q_x}{\frac{1}{v^2} - p_x} = 0.1827$$

$$\text{Var}(\ddot{Y}_x) = \frac{^2A_x - A_x^2}{(1-v)^2} = \frac{0.1827 - (0.3158)^2}{(1-(1.065)^{-1})^2} \approx 22.269.$$

Example 5.3. *Assume $i = 6\%$ and the de Moivre model with terminal age 100. Find $\ddot{a}_{30}$.*

Solution: Method (3) is better here. Formula: $\ddot{Y}_x = \frac{1-Z_x}{1-v}$. $\quad \ddot{a}_x = \frac{1-A_x}{1-v}$.

$$A_x = E(v^{K_x}) = \sum_{k=1}^{\omega-x} v^k \frac{1}{\omega - x} = \frac{v(1-v^n)}{1-v} \frac{1}{n}\Big|_{v=1/1.06, n=w-x=70} = 0.2340649124.$$

Hence, $\ddot{a}_{30} = \frac{1-A_{30}}{1-v} = \frac{1-0.2340649124}{1-(1.06)^{-1}} = 13.53151988.$

Example 5.4. *John, age 65, has \$750,000 in his retirement account. An insurance company offers a whole life due annuity to John which pays \$P at the beginning of the year while (65) is alive for \$750,000. The annuity is priced assuming that $i = 6\%$ and the life table for the USA population in 2004 (see pages 605). The insurance company charges John 30% more of the APV of the annuity. Calculate P.*

Solution: $750000 = (1.3)P\ddot{a}_x = 1.3P\frac{1-A_{65}}{1-v} = 1.3P\frac{1-0.376}{1-v}$ (from the life table)

$750000 = 1.3P \times 11.022 => P = \frac{750000}{(1.3)(11.022)} \approx 52341.$

Denote ${}^m\ddot{a}_x = \ddot{a}_x(v^m)$. **Notice that,** ${}^2\ddot{a}_x = \frac{1-{}^2A_x}{1-v^2} \neq E((\ddot{Y}_x)^2)$**.**

Example 5.5. *Suppose that $i = 0.075$, $\ddot{a}_x = 8.6$ and ${}^2\ddot{a}_x = 5.6$.* $\mathrm{Var}(\ddot{Y}_x)$=*?*

Solution: Formula: $\ddot{Y}_x = \frac{1-Z_x}{1-v}$. $\mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{(1-v)^2}$? or ${}^2\ddot{a}_x - (\ddot{a}_x)^2$?

Given conditions: (1) $\ddot{a}_x = \frac{1-A_x}{1-v} = 8.6$. (2) ${}^2\ddot{a}_x = \frac{1-A_x(v^2)}{1-v^2} = 5.6$.

Eq.(1) => $A_x = 1 - (1-v)8.6\Big|_{v=1/1.075} = 0.4$.

Eq.(2) => ${}^2A_x = A_x(v^2) = 1 - (1-v^2)5.6\Big|_{v=1/1.075} = 0.246$. Hence,

$$\mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{(1-v)^2} \approx \frac{0.246 - (0.4)^2}{(1-(1.075)^{-1})^2} \approx 17.640.$$

Example 5.6. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

An 80–year old buys a due life annuity which will pay \$50000 at the beginning of the year. Let $i = 6.5\%$. The single benefit premium for this annuity = ?

Solution: $50000\ddot{a}_x = ?$

Formula: $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = \frac{1-Z_x}{1-v}$. $Z_x = v^{K_x}$.

3 ways:

(i) $\ddot{a}_x = E(\frac{1-v^{K_x}}{1-v}) = \sum_{k=1}^{\infty} \frac{1-v^k}{1-v} f_{K_x}(k) = \sum_{k=1}^{\infty} \frac{1-v^k}{1-v} \frac{d_{x+k-1}}{\ell_x}$ (as $f_{K_x}(k) = \frac{d_{x+k-1}}{\ell_x}$.)

(ii) $\ddot{a}_x = \frac{1-A_x}{d}$.

(iii) $\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x$ (${}_kp_x = \frac{\ell_{x+k}}{\ell_x}$).

Either way is fine, most of the time, use the last two ways.

(ii) $\ddot{a}_x = \frac{1-A_x}{d}$, $A_x = \sum_k v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k \frac{d_{x+k-1}}{\ell_x}$.

By previous example, $A_{80} = 0.816$.

$$\ddot{a}_{80} = \frac{1-A_{80}}{d} \approx \frac{1-(0.816)}{1-1/1.065} \approx 3.012. \text{ Hence, } (50000)\ddot{a}_{80} \approx (50000)(3.01165) \approx 150582.71.$$

(iii) $\ddot{a}_x = \sum_{k=0}^{\infty} v^k \frac{\ell_{x+k}}{\ell_x}$.

$$=1 + (1.065)^{-1}\frac{217}{250} + (1.065)^{-2}\frac{161}{250} + (1.065)^{-3}\frac{107}{250} + (1.065)^{-4}\frac{62}{250} + (1.065)^{-5}\frac{28}{250}$$

≈ 3.012. Hence, $(50000)\ddot{a}_{80} \approx (50000)(3.01165) \approx 150582.71$.

Example 5.7. *An insurance company issues 800 identical due annuities to independent lives aged* 65. *Each of this annuities provides an annual payment of 30000. Suppose that*

$p_{x+k} = 0.95$ *for each integer* $k \geq 0$*, and* $i = 7.5\%$*. Using the central limit theorem, estimate the initial fund needed at time zero in order that the probability that the present value of the random loss for this block of policies exceeds this fund is 1%.*

Solution: Let $\ddot{Y}_{x,1}, \dots, \ddot{Y}_{x,800}$ be the present value per unit face value for 800 due annuities.
The present value of total payment is $Y = \sum_{j=1}^{800} 30000\ddot{Y}_{x,j}$.
The fund needed is $Q = E(Y) + z_{0.01}\sigma_Y$, where $\Phi(z_{0.01}) = 0.99$, why ?

$P(\overline{Y} \leq t) \approx \Phi(\frac{t-E(\overline{Y})}{\sigma_{\overline{Y}}}) = \Phi(z_\alpha) = 1-\alpha$ by [22] =>

$t = E(\overline{Y}) + z_\alpha\sigma_{\overline{Y}}$ and $z_{0.01} = 2.326$ or 2.33.

Formulas: $\ddot{Y}_x = \frac{1-Z_x}{1-v}$. (1) $\ddot{a}_x = \frac{1-A_x}{1-v}$. (2) $\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x$, **which one ?**

Since we need to compute $V(\ddot{Y}_x)$, first way is simpler, then need [8]:
$f_{K_x}(k) = {}_{k-1}|q_x \ = {}_{k-1}p_x \cdot q_{x+k-1} = (\prod_{j\geq 0}^{k-2} p_{x+j})q_{x+k-1}$ **which ?**

$$A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k p_x^{k-1} q_x = \sum_{k=1}^{\infty} v^k p_x^k \frac{q_x}{p_x} = \frac{q_x}{p_x}\frac{vp_x}{1-vp_x} = \frac{q_x}{\frac{1}{v}-p_x}$$

$$A_x = \frac{q_x}{\frac{1}{v}-p_x} = \frac{0.05}{1.075-0.95} = 0.4, \quad {}^2A_x = \frac{q_x}{\frac{1}{v^2}-p_x} = \frac{0.05}{1.075^2-0.95} = 0.2444988,$$

$$\ddot{a}_x = \frac{1-A_x}{1-v} = 8.6,$$

$$\mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{(1-v)^2} = \frac{0.2444988-(0.4)^2}{(1-1/1.075)^2} = 17.35981.$$

$E(Y) = E\left[\sum_{j=1}^{800} 30000\ddot{Y}_{x,j}\right] = (30000)(800)(8.6) = 206400000,$

$\sigma_Y^2 = \mathrm{V}\left(\sum_{j=1}^{800} 30000\ddot{Y}_{x,j}\right) = n30000^2\mathrm{V}\left(\ddot{Y}_{x,j}\right) = (800)(30000)^2(17.3598) = 12499063200000.$

$Q = E(Y) + 2.326\sigma_Y = 206400000 + (2.326)\sqrt{12499063200000} = 214623343.70.$

Theorem 5.2. *(*Iterative formula for the APV of a due annuity*)* $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$. *[18]*

Example 5.8. *Suppose that* $\ddot{a}_x = \ddot{a}_{x+1} = 10$ *and* $q_x = 0.01$*. Find* i*.*

Solution: Formula [17]: $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$ and $v = \frac{1}{1+i}$ =>

$10 = 1 + \frac{1}{1+i}(0.99)(10)$ => $i = \frac{(0.99)(10)}{10-1} - 1 = 10\%$.

Theorem 5.3. *For the constant force of mortality model,*
$\ddot{a}_x = \frac{1}{1-vp_x} = \frac{1+i}{i+q_x} = \frac{1}{1-e^{-(\delta+\mu)}}$*, where* $q_x = 1-e^{-\mu}$*.*

Proof. 3 ways: $\ddot{a}_x = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} v^j f_{K_x}(k) = \frac{1-A_x}{1-v} = \sum_{k=0}^{\infty} v^k {}_kp_x$.

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \sum_{k=0}^{\infty} e^{-k\delta}e^{-k\mu} = \sum_{k=0}^{\infty}(e^{-\delta-\mu})^k = \sum_{k=0}^{\infty} x^k = \left.\frac{1-x^{\infty+1}}{1-x}\right|_{x=e^{-\delta-\mu}} = \frac{1}{1-e^{-\mu-\delta}}.$$

Theorem 5.4.

Theorem 5.5.

Theorem 5.6.

Theorem 5.7.

Example 5.9.

Example 5.10.

Example 5.11.

Theorem 5.8.

Example 5.12.

Theorem 5.9.

Theorem 5.10.

Theorem 5.11.

Example 5.13.

Theorem 5.12.

Example 5.14.

Example 5.15.

5.1.2 Whole life immediate annuity

Definition 5.4. *A* **whole life discrete immediate annuity** *is a series payments made at the end of each year, while an individual is alive. Its present value and APV with unit payment are denoted by Y_x and a_x, respectively. $Y_x = \sum_{k\geq 1}^{K_x-1} v^k$,*

If $T_x = 0.5$ then $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k =$? $Y_x =$? **Class exercise.**
If $T_x = 1.1$ then $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k =$? $Y_x =$?

Theorem 5.13. *$Y_x = \ddot{Y}_x - 1$, $a_x = \ddot{a}_x - 1$ and $Var(Y_x) = Var(\ddot{Y}_x)$.*

Example 5.16. *Suppose that $p_{x+k} = 0.97$, $\forall\ k \geq 0$, and $i = 6.5\%$. $a_x =$? $\mathrm{Var}(Y_x)=$?*

Solution: By Ex 5.2, $\ddot{a}_x \approx 11.21$ and $\mathrm{Var}(\ddot{Y}_x) \approx 22.27$ => Answers ?

$$a_x = \ddot{a}_x - 1 = 10.21 \text{ and } \mathrm{Var}(Y_x) = \mathrm{Var}(\ddot{Y}_x) = 22.27.$$

Definition 5.5. $Y_x = a_{\overline{K_x-1}|} = \sum_{k\ge 1}^{K_x-1} v^k$, *where* $a_{\overline{n}|} = a_{\overline{n}|i} = \sum_{k=1}^{n} v^k$.

Definition 5.6. $\ddot{a}_{\overline{n}|} = \sum_{k=0}^{n} v^k$.

Theorem 5.14. $a_x = vp_x\ddot{a}_{x+1} = vp_x(1+a_{x+1})$.

5.1.3 Whole life continuous annuity

Recall $\overline{a}_{\overline{n}|i} = \int_0^n v^t dt$.

Definition 5.7. *A* **whole life continuous annuity** *is a continuous flow of payments with constant rate made while an individual is alive. Its present value and APV with unit rate are denoted by* $\overline{Y}_x$ *and* $\overline{a}_x$*, respectively.* $\overline{Y}_x = \int_0^{T_x} v^t dt$ $(= \overline{a}_{\overline{T_x}|})$ *and* ${}^m\overline{a}_x = \overline{a}_x(v^m)$

$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k$, $Y_x = \sum_{k=0}^{K_x-1} v^k - 1$ and $\overline{Y}_x = \int_0^{T_x} v^t dt$.
Like Z_x, $\overline{Z}_x$ *etc.*, they are all of the form $g(T_x)$ or $g(K_x)$.

Theorem 5.15. *(i) If* $\delta = 0$ *(i.e.,* $v = 1$*) then* $\overline{Y}_x = T_x$ *and* $\overline{a}_x = \mathring{e}_x$*.*
(ii) If $\delta \ne 0$ *then* $\overline{Y}_x = \frac{1-\overline{Z}_x}{\delta}$, $\overline{a}_x = \frac{1-\overline{A}_x}{\delta} = \int_0^\infty v^t {}_tp_x dt$ and $\mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}$.

Do we have $\mathrm{Var}(\overline{Y}_x) = E((\overline{Y}_x)^2) - (\overline{a}_x)^2$?

Example 5.17. *Let* $v = 0.92$*, and the force of mortality be* $\mu_{x+t} = 0.02$*, for* $t \ge 0$*. Find (i) the density of* $\overline{Y}_x$*, (ii) the first quartile of* $\overline{Y}_x$*, (iii)* $\overline{a}_x$ *and* $\mathrm{Var}(\overline{Y}_x)$*.*

Solution: (i) Given $f_{T_x}(t) = 0.02e^{-0.02t}$, $t \ge 0$, $=> f_{\overline{Y}_x}(y) = f_{T_x}\left(h^{-1}(y)\right)\left|\frac{d}{dy}h^{-1}(y)\right|$, where
$\overline{Y}_x = \frac{1-v^{T_x}}{\delta} = \frac{1-(0.92)^{T_x}}{-\ln(0.92)} = h(T_x)$ and

$$h^{-1}(y) = \frac{\ln(1+y\ln 0.92)}{\ln 0.92},\ \underline{y \in (0, 1/\delta]} \qquad \text{(by [20] in 447)}$$

as $y = \frac{1-v^t}{\delta} = h(t) => v^t = 1 - y\delta => t = \frac{\ln(1+y\ln v)}{\ln v} = h^{-1}(y)$, as $\delta = -\ln v$, (1)

$$t \to T_x \in (0,\infty) \text{ and } y \to \overline{Y}_x = \frac{1-v^{T_x}}{\delta} \in (0, \frac{1-v^\infty}{\delta}). \qquad (2)$$

$$\begin{aligned}
f_{\overline{Y}_x}(y) =& f_{T_x}\left(\frac{\ln(1+y\ln(0.92))}{\ln(0.92)}\right)\left|\frac{d}{dy}\frac{\ln(1+y\ln(0.92))}{\ln(0.92)}\right| \qquad = f_{T_x}(\frac{\ln(1-\delta y)}{-\delta})\left|(\frac{\ln(1-\delta y)}{-\delta})'_y\right| \\
=&(0.02)\exp\left(-(0.02)\frac{\ln(1+y\ln(0.92))}{\ln(0.92)}\right)\frac{1}{\ln 0.92}\cdot\frac{1}{1+y\ln(0.92)}\cdot\ln 0.92 \\
=&(0.02)\exp\left(\ln[(1+y\ln(0.92))^{\frac{-(0.02)}{\ln(0.92)}}]\right)\frac{1}{1+y\ln(0.92)} \\
=&(0.02)(1+(\ln 0.92)y)^{-\frac{0.02}{\ln(0.92)}-1} \textbf{ done ?}
\end{aligned}$$

$0 < y < \frac{-1}{\ln(0.92)}$ **Why ??** (see Eq.(1) and Eq.(2))

(ii) Two ways for finding η_p, the $100p-$th percentile of $\overline{Y}_x$:

(1) to solve $p = P(\overline{Y}_x \le \eta_p)$ with $p = 0.25$.

(2) $\eta_p = \begin{cases} h(\xi_p) & \text{if } h\uparrow \\ h(\xi_{1-p}) & \text{if } h\downarrow \end{cases}$, where ξ_p is the $100p-$th percentile of T_x. Now $h(t)\uparrow$ in t.

Second way is easier here : First solve $p = \mathbb{P}\{T_x \le \xi_p\} = 1 - e^{-0.02\xi_p}$.

So, $\xi_p = -\frac{\ln(1-p)}{0.02}$. We have that $\overline{Y}_x = \frac{1-(0.92)^{T_x}}{-\ln(0.92)} = h(T_x)$.

So, the 100pth percentile of $\overline{Y}_x$ is $h(\xi_p) = \frac{1-(0.92)^{\xi_p}}{-\ln(0.92)} = \frac{1-(0.92)^{-\frac{\ln(1-p)}{0.02}}}{-\ln(0.92)}$, $p = ?$

The first quartile of $\overline{Y}_x$ is $\frac{1-(0.92)^{-\frac{\ln(1-0.25)}{0.02}}}{-\ln(0.92)} = 8.378536891$.

First way: $P(\overline{Y}_x \le t) = \int_0^t f_{\overline{Y}_x}(y)dy$
$= 0.02\int_0^t (1+(\ln 0.92)y)^{-\frac{0.02}{\ln(0.92)}-1}dy$
$= 0.02\int_0^t (1+ay)^b dy$ $\quad((a,b) = ?)$
$= 0.02\int_1^{1+at} u^b du/a$ $\quad(u = ?)$
$= 0.02\frac{u^{b+1}}{(b+1)a}\Big|_1^{1+at}$ $=\ldots = 1-(1+(\ln 0.92)t)^{-\frac{0.02}{\ln(0.92)}} = p = 0.25$.

Solving $1-(1+(\ln 0.92)\eta_{0.25})^{-\frac{0.02}{\ln(0.92)}} = 0.25$. yields $\eta_{0.25} = 8.378536891$.

(iii) $E(\overline{Y}_x) =$ **?** $\overline{Y}_x = \int_0^{T_x} v^t dt = \frac{1-\overline{Z}_x}{\delta}$, $\quad(\delta = -\ln v)$.

$E(\overline{Y}_x) = \int t f_{\overline{Y}_x}(t)dt = \int\int_0^y v^t dt f_{T_x}(y)dy = \frac{1-\overline{A}_x}{\delta} = \int_0^\infty v^t {}_tp_x dt$. **Are they feasible ?**
The last two are simpler for $E(\overline{Y}_x)$, and the 3rd is better for $\sigma^2_{\overline{Y}_x}$.

$$\overline{A}_x = E(v^{T_x}) = \int_0^\infty v^t f_{T_x}(t)dt = \int_0^\infty e^{-\delta t}\mu e^{-\mu t}dt = \frac{\mu}{\mu+\delta}\int_0^\infty \underbrace{(\delta+\mu)}_{why?} e^{-(\delta+\mu)t}dt = \frac{\mu}{\delta+\mu}$$

$$\overline{A}_x = \frac{\mu}{\delta+\mu} = \frac{\mu}{\underbrace{-\ln v}_{why\ do\ this\ ?} + \mu} = \frac{0.02}{-\ln(0.92)+0.02} = 0.1934580068,$$

$${}^2\overline{A}_x = \overline{A}_x(v^2) = \frac{0.02}{-\ln(0.92^2)+0.02} = 0.1070874674,$$

$$\overline{a}_x = \frac{1-\overline{A}_x}{-\ln v} = \frac{1-0.1934580068}{-\ln(0.92)} = 9.672900337, \qquad Var(\overline{Y}_x) =?$$

$$Var(\overline{Y}_x) = \frac{\mathrm{Var}(\overline{Z}_x)}{\delta^2} = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{(-\ln v)^2} = \frac{0.1070874674-(0.1934580068)^2}{(-\ln(0.92))^2} \approx 10.02.$$

A by-product: $\overline{a}_x = \frac{1-\overline{A}_x}{\delta} = \frac{1-\frac{\mu}{\delta+\mu}}{\delta} = \frac{1}{\mu+\delta}$

Example 5.18.

Example 5.19. *Let $\delta = 0.05$ and ${}_tp_x = (0.01)te^{-0.1t}$, $t \ge 0$. Calculate $\overline{a}_x$.*

Solution: Formulas: $\overline{a}_x = \frac{1-\overline{A}_x}{\delta} = \int_0^\infty v^t \cdot {}_tp_x\, dt$ and $v = e^{-\delta}$.
Which is better here ?

$$\begin{aligned}
\overline{a}_x =& \int_0^\infty e^{-(0.05)t}(0.01)te^{-0.1t}\,dt = 0.01\int_0^\infty te^{-0.15t}\,dt \\
=& 0.01\Gamma(\alpha)\beta^\alpha \int_0^\infty \frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^\alpha}\,dt \quad (see[23]) \\
=& \frac{0.01\Gamma(2)}{0.15^2}\int_0^\infty \frac{(0.15)^2 t^{2-1}e^{-0.15t}}{\Gamma(2)}\,dt = \frac{0.01}{(0.15)^2} \approx 0.444.
\end{aligned}$$

Skip to Ex. 5.23.

Definition 5.8.

Definition 5.9.

Theorem 5.16.

Theorem 5.17.

Theorem 5.18. $\overline{a}_x = \int_0^\infty v^t \cdot {}_tp_x\,dt = \int_0^\infty {}_tE_x\,dt.$

Theorem 5.19.

Theorem 5.20.

Corollary 5.1.

Corollary 5.2.

Theorem 5.21.

Example 5.20.

Example 5.21.

Skip to next section.

Theorem 5.22. *If ξ_p is a p–th quantile of T_x, then given $b, \delta > 0$, the p–th quantile of $b\frac{1-e^{-\delta T_x}}{\delta}$ is $b\frac{1-e^{-\delta\xi_p}}{\delta}$.*

Theorem 5.23.

Example 5.22. *Assume $i = 6\%$ and de Moivre model with $\omega = 100$.*
(i) Calculate the 30th percentile of $\overline{Y}_{30}$.
(ii) Calculate the APV and variance of $\overline{Y}_{30}$.

Solution: (i) Two ways since $\overline{Y}_{30} = \frac{1-(1.06)^{-T_{30}}}{-\ln(1/1.06)}(= h(T_x))$:

(1) based on $F_{\overline{Y}_x}(t)$, (2) based on $F_{T_x}(t)$.
(2) is easier, as it is U(a,b).

$h(t) = \frac{1-(1.06)^{-t}}{-\ln(1/1.06)} \uparrow$ in t.
Let $\xi_{0.30}$ the 30th percentile of T_{30}.
$0.3 = F_{T_{30}}(\xi_{0.30}) = \frac{\xi_{0.30}}{70}$.
So, $\xi_{0.30} = 21$. The 30th percentile of $\overline{Y}_{30}$ is $h(21) = \frac{1-(1.06)^{-21}}{-\ln(1/1.06)} = 12.11357171$.

(ii) $\overline{a}_{30} = \frac{1-\overline{A}_{30}}{\delta} = \int v^t {}_tp_x dt$. **Which equation is better here ?**
(Since we need to compute $\sigma^2_{\overline{Y}_x}$, the first one is better).

$\overline{A}_{30} = \int_0^{w-x} v^t \frac{1}{w-x} dt = \frac{v^t}{(w-x)\ln v}\Big|_0^{w-x} = \frac{1-(v)^{70}}{-(70)\ln(v)} = \frac{1-(1.06)^{-70}}{(70)\ln(1.06)} \approx 0.2410$

${}^2\overline{A}_{30} = \frac{1-(v^2)^{70}}{-(70)\ln(v^2)} = 0.1225492409$

$\overline{a}_{30} = \frac{1-\overline{A}_{30}}{\delta} = \frac{1-0.2410186701}{\ln(1.06)} = 13.02549429$.

$\text{Var}(\overline{Y}_{30}) = Var(\frac{1-\overline{Z}_{30}}{\delta}) = \frac{{}^2\overline{A}_{30}-(\overline{A}_{30})^2}{\delta^2} = \frac{0.1225492409-(0.2410186701)^2}{(\ln(1.06))^2} = 207.8908307$.

Skip ends here

Example 5.23. *An actuary models the future lifetime T_x of a life age x as follows. T_x given μ has constant of mortality μ. μ has a density function $f_\mu(t) = 1250(t-0.01)$, for $0.01 \le t \le 0.05$. $\delta = 0.06$. Calculate $\overline{a}_x$.*

Solution: Q: $f_{T_x}(t) = \mu e^{-\mu t}$**,** $t > 0$ **or** $f_{T_x|\mu}(t|\mu) = \mu e^{-\mu t}$**,** $t > 0$ **??**

$\overline{a}_x = E(\overline{Y}_x) = E[E[\overline{Y}_x|\mu]]$
and $E[\overline{Y}_x|\mu] = \int_0^\infty v^t e^{-\mu t} dt = \int_0^\infty e^{-(\delta+\mu)t} dt = \frac{1}{\mu+\delta}$.

$$\begin{aligned}
\overline{a}_x = E\left[\frac{1}{\mu+0.06}\right] &= \int_{0.01}^{0.05} \frac{1}{t+0.06} 1250(t-0.01)\, dt \\
=&1250 \int_{0.01}^{0.05} \frac{t+0.06-0.07}{t+0.06}\, dt \\
=&1250 \int_{0.01}^{0.05} 1 - \frac{0.07}{t+0.06}\, dt = 1250\Big[\int_{0.01}^{0.05} 1\, dt - \int_{0.01}^{0.05} \frac{0.07}{t+0.06}\, dt\Big] \\
=&1250\,(t-(0.07)\ln(t+0.06))\Big|_{0.01}^{0.05} \\
=&10.45130167.
\end{aligned}$$

Example 5.24.

Skip to next section

5.2 Deferred annuities.

(Quiz this week: 450 [1]-[12], [14]-[16], [17] $\ddot{Y}_x$)

5.2.1 Due n–year deferred annuity.

Definition 5.10. *A* **due n–year deferred annuity** *guarantees payments made at the beginning of the year while an individual is alive starting in n years. Its present value and APV for (x) with unit payment are denoted by ${}_n|\ddot{Y}_x$ and ${}_n|\ddot{a}_x$, respectively.*

Definition 5.11. ${}_n|\ddot{Y}_x = \sum_{k\geq n}^{K_x-1} v^k$.

Theorem 5.24. ${}_n|\ddot{Y}_x = v^n \sum_{k=0}^{K_x-n-1} v^k$.

Theorem 5.25. *If* $i \neq 0$, ${}_n|\ddot{Y}_x = \frac{v^n - v^{K_x}}{1-v} I(K_x > n) \ \ (= \frac{Z_{x:\overline{n}|}^{\ 1} - {}_n|Z_x}{1-v})$

Example 5.25.

Theorem 5.26.

5 ways for $E({}_n|\ddot{Y}_x)$: ${}_n|\ddot{a}_x = \sum_t t f_{{}_n|\ddot{Y}_x}(t) = \sum_{k=n+1}^{\infty} \frac{v^n - v^k}{1-v} f_{K_x}(k)$ and

Theorem 5.27. ${}_n|\ddot{a}_x = \frac{A_{x:\overline{n}|}^{\ 1} - {}_n|A_x}{1-v} = \sum_{k=n}^{\infty} v^k \cdot {}_kp_x = {}_nE_x \ddot{a}_{x+n}$. **All in formulas sheet**.

(The last three are more useful, the third is from the definitions in [14], and the last one is included in the formula sheet [18].

Example 5.26. *An insurer offers a 10–year deferred life annuity–due to a (55) with an annual payment of \$30000. Mortality follows the life table for the US population in 2004 (see pages 605). The annual effective rate of interest is 6%. Find the APV of this life annuity.*

Solution: $(30000)_{10}|\ddot{a}_{55} = ?$ life table gives ${}_nE_x$ and $\ddot{a}_{x+n}$

$$ {}_n|\ddot{a}_x = \sum_t t f_{{}_n|\ddot{Y}_x}(t) = \sum_{k=n+1}^{\infty} \frac{v^n - v^k}{1-v} f_{K_x}(k) = \frac{A_{x:\overline{n}|}^{\ 1} - {}_n|A_x}{1-v} = \sum_{k=n}^{\infty} v^k \cdot {}_kp_x = {}_nE_x \ddot{a}_{x+n}. \quad (1) $$

Which way ? $n = ?$ $x = ?$
$(30000)_{10}|\ddot{a}_{55} = (30000)_{10}E_{55}\ddot{a}_{65} = (30000)(0.508011685)(11.022302) \approx 167983.75$

Theorem 5.28.

Example 5.27. *Let* $A_{x:\overline{n}|}^{\ 1} = 0.3$, $A_{x+n} = 0.6$ *and* $i = 0.05$. ${}_n|\ddot{a}_x = ?$

Solution: 5th way: ${}_n|\ddot{a}_x = {}_nE_x \ddot{a}_{x+n}$, $\ddot{a}_{x+n} = \frac{1-A_{x+n}}{1-v}$. $A_{x+n} = 0.6$. **Q:** ${}_nE_x =$**??**
${}_nE_x = A_{x:\overline{n}|}^{\ 1} = 0.3$.
${}_n|\ddot{a}_x = {}_nE_x \ddot{a}_{x+n} = (0.3)\frac{1-0.6}{1-1/1.05} = 2.52$.

Theorem 5.29. $E\left[\left({}_n|\ddot{Y}_x\right)^2\right] = {}^2_nE_x \cdot E\left[\left(\ddot{Y}_{x+n}\right)^2\right]$ **Ignore it.**

$E\left[\left({}_n|\ddot{Y}_x\right)^2\right] = v^{2n}\cdot{}_np_x\frac{2\ddot{a}_{x+n}-(2-d)\cdot{}^2\ddot{a}_{x+n}}{d} = \frac{v^{2n}\cdot{}_np_x-2v^n\cdot{}_n|A_x+{}_n|A_x(v^2)}{(1-v)^2}$

$V({}_n|\ddot{Y}_x) = \frac{v^{2n}\cdot{}_np_x\cdot{}_nq_x-2v^n\cdot{}_nq_x\cdot{}_n|A_x+{}^2_n|A_x-{}_n|A_x^2}{(1-v)^2}$

Example 5.28. *Let* $v = 0.91$ *and* $p_{x+k} = 0.97$ *for* $k \ge 0$. *Find* ${}_{40}|\ddot{a}_x$ *and* $\text{Var}({}_{40}|\ddot{Y}_x)$.

Solution: ${}_n|\ddot{a}_x = {}_nE_x\cdot\ddot{a}_{x+n}$, $\quad {}_nE_x = v^n{}_np_x$,
${}_np_x = p_xp_{x+1}\cdots p_{x+n-1} = p_x^n$ **(due to** $p_{x+k} = 0.97$**,** $k \ge 0$),
$\ddot{a}_{x+n} = \sum_{k=0}^{\infty} v^k{}_kp_{x+n} = \sum_{k=0}^{\infty} v^kp_x^k = \frac{1}{1-vp_x}$,

$$ {}_{40}|\ddot{a}_x = {}_nE_x\cdot\ddot{a}_{x+n} = v^np_x^n\frac{1}{1-vp_x}\Big|_{v=0.91,p_x=0.97,n=40} = 0.05797317039. \qquad (0)$$

Q: Note $V({}_n|\ddot{Y}_x) = E(({}_n|\ddot{Y}_x)^2) - ({}_n|\ddot{a}_x)^2$. **How to compute** $E(({}_n|\ddot{Y}_x)^2)$ **?**
Ans: (1) **Th. 5.29, or (2) the simple and recommended way:**

$$\begin{aligned}
E(({}_n|\ddot{Y}_x)^2) =& E((\frac{v^n - v^{K_x}}{1-v}I(K_x > n))^2) && (1)\\
=& E(\frac{v^{2n} - 2v^{n+K_x} + v^{2K_x}}{(1-v)^2}I(K_x > n))\\
=& \frac{1}{(1-v)^2}[v^{2n}{}_np_x - 2v^nE(v^{K_x}I(K_x > n)) + E(v^{2K_x}I(K_x > n))]\\
E(({}_n|\ddot{Y}_x)^2) =& \frac{1}{(1-v)^2}[v^{2n}{}_np_x - 2v^n{}_n|A_x + {}_n|A_x(v^2)] \qquad {}_n|A_x = {}_nE_xA_{x+n} && (2)\\
=& \frac{1}{(1-v)^2}[v^{2n}{}_np_x - 2v^nv^n{}_np_xA_{x+n} + v^{2n}{}_np_xA_{x+n}(v^2)]\Big|_{v=0.91,p_x=0.97} && (\textit{noting}\\
& A_x = \sum_{k=1}^{\infty} v^kf_{K_x}(k) = \sum_{k=1}^{\infty} v^kq_xp_x^{k-1} = \frac{vq_x}{1-vp_x} \text{ (see [14]) and } {}_np_x = p_x^n)\\
=& 0.01275747\\
\text{Var}({}_{40}|\ddot{Y}_x) =& 0.01275747 - (0.05797317)^2 = 0.009396582
\end{aligned}$$

Neither Eq.(2) nor Th 5.29 is easy to memorize.
Just learn to expand Eq.(1).

Theorem 5.30.

Theorem 5.31.

Example 5.29.

Example 5.30.

Theorem 5.32. ${}_n|\ddot{a}_x = vp_x\cdot{}_{n-1}|\ddot{a}_{x+1}$. *[18]*

Example 5.31. *Using $i = 0.05$ and a certain life table ${}_{10}|\ddot{a}_{30} = 7.48$. Suppose that an actuary revises this life table and changes p_{30} from 0.95 to 0.96. Other values in the life table are unchanged. Find ${}_{10}|\ddot{a}_{30}$ using the revised life table.*

Solution: Formula: ${}_{n}|\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_kp_x$ and

[18]: ${}_{n}|\ddot{a}_x = vp_x \cdot {}_{n-1}|\ddot{a}_{x+1} = v \overbrace{p_x}^{changed} \sum_{j=n-1}^{\infty} v^j \cdot \overbrace{{}_jp_{x+1}}^{unchanged}$

${}_{10}|\ddot{a}_{30}^{old} = \overbrace{vp_{30}^{old}}^{known} \cdot {}_{10-1}|\ddot{a}_{30+1} = 7.48. \Rightarrow {}_{9}|\ddot{a}_{31} = \frac{(1.05)(7.48)}{0.95} = 8.267368421.$

${}_{10}|\ddot{a}_{30}^{new} = vp_{30}^{new} \cdot {}_{10-1}|\ddot{a}_{30+1} = ?$

$${}_{10}|\ddot{a}_{30} = vp_x \cdot {}_{9}|\ddot{a}_{31} = (1.05)^{-1}(0.96)(8.267368421) = 7.558736842.$$

Skip to next section

Example 5.32.

Theorem 5.33.

Theorem 5.34.

5.2.2 Immediate n–year deferred annuity.

Example 5.33.

Theorem 5.35.

Theorem 5.36.

Definition 5.12. *An* **immediate n–year deferred annuity** *guarantees payments made at the end of each year, while an individual is alive starting n years from now. The present value and APV of an immediate n–year deferred annuity for (x) with unit payment are denoted by ${}_{n}|Y_x$ and ${}_{n}|a_x$, respectively.*

Definition 5.13. *Immediate n–year deferred annuity ${}_{n}|Y_x = \sum_{k\geq n+1}^{K_x-1} v^k$.*

Theorem 5.37. ${}_{n}|Y_x = {}_{n+1}|\ddot{Y}_x$ *(v.s. ${}_{n}|\ddot{Y}_x = \sum_{k\geq n}^{K_x-1} v^k$).*

5.2.3 Continuous n–year deferred annuity.

Definition 5.14. *An* **n–year deferred continuous annuity** *guarantees a continuous flow of payments while the individual is alive starting in n years. Its present value and APV with unit payment are denoted by ${}_{n}|\overline{Y}_x$ and ${}_{n}|\overline{a}_x$, respectively. ${}_{n}|\overline{Y}_x = \int_n^{T_x} v^s\, ds I(T_x > n)$.*

$$_n|\overline{Y}_x = \int_n^{T_x} v^s\, ds I(T_x > n) = \begin{cases} \frac{v^n - v^{T_x}}{\delta} I(T_x > n) & \text{if } v \neq 1. \\ (T_x - n) I(T_x > n) & \text{if } v = 1. \end{cases}$$

If $v \neq 1$, **4 ways for computing** $_n|\overline{a}_x = E(_n|\overline{Y}_x)$:

$_n|\overline{a}_x = \int t f_{_n|\overline{Y}_x}(t)dt = \int_n^\infty \frac{v^n - v^t}{-\ln v} f_{T_x}(t)dt = \frac{A_{x:\overset{1}{\overline{n}|}} - {}_n|\overline{A}_x}{-\ln v} = \int_n^\infty v^t{}_tp_x dt.$

Most of the time, it is simpler to just make use of $_n|\overline{Y}_x = \frac{v^n - v^{T_x}}{-\ln v} I(T_x > n)$ for $V(_n|\overline{Y}_x)$.

Example 5.34. *Assume $v = 0.91$ and de Moivre's model with terminal age 100. $_{20}|\overline{a}_{40}$=?*

Solution : $_{20}|\overline{Y}_{40} = \frac{v^n - v^{T_x}}{-\ln v} I(T_x > n)$ $(= g(T_x))$. $(v =?\ w - x =?\ n =?)$

$$\begin{aligned} _{20}|\overline{a}_{40} &= \int_0^\infty \frac{(v^t - v^n) I(t > n)}{\ln v} \frac{I(t \in (0, w - x))}{w - x} dt = \int_n^{w-x} \frac{v^t - v^n}{\ln v} \frac{1}{w - x} dt \\ &= \frac{1}{w - x} \frac{1}{\ln v} \Big[\frac{v^t}{\ln v} - v^n t\Big]_n^{w-x} \qquad \approx 0.794 \end{aligned}$$

Example 5.35. *Suppose that $v = 0.92$, and the force of mortality is $\mu_{x+t} = 0.02$, for $t \geq 0$. Find $_{20}|\overline{a}_x$ and* $\mathrm{Var}(_{20}|\overline{Y}_x)$.

Solution: Formula: (Th 5.43.) $_n|\overline{a}_x = \frac{A_{x:\overset{1}{\overline{n}|}} - {}_n|\overline{A}_x}{-\ln v} = \int_n^\infty v^t{}_tp_x dt.$

$$_n|\overline{a}_x = \int_n^\infty v^t{}_tp_x dt = \int_n^\infty v^t e^{-\mu t} dt = \int_n^\infty e^{-(\mu+\delta)t} dt = \frac{e^{-(\mu+\delta)n}}{\mu + \delta}\Bigg|_{n=20, \delta=-\ln 0.92, \mu=0.02} \approx 1.22$$

$$\begin{aligned} E((_n|\overline{Y}_x)^2)) &= \int_0^\infty (\frac{v^n - v^t}{\delta} I(t > n))^2 f_{T_x}(t)dt \qquad (= E(g(T_x))) \ \ \textbf{next step}?? \\ &= \int_n^\infty (\frac{v^n - v^t}{\delta})^2 f_{T_x}(t)dt = \int_n^\infty (\frac{v^{2n} - 2v^{n+t} + v^{2t}}{\delta^2}) f_{T_x}(t)dt \\ &= \int_n^\infty (\frac{e^{-2n\delta} - 2e^{-(n+t)\delta} + e^{-2\delta t}}{\delta^2}) \mu e^{-\mu t} dt \\ &= \frac{\mu}{\delta^2}\Big[e^{-2n\delta} \int_n^\infty e^{-\mu t} dt - 2e^{-n\delta} \int_n^\infty e^{-(\delta+\mu)t} dt + \int_n^\infty e^{-(2\delta+\mu)t} dt\Big] \\ &= 2.472240188 \qquad (as \int e^{at} dt = \frac{e^{at}}{a} + c). \end{aligned}$$

$\mathrm{Var}(_{20}|\overline{Y}_x) = 2.472240188 - (1.223476036)^2 = 0.9753465773.$

Theorem 5.38.

Theorem 5.39.

Theorem 5.40.

Theorem 5.41.

Theorem 5.42.

Theorem 5.43.

Theorem 5.44.

Theorem 5.45.

Theorem 5.46.

Theorem 5.47.

Theorem 5.48.

Theorem 5.49.

Theorem 5.50.

Definition 5.15.

Definition 5.16.

5.3 Temporary annuities.

5.3.1 Due n–year temporary annuity.

Definition 5.17. *A* **due n–year temporary annuity** *guarantees payments made at the beginning of the year while an individual is alive for at most n payments.* $\ddot{Y}_{x:\overline{n}|}=\sum_{k=0}^{K_x\wedge n-1}v^k$. *Its present value and APV for (x) with unit payment are denoted by $\ddot{Y}_{x:\overline{n}|}$ and $\ddot{a}_{x:\overline{n}|}$, respectively.*

Theorem 5.51. *(i)* $\ddot{Y}_{x:\overline{n}|}=\sum_{k=0}^{K_x\wedge n-1}v^k \qquad\qquad =\ddot{a}_{\overline{\min(K_x,n)}|}$,
(ii) If $i\neq 0$, $\ddot{Y}_{x:\overline{n}|}=\frac{1-v^{K_x\wedge n}}{1-v}(=\frac{1-Z_{x:\overline{n}|}}{d})$
(iii) If $i=0$, $\ddot{Y}_{x:\overline{n}|}=\min(K_x,n)$.

Theorem 5.52. $\ddot{a}_{x:\overline{n}|}=\sum_{k=1}^{n}\frac{1-v^k}{1-v}\cdot{}_{k-1}|q_x+\frac{1-v^n}{1-v}\cdot{}_np_x$ *(***easy to get right***)*
$=\sum_{k=1}^{n-1}\frac{1-v^k}{1-v}\cdot{}_{k-1}|q_x+\frac{1-v^n}{1-v}\cdot{}_{n-1}p_x$ *(***faster** *sometimes).*

Theorem 5.53. *If* $i\neq 0$, $\ddot{a}_{x:\overline{n}|}=\frac{1-A_{x:\overline{n}|}}{d}$ and $\mathrm{Var}(\ddot{Y}_{x:\overline{n}|})=\frac{{}^2A_{x:\overline{n}|}-(A_{x:\overline{n}|})^2}{d^2}$.

The last formula is due to $V(a+bX)=b^2V(X)=\frac{V(Z_{x:\overline{n}|})}{d^2}$ as $\ddot{Y}_{x:\overline{n}|}=\frac{1-Z_{x:\overline{n}|}}{d}$.
But ${}_n|\ddot{Y}_x=\frac{v^nI(K_x>n)-v^{K_x}I(K_x>n)}{d}$, *i.e.* $Z=\frac{Y+X}{d}$.
Thus $V(Z)=\frac{V(X)+V(Y)+2Cov(X,Y)}{d^2}\neq\frac{A_{x:\overline{n}|}^{1}(v^2)-(A_{x:\overline{n}|}^{1}(v))^2+{}_n|A_x(v^2)-({}_n|A_x(v))^2}{d^2}$.

Example 5.36. *Assume* $v=0.91$ *and de Moivre's model with* $w=100$. $\ddot{a}_{40:\overline{20}|}$ *=?* $\sigma^2_{\ddot{Y}_{40:\overline{20}|}}$ *=?*

Solution: 3 common ways for APV:
$\ddot{a}_{x:\overline{n}|}=\sum_{k=1}^{n}\frac{1-v^k}{1-v}\cdot{}_{k-1}|q_x+\frac{1-v^n}{1-v}\cdot{}_np_x=\frac{1-A_{x:\overline{n}|}}{d}=\sum_{k=0}^{n-1}v^k{}_kp_x$.

It is better to use $\ddot{Y}_{40:\overline{20}|}=\frac{1-v^{K_x\wedge n}}{d}=\frac{1-Z_{x:\overline{n}|}}{d}$, $d=1-v$.

$$A_{x:\overline{n}|}=\sum_{k=1}^{n}v^k\cdot{}_{k-1}|q_x+v^n\cdot{}_np_x=\sum_{k=1}^{n}v^k/60+v^n(1-n/60)=v\frac{1-v^n}{1-v}\frac{1}{60}+v^n\frac{40}{60}\approx 0.244,$$
$$\ddot{a}_{40:\overline{20}|}=\frac{1-A_{40:\overline{20}|}}{d}\approx\frac{1-0.244}{0.09}\approx 8.399$$
$$\sigma^2_{\ddot{Y}_{40:\overline{20}|}}=\frac{A_{40:\overline{20}|}(v^2)-(A_{40:\overline{20}|}(v))^2}{(1-v)^2}=\frac{A_{40:\overline{20}|}(0.91^2)-0.244^2}{(1-0.91)^2}=\cdots$$

Example 5.37.

Theorem 5.54.

Theorem 5.55. $\ddot{Y}_{x:\overline{n}|}=\sum_{k=0}^{n-1}Z_{x:\overline{k}|}^{\ 1}$ *and* $\ddot{a}_{x:\overline{n}|}=\sum_{k=0}^{n-1}v^k{}_kp_x$.

Theorem 5.56. *If* $i=0$, $\ddot{a}_{x:\overline{n}|}=1+e_{x:\overline{n-1}|}$.

Example 5.38.

Example 5.39. *If* $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$, *find* $\mathrm{E}(\ddot{Y}_{x:\overline{3}|})$ *and* $\mathrm{Var}(\ddot{Y}_{x:\overline{3}|})$.

Solution: Three ways to find $\ddot{a}_{x:\overline{3}|}$ $= E(\sum_{k=0}^{K_x\wedge 3-1} v^k) = E(\frac{1-v^{K_x\wedge 3}}{1-v})$.

$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} \frac{1-v^k}{1-v} \cdot {}_{k-1}|q_x + \frac{1-v^n}{1-v} \cdot {}_np_x = \frac{1-A_{x:\overline{n}|}}{d} = \sum_{k=0}^{n-1} v^k {}_kp_x$.

(1) $\ddot{a}_{x:\overline{3}|} = \sum_{k=1}^{n-1} \frac{1-v^k}{1-v} \cdot {}_{k-1}|q_x + \frac{1-v^n}{1-v} \cdot {}_{n-1}p_x$ **Is it right ?** $n = ?$

${}_0|q_x = ?$

${}_1|q_x = p_xq_{x+1}$ and ${}_2p_x = p_xp_{x+1}$.

$\ddot{a}_{x:\overline{3}|} = \frac{1-v^1}{1-v} \cdot {}_{1-1}|q_x + \frac{1-v^2}{1-v} \cdot {}_{2-1}|q_x + \frac{1-v^3}{1-v} \cdot {}_2p_x$

$= (1)(0.02) + \frac{1-0.92^2}{1-0.92}(0.98)(0.03) + \frac{1-0.92^3}{1-0.92}(0.98)(0.97) \approx 2.71.$

(2) $\ddot{a}_{x:\overline{3}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$

$= (1) + (0.92)(0.98) + (0.92)^2(0.98)(0.97) \approx 2.71.$

(3) Often use $\ddot{Y}_{x:\overline{3}|} = \frac{1-Z_{x:\overline{3}|}}{1-v}$ $\quad$ ($\ddot{a}_{x:\overline{3}|} = \frac{1-A_{x:\overline{3}|}}{1-v}$ and $\mathrm{Var}(\ddot{Y}_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}(v^2)-(A_{x:\overline{n}|})^2}{(1-v)^2}$).

$$A_{x:\overline{3}|} = E(v^{K_x\wedge n}) = \sum_{k=1}^{n-1} v^k \cdot {}_{k-1}|q_x + v^n \cdot {}_{n-1}p_x$$

$$= (0.92)(0.02) + (0.92)^2(0.98)(0.03) + (0.92)^3(0.98)(0.97) \approx 0.78$$

$$\ddot{a}_{x:\overline{3}|} \approx \frac{1-0.78}{1-0.92} \approx 2.71.$$

$${}^2A_{x:\overline{3}|} = (0.92)^2(0.02) + (0.92)^4(0.98)(0.03) + (0.92)^6(0.98)(0.97) \approx 0.61$$

$$\mathrm{Var}(\ddot{Y}_{x:\overline{3}|}) \approx \frac{0.61-(0.78)^2}{(1-0.92)^2} \approx 0.0798398$$

Theorem 5.57. $\ddot{Y}_x = \ddot{Y}_{x:\overline{n}|} + {}_n|\ddot{Y}_x$. **Formula #18**.

Example 5.40. *An insurer offers a 20–year temporary life annuity–due to lives age* (60) *with an annual payment of \$40000. Mortality follows the life table for the US population in 2004 (see pages 605) The annual effective rate of interest is* 6%. *Calculate the APV of this life annuity.*

Solution: The tables give $\ddot{a}_x$ and ${}_nE_x$.
Formulas #18 leads $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_n|\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x\ddot{a}_{x+n}$.
The APV of this life annuity is
$(40000)\ddot{a}_{60:\overline{20}|} = (40000)\left(\ddot{a}_{60} - {}_{20}E_{60}\ddot{a}_{80}\right)$
$= (40000)\left(12.154122 - (0.190986505)(7.026210)\right) = 432488.4283.$

Definition 5.18. *The* **actuarial accumulated value** *at time* n *of an* n*–year temporary due annuity is* $\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x}$.

$\ddot{s}_{x:\overline{n}|}$ is the actuarial future value of an n–year due life insurance policy to (x).

Theorem 5.58.

Theorem 5.59. *Under constant force of mortality,* $\ddot{a}_{x:\overline{n}|} = \frac{1-v^n p_x^n}{1-vp_x}$. **Ignore rest.**

Proof. $\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = \sum_{k=0}^{n-1} v^k p_x^k = \frac{1-(vp_x)^n}{1-vp_x}$. ∎

Theorem 5.60. $\ddot{a}_{x:\overline{n}|} = 1 + vp_x \ddot{a}_{x+1:\overline{n-1}|}$.

Definition 5.19.

5.3.2 Immediate n–year temporary annuity.

Definition 5.20. *An* **immediate n–year temporary annuity** *guarantees payments made at the end of the year, while an individual is alive for n years. Its present value and APV for* (x) *with unit payment are denoted by* $Y_{x:\overline{n}|}$ *and* $a_{x:\overline{n}|}$*, respectively.* $Y_{x:\overline{n}|} = \sum_{k\geq 1}^{n\wedge(K_x-1)} v^k$ *(v.s.* $\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{(n\wedge K_x)-1} v^k$*).*

Theorem 5.61. $Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n+1}|} - 1$, $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1$ *and* $\mathrm{Var}(Y_{x:\overline{n}|}) = \mathrm{Var}(\ddot{Y}_{x:\overline{n+1}|})$.

Example 5.41. *Let* $p_x = 0.98$, $p_{x+1} = 0.97$, *and* $v = 0.92$. *Find* $a_{x:\overline{2}|}$ *and* $\mathrm{Var}(Y_{x:\overline{2}|})$.

Solution: $Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n+1}|} - 1$.
$\ddot{a}_{x:\overline{3}|} = 2.70618784$ and $\mathrm{Var}(\ddot{Y}_{x:\overline{3}|}) \approx 0.0798398$, (by Ex5.38),
$a_{x:\overline{2}|} = 1.70618784$ and $\mathrm{Var}(Y_{x:\overline{2}|}) \approx 0.0798398$.

Definition 5.21.

Theorem 5.62.

Theorem 5.63.

Theorem 5.64.

Example 5.42.

Example 5.43.

Theorem 5.65.

Theorem 5.66.

Theorem 5.67.

Theorem 5.68.

Theorem 5.69. $a_{x:\overline{n}|} = vp_x\ddot{a}_{x+1:\overline{n}|} = vp_x(1 + a_{x+1:\overline{n-1}|})$.

5.3.3 Continuous n–year temporary annuity.

Definition 5.22. *An n–***year temporary continuous annuity** *guarantees a continuous flow of payments at a constant rate for n years while the individual is alive. Its present value and APV for (x) with unit rate are denoted by $\overline{Y}_{x:\overline{n}|}$ and $\overline{a}_{x:\overline{n}|}$, respectively.*

Definition 5.23. $\overline{Y}_{x:\overline{n}|} = \int_0^{T_x\wedge n} v_t\,dt$. *If $v_t = v^t$, then* $\overline{Y}_{x:\overline{n}|} = \int_0^{T_x\wedge n} v^t\,dt = \overline{a}_{\overline{\min(T_x,n)}|}$.

Theorem 5.70. *If $\delta \neq 0$, then* $\overline{Y}_{x:\overline{n}|} = \frac{1-v^{T_x\wedge n}}{-\ln v} = \frac{1-\overline{Z}_{x:\overline{n}|}}{\delta}$.

Example 5.44. *Suppose that $\delta = 6\%$ and deaths are $U(0, 105)$.* $\overline{a}_{65:\overline{20}|}$ *=?*

Solution: 3 common ways for $\overline{a}_{x:\overline{n}|}$

$= \int_0^n \frac{1-v^t}{-\ln v}\cdot f_{T_x}(t)\,dt + \frac{1-v^n}{-\ln v}\mathbb{P}\{T_x > n\} = \int_0^n v^t \cdot {}_tp_x\,dt = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$ if $\delta \neq 0$.

Either way works, but the last one $\frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$ **is better here.**

$$\overline{A}_{65:\overline{20}|} = E(v^{n\wedge T_x}) = \frac{\int_0^{20} v^t dt}{40} + v^{20}\cdot {}_{20}p_{65} = \frac{1-e^{-(20)(0.06)}}{(40)(0.06)} + e^{-(20)(0.06)}\frac{20}{40} = 0.4417661844,$$

$$\overline{a}_{65:\overline{20}|} = \frac{1-\overline{A}_{65:\overline{20}|}}{\delta} = \frac{1-0.4417661843}{(0.06)} = 9.303896928.$$

Jump to Example 5.47 first.

Theorem 5.71. *Under constant force of mortality,* $\overline{a}_{x:\overline{n}|} = \frac{1-e^{-n(\mu+\delta)}}{\mu+\delta}$.

Theorem 5.72. $\overline{a}_{x:\overline{m+n}|} = \overline{a}_{x:\overline{n}|} + {}_mE_x\cdot\overline{a}_{x+m:\overline{n}|}$.

Theorem 5.73.

Theorem 5.74.

Theorem 5.75.

Theorem 5.76.

Theorem 5.77.

Theorem 5.78. $\overline{Y}_x = \overline{Y}_{x:\overline{n}|} + {}_{n|}\overline{Y}_x$ *and* $\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n|}\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_nE_x\overline{a}_{x+n}$.

Example 5.47. If (i) $\mu_x(t) = \begin{cases} 0.01 & \text{if } 0 \le t < 10, \\ 0.03 & \text{if } 10 \le t. \end{cases}$ (ii) $\delta(t) = \begin{cases} 0.06 & \text{if } 0 \le t < 10, \\ 0.04 & \text{if } 10 \le t. \end{cases}$

Calculate $\overline{a}_{x:\overline{15}|}$ (= $E(v_{T_x\wedge 15})$ **why not** $E(v^{T_x\wedge 15})$ **?**)

Solution: $\overline{a}_{x:\overline{n}|} = \int_0^n \int_0^{y\wedge n} v_t dt f_{T_x}(y)dy = \int_0^n v_t * {}_tp_x dt = \ddot{a}_{x:\overline{10}|} + {}_{10}E_x \ddot{a}_{x+10:\overline{5}|}$ **??** [18]

(3) $\overline{a}_{x:\overline{15}|} = \overline{a}_{x:\overline{10}|} + {}_{10}E_x \overline{a}_{x+10:\overline{5}|}$. $\quad {}_{10}E_x = v_{10} \times {}_{10}p_x = \begin{cases} e^{-0.06(10)}e^{-0.01(10)} & ? \\ e^{-0.04(10)}e^{-0.03(10)} & ? \end{cases}$

Formula [5]: $S_X(x) = \exp(-\int_0^x \mu(t)dt)$. $\qquad$ [17] $\overline{a}_{x:\overline{10}|} = \int_0^{10} v^t {}_tp_x dt$

(2) ${}_tp_x = \exp(-\int_0^t \mu_x(u)du)$. => ${}_tp_x = \begin{cases} \exp(-\int_0^t 0.01dy) & if\ t \in (0, 10] \\ \exp(-(\int_0^{10} 0.01dy + \int_{10}^t 0.03dy)) & if\ t > 10, \end{cases}$

$$v_t = \begin{cases} \exp(-\int_0^t 0.06dy) & if\ t \in (0, 10] \\ \exp(-(\int_0^{10} 0.06dy + \int_{10}^t 0.04dy)) & if\ t > 10 \end{cases}$$

and $f_{T_x}(t) = \begin{cases} 0.01\exp(-\int_0^t 0.01dy) & if\ t \in (0, 10] \\ 0.03\exp(-(\int_0^{10} 0.01dy + \int_{10}^t 0.03dy)) & if\ t > 10, \end{cases}$

$$\begin{aligned}
\overline{a}_{x:\overline{15}|} &= \int_0^{15} v_t * {}_tp_x dt = \int_0^{10} v_t * {}_tp_x dt + \int_{10}^{15} v_t * {}_tp_x dt \\
&= \int_0^{10} e^{-0.06t}e^{-0.01t}dt + \int_{10}^{15} e^{-0.06(10)-0.04(t-10)}e^{-0.01(10)-0.03(t-10)}dt \\
&= \int_0^{10} e^{-0.07t}dt + \int_{10}^{15} e^{-0.2-0.04t}e^{0.2-0.03t}dt \\
&= \int_0^{10} e^{-0.07t}dt + \int_{10}^{15} e^{-0.07t}dt = \frac{e^{-0.07t}}{-0.07}\bigg|_0^{15} = 9.286603584
\end{aligned}$$

$$\begin{aligned}
(3) \qquad \overline{a}_{x:\overline{15}|} =& \overline{a}_{x:\overline{10}|} + {}_{10}E_x \overline{a}_{x+10:\overline{5}|} = \int_0^{10} e^{-0.06t} {}_tp_x dt + v_{10} \cdot {}_{10}p_x \int_0^5 e^{-0.04t} {}_tp_{x+10} dt \\
=& \int_0^{10} e^{-0.06t}e^{-0.01t}dt + e^{-0.06(10)}e^{-0.01(10)} \int_0^5 e^{-0.04t}e^{-0.03t}dt \\
=& \frac{1 - e^{-10(0.01+0.06)}}{0.01 + 0.06} + e^{-10(0.01+0.06)} \frac{1 - e^{-5(0.03+0.04)}}{0.03 + 0.04} \\
=& \frac{1 - e^{-0.7}}{0.07} + e^{-0.7}\frac{1 - e^{-0.35}}{0.07} = 9.286603584.
\end{aligned}$$

The <u>actuarial accumulated value at time n of an n–year temporary continuous annuity</u> is

$$\overline{s}_{x:\overline{n}|} = \frac{\overline{a}_{x:\overline{n}|}}{{}_nE_x} = \frac{\overline{a}_{x:\overline{n}|}}{v^n \cdot {}_np_x} = \frac{\int_0^n v^t \cdot {}_tp_x\, dt}{v^n \cdot {}_np_x} = \int_0^n \frac{1}{v^{n-t} \cdot {}_{n-t}p_{x+t}}\, dt$$
$$= \int_0^n \frac{1}{{}_{n-t}E_{x+t}}\, dt.$$

$\frac{1}{{}_{n-t}E_{x+t}}$ is the actuarial factor from time t to time n for a live age x.

Theorem 5.79. $\overline{a}_x = \overline{a}_{x:\overline{1}|} + vp_x\overline{a}_{x+1}$.

Example 5.45. *Suppose that $\overline{a}_x = 10$, $q_x = 0.02$ and $\delta = 0.07$. Deaths are uniformly distributed within each year of age. Find $\overline{a}_{x+1}$.*

Solution: Given $10 = \overline{a}_x = \overline{a}_{x:\overline{1}|} + vp_x\overline{a}_{x+1}$, $=>$ $\frac{10-\overline{a}_{x:\overline{1}|}}{vp_x} = \overline{a}_{x+1}$, where $\overline{a}_{x:\overline{1}|} = \frac{1-\overline{A}_{x:\overline{1}|}}{\delta}$,

$$\begin{aligned}
\overline{A}_{x:\overline{1}|} =& \overline{A}^{1}_{x:\overline{1}|} + \overline{A}_{x:\overline{1}|}^{\ \ 1} \qquad \overline{A}^{1}_{x:\overline{1}|} = \int_0^1 v^t q_x dt = q_x\frac{v^t}{\ln v}\Big|_0^1 = q_x\frac{v-1}{\ln v}\\
=& q_x\frac{v-1}{\ln v} + vp_x \qquad \textit{and} \quad \overline{A}_{x:\overline{1}|}^{\ \ 1} = v^n{}_np_x = v^1 * {}_1p_x\\
\overline{a}_{x+1} =& \frac{10-\frac{1-\overline{A}_{x:\overline{1}|}}{\delta}}{vp_x} = \frac{10-\frac{1-(q_x\frac{v-1}{\ln v}+vp_x)}{\delta}}{vp_x}\Bigg|_{v=?,p_x=?,\delta=?,q_x=?} \approx 9.897
\end{aligned}$$

Definition 5.24. Skip next Example.

Example 5.46. *Suppose that $\delta = 0.08$, and the force of mortality is $\mu_{x+t} = 0.01$, for $t \geq 0$. Find $\overline{a}_{x:\overline{10}|}$ and $\mathrm{Var}(\overline{Y}_{x:\overline{10}|})$.*

Solution: Formula: $\overline{Y}_{x:\overline{n}|} = \frac{1-v^{T_x\wedge n}}{\delta} = \frac{1-\overline{Z}_{x:\overline{n}|}}{\delta}$, or $\overline{a}_{x:\overline{n}|} = \frac{1-e^{-n(\mu+\delta)}}{\mu+\delta}$ (by Theorem 5.78). It is more convenience to use the first one here. From Chapter 4,

$$\begin{aligned}
\overline{A}_{x:\overline{n}|} =& \mu\frac{1-e^{-n(\mu)}v^n}{\mu-\ln v} + e^{-n\mu}v^n \qquad \delta = -\ln v.\\
\overline{A}_{x:\overline{10}|} =& \frac{(0.01)(1-e^{-(10)(0.01+0.08)})}{0.01+0.08} + e^{-(10)(0.01+0.08)} = 0.4725063642,\\
{}^2\overline{A}_{x:\overline{10}|} =& \overline{A}_{x:\overline{10}|}(v^2)\\
=& \frac{(0.01)(1-e^{-(10)(0.01+(2)0.08)})}{(0.01+(2)0.08)} + e^{-(10)(0.01+(2)0.08)} = 0.6634579217,\\
\overline{a}_{x:\overline{10}|} =& \frac{1-\overline{A}_{x:\overline{10}|}}{\delta} = \frac{1-0.4725063642}{0.08} = 6.593670447.\\
Var(\overline{Y}_{x:\overline{n}|}) =& \frac{\mathrm{Var}(\overline{Z}_{x:\overline{10}|})}{\delta^2} = \frac{{}^2\overline{A}_{x:\overline{10}|} - (\overline{A}_{x:\overline{10}|})^2}{\delta^2}\\
=& \frac{0.6634579217-(0.4725063642)^2}{(0.08)^2} = 68.78057148.
\end{aligned}$$

Ignore this page !

Remark. $f_{T_x}(t) = \mu e^{-\mu t}$? $\overline{(IA)}_{x:\overline{15}|} =$? (additional homework).

$$ {}_tp_x = \begin{cases} e^{-0.01t} & if\ t \in [0,10) \\ e^{0.2-0.03t} & if\ t \geq 10 \end{cases} \text{ and } f_{T_x}(t) = \begin{cases} 0.01e^{-0.01t} & if\ t \in [0,10) \\ 0.03e^{-0.03t}\underline{e^{0.2}} & if\ t \geq 10 \end{cases} $$

$$ \overline{(IA)}_{x:\overline{15}|} = E((T_x \wedge 15)v_{T_x \wedge 15}) = \int_0^\infty (t \wedge 15)v_{t\wedge 15} f_{T_x}(t)dt $$

$$ =(\int_0^{10} + \int_{10}^{15} + \int_{15}^{\infty})(t \wedge 15)v_{t\wedge 15} f_{T_x}(t)dt $$

$$ = \int_0^{10} te^{-0.06t}0.01e^{-0.01t}dt + \int_{10}^{15} te^{-0.04t}0.03e^{-0.03t}\overbrace{e^{0.2}}^{why\ ?}dt + \underbrace{\int_{15}^{\infty} 15e^{-0.04(15)}0.03e^{-0.03t}\overbrace{e^{0.2}}^{why\ ?}dt}_{\text{most of you missed it}} $$

5.4 n–year certain life annuity.

Remark: Exam 3 covers upto §5.3.

Quiz on Friday: Formulae of 450 (all except [13] and [19]).

5.4.1 n–year certain life annuity–due

Definition 5.25. *An n–year certain due life annuity pays at the beginning of the year while either the individual is alive or the number of payments does not exceed n. Its present value and APV for (x) with unit payment are denoted by $\ddot{Y}_{\overline{x:\overline{n}|}}$ and $\ddot{a}_{\overline{x:\overline{n}|}}$, respectively.*

Under this annuity, the first n payments are certain to be paid.

Definition 5.26. $\ddot{Y}_{\overline{x:\overline{n}|}} = \sum_{k=0}^{n\vee K_x-1} v^k \quad (= \ddot{a}_{\overline{\max(n,K_x)}|})$. $\qquad \ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{n\wedge K_x-1} v^k$

Definition 5.27. $\ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v}$. *[17]*

Theorem 5.80. $\ddot{Y}_{\overline{x:\overline{n}|}} = \sum_{k=0}^{n-1} v^k + \sum_{k\geq n}^{K_x-1} v^k = \frac{1-v^n}{1-v} + {}_n|\ddot{Y}_x \quad = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} Z_{x:\overset{1}{\overline{k}|}}$,

$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} \cdot {}_nq_x + \sum_{k=n+1}^{\infty} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x = \ddot{a}_{\overline{n}|} + {}_n|\ddot{a}_x = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k \cdot {}_kp_x$,

$\mathrm{Var}(\ddot{Y}_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_n|\ddot{Y}_x)$.

Proof.
$$\ddot{Y}_{\overline{x:\overline{n}|}} = \begin{cases} \sum_{k=0}^{n-1} v^k & \text{if } K_x \leq n \\ \sum_{k=0}^{K_x-1} v^k & \text{if } K_x > n \end{cases} = \begin{cases} \sum_{k=0}^{n-1} v^k & \text{if } K_x \leq n \\ \sum_{k=0}^{n-1} v^k + \sum_{k=n}^{K_x-1} v^k & \text{if } K_x > n \end{cases} \tag{1}$$

$$= \sum_{k=0}^{n-1} v^k + I(K_x > n) \sum_{k=n}^{K_x-1} v^k = \frac{1-v^n}{1-v} + I(K_x > n) \sum_{k=n}^{K_x-1} v^k = \ddot{a}_{\overline{n}|} + {}_n|\ddot{Y}_x. \tag{2}$$

$$\begin{aligned} E(\ddot{Y}_{\overline{x:\overline{n}|}}) =& E(g(K_x)) = \sum_{j=1}^{\infty} g(j) f_{K_x}(j) = \sum_{j=1}^{n} g(j) f_{K_x}(j) + \sum_{j>n}^{\infty} g(j) f_{K_x}(j) \\ =& \sum_{j=1}^{n} \sum_{k=0}^{n-1} v^k f_{K_x}(j) + + \sum_{j>n}^{\infty} \sum_{k=0}^{j-1} v^k f_{K_x}(j) \qquad \text{(by Eq.(1))} \\ =& \frac{1-v^n}{1-v} P(K_x \leq n) + \sum_{j>n}^{\infty} \sum_{k=0}^{j-1} v^k f_{K_x}(j) = \ddot{a}_{\overline{n}|} + {}_n|\ddot{a}_x. \qquad \text{(by Eq.(2))}. \end{aligned}$$

Example 5.47. *A special pension plan pays \$30000 at the beginning of the year guaranteed for 10 years and continuing thereafter per life. Suppose that $i = 0.06$ and mortality follows the life table for the USA population in 2004. Calculate the APV of this annuity for (65).*

Solution: The APV of this insurance is $A = 30000\ddot{a}_{\overline{x:\overline{n}|}} = ?$ $\qquad (n, x) = ?$

$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_n|\ddot{a}_x$, $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{1-v}$, and ${}_n|\ddot{a}_x = {}_nE_x\ddot{a}_{x+n}$. $\qquad$ [18]

${}_{10}|\ddot{a}_{65} = {}_{10}E_{65}\ddot{a}_{75} = (0.447480378)(8.412220) = 3.764303385$.

$A = (30000)(\frac{1-v^{10}}{1-v}\Big|_{v=1/1.06} + {}_{10}|\ddot{a}_{65}) = 346979.8698$.

Example 5.48. *Calculate* $\ddot{a}_{\overline{x:\overline{3|}}}$ *if*
(i) $v = 0.94$*; (ii)* $p_x = 0.99$*; (iii)* $p_{x+1} = 0.95$*; (iv)* $\ddot{a}_x = 5.6$.

Solution: Formula [17]: $\ddot{a}_{\overline{x:\overline{n|}}} = \ddot{a}_{\overline{n|}} + {}_n|\ddot{a}_x = \overbrace{\frac{1-v^n}{1-v}}^{known} + \overbrace{{}_n|\ddot{a}_x}^{=?} = ?$ (1)

$$\begin{aligned}
\textit{Given } 5.6 = \quad & \ddot{a}_x = \ddot{a}_{x:\overline{n|}} + {}_n|\ddot{a}_x \quad (\text{as } \ddot{Y}_x = \ddot{Y}_{x:\overline{n|}} + {}_n|\ddot{Y}_x \text{ (by Formula \#18)}) \\
& => \quad {}_n|\ddot{a}_x = 5.6 - \ddot{a}_{x:\overline{n|}} \qquad (2) \\
& formula[17] \quad \ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^k {}_kp_x = 1 + vp_x + v^2 \underbrace{{}_2p_x}_{=?} \\
& \ddot{a}_{x:\overline{n|}} = 1 + vp_x + v^2 p_x p_{x+1} \quad \textit{thus } {}_n|\ddot{a}_x \textit{ known by } (2)
\end{aligned}$$

$$\ddot{a}_{\overline{x:\overline{3|}}} = \overbrace{\frac{1-v^n}{1-v} + {}_3|\ddot{a}_x}^{by\ (1)} = \frac{1-v^n}{1-v} + \overbrace{5.6 - \ddot{a}_{x:\overline{3|}}}^{by\ (2)} = \overbrace{\frac{1-v^n}{1-v} + 5.6 - (1 + vp_x + v^2 p_x p_{x+1})}^{(v, p_x, p_{x+1})=?} = 5.66.$$

5.4.2 n–year certain life annuity–immediate

Definition 5.28. *An* n*–***year certain life annuity–immediate** *pays at the end of the year while either the individual is alive or the number of payments does not exceed* n*. Its present value and APV for* (x) *with unit payment are denoted by* $Y_{\overline{x:\overline{n|}}}$ *and* $a_{\overline{x:\overline{n|}}}$ *, respectively.*

Definition 5.29. $Y_{\overline{x:\overline{n|}}} = \sum_{k=1}^{n \vee (K_x - 1)} v^k$.

Under the n–year certain (due or immediate) life annuity, the first n payments are guaranteed.

Theorem 5.81. $Y_{\overline{x:\overline{n|}}} = \ddot{Y}_{\overline{x:\overline{n+1|}}} - 1$.

Definition 5.30.

5.4.3 n–year certain life continuous annuity

Definition 5.31. *An* n*–***year certain continuous life annuity** *makes continuous payments while either an individual is alive or the # of years of payments does not exceed* n*. Its present value and APV for* (x) *with unit rate are denoted by* $\overline{Y}_{\overline{x:\overline{n|}}}$ *and* $\overline{a}_{\overline{x:\overline{n|}}}$ *, respectively.*

Definition 5.32. $\overline{Y}_{\overline{x:\overline{n|}}} = \int_0^{n \vee T_x} v^t dt$ *and* $\overline{a}_{\overline{n|}} = \int_0^n v^t dt$.

Definition 5.33. $(\ddot{a}_{\overline{n|}} = \sum_{i=0}^{n-1} v^i$ *and* $a_{\overline{n|}} = \sum_{i=1}^{n} v^i)$

Definition 5.34.

Theorem 5.82. $\overline{Y}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{n\vee T_x}|} = \int_0^{n\vee T_x} v^t dt = \overline{a}_{\overline{n}|} + {}_n|\overline{Y}_x$,

$$\overline{a}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{n}|} \cdot {}_nq_x + \int_n^\infty \overline{a}_{\overline{t}|} \cdot f_{T_x}(t)\,dt = \overline{a}_{\overline{n}|} + {}_n|\overline{a}_x = \overline{a}_{\overline{n}|} + \int_n^\infty v^t \cdot {}_tp_x\,dt,$$

$$\mathrm{Var}(\overline{Y}_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_n|\overline{Y}_x).$$

Remark. $E(\overline{Y}_{\overline{x:\overline{n}|}}) = E(\int_0^{n\vee T_x} v^t dt) = E(g(T_x)) = \int_0^\infty g(y) f_{T_x}(y)dy$

$$= \int_0^n g(y)f_{T_x}(y)dy + \int_n^\infty g(y)f_{T_x}(y)dy = \int_0^n \int_0^n v^t dt f_{T_x}(y)dy + \int_n^\infty \int_0^y v^t dt f_{T_x}(y)dy$$

$$= \int_0^n \frac{v^t}{\ln v}\bigg|_0^n f_{T_x}(y)dy + \int_n^\infty \frac{v^t}{\ln v}\bigg|_0^y f_{T_x}(y)dy$$

$$= \frac{1-v^n}{-\ln v}{}_nq_x + \int_n^\infty \frac{1-v^y}{-\ln v} f_{T_x}(y)dy = \overline{a}_{\overline{n}|} \cdot {}_nq_x + \int_n^\infty \overline{a}_{\overline{y}|} f_{T_x}(y)dy$$

$$= E(\overline{a}_{\overline{n}|} + {}_n|\overline{Y}_x) = \overline{a}_{\overline{n}|} + {}_n|\overline{a}_x = \overline{a}_{\overline{n}|} + \int_n^\infty v^t \cdot {}_tp_x\,dt,$$

$\mathrm{Var}(\overline{Y}_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_n|\overline{Y}_x)$.

5.5 Contingencies paid m times a year.

In this section, we will consider the case of life insurance paid m times a year. In the unit case, a payment of \$1 is made each year. Hence, each m–thly payment is $\frac{1}{m}$.

For a period of length $\frac{1}{m}$:

(i) the discount factor is $\underline{v^{(m)} = v^{1/m} = (1+i)^{-1/m}}$ v.s. $\frac{1}{1+i}$.

(ii) the interest factor is $(1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$ v.s. $1+i$.

(iii) the effective rate of interest is $(1+i)^{1/m} - 1 = \frac{i^{(m)}}{m}$ v.s. i.

(iv) the effective rate of discount is $1 - v^{1/m} = \frac{d^{(m)}}{m}$ v.s. $d = 1 - v$.

Remark. It suffices to remember $v^{(m)} = v^{1/m}$, and ignore the rest notations. Then each of insurance and annuity can be viewed as

(1) the time unit is a $\frac{1}{m}$ year rather than a year,
(2) the unit paid is not one but $1/m$,
(3) the discount factor is not v but $v^{1/m}$.
(4) $f_{K_x}(k) = {}_{k-1}|q_x$ is replaced by $f_{J_x^{(m)}}(k) = {}_{\frac{k-1}{m}}|_{\frac{1}{m}}q_x$, where
$J_x^{(m)} = k$ if $T_x \in (\frac{k-1}{m}, \frac{k}{m}]$.

5.5.1 Whole life due annuity paid m times a year.

Definition 5.35. *A whole life due annuity with payments paid m times a year is a series payments made at the beginning of each m–thly time interval while an individual is alive. Its present value and APV for (x) with unit annual payment are denoted by* $\ddot{Y}_x^{(m)}$ *and* $\ddot{a}_x^{(m)}$. *respectively.*

Definition 5.36. $\ddot{Y}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{J_x^{(m)}-1} v^{k/m}$ *(compare to* $\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k$*).*

Theorem 5.83. *If* $v \neq 1$*, then* $\ddot{Y}_x^{(m)} = \frac{1}{m}\frac{1-(v^{1/m})^{J_x^{(m)}}}{1-v^{1/m}} = \frac{1-Z_x^{(m)}}{m(1-v^{1/m})}$, $Z_x^{(m)} = ?$

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1} v^{j/m} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{1}{m} \times \frac{1-A_x^{(m)}}{1-v^{1/m}} = \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x.$$

Example 5.49. *Let* $\mu_x(t) = 0.03$*,* $t \geq 0$*, and* $\delta = 0.06$*. Calculate* $\ddot{a}_x^{(12)}$ *and* $\mathrm{Var}(\ddot{Y}_x^{(12)})$*.*

Solution: $\mathrm{Var}(\ddot{Y}_x^{(12)}) = \frac{V(Z_x^{(m)})}{(m(1-v^{1/m}))^2}$ **??**

$\ddot{a}_x^{(m)} = \frac{1}{m} \times \frac{1-A_x^{(m)}}{1-v^{1/m}}$. Need to compute $A_x^{(m)} = E((v^{1/m})^{J_x^{(m)}})$.

$$A_x^{(m)} = \sum_{k=1}^{\infty}(v^{1/m})^k {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \sum_{k=1}^{\infty} v^{k/m}(e^{-(k-1)\mu/m} - e^{-k\mu/m}) \qquad P(T_x \in (a,b]) = {}_ap_x - {}_bp_x$$

$$= \sum_{k=1}^{\infty} v^{(k-1)/m} v^{1/m} e^{-(k-1)\mu/m}(1-e^{-\mu/m})$$

$$= v^{1/m}(1-e^{-\mu/m})\sum_{k-1=0}^{\infty} v^{\frac{k-1}{m}} e^{-(k-1)\mu/m}$$

$$= v^{1/m}(1-e^{-\mu/m})\sum_{j=0}^{\infty} v^{\frac{j}{m}} e^{-j\mu/m}$$

$$= v^{1/m}(1-e^{-\mu/m})\sum_{j=0}^{\infty}(v^{1/m}e^{-\mu/m})^j \qquad \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

$$= v^{1/m}(1-e^{-\mu/m})\frac{1}{1-v^{1/m}e^{-\mu/m}} \qquad \textit{or} = \frac{(1-e^{-\mu/m})}{\frac{1}{v^{1/m}} - e^{-\mu/m}} = \frac{{}_{\frac{1}{m}}q_x}{\frac{1}{v^{(m)}} - {}_{\frac{1}{m}}p_x}$$

$$A_x^{(12)} = \frac{(1-e^{-\mu/m})}{\frac{1}{v^{1/m}} - e^{-\mu/m}} = \frac{1-e^{-0.03/12}}{e^{0.06/12} - e^{-0.03/12}} = 0.3325003481,$$

$$\ddot{a}_x^{(12)} = \frac{1}{m}\frac{1-A_x^{(m)}}{1-v^{1/m}}\bigg|_{m=12, v=e^{-0.06}, A_x^{(m)}=0.33} \approx 11.15282986,$$

$${}^2A_x^{(12)} = A_x^{(12)}(v^2) = \frac{1-e^{-0.03/12}}{e^{0.12/12} - e^{-0.03/12}} = 0.1990012521,$$

$$\mathrm{Var}(\ddot{Y}_x^{(12)}) = \frac{{}^2A_x^{(12)} - (A_x^{(12)})^2}{(m(1-v^{1/12}))^2} = \frac{0.1990012521 - (0.3325003481)^2}{(0.05985024969)^2} \approx 24.691.$$ **(skip 1 page)**

Additional hw: Suppose that $i = 6\%$ and

k	1	2	4
$\mathbb{P}\{K_x = k\}$	0.3	0.1	0.6

Find $\ddot{a}_x$ and $\mathrm{Var}(\ddot{Y}_x)$.

Mimic my notes on Example 5.1, using all 4 ways.

$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k$.

(1) $\ddot{a}_x = \sum_y y f_{\ddot{Y}_x}(y) = 1 * 0.3 + (1+v)0.1 + (1+v+v^2+v^3)0.6 = 2.698.$

(4) $\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = 1 + 0.7v + 0.6v^2 + 0.6v^3 \approx 2.698$

5.5.2 Whole life immediate annuity paid m times a year.

Definition 5.37. *A whole life immediate annuity with payments paid m times year is a series payments made at the end m–thly time interval while an individual is alive. Its present value and APV for (x) with unit annual payment are denoted by $Y_x^{(m)}$ and $a_x^{(m)}$, respectively.*

A whole life immediate annuity paid m times a year makes payments at times $\frac{1}{m}, \frac{2}{m}, \ldots \frac{J_x^{(m)}-1}{m}$.

Definition 5.38. $Y_x^{(m)} = \frac{1}{m}\sum_{k=1}^{J_x^{(m)}-1} v^{k/m}$ *(v.s.* $\ddot{Y}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{J_x^{(m)}-1} v^{k/m}$*).*

Theorem 5.84. $Y_x^{(m)} = \ddot{Y}_x^{(m)} - \frac{1}{m}$ *(compare to* $Y_x = \ddot{Y}_x - 1$*).*

Definition 5.39.

Definition 5.40.

5.5.3 Due n–year temporary annuity paid m times a year.

Definition 5.41. *The present value and APV of an n–year temporary life due annuity for (x) with unit annual payment paid m times a year is denoted by $\ddot{Y}_{x:\overline{n}|}^{(m)}$ and $\ddot{a}_{x:\overline{n}|}^{(m)}$, respectively.*
$\ddot{Y}_{x:\overline{n}|}^{(m)} = \frac{1}{m}\sum_{k=0}^{\min(J_x^{(m)},nm)-1} v^{(m)k}$. *v.s.* $\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{K_x\wedge n-1} v^k$.

Theorem 5.85. *If $v \neq 1$ then* $\ddot{Y}_{x:\overline{n}|}^{(m)} = \frac{1-Z_{x:\overline{n}|}^{(m)}}{m(1-v^{(m)})}$.

Example 5.50. *Let $\mu_x(t) = 0.03$, $t \geq 0$, and $\delta = 0.06$. Calculate $\ddot{a}_{x:\overline{10}|}^{(12)}$ and $\mathrm{Var}(\ddot{Y}_{x:\overline{10}|}^{(12)})$.*

Solution: Formula: $\ddot{Y}^{(m)}_{x:\overline{n}|} = \frac{1-Z^{(m)}_{x:\overline{n}|}}{m(1-v^{(m)})}$. $\quad \ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{m(1-v^{(m)})} = ?$ $\quad v^{(m)} = v^{1/m}$

$$\begin{aligned}
need \;\; A^{(m)}_{x:\overline{n}|} =& E((v^{(m)})^{J^{(m)}_x \wedge mn}) = ??\\
& (v^{(m)})^{J^{(m)}_x \wedge mn} = (v^{(m)})^{J^{(m)}_x \wedge mn} I(J^{(m)}_x \le mn) + (v^{(m)})^{J^{(m)}_x \wedge mn} I(J^{(m)}_x > mn)\\
&= (v^{(m)})^{J^{(m)}_x} I(J^{(m)}_x \le mn) + (v^{(m)})^{mn} I(J^{(m)}_x > mn) \quad (v^{(m)})^{mn} = v^{(m)mn} \; or \; v^n?\\
&= (v^{(m)})^{J^{(m)}_x} - (v^{(m)})^{J^{(m)}_x} I(J^{(m)}_x > mn) + v^n I(J^{(m)}_x > mn)\\
& \qquad I(J^{(m)}_x = k) = I(T_x \in (\frac{k-1}{m}, \frac{k}{m}]) \quad by \; definition\\
& \qquad I(J^{(m)}_x = mn+1) = I(T_x \in (\frac{mn+1-1}{m}, \frac{mn+1}{m}]) = I(T_x \in [n, n+\frac{1}{m})\\
&= (v^{(m)})^{J^{(m)}_x} - (v^{(m)})^{J^{(m)}_x} I(J^{(m)}_x > mn) + v^n I(T_x > n)\\
=> \quad A^{(m)}_{x:\overline{n}|} =& A^{(m)}_x - {}_{n|}A^{(m)}_x + v^n {}_n p_x\\
=& A^{(m)}_x - {}_nE_x A^{(m)}_{x+n} + v^n {}_n p_x \qquad [14]: \;\; {}_{n|}A_x = {}_nE_x A_{x+n}\\
=& (1 - {}_nE_x) A^{(m)}_{x+n} + v^n {}_n p_x \;\; why \; ?? \qquad {}_nE_x = v^n {}_n p_x \quad A_x = A_{x+n} \; ?\\
=& (1 - e^{-n(\mu+\delta)}) A^{(m)}_x + e^{-n(\mu+\delta)} \qquad A^{(m)}_x = \frac{(1-e^{-\mu/m})}{\frac{1}{v^{1/m}} - e^{-\mu/m}} \quad \text{by Example 5.49,}\\
=& (1 - e^{-n(\mu)} v^n) A^{(m)}_x + e^{-n(\mu)} v^n \qquad \textbf{why do this ?}
\end{aligned}$$

$$A^{(12)}_{x:\overline{10}|} = (1 - v^n e^{-n\mu}) \frac{(1-e^{-\mu/m})}{\frac{1}{v^{1/m}} - e^{-\mu/m}} + v^n e^{-n\mu} \Big|_{n=10, m=12, v=e^{-0.06}, \mu=0.03} = 0.3507389233.$$

$$\ddot{a}^{(12)}_{x:\overline{10}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{m(1-v^{(m)})} = \frac{1 - 0.3507389233}{0.05985024969} = 10.84809303,$$

$$A^{(12)}_{x:\overline{10}|}(v^2) = (1 - v^n e^{-n\mu}) \frac{(1-e^{-\mu/m})}{\frac{1}{v^{1/m}} - e^{-\mu/m}} + v^n e^{-n\mu} \Big|_{n=10, m=12, v=e^{-0.06\times 2}, \mu=0.03} = 0.2110126904.$$

$\mathrm{V}(\ddot{Y}^{(12)}_{x:\overline{10}|}) = \frac{A^{(12)}_{x:\overline{10}|}(v^2) - (A^{(12)}_{x:\overline{10}|})^2}{(m(1-v^{1/12}))^2} = 24.56549725.$ **(jump 10 pages to §5.7)**

5.5.4 Immediate n–year temporary annuity paid m times a year.

Definition 5.42. *The present value and APV of an n–year temporary life immediate annuity for (x) with unit annual payment paid m times a year are denoted by $Y^{(m)}_{x:\overline{n}|}$ and $a^{(m)}_{x:\overline{n}|}$. respectively.*

Definition 5.43. $Y^{(m)}_{x:\overline{n}|} = \frac{1}{m} \sum_{k=1}^{\min(J^{(m)}_x - 1, nm)} v^{k/m}$.

Theorem 5.86. *If $v \neq 1$ then $Y^{(m)}_{x:\overline{n}|} = \ddot{Y}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m} Z_{x:\overline{n}|}^{\;1}$.*

5.5.5 Due n–year deferred annuity paid m times a year.

Definition 5.44. *The present value and APV of a due n–year deferred annuity for (x) with unit annual payment paid m times a year are denoted by ${}_{n|}\ddot{Y}_x^{(m)}$ and ${}_{n|}\ddot{a}_x^{(m)}$, respectively.*

Definition 5.45. ${}_{n|}\ddot{Y}_x^{(m)} = \frac{1}{m}\sum_{k\geq nm}^{J_x^{(m)}-1} v^{k/m}$.

Theorem 5.87. *If $v \neq 1$ then*

$$
\begin{aligned}
{}_{n|}\ddot{Y}_x^{(m)} &= \frac{Z_{x:\overline{n}|}^{\ \ 1} - {}_{n|}Z_x^{(m)}}{m(1-v^{(m)})} = \frac{Z_{x:\overline{n}|}^{\ \ 1} - {}_{n|}Z_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=nm}^{\infty} Z_{x:\frac{k}{m}|}^{\ \ 1},\\
{}_{n|}\ddot{a}_x^{(m)} &= \frac{1}{m}\sum_{k=nm+1}^{\infty} v^n \ddot{a}_{\overline{k-nm}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{A_{x:\overline{n}|}^{\ \ 1} - {}_{n|}A_x^{(m)}}{m(1-v^{(m)})}\\
&= \frac{1}{m}\sum_{k=nm}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x = {}_nE_x \cdot \ddot{a}_{x+n}^{(m)},\\
\ddot{a}_x^{(m)} &= \ddot{a}_{x:\overline{n}|}^{(m)} + {}_{n|}\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_nE_x\ddot{a}_{x+n}^{(m)}.
\end{aligned}
$$

Example 5.51. *Suppose that $\mu_x(t) = 0.03$, $t \geq 0$, and $\delta = 0.06$. ${}_{10|}\ddot{a}_x^{(12)}$=?*

Solution: Formula [18] $\underline{\ddot{Y}_x} = \ddot{Y}_{x:\overline{n}|} + {}_{n|}\ddot{Y}_x$. Similarly, $\ddot{a}_x^{(12)} = \ddot{a}_{x:\overline{10}|}^{(12)} + {}_{10|}\ddot{a}_x^{(12)}$. By Example 5.49 (see page 161)

$\ddot{a}_x^{(12)} = 11.15282986.$

By Example 5.50(see page 162) $\ddot{a}_{x:\overline{10}|}^{(12)} = 10.84809303$. Hence,

$$ {}_{10|}\ddot{a}_x^{(12)} = 11.15282986 - 10.84809303 = 0.30473683. $$

5.5.6 Immediate n–year deferred annuity paid m times a year.

Definition 5.46. *The present value and APV of a immediate n–year deferred annuity for (x) with unit annual payment paid m times a year are denoted by ${}_{n|}Y_x^{(m)}$ and ${}_{n|}a_x^{(m)}$, respectively.* ${}_{n|}\ddot{Y}_x^{(m)} = \frac{1}{m}\sum_{k>nm}^{J_x^{(m)}-1} v^{k/m}$.

Definition 5.47.

Theorem 5.88. *If $v \neq 1$ then ${}_n|Y_x^{(m)} = {}_n|\ddot{Y}_x^{(m)} - \frac{1}{m} Z_{x:\overline{n}|}^{\;1}$.*

$$
\begin{aligned}
{}_n|Y_x^{(m)} =& \frac{1}{m} v^n a_{\overline{J_x^{(m)}-nm-1}|\frac{i^{(m)}}{m}} I(J_x^{(m)} > nm+1) = \frac{1}{m}\sum_{k=nm+1}^{\infty} Z_{x:\overline{\frac{k}{m}}|}^{\;1} = {}_n|\ddot{Y}_x^{(m)} - \frac{1}{m} Z_{x:\overline{n}|}^{\;1},\\
{}_n|a_x^{(m)} =& \frac{1}{m}\sum_{k=nm+2}^{\infty} v^n a_{\overline{k-nm-1}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x\\
=& \frac{1}{m}\sum_{k=nm+1}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x = {}_nE_x \cdot a_{x+n}^{(m)} = {}_n|\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x,\\
a_x^{(m)} =& a_{x:\overline{n}|}^{(m)} + {}_n|a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_nE_x a_{x+n}^{(m)}.
\end{aligned}
$$

5.6 Non–level payments annuities

In general, we can consider the case of life annuities with varying payments and general discount rates.
Recall that v_t is the t–year discount factor.

The force of interest is $\delta_t = -\frac{d}{dt}\ln v_t = \frac{-\frac{d}{dt}v_t}{v_t}$.

$v_t = e^{-\int_0^t \delta_s\, ds}$.
Under compound interest: $v_t = v^t = (1+i)^{-t}$.
Due life annuity present value:

	Level payment with compound interest rate	$non - level$
whole	$Y = \sum_{j=0}^{K_x-1} cv^j$	$Y = \sum_{j=0}^{K_x-1} c_j v_j$
defer.	$Y = \sum_{j=n}^{K_x-1} cv^j$	$Y = \sum_{j\geq n}^{K_x-1} c_j v_j$
tem.	$Y = \sum_{j=0}^{K_x\wedge n-1} cv^j$	$Y = \sum_{j=0}^{K_x\wedge n-1} c_j v_j$
cer.	$Y = \sum_{j=0}^{K_x\vee n-1} cv^j$	$Y = \sum_{j=0}^{K_x\vee n-1} c_j v_j$

This gives one way to compute APV by $E(g(K_x)) = \sum_k g(k) f_{K_x}(k)$.
Another way is as follows.

	Level payment with compound interest rate	$non - level$
whole	$APV = \sum_{j=0}^{\infty} cv^j \cdot {}_jp_x$	$APV = \sum_{j=0}^{\infty} c_j v_j \cdot {}_jp_x$
defer.	$APV = \sum_{j\geq n}^{\infty} cv^j \cdot {}_jp_x$	$APV = \sum_{j\geq n}^{\infty} c_j v_j \cdot {}_jp_x$
tem.	$APV = \sum_{j=0}^{n-1} cv^j \cdot {}_jp_x$	$APV = \sum_{j=0}^{n-1} c_j v_j \cdot {}_jp_x$

$$
e.g.\quad E(\sum_{j=0}^{K_x-1} c_j v_j) = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} c_j v_j \cdot f_{K_x}(k) = \sum_{j=0}^{\infty} c_j v_j \cdot {}_jp_x. \quad (= \textstyle\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} c_j v_j \cdot f_{K_x}(k))
$$

$$
\textstyle\sum_{k=j+1}^{\infty} f_{K_x}(k) = {}_jp_x
$$

Summary for Non-level Annuity:

	due present value	APV	continuous present value	APV
whole	$\sum_{j=0}^{K_x-1} c_j v_j$	$\sum_{j=0}^{\infty} c_j v_j \cdot {}_j p_x$	$\int_0^{T_x} c_t v_t dt$	$\int_0^{\infty} c_t v_t \cdot {}_t p_x dt$
defer.	$\sum_{j\geq n}^{K_x-1} c_j v_j$	$\sum_{j\geq n}^{\infty} c_j v_j \cdot {}_j p_x$	$\int_n^{T_x} c_t v_t dt$	$\int_n^{\infty} c_t v_t \cdot {}_t p_x dt$
tem.	$\sum_{j=0}^{K_x\wedge n-1} c_j v_j$	$\sum_{j=0}^{n-1} c_j v_j \cdot {}_j p_x$	$\int_0^{T_x\wedge n} c_t v_t dt$	$\int_o^{n} c_t v_t \cdot {}_t p_x dt$
cer.	$\sum_{j=0}^{K_x\vee n-1} c_j v_j$		$\int_0^{T_x\vee n} c_t v_t dt$	

Example 5.52. *Suppose that a special 3–year temporary life annuity due makes a payment of* $10^4 + 10^3(k-1)$ *at the beginning of year* k, $k = 1, 2, 3$. *The effective annual rate of interest earned in the first and second year are* 6.5% *and* 6%, *respectively. We have that* $p_x = 0.98$ *and* $p_{x+1} = 0.95$. *Let* Y *be the present value for this life annuity. Calculate* $E[Y]$ *and* $\mathrm{Var}(Y)$.

Solution. $Y = \sum_{j=0}^{K_x\wedge n-1} c_j v_j$. $n = ?$
(1) **Standard way:** $E(Y) = \sum_k g(k) f_{K_x}(k)$,

$$g(k) = c_k v_k \text{ or } \sum_{j=0}^{k\wedge n-1} c_j v_j \textbf{ ???}$$

(2) $E(Y) = \sum_{j=0}^{n-1} c_j v_j ({}_j p_x)$, try the 1st.

$E(Y) = \sum_k g(k) f_{K_x}(k) = g(1) f_{K_x}(1) + g(2) f_{K_x}(2) + \sum_{k=3}^{\infty} g(k) f_{K_x}(k)$ **why** $k = 3$ **?**
$c_0 = 10000$, $c_1 = 11000$, $c_2 = 12000$,

$v_0 = ?$ $v_1 = 1.065^{-1}$ and $v_2 = (1.065 * 1.06)^{-1}$? or $v_2 = (1.06)^{-1}$?

$$\text{If } K_x = 1,\ Y = g(K_x) = \sum_{j=0}^{K_x\wedge 3-1} [10^4 + 10^3(j)] v_j = 10000,$$

If $K_x = 2$, $Y = g(K_x) = (10000) + (11000)(1.065)^{-1} = 20328.6385$,
If $K_x > 2$, $Y = g(K_x) = (10000) + (11000)(1.065)^{-1} + (12000)(1.065)^{-1}(1.06)^{-1} = 30958.45513$,

$f_{K_x}(1) = \mathbb{P}\{K_x = 1\} = q_x = (0.02)$,
$f_{K_x}(2) = \mathbb{P}\{K_x = 2\} = p_x q_{x+1} = (0.98)(0.05) = 0.049$,
$\sum_{j>2} f_{K_x}(1) = \mathbb{P}\{K_x > 2\} = p_x p_{x+1} = (0.98)(0.95) = 0.931$.
$E[Y] = E(g(K_x)) = \sum_{k=1}^{\infty} g(k) f_{K_x}(k)$ is simpler here due to $V(Y)$.

$$E[Y] = (10000)(0.02) + (20328.6385)(0.049) + (30958.45513)(0.931) = 30018.42501,$$
$$E[Y^2] = (10000)^2(0.02) + (20328.6385)^2(0.049) + (30958.45513)^2(0.931) = 914543977.5,$$
$$\mathrm{Var}(Y) = 914543977.5 - (30018.42501)^2 = 13438137.42.$$

Theorem 5.89. *Assume that the* t*–year discount factor is* v_t. *The actuarial present value* *of a whole life annuity due with payments $c_0, c_1, \ldots,$ is*

$$E(\sum_{j=0}^{K_x-1} c_j v_j) = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} c_j v_j \cdot f_{K_x}(k) = \sum_{k=0}^{\infty} c_k v_k \cdot {}_k p_x.$$

Proof. $\sum_{k=1}^{\infty}\sum_{j=0}^{k-1} c_j v_j f_{K_x}(k) = \sum_{j=0}^{\infty}\sum_{k>j}^{\infty} c_j v_j f_{K_x}(k)$ (as $0 \le j < k < \infty$) $= \sum_{j=0}^{\infty} c_j v_j \sum_{k>j}^{\infty} f_{K_x}(k) = \sum_{j=0}^{\infty} c_j v_j \cdot {}_j p_x$ ∎

Corollary 5.3. *A unit annually increasing due whole life annuity has payments $1, 2, \ldots,$ at the beginning of the year and actuarial present value*

$$(I\ddot{a})_x = E(\big(I\ddot{Y}\big)_x) = E(\sum_{k=0}^{K_x-1} \underbrace{(k+1)v^k}_{=c_k v_k}) = \sum_{k=0}^{\infty} \underbrace{(k+1)v^k}_{=c_k v_k} \cdot {}_k p_x.$$

Example 5.53. *An insurer offers a whole life annuity–due to (x) with annual payments. The first payment is \$1000. To take in account inflation, each payment is 4% higher than the previous one. The force of mortality is $\mu = 0.01$. Calculate the APV of this annuity using the annual effective rate of interest of 7%.*

Solution: $Y = \sum_{j=0}^{K_x-1} c_j v_j$, $v_j = v^j$, and $c_j = (1000)(1.04)^j$.
Two ways: (1) Standard: $E(Y) = E(g(X)) = \sum_k g(k) f_X(k)$, where $X = K_x$;
(2) $E(Y) = \sum_{j=0}^{\infty} c_j v_j \cdot {}_j p_x$

$$(1) \quad Y = \sum_{j=0}^{K_x-1} c_j v_j = \sum_{j=0}^{K_x-1} 10^3 (1.04)^j (1.07)^{-j} = 10^3 \sum_{j=0}^{K_x-1} p^j = 10^3 \frac{1-p^{K_x}}{1-p}\Big|_{p=\frac{1.04}{1.07}}$$

$$E(Y) = 10^3 \frac{1-E(p^{K_x})}{1-p} =??$$

$$E(p^{K_x}) = E(g(K_x)) = \sum_k g(k) f_{K_x}(k)$$

$$= \sum_{k=1}^{\infty} p^k (e^{-(k-1)\mu} - e^{-k\mu}) = \sum_{k=1}^{\infty} p^k e^{-(k-1)\mu}(1-e^{-\mu}) = p(1-e^{-\mu}) \sum_{k-1=0}^{\infty} (pe^{-\mu})^{k-1}$$

$$= p(1-e^{-\mu}) \sum_{j=0}^{\infty} (pe^{-\mu})^j = p(1-e^{-\mu}) \frac{1}{1-pe^{-\mu}}$$

$$E(Y) = 10^3 \frac{1-E(p^{K_x})}{1-p} = 10^3 \frac{1-p(1-e^{-\mu})\frac{1}{1-pe^{-\mu}}}{1-p}\Bigg|_{\mu=0.01, p=\frac{1.04}{1.07}} \approx 26519.17.$$

$$(2) \quad E(Y) = \sum_{k=0}^{\infty} c_k v_k \cdot {}_k p_x = \sum_{k=0}^{\infty} (1000)(1.04)^k (1.07)^{-k} e^{-k(0.01)} = 1000 \sum_{k=0}^{\infty} x^k \quad (x = ?)$$

$$= 1000 \cdot \frac{1}{1-x} = \frac{1000}{1-(1.04)(1.07)^{-1} e^{-0.01}} = 26519.17.$$

Q: (I) Which is simpler ?
(II) Why the standard way ?

Example 5.54. *Suppose that* $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. *Find* $(I\ddot{a})_x$.

Solution: $\left(I\ddot{Y}\right)_x = \sum_{k=0}^{K_x-1} \underbrace{(k+1)v^k}_{=c_k v_k})$, $c_k = k+1$, $v_k = v^k$.

Using that $\sum_{k=1}^{\infty} kx^{k-1} = (\frac{1-x^{\infty+1}}{1-x})'_x = \frac{1}{(1-x)^2}$, #16, and by the second way,

$$\begin{aligned}
(I\ddot{a})_x =& \sum_{k=0}^{\infty}(k+1)v^k \cdot {}_kp_x = \sum_{k=0}^{\infty}(k+1)e^{-0.07k}e^{-0.05k} \\
=& \sum_{k+1=1}^{\infty+1}(k+1)e^{-0.12(k+1-1)} = \sum_{j=1}^{\infty} je^{-0.12(j-1)} \qquad (j = k+1) \\
=& \frac{1}{(1-e^{-0.12})^2} = 78.20450201.
\end{aligned}$$

The standard way works too, but little bit more complicated.

$$\begin{aligned}
E(\left(I\ddot{Y}\right)_x) =& E(\sum_{k=0}^{K_x-1} \underbrace{(k+1)v^k}_{=c_k v_k}) = E(\sum_{j=1}^{K_x}(j)v^{j-1}) \qquad j = k+1 \text{ to use } [16] \\
=& E((\frac{1-v^{K_x+1}}{1-v})') \\
=& E(\frac{(1-v)(-K_x-1)v^{K_x} - (1-v^{K_x+1})(-1)}{(1-v)^2}) \\
=& E(\frac{1-(1-v+v)v^{K_x} - (1-v)K_x v^{K_x}}{(1-v)^2}) \\
=& \frac{1}{(1-v)^2}E(1-v^{K_x} - (1-v)K_x v^{K_x}) \\
=& \frac{1}{(1-v)^2}(1-E(v^{K_x}) - (1-v)E(K_x v^{K_x}))\cdots \qquad {}_tp_x = e^{-\mu t},\ t > 0
\end{aligned}$$

Corollary 5.4. *A unit annually increasing* n*–year temporary due life unit annuity has payments* $1, 2, \ldots, n$ *at the beginning of the year and actuarial present value* $(I\ddot{a})_{x:\overline{n}|} = E(\sum_{k=0}^{K_x\wedge n-1}(k+1)v^k) = \sum_{k=0}^{n-1}(k+1)v^k \cdot {}_kp_x$.

Example 5.55. *Suppose that* $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. *Find* $(I\ddot{a})_{x:\overline{10}|}$.

Sol: $\left(I\ddot{Y}\right)_{x:\overline{10}|} = \sum_{j=0}^{K_x\wedge 10-1}(j+1)v^j$. $c_j = j+1$, $v_j = v^j$.
Two ways too: (1) standard way, (2) By formula [2],
Formula [2]: $E(H(X)) = \sum_{k=1}^{\infty}(H(k) - H(k-1))P(X \geq k)$, where $H(K_x) = \sum_{j=0}^{K_x\wedge n-1}(j+1)v^j$.

$$H(k) - H(k-1) = \sum_{j=0}^{k\wedge n-1}(j+1)v^j - \sum_{j=0}^{(k-1)\wedge n-1}(j+1)v^j = \begin{cases} kv^{k-1} & if\ 1 \le k \le n \\ 0 & if\ k > n \end{cases}$$

$$\begin{aligned}
(I\ddot{a})_{x:\overline{10}|} &= \sum_{k=1}^{\infty} P(K_x \ge k)(H(k) - H(k-1)) = \sum_{k=1}^{10} {}_{k-1}p_x k v^{k-1} \\
&= \sum_{k=1}^{10} k e^{-0.05(k-1)} e^{-0.07(k-1)} \\
&= \sum_{k=1}^{10} k x^{k-1} \qquad x = ? \\
&= \left(\frac{1-x^{n+1}}{1-x}\right)'_x \\
&= \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}\Big|_{x=e^{-0.12}, n=10} = 28.01415775.
\end{aligned}$$

$$\begin{aligned}
\textbf{Standardway}: (I\ddot{a})_{x:\overline{10}|} &= E(g(K_x)) = \sum_{j=1}^{\infty}\Big(\sum_{k=0}^{j\wedge 10-1}(k+1)v^k\Big) f_{K_x}(j) \\
&= \sum_{j=1}^{9}\Big(\sum_{k=0}^{j\wedge 10-1}(k+1)v^k\Big) f_{K_x}(j) + \sum_{j=10}^{\infty}\Big(\sum_{k=0}^{j\wedge 10-1}(k+1)v^k\Big) f_{K_x}(j) \\
&= \sum_{j=1}^{9}\Big(\sum_{k=0}^{j-1}(k+1)v^k\Big)(e^{-\mu(j-1)} - e^{-\mu j}) + \sum_{j=10}^{\infty}\Big(\sum_{k=0}^{10-1}(k+1)v^k\Big) f_{K_x}(j) \\
&= \sum_{j=1}^{9}\Big(\sum_{k=0}^{j-1}(k+1)v^k\Big)(e^{-\mu(j-1)} - e^{-\mu j}) + \Big(\sum_{k=0}^{10-1}(k+1)v^k\Big) P(K_x \ge 9) \\
&= \sum_{j=1}^{9}\Big(\frac{1-x^j}{1-x}\Big)'_x\Big|_{x=v}(e^{-\mu(j-1)} - e^{-\mu j}) + \Big(\frac{1-x^{10}}{1-x}\Big)'_x\Big|_{x=v} {}_9p_x \cdots
\end{aligned}$$

Corollary 5.5. *A unit annually decreasing n–year temporary due life annuity has payments $n, n-1, \ldots, 1$ at the beginning of the year and actuarial present value* $(D\ddot{a})_{x:\overline{n}|} = E(\sum_{k=0}^{K_x\wedge n-1}(n-k)v^k) = \sum_{k=0}^{n-1}(n-k)v^k \cdot {}_kp_x$.

Example 5.56. *Suppose that $\mu_x(t) = 0.05$, $t \ge 0$, $\delta = 0.07$. Find $(D\ddot{a})_{x:\overline{10}|}$.*

Solution: $c_k = n-k$, $k = 0, 1, 2, \ldots$ $v_k = v^k$.

Two ways too: (1) By Corollary 5.5, (2) standard way. $(D\ddot{a})_{x:\overline{10}|} = \sum_{k=0}^{9}(10-k)v^k \cdot {}_kp_x$

$$
\begin{aligned}
=&\sum_{k=0}^{9}(10-k)e^{-0.05k}e^{-0.07k} = \sum_{k=0}^{9}(11-(k+1))e^{-0.12k} = \sum_{k=0}^{9}11x^k - \sum_{k=0}^{9}(k+1)x^k \\
=&11\frac{1-x^{10}}{1-x} - \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}\Big|_{x=e^{-0.12},n=10} = 39.96332152.
\end{aligned}
$$

Theorem 5.90. *Assume that the t–year discount factor is v_t. The actuarial present value of a whole life annuity immediate with annual payments of $c_1, c_2, \ldots,$*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k} c_j v_j \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} c_k v_k \cdot {}_kp_x.$$

Corollary 5.6. *A unit annually increasing immediate whole life annuity has payments $1, 2, \ldots,$ at the end of the year and actuarial present value*
$(Ia)_x = E(\sum_{k=1}^{K_x-1} kv^k) = \sum_{k=1}^{\infty} kv^k \cdot {}_kp_x.$

Example 5.57. *Suppose that $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. Find $(Ia)_x$.*

Solution: Using that $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$, $x \neq 1$,
$(Ia)_x = \sum_{k=1}^{\infty} kv^k \cdot {}_kp_x$

$$= \sum_{k=1}^{\infty} ke^{-0.05k}e^{-0.07k} = e^{-0.12}\sum_{k=1}^{\infty} ke^{-0.12(k-1)} = \frac{e^{-0.12}}{(1-e^{-0.12})^2} = 69.36117108.$$

Corollary 5.7. *A unit annually increasing n–year temporary life immediate annuity has payments $1, 2, \ldots, n$ at the end of the year and actuarial present value*
$(Ia)_{x:\overline{n}|} = E(\sum_{k=1}^{(K_x-1)\wedge n} kv^k) = \sum_{k=1}^{n} kv^k \cdot {}_kp_x.$

Corollary 5.8. *A unit annually decreasing n–year temporary life immediate annuity has payments $n, n-1, \ldots, 1$ at the end of the year and actuarial present value*
$(Da)_{x:\overline{n}|} = E(\sum_{k=1}^{(K_x-1)\wedge n}(n-k+1)v^k) = \sum_{k=1}^{n}(n+1-k)v^k \cdot {}_kp_x.$

Example 5.58.

Example 5.59.

Example 5.60.

Theorem 5.91. *Assume that the t–year discount factor is v_t. A continuous whole life annuity with rate of payments $c(t)$ has an actuarial present value of $\int_0^{\infty}\int_0^t c(s)v_s\,ds\, f_{T_x}(t)\,dt = \int_0^{\infty} c(t)\cdot v_t \cdot {}_tp_x\,dt$.*

Theorem 5.92.

Corollary 5.9. *A continuously increasing whole life unit annuity paid at the time of death has an actuarial present value of* $\left(\overline{I}\overline{a}\right)_x = E(\int_0^{T_x} tv^t dt) = \int_0^\infty t \cdot v^t \cdot {}_tp_x \, dt$.

Corollary 5.10. *A continuously increasing n–year temporary life unit annuity paid at the time of death with rate of payments t has an actuarial present value of* $\left(\overline{I}\overline{a}\right)_{x:\overline{n}|} = E(\int_0^{T_x \wedge n} tv^t dt) = \int_0^n t \cdot v^t \cdot {}_tp_x \, dt$.

Example 5.61. *Suppose that* $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. *Find* $\left(\overline{I}\overline{a}\right)_{x:\overline{15}|}$.

Solution : $c_t = t$ and $v_t = v^t$.
Two ways:

(1) $\left(\overline{I}\overline{a}\right)_{x:\overline{15}|} = \int_0^\infty \int_0^{t \wedge n} xv^x dx \mu e^{-\mu t} dt$.

(2) $\left(\overline{I}\overline{a}\right)_{x:\overline{15}|} = \int_0^{15} te^{-t(0.05)} e^{-t(0.07)} \, dt = 37.30299396$.

Example 5.62.

Corollary 5.11. *A continuously decreasing n–year temporary life unit annuity paid at the time of death has an actuarial present value of* $\left(\overline{D}\overline{a}\right)_{x:\overline{n}|} = E(\int_0^{n \wedge T_x} (n-t) v^t dt) = \int_0^n (n-t) \cdot v^t \cdot {}_tp_x \, dt$.

Corollary 5.12. *An annually increasing whole life unit annuity paid at the time of death has an actuarial present value of* $(I\overline{a})_x = E(\int_0^{T_x} \lceil t \rceil v^t dt) = \int_0^\infty \lceil t \rceil \cdot v^t \cdot {}_tp_x \, dt$.

Example 5.63. *Suppose that* $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. *Find* $(I\overline{a})_x$.

Solution: $c_t = \lceil t \rceil$ and $v_t = v^t$.

$$
\begin{aligned}
(I\overline{a})_x &= \int_0^\infty \lceil t \rceil \cdot e^{-t(0.07)} e^{-t(0.05)} \, dt = \sum_{k=1}^\infty \int_{k-1}^k ke^{-t(0.12)} \, dt \\
&= \sum_{k=1}^\infty k \frac{e^{-(k-1)(0.12)} - e^{-k(0.12)}}{0.12} = \frac{1}{0.12} \sum_{k=1}^\infty ke^{-(k-1)(0.12)} (1 - e^{-0.12}) \\
&= \frac{1 - e^{-0.12}}{0.12} \frac{1}{(1-x)^2} \Big|_{x = e^{-0.12}} = 73.69442445.
\end{aligned}
$$

Example 5.64.

Example 5.65.

Corollary 5.13. *An annually increasing n–year temporary life unit annuity paid at the* *time of death has an actuarial present value of* $(I\overline{a})_{x:\overline{n}|} = E(\int_0^{n\wedge T_x} \lceil t \rceil v^t dt) = \int_0^n \lceil t \rceil \cdot v^t \cdot {}_tp_x\, dt$.

Corollary 5.14. *An annually decreasing n–year temporary life unit annuity paid at the time of death has an actuarial present value of* $(D\overline{a})_{x:\overline{n}|} = E(\int_0^{n\wedge T_x} \lceil n-t \rceil v^t dt) = \int_0^n \lceil n-t \rceil \cdot v^t \cdot {}_tp_x\, dt$.

Example 5.66. *A pension plan pays continuous payments for the remaining lifetime of a life aged* (65)*. The rate of payments is* 50000 *a year. Suppose that the force of mortality is* 0.01*. The force of interest is 0.08 for deposits made in the next 10 years and 0.06 for deposits made after 10 years. Find the actuarial present value of this pension plan.*

Solution : $Y = \int_0^{T_x} c_t v_t dt$, where $c_t = 50000 = c$, and

$$v_t = e^{-\int_0^t \delta_s\, ds} = \begin{cases} e^{-0.08t} & \text{if } 0 \le t \le 10, \\ e^{-(0.08)(10)-(0.06)(t-10)} & \text{if } 10 < t. \end{cases}$$

$$\begin{aligned}
(1)\quad & E(\int_0^{T_x} c_t v_t dt) = \int_0^\infty H'(t) \cdot {}_tp_x\, dt \quad (\#1\ H(t) = \int_0^t c_s v_s ds) \\
=& \int_0^\infty c_t v_t \cdot {}_tp_x\, dt \\
=& \int_0^{10} (50000) e^{-0.08t} e^{-0.01t}\, dt + \int_{10}^\infty (50000) e^{-(0.08)(10)-(0.06)(t-10)} e^{-0.01t}\, dt \\
=& \frac{50000}{0.09} \int_0^{10} 0.09 e^{-0.09t}\, dt + \int_{10}^\infty 50000 e^{-0.9-(0.07)(t-10)}\, dt \\
=& \frac{(50000)(1-e^{-(0.09)(10)})}{0.09} + \frac{50000e^{-0.9}}{0.07} \int_0^\infty 0.07 e^{-0.07t}\, dt \\
=& \frac{(50000)(1-e^{-0.9})}{0.09} + \frac{(50000)e^{-0.9}}{0.07} \\
=& 620090.4.
\end{aligned}$$

Direct way: $E(\int_0^{T_x} cv_t dt) = E(g(T_x)) = \int_0^\infty g(y) f_{T_x}(y) dy = \int_0^\infty \int_0^y cv_t dt 0.01e^{-0.01y} dy$.

$$g(y) = \begin{cases} \int_0^y ce^{-0.08t} dt & \text{if } y \in [0,10] \\ \int_0^{10} ce^{-0.08t} dt + \int_{10}^y ce^{-0.08(10)-0.06(t-10)} dt & \text{if } y > 10 \end{cases}$$

$$g(y) = \begin{cases} \int_0^y ce^{-0.08t} dt & \text{if } y \in [0,10] \\ \int_0^{10} ce^{-0.08t} dt + \int_{10}^y ce^{-0.2} e^{-0.06t} dt & \text{if } y > 10 \end{cases}$$

$$= \begin{cases} \frac{c}{0.08}[1-e^{-0.08y}] & \text{if } y \in [0,10] \\ \frac{c}{0.08}[1-e^{-0.8}] + \frac{ce^{-0.2}}{0.06}[e^{-0.6} - e^{-0.06y}] & \text{if } y > 10 \end{cases}$$

$$
\begin{aligned}
&E(\int_0^{T_x} cv_t dt) \\
&= \int_0^{10} \frac{c}{0.08}[1 - e^{-0.08y}]0.01e^{-0.01y}dy + \int_{10}^{\infty} \{\frac{c}{0.08}[1 - e^{-0.8}] + \frac{ce^{-0.2}}{0.06}[e^{-0.6} - e^{-0.06y}]\}0.01e^{-0.01y}dy \\
&= \int_0^{10} a_1 e^{-0.01y} + a_2 e^{-0.09y}dy + \int_{10}^{\infty} a_3 e^{0.01y} - a_4 e^{-0.07y}dy
\end{aligned}
$$

5.7 Computing present values from a life table.

5.7.1 Whole life annuities.

Example 5.67. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Suppose that $i = 6.5\%$. Calculate $\ddot{a}_{80}^{(12)}$, $a_{80}^{(12)}$ and $\overline{a}_{80}$ under UDD.

Remark. Each of insurance and annuity can be viewed as

(1) the time unit is a $\frac{1}{m}$ year rather than a year,

(2) the unit paid is not one but $1/m$,

(3) the discount factor is not v but $v^{(m)}$ $(= v^{1/m})$.

(4) $f_{K_x}(k) = {}_{k-1|}q_x$ is replaced by $f_{J_x^{(m)}}(k) = {}_{\frac{k-1}{m}|\frac{1}{m}}q_x$, where $J_x^{(m)} = k$ if $T_x \in (\frac{k-1}{m}, \frac{k}{m}]$.

We often linearly interpolate ℓ_x (UDD): $\quad \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$ for $t \in [0,1]$.

Solution : $\quad \ddot{Y}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{J_x^{(m)}-1}(v^{1/m})^k = \frac{1}{m}\frac{1-(v^{1/m})^{J_x^{(m)}}}{1-v^{1/m}}, \qquad \ddot{Y}_x = \frac{1-(v^{K_x})}{1-v}$

need to know $E(v^{J_x^{(m)}/m}) = \sum_{k=1}^{\infty} v^{k/m}\,{}_{\frac{k-1}{m}|\frac{1}{m}}q_x \quad m = ? \qquad (f_{J_x^{(m)}}(k) = {}_{\frac{k-1}{m}|\frac{1}{m}}q_x)$

$$
\begin{aligned}
&= \sum_{j=1}^{6}\sum_{h=1}^{m} v^{\frac{m(j-1)+h}{m}}\,{}_{\frac{m(j-1)+h-1}{m}|\frac{1}{m}}q_x && \text{UDD within each year}\\
&= \sum_{j=1}^{6}\sum_{h=1}^{m} v^{j-1+\frac{h}{m}}\frac{1}{m}\frac{\ell_{x+j-1}-\ell_{x+j}}{\ell_x} && ({}_{\frac{m(j-1)+h-1}{m}|\frac{1}{m}}q_x = \frac{{}_{j-1|}q_x}{m} = \frac{1}{m}\frac{d_{x+j}}{\ell_x},)\\
&= v^{-1}\frac{1}{m}\sum_{j=1}^{6}(\sum_{h=1}^{m} v^{\frac{h}{m}})v^j\frac{\ell_{x+j-1}-\ell_{x+j}}{\ell_x}\\
&= \frac{1}{mv}(\sum_{h=1}^{m}(v^{\frac{1}{m}})^h)\sum_{j=1}^{6} v^j\frac{\ell_{x+j-1}-\ell_{x+j}}{\ell_x} && (\sum_{j=1}^{6} v^j\frac{\ell_{x+j-1}-\ell_{x+j}}{\ell_x} = E(v^{K_x}))\\
&= \frac{1}{mv}(v^{1/m}\frac{1-(v^{1/m})^m}{1-v^{1/m}})A_x && (A_x = E(v^{K_x}) \approx 0.82 \quad (Ex.4.9))\\
&\approx \frac{1}{mv}v^{1/m}\frac{1-v}{1-v^{1/m}}0.82
\end{aligned}
$$

$$
\begin{aligned}
\ddot{a}_x^{(12)} &= \frac{1}{m}\frac{1-E(v^{J_x^{(m)}/m})}{1-v^{1/m}}\\
&\approx \frac{1}{m}\frac{1-\frac{1}{mv}v^{1/m}\frac{1-v}{1-v^{1/m}}0.82}{1-v^{1/m}}\Big|_{m=12,v=1/1.06} \approx 2.54. && a_{80}^{(12)} = \ddot{a}_{80}^{(12)} - 1/m \approx 2.48.\\
\overline{a}_{80} &= E(\frac{1-\overline{Z}_x}{\delta}) = \frac{1-\overline{A}_x}{\delta} \approx \frac{1-0.84}{\delta}, && \approx 2.502 \quad as
\end{aligned}
$$

$$
\overline{A}_x = \int v^t f_{T_x}(t)dt = \sum_{k=1}^{6}\int_{k-1}^{k} v^t f_{T_x}(t)dt = \sum_{k=1}^{6}\int_{k-1}^{k} v^t\frac{d_{x+k-1}}{\ell_x}dt = \sum_{k=1}^{6}\frac{v^{k-1}-v^k}{-\ln v}\frac{d_{x+k-1}}{\ell_x} \approx 0.84.
$$

Skip to 5.7.4.

Theorem 5.93.

5.7.2 Deferred annuities

Theorem 5.94. *Under a uniform distribution of deaths within each year,*

$$
\begin{aligned}
{}_{n}|\ddot{a}_x^{(m)} &= \frac{{}_nE_x - \frac{i}{i^{(m)}}\cdot {}_n|A_x}{d^{(m)}} = \frac{id}{i^{(m)}d^{(m)}}\cdot {}_n|\ddot{a}_x + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}\cdot {}_nE_x \\
&= \alpha(m)\cdot {}_n|\ddot{a}_x - \beta(m)\cdot {}_nE_x. \\
{}_n|a_x^{(m)} &= {}_n|\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x = \frac{v^{1/m}\cdot {}_nE_x - \frac{i}{i^{(m)}}\cdot {}_n|A_x}{d^{(m)}} \\
&= \frac{id}{i^{(m)}d^{(m)}}\cdot {}_n|a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}\cdot {}_nE_x, \\
{}_n|\overline{a}_x &= \frac{{}_nE_x - \frac{i}{\delta}\cdot {}_n|A_x}{\delta} = \frac{id}{\delta^2}\cdot {}_n|\ddot{a}_x + \frac{\delta - i}{\delta^2}{}_nE_x \\
&= \alpha(\infty)\cdot {}_n|\ddot{a}_x - \beta(\infty)\cdot {}_nE_x.
\end{aligned}
$$

Proof. For a deferred life annuity due, using that ${}_n|\ddot{a}_x^{(m)} = {}_nE_x\ddot{a}_{x+n}^{(m)}$ (see page 164), $\ddot{a}_{x+n}^{(m)} = \frac{1-A_{x+n}^{(m)}}{d^{(m)}}$ (see page 161), $A_x^{(m)} = \frac{i}{i^{(m)}}A_x$ (see page **??**), ${}_nE_xA_{x+n} = {}_n|A_x$ (see page 106), and ${}_n|\ddot{a}_x = \frac{{}_nE_x - {}_n|A_x}{d}$ (see page 146), we get that

$$
\begin{aligned}
&{}_n|\ddot{a}_x^{(m)} = {}_nE_x\ddot{a}_{x+n}^{(m)} = {}_nE_x\frac{1-A_{x+n}^{(m)}}{d^{(m)}} = {}_nE_x\frac{1-\frac{i}{i^{(m)}}A_{x+n}}{d^{(m)}} \\
=&\frac{{}_nE_x - \frac{i}{i^{(m)}}\cdot {}_n|A_x}{d^{(m)}} \\
=&\frac{{}_nE_x - \frac{i}{i^{(m)}}\cdot ({}_nE_x - d\cdot {}_n|\ddot{a}_x)}{d^{(m)}} \\
=&\frac{id}{i^{(m)}d^{(m)}}\cdot {}_n|\ddot{a}_x + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}\cdot {}_nE_x.
\end{aligned}
$$

For a deferred life annuity immediate, using that ${}_n|a_x^{(m)} = {}_n|\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x$ (see page 164), ${}_n|\ddot{a}_x^{(m)} = {}_nE_x\ddot{a}_{x+n}^{(m)}$ (see page 164), $\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}}$ (see page 161), $A_x^{(m)} = \frac{i}{i^{(m)}}A_x$ (see page **??**)

$$
\begin{aligned}
&{}_n|a_x^{(m)} = {}_n|\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x \\
=&{}_nE_x\ddot{a}_{x+n}^{(m)} - \frac{1}{m}{}_nE_x = {}_nE_x\frac{1-A_{x+n}^{(m)}}{d^{(m)}} - \frac{1}{m}{}_nE_x \\
=&{}_nE_x\frac{1-\frac{i}{i^{(m)}}A_{x+n}}{d^{(m)}} - \frac{1}{m}{}_nE_x = \frac{{}_nE_x(1-\frac{d^{(m)}}{m}) - {}_nE_x\frac{i}{i^{(m)}}A_{x+n}}{d^{(m)}} \\
=&\frac{v^{1/m}\cdot {}_nE_x - \frac{i}{i^{(m)}}\cdot {}_n|A_x}{d^{(m)}}
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{n}|a_x^{(m)} &= {}_nE_x a_{x+n}^{(m)} = {}_nE_x \left(\frac{id}{i^{(m)}d^{(m)}} a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}} \right) \\
&= \frac{id}{i^{(m)}d^{(m)}} \cdot {}_{n}|a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}} \cdot {}_nE_x
\end{aligned}
$$

For a deferred continuous life annuity, using that ${}_n|\overline{a}_x = {}_nE_x\overline{a}_{x+n}$ (see page 150), $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$ (see page 142), $\overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta}A^1_{x:\overline{n}|}$ (see page 133), ${}_nE_xA_{x+n} = {}_n|A_x$ (see page 106), and ${}_n|\ddot{a}_x = \frac{{}_nE_x - {}_n|A_x}{d}$ (see page 146), we get that

$$
\begin{aligned}
{}_n|\overline{a}_x &= {}_nE_x\overline{a}_{x+n} = {}_nE_x \frac{1-\overline{A}_{x+n}}{\delta} \\
&= {}_nE_x \frac{1-\frac{i}{\delta}A_{x+n}}{\delta} = \frac{{}_nE_x - \frac{i}{\delta}\cdot {}_n|A_x}{\delta} \\
&= \frac{{}_nE_x - \frac{i}{\delta}\cdot ({}_nE_x - d\cdot {}_n|\ddot{a}_x)}{\delta} = \frac{id}{\delta^2}\cdot {}_n|\ddot{a}_x + \frac{\delta - i}{\delta^2}\cdot {}_nE_x.
\end{aligned}
$$

■

5.7.3 Temporary annuities

Theorem 5.95. *Under a uniform distribution of deaths within each year,*

$$
\begin{aligned}
\ddot{a}^{(m)}_{x:\overline{n}|} &= \frac{1-{}_nE_x - \frac{i}{i^{(m)}}A^1_{x:\overline{n}|}}{d^{(m)}} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}|} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}(1-{}_nE_x), \\
a^{(m)}_{x:\overline{n}|} &= \frac{id}{i^{(m)}d^{(m)}}a_{x:\overline{n}|} + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}(1-{}_nE_x), \\
\overline{a}_{x:\overline{n}|} &= \frac{1-{}_nE_x - \frac{i}{\delta}A^1_{x:\overline{n}|}}{\delta} = \frac{id}{\delta^2}\ddot{a}_{x:\overline{n}|} + \frac{\delta - i}{\delta^2}(1-{}_nE_x).
\end{aligned}
$$

Proof. For a n–year temporary life annuity due, using that $\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$ (see page 162), $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + {}_nE_x$ (see page 106), $A^{(m)}_x = \frac{i}{i^{(m)}}A_x$ (see page **??**), $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ (see page 151), we get that

$$
\begin{aligned}
\ddot{a}^{(m)}_{x:\overline{n}|} &= \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}} = \frac{1-{A^1_{x:\overline{n}|}}^{(m)} - {}_nE_x}{d^{(m)}} \\
&= \frac{1-{}_nE_x - \frac{i}{i^{(m)}}A^1_{x:\overline{n}|}}{d^{(m)}} \\
&= \frac{1-{}_nE_x - \frac{i}{i^{(m)}}\left(1-d\ddot{a}_{x:\overline{n}|} - {}_nE_x\right)}{d^{(m)}} \\
&= \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}|} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}(1-{}_nE_x)
\end{aligned}
$$

For a n–year temporary life annuity immediate, using that $\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$ (see page 162), $A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$ (see page 106), $A^{(m)}_x = \frac{i}{i^{(m)}}A_x$ (see page **??**), $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ (see page 151), we get that

$$\begin{aligned}
&a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m}{}_nE_x \\
=&\frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}|} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \frac{1}{m} + \frac{1}{m}{}_nE_x \\
=&\frac{id}{i^{(m)}d^{(m)}}(1+a_{x:\overline{n}|}-{}_nE_x) + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \frac{1}{m}(1-{}_nE_x) \\
=&\frac{id}{i^{(m)}d^{(m)}}a_{x:\overline{n}|} + (1-{}_nE_x)\left(\frac{id}{i^{(m)}d^{(m)}} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}} - \frac{1}{m}\right).
\end{aligned}$$

We have that

$$\begin{aligned}
&\frac{id}{i^{(m)}d^{(m)}} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}} - \frac{1}{m} = \frac{id + i^{(m)} - i + \frac{i^{(m)}d^{(m)}}{m}}{i^{(m)}d^{(m)}} \\
=&\frac{i^{(m)}\left(1-\frac{d^{(m)}}{m}\right) - i(1-d)}{i^{(m)}d^{(m)}} \\
=&\frac{i^{(m)}v^{1/m} - iv}{i^{(m)}d^{(m)}} = \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}.
\end{aligned}$$

Hence,

$$a^{(m)}_{x:\overline{n}|} = \frac{id}{i^{(m)}d^{(m)}}a_{x:\overline{n}|} + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}(1-{}_nE_x).$$

For a n–year temporary life continuous annuity, using that $\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$ (see page 154), $\overline{A}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{n}|} + {}_nE_x$ (see page 106), $\overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta}A^{1}_{x:\overline{n}|}$ (see page 133), $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ (see page 151), we get that

$$\begin{aligned}
&\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta} = \frac{1-\overline{A}^{1}_{x:\overline{n}|} - {}_nE_x}{\delta} = \frac{1-{}_nE_x - \frac{i}{\delta}A^{1}_{x:\overline{n}|}}{\delta} \\
=&\frac{1-{}_nE_x - \frac{i}{\delta}\left(1-d\ddot{a}_{x:\overline{n}|} - {}_nE_x\right)}{d^{(m)}} = \frac{id}{\delta^2}\ddot{a}_{x:\overline{n}|} + \frac{\delta-i}{\delta^2}(1-{}_nE_x).
\end{aligned}$$

■

Connect to §5.7.1.

It is quite tedious to compute $\ddot{a}_x^{(m)}$ etc, as it depends on the assumptions on f_{T_x}. There are two approaches to approximate it, which leads to simpler formulas.

(1) Linear interpolation of the actuarial discount factor.

(2) Woolhouse's formula.

5.7.4 Linear interpolation of the actuarial discount factor.

v^t is t-year discount factor,

${}_tE_x = v^t \cdot {}_tp_x (= A_{x:\overset{1}{\overline{t|}}})$ is also called the actuarial (t-year) discount factor.

$\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$, $t \in [0,1]$ $----$ linear interpolation of ℓ_x or $S_{T(x)}(t)$;

${}_{k+t}E_x = (1-t)_kE_x + t \cdot {}_{k+1}E_x$, $t \in [0,1]$ $--$ linear interpolation of ${}_tE_x$

$$ {}_{k+\frac{j}{m}}E_x = (1-\frac{j}{m})_kE_x + \frac{j}{m}{}_{k+1}E_x \qquad (= {}_kE_x + \frac{j}{m}({}_{k+1}E_x - {}_kE_x)),\ j = 0,1,\ldots,m-1. \quad (1) $$

The actuarial discount factor ${}_tE_x$ appears in annuities computations:

$$ \ddot{a}_x^{(m)} = \frac{1}{m}\sum_{i=0}^{\infty} \underbrace{v^{\frac{i}{m}} \cdot {}_{\frac{i}{m}}p_x}_{{}_{\frac{i}{m}}E_x} = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1} \underbrace{v^{k+\frac{j}{m}} \cdot {}_{k+\frac{j}{m}}p_x}_{{}_{k+\frac{j}{m}}E_x}. \quad (2) $$

Theorem 5.96. *Assuming that ${}_{k+\frac{j}{m}}E_x$ is linear in j, then*

(I) $\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$, *(II)* $\ddot{a}_{x:\overline{n|}}^{(m)} = \ddot{a}_{x:\overline{n|}} - \frac{m-1}{2m}(1 - {}_nE_x)$, *(III)* ${}_{n|}\ddot{a}_x^{(m)} = {}_{n|}\ddot{a}_x - \frac{m-1}{2m} \cdot {}_nE_x$.

Letting $m \to \infty$ in Theorem 5.96 (see page 178), we get that:

Theorem 5.97. *Assuming that ${}_{k+t}E_x$ is linear in t, $0 \le t \le 1$,*

$$ \overline{a}_x = \ddot{a}_x - \frac{1}{2}, \qquad \ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} $$

$$ \overline{a}_{x:\overline{n|}} = \ddot{a}_{x:\overline{n|}} - \frac{1}{2}(1 - {}_nE_x), \qquad \ddot{a}_{x:\overline{n|}}^{(m)} = \ddot{a}_{x:\overline{n|}} - \frac{m-1}{2m}(1 - {}_nE_x) $$

$$ {}_{n|}\overline{a}_x = {}_{n|}\ddot{a}_x - \frac{1}{2} \cdot {}_nE_x \qquad {}_{n|}\ddot{a}_x^{(m)} = {}_{n|}\ddot{a}_x - \frac{m-1}{2m} \cdot {}_nE_x. $$

Remark. In reality, the linear assumption in the previous theorem may not be true, then both the RHS and LHS formulas are approximations.

5.7.5 Woolhouse's formula

The Euler-Maclaurin Formula:

$$\begin{aligned}\int_a^b g(x)dx =& h(\sum_{i=0}^{N} g(a+ih) - \frac{g(a)+g(b)}{2})\\ &+\sum_{j=1}^{k-1}\frac{B_{2j}}{(2j)!}h^{2j}(g^{(2j-1)}(a)-g^{(2j-1)}(b))-(b-a)\frac{B_{2k}}{(2k)!}h^{2k}g^{(2k)}(\xi),\ \text{where } k>1,\\ &Nh=b-a,\ B_i \text{ is the } i\text{-th Bernoulli number, } i.e.,\ B_1=1/6,\ B_2=1/30,\ ...\end{aligned}$$

$$\begin{aligned}\int_a^b g(x)dx =& h(\sum_{i=0}^{N} g(a+ih) - \frac{g(a)+g(b)}{2})\\ &+\frac{h^2}{12}(g'(a)-g'(b))-\frac{h^4}{720}(g''(a)-g''(b))+\cdots\end{aligned}$$

5.7.5.1. Woolhouse's formula:

$$\int_0^\infty g(x)dx = h(\sum_{i=0}^{\infty} g(ih)-\frac{g(0)}{2})+\frac{h^2}{12}g'(0)-\frac{h^4}{720}g'''(0)+\cdots,\quad \text{if } g^{(k)}(\infty)=0 \text{ for } k=0,\ 1,\ ...$$

$h=1/m$, and $m\in\{1,\ 2,\ 3,\ ...\}$. Its application to whole life annuity:

(1) $\overline{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta+\mu_x) \approx \ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}(\delta+\mu_x),$ [13] *it yields*

(2) $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta+\mu_x).$

Reason: $\overline{a}_x = \int_0^\infty v^t{}_tp_x dt = \int_0^\infty g(t)dt$, where
$g(t) = v^t{}_tp_x = e^{-\delta t}{}_tp_x.$
$g'(t) = -{}_tp_x\delta e^{-\delta t} - v^t({}_tp_x)' = -{}_tp_x\delta e^{-\delta t} - v^t{}_tp_x\mu_{x+t}$
$g'(0) = -(\delta+\mu_x)$ and
$g(0) = v^0{}_0p_x = 1.$
Letting $h=1$, then

$$\overline{a}_x = \int_0^\infty g(t)dt = \sum_{i=0}^\infty g(i) - \frac{1}{2} - \frac{1}{12}(\delta+\mu_x)+\cdots \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta+\mu_x).$$

Letting $h=\frac{1}{m}$, then

$$\begin{aligned}\overline{a}_x =& \frac{1}{m}(\sum_{i=0}^\infty g(i/m) - \frac{1}{2}) - \frac{1/m^2}{12}(\delta+\mu_x)+\cdots\\ \approx& \frac{1}{m}\sum_{i=0}^\infty v^{i/m}{}_{i/m}p_x - \frac{1}{2m} - \frac{1}{12m^2}(\delta+\mu_x)\\ =& \ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}(\delta+\mu_x).\end{aligned}$$

Its application to term annuity can be simplified by

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x - v^n {}_np_x \ddot{a}^{(m)}_{x+n}.$$

Example 5.68. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

with $i =$ 6.5%. *We already got* $\ddot{a}_{80} = 3.011654244$ *(from Ex. 5.6).*
(1) Calculate $\ddot{a}^{(12)}_{80}$ *assuming that the actuarial discount factor is linear.*
(2) Calculate $\ddot{a}^{(12)}_{80}$ *using Woolhouse's formula assuming* $\mu_x(t) = 0.05$, $t > 0$.

Solution: Using UDD (in Ex. 5.6 and Ex. 5.67), $\ddot{a}_{80} \approx 3.012$, and $\ddot{a}^{(12)}_{80} \approx 2.544$.

(1) (I) *in* $Th.5.96 \Rightarrow \quad \ddot{a}^{(12)}_{80} \approx \ddot{a}_x - \dfrac{m-1}{2m} = 3.011654244 - \dfrac{12-1}{(2)(12)} = 2.553.$

(2) $\ddot{a}^{(12)}_{80} \approx \ddot{a}_x - \dfrac{m-1}{2m} - \dfrac{m^2-1}{12m^2}(\delta + \mu_x) = \ddot{a}_x - \dfrac{m-1}{2m} - \dfrac{m^2-1}{12m^2}(\log(1+i) + \mu_x) \quad = 2.544$

B1. $P(g(K_x) = g(k)) = P(K_x = k) = f_{g(K_x)}(g(k)) = f_{K_x}(k)$

B3. (25 pts) You are given:
$\delta = 0.06$ and $\mu_x(t) = \begin{cases} 0.01 & \text{if } 0 \le t < 10; \\ 0.04 & \text{if } 10 \le t. \end{cases}$ Calculate $\overline{IA}_{x:\overline{15}|}$.

Ans. $\overline{A}_{x:\overline{15}|} = E(v^{T_x \wedge 15})$

$\overline{IA}_x = E(T_x v^{T_x})$

$\overline{IA}_{x:\overline{15}|} = E((T_x \wedge 15)v^{T_x \wedge 15}) = (\int_0^{10} + \int_{10}^{15} + \int_{15}^{\infty})(t \wedge 15)v^{t \wedge 15} f_{T_x}(t)dt.$

$S_{T_x}(t) = \exp(-\int_0^t \mu_x(y)dy) \neq \exp(-t\mu_x(t))$ if $\mu_x(t)$ is not constant.

$f_{T_x}(t) = -S'(t).$

$\int_{15}^{\infty}(t \wedge 15)v^{t \wedge 15} f_{T_x}(t)dt = \int_{15}^{\infty}(15)v^{15} f_{T_x}(t)dt.$

CHAPTER 6

Benefit Premiums

6.1 Funding a liability.

When an insurance takes an insuree it assumes a liability. Suppose that an insurer has to pay a liability consisting of a unique payment of L at time n and an effective annual rate of interest of i.

The **net single premium** of an insurance product is the APV of the benefit payments for this insurance product. Usually, insurance products are funded periodically while the contract is in hold. These payments are made while the individual is alive and the obligations of the contract are not expired. Payments made to fund an insurance contract are called **benefit premiums**, which usually are made annually. The **annual premium** (also called the **net annual premium** and the **benefit annual premium**) is the amount which an insurance company allocates to fund an insurance product. We usually consider funding as follows.

Definition 6.1. *An insurance product is funded according with the* **equivalence principle** *if the APV's of the funding scheme and of the contingent benefits agree.*

The annual premium found under the equivalence principle is the basis to asses an insurance product. Costs and commissions have to be taken into account to determine the contract to be offered to a customer. The value of each payment in an insurance contract is called a **contract premium** (v.s. the **annual benifit premium**, against the **net single premium**).

Definition 6.2. *The* **loss** *of an insurance product is the excess of the present value at issue of benefit payments over the present value of funding.*

The loss of an insurance contract is the present value at issue of the net outflow for this contract. The loss is a random variable. It refers to either with or without face value.

6.2 Fully discrete benefit premiums.

In this section, we will consider the funding of insurance products paid at the end of the year of death with annual benefits premiums made at the beginning of the year. The funding is made as far as the individual is alive and the term of the insurance has not expired.

6.2.1 Whole life insurance. (for the annual benifit premium)

Definition 6.3. *Let* $L_x = v^{K_x} - P\sum_{k=0}^{K_x-1} v^k$ *be the loss random variable (rv) for a unit whole life insurance paid at the end of the year of death (present value*$= 1\cdot v^{K_x}$*) funded with an annual benefit premium* ($P\sum_{k=0}^{K_x-1} v^k$) *at the beginning of the year while the individual is alive. This insurance contract is called a* **fully discrete whole life insurance**.

Remark. $L_x = v^{K_x} - P\sum_{k=0}^{K_x-1} v^k$ is used in definitions or theorems. In general, the payment is not a unit but B unit, then the loss is

$L_x = B(v^{K_x} - P\sum_{k=0}^{K_x-1} v^k)$ or $L_x = Bv^{K_x} - P_c\sum_{k=0}^{K_x-1} v^k$, where $P_c = BP$.

Under compound interest: $v = (1+i)^{-1} = 1-d = e^{-\delta}$, i is the annual effective rate of interest, δ is force of interest, v is the annual discount factor, d is the annual discount rate.

$L_x = v^{K_x} - P\sum_{k=0}^{K_x-1} v^k \downarrow$ in K_x. (L_x=whole life insurance - P$\times$ whole life annuity) ?
To make a profit, an insurer would like that insurees will die as late as possible.

Theorem 6.1. *For a fully discrete whole life insurance,*
(i) $L_x = v^{K_x} - P\sum_{k=0}^{K_x-1} v^k = Z_x - P\ddot{Y}_x = Z_x\left(1+\frac{P}{d}\right) - \frac{P}{d}$. $\quad (\ddot{Y}_x = \frac{1-v^{K_x}}{1-v})$
(ii) $E(L_x) = A_x - P\ddot{a}_x = A_x\left(1+\frac{P}{d}\right) - \frac{P}{d}$.
(iii) $\mathrm{V}(L_x) = \left(1+\frac{P}{d}\right)^2\left({}^2A_x - A_x{}^2\right)$.

Example 6.1. *Consider a fully discrete whole life insurance with face value \$10,000. The annual benefit premium paid at the beginning of each year which the insuree is alive is* \$46. *Suppose that the force of mortality is* 0.005*. The force of interest is* $\delta = 0.075$

(i) John entered this contract and died 10 years, 5 months and 5 days after the issue of this contract. Find the insurer's loss at the time of the issue of the policy.

(ii) Peter entered this contract and died 42 years, 2 months and 20 days after the issue of this contract. Find the insurer's loss at the time of the issue of the policy.

(iii) Calculate the probability that the loss at issue is positive.

Solution: $v = e^{-\delta} = e^{-0.075}$ and $S_{T_x}(t) = e^{-0.005t}$, $t > 0$. $L_x = Z_x - P\ddot{Y}_x$.
(i) $T_x = ?$ $\qquad K_x = \lceil 10+ \rceil = ?$

$$L_x = \underbrace{B}_{=10^4} v^{K_x} - \underbrace{P_c}_{=46}\sum_{k=0}^{K_x-1} v^k = (10^4 v^{11} - (46)\tfrac{1-v^{11}}{1-v})\Big|_{v=e^{-0.075}} = 4024.7.$$

(ii) $K_x = \lceil 42+ \rceil = ?$

$$L_x = (10^4 v^{43} - (46)\tfrac{1-v^{43}}{1-v})\Big|_{v=e^{-0.075}} = -213.75.$$

(iii) $0 < v^{K_x}(1+P/d) - P/d$ $(=L_x)$,
$P/d < v^{K_x}(1+P/d)$,
$\frac{P/d}{1+P/d} < v^{K_x}$, $=>$ $\ln\frac{P/d}{1+P/d} < K_x \ln v$,

$$K_x > \frac{\ln\left(\frac{P}{P+d}\right)}{\ln v} \approx 37.5 \text{ or } K_x < \frac{\ln\left(\frac{P}{P+d}\right)}{\ln v} \approx 37.5 \text{ ????} \tag{1}$$

$$P(L_x > 0) = \begin{cases} P(K_x < 37.5) = 1 - e^{-0.005\times 37.5}?? \\ P(K_x < 37.5) = 1 - e^{-0.005\times 38}?? \\ P(K_x < 37.5) = 1 - e^{-0.005\times 37}?? \end{cases} \qquad \textbf{Class Exercise.}$$

Example 6.2. *Questions:* $f_{T_x} = ?$ $f_{K_x} = ?$ $f_{L_x} = ?$

Solution: $f_{T_x}(t) = -S'_{T_x}(t) = \mu e^{-\mu t}$, $f_{K_x}(k) = P(T_x \in (k-1, k]) = e^{-\mu(k-1)} - e^{-\mu(k)}$,
$f_{L_x}(y) = P(L_x = y) = P(v^{K_x}(1+\frac{P}{d}) - \frac{P}{d} = y) = P(K_x = g(y)) = f_{K_x}(g(y))$, $g(y) = ??$
Class Exercise.

$\frac{\ln\frac{P+yd}{P+d}}{\ln v}$

Theorem 6.2. *The probability that the loss L_x is positive is ${}_{k_0}q_x$, where $k_0 = \left\lceil \frac{\ln\left(\frac{P}{P+d}\right)}{\ln v} \right\rceil - 1$, i.e., $k_0 < \frac{\ln\left(\frac{P}{P+d}\right)}{\ln v} \le k_0 + 1$, (see Eq.(1)).* due to $P(L_x > 0) = P(v^{K_x}(1+\frac{P}{d}) - \frac{P}{d} > 0) = \cdots$.

Definition 6.4. *The benefit premium of a fully discrete whole life insurance funded under the equivalence principle $E(L_x) = 0$ is $P_x = A_x/\ddot{a}_x$ from $E(L_x) = A_x - P_x\ddot{a}_x = 0$.*

Example 6.3. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

An 80–year old individual signs a whole life policy insurance which will pay \$50000 at the end of the year of his death. The insuree will make level benefit premiums at the beginning of the year while he is alive. Suppose that $i = 6.5\%$.
(i) The net single premium for this policy ?
(ii) Benefit annual premium for this policy using the equivalence principle ?
(iii) Skip
(iv) Find the df f_L of the loss L when the benefit premium follows the equivalence principle.
(v) Find the probability that the loss is positive.
(vi) Find the variance of the loss.

Solution: (i) The net single premium for this policy is
$(50000)A_{80} \approx 50000(0.81619) = 40809.5$, as
$A_{80} = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k {}_{k-1}|q_x = \sum_{k=1}^{\infty} v^k d_{x+k-1}/\ell_x \approx 0.81619$ (derived before).
(ii) The benefit annual premium for this policy using the equivalence principle ($P = \frac{A_x}{\ddot{a}_x}$) is
$P = (50000)P_{80} = 50000\frac{A_{80}}{\ddot{a}_{80}} \approx (50000)\frac{0.816}{3.012} \approx 13550.52822$ as $\ddot{a}_{80} \approx 3.012$ derived before.
Or $\ddot{a}_{80} = E(\sum_{k=0}^{K_x-1} v^k) = \frac{1-A_{80}}{1-v} = \frac{1-0.81619}{1-v} = 3.011654244.$
(iv) The loss is $L = 50000L_x$, where $L_x = v^{K_x} - P_x\frac{1-v^{K_x}}{1-v}$.

$$L = (50000)v^{K_{80}} - (13550.53)\frac{1-v^{K_{80}}}{1-v} = (272020.19)v^{K_{80}} - 222020.19 = g(K_x).$$

$$f_{L_x}:\ \mathbb{P}\{L = (272020.19)v^k - 222020.19\} = P(K_x = k) = \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}},$$

$$\mathbb{P}\{L = 33397.83\} = \frac{250-217}{250} = \frac{33}{250}, \qquad = P(K_x = 1)$$
$$\mathbb{P}\{L = 17808.94\} = \frac{217-161}{250} = \frac{56}{250}, \qquad = P(K_x = 2)$$
$$\mathbb{P}\{L = 3171.48\} = \frac{161-107}{250} = \frac{54}{250}, \qquad = P(K_x = 3)$$
$$\mathbb{P}\{L = -10572.62\} = \frac{107-62}{250} = \frac{45}{250}, \qquad = P(K_x = 4)$$
$$\mathbb{P}\{L = -23477.87\} = \frac{62-28}{250} = \frac{34}{250}, \qquad = P(K_x = 5)$$
$$\mathbb{P}\{L = -35595.47\} = \frac{28-0}{250} = \frac{28}{250}. \qquad = P(K_x = 6)$$

$f_L(3171.48) = f_{K_x}(?)$

(v) By (iv), $P(L>0)=\frac{33}{250}+\frac{56}{250}+\frac{54}{250}=0.572$.
Alternatively, formula: $P(L>0)={}_{k_o}q_x$, where

$$k_0=\left\lceil\frac{\ln\left(\frac{P}{P+d}\right)}{\ln v}\right\rceil-1=\left\lceil\frac{\ln\left(\frac{13550.52822/50000}{(13550.52822/50000)+(0.065/1.065)}\right)}{-\ln(1.065)}\right\rceil-1\approx\lceil 3.2\rceil-1=3.$$

$P(L>0)={}_3q_x=\frac{250-107}{250}=0.572$.

(vi) **3 ways for $V(L)$ here:**
(1) $L=v^{K_x}-P\frac{1-v^{K_x}}{1-v}=Z_x(1+\frac{P}{d})-\frac{P}{d}$ and $V(L)=(A_x(v^2)-(A_x(v))^2)(1+P/d)^2$.
(2) $V(L)=E(L^2)$ by f_L, **why not $V(L)=E(L^2)-(E(L))^2$?**

$$\begin{aligned}E[L^2]=&(33397.8286)^2\frac{33}{250}+(3171.4767)^2\frac{56}{250}+(17808.9352)^2\frac{54}{250}\\&+(-10572.6158)^2\frac{45}{250}+(-23477.8670)^2\frac{34}{250}+(-35595.4738)^2\frac{28}{250}=457444048.\end{aligned}$$

(3) The third way is the next theorem, which is not important.

Theorem 6.3. *If a fully discrete whole life insurance is funded using the equivalence principle, then* $P_x=\frac{A_x}{\ddot a_x}=\frac{dA_x}{1-A_x}=\frac{1}{\ddot a_x}-d$ *and* $\mathrm{V}(L_x)=\frac{{}^2A_x-{A_x}^2}{(1-A_x)^2}=\frac{{}^2A_x-{A_x}^2}{(d\ddot a_x)^2}$.

About midterm B1. See Example 6.3 above.

B3. $\delta=0.06$ and $\mu_x(t)=\begin{cases}0.01 & \text{if } 0\le t<10\\ 0.04 & \text{if } 10\le t.\end{cases}$ $\overline{IA}_{x:\overline{15}|}=E([T_x\wedge 15]v^{T_x\wedge 15})$

Hint: $S(t)=exp(-\int_0^t\mu(x)dx)$, is not Exp($\mu$) and $\int_0^\infty=\int_0^{10}+\int_{10}^{15}+\int_{15}^\infty$.

Example 6.4. Skip this example. Quiz on Dec. 4: 450: 13, 18, 19

Example 6.5. *Michael is 50 years old and purchases a whole life insurance policy with face value of \$100,000 payable at the end of the year of death. This policy will be paid by level benefit annual premiums at the beginning of each year while Michael is alive. Assume that $i=6\%$ and death is modeled using de Moivre's model with terminal age 100.*
(i) The net single premium for this policy ?
(ii) The benefit annual premium for this policy ?
(iii) The variance of the present value of the loss for this insurance contract ?

Solution: (i) The net single premium $E(Z)=?$ where $Z=10^5Z_x=10^5v^{K_x}$.
$E(Z)=10^5\sum_k v^kf_{K_x}(k)=10^5A_x=10^5\sum_{k=1}^{w-x}v^k\frac{1}{w-x}=10^5\frac{v}{50}\frac{1-v^{50}}{1-v}\approx 0.31524\times 10^5$.
(ii) The benefit annual premium for this policy is 10^5P_x

$$A_x-P_x\ddot a_x=0 => P_x=A_x/\ddot a_x=\left.\frac{A_x}{\frac{1-A_x}{1-v}}\right|_{A_x\approx 0.32,v=1/1.06}\approx 0.0260581 => \quad 10^5P_x\approx 2605.81.$$

(iii) $L_x=B(Z_x-P_x\frac{1-Z_x}{1-v})=B(Z_x(1+P_x/d)-P_x/d)$ and thus
$V(L_{50})=(100000)^2(A_x(v^2)-{A_x}^2)(1+P_x/d)^2=880929379.5$, as $A_x(v)$ is $\frac{v}{50}\frac{1-v^{50}}{1-v}$.

Theorem 6.4. *Under constant force of mortality μ for life insurance funded through the equivalence principle, $P_x=vq_x$.* *(No need to remember, just derive it).*

Besides the equivalence principle, there are other ways to determine annual benefit premiums. Unless said otherwise, we will assume that the equivalence principle is used. Often, the annual benefit premium in an insurance contract is bigger than the annual benefit premium obtained using the equivalence principle. The **risk charge** (or **security loading**) is the excess of the benefit annual premium over the benefit annual premium found using the equivalence principle.

3 types of problems:
(1) P_x or $BP_x = ?$
(2) Variance of L_x ?
(3) Percentile or probabililty related to L_x or P_x etc.

Definition 6.5. *The* **100α–th percentile annual premium** *for an insurance product is the largest premium making the probability that a loss results is less than or equal to α.*

The percentile annual premium can be found using either
(1) only one policy (see Example 6.6) or
(2) an aggregate of policies (see Example 6.7).

Example 6.6. *Michael is 50 year old and purchases a whole life insurance policy with face value of \$100,000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Michael is alive. Assume that $i = 6\%$ and death is modeled using de Moivre's model with terminal age 100. Find*
(i) the benefit annual premium if the probability of a loss is at most 0.25;
(ii) the benefit annual premium using the equivalence principle;
(iii) P(a positive loss for the benefit annual premium using the equivalence principle).

Solution: (i) Solve BP_α, $\alpha = 0.25$, $B = ?$ $P_\alpha = ?$

Solve for $P = P_\alpha$ from $v^{k_\alpha} - P\frac{1-v^{k_\alpha}}{1-v} = 0$, where k_α satisfies

$$\mathbb{P}(L_x > 0) = \mathbb{P}(K_x < k_\alpha) \le \alpha < \mathbb{P}(K_x \le k_\alpha), \text{ and } L_x = v^{K_x} - P\frac{1 - v^{K_x}}{1 - v}. \tag{1}$$

Key steps: (a) Solve for k_α from $v^{k_\alpha} - P\frac{1-v^{k_\alpha}}{1-v} = 0$ and Eq. (1),
(b) Setting $L_x = v^{k_\alpha} - P_\alpha\frac{1-v^{k_\alpha}}{1-v} = 0$ yields $P_\alpha = v^{k_\alpha}\frac{1-v}{1-v^{k_\alpha}}$.

Eq. (1) yields integer k_α

$$\mathbb{P}(T_x \le k_\alpha - 1) = \mathbb{P}(K_x < k_\alpha) \le \alpha < \mathbb{P}(K_x \le k_\alpha) = \mathbb{P}(T_x \le k_\alpha). \tag{2}$$

Now two possibilities: $\alpha = 0.25 = \mathbb{P}\{T_{50} \le t\} = \frac{t}{50}$ **Why ??** $=> t = 12.5$??
Or $\alpha = 0.25 = \mathbb{P}\{T_{50} \le k_\alpha\} = \frac{k_\alpha}{50} => k_\alpha = 12.5$?? **Class exercise.**
Hence, by Eq. (2), $\mathbb{P}\{K_{50} < 13\} \le \mathbb{P}(T_{50} \le 12.5) = 0.25 < \mathbb{P}\{K_{50} \le 13\}$.
(a) $k_{0.25} = 13$ and (b) $P_{0.25} = \frac{v^{k_\alpha}(1-v)}{1-v^{k_\alpha}}\Big|_{k_\alpha=13, v=1/1.06} = 0.04996236$.
The benefit annual premium is $BP_{0.25} = 10^5(0.04996236) = 4996.236$.
(ii) equivalent principle: $BP = ?$ $\qquad A_x = P\ddot{a}_x. => P = A_x/\ddot{a}_x$.
$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \frac{1-A_x}{1-v}$ **?????** and $A_{50} = \sum_{k=1}^{w-x} v^k \frac{1}{w-x} = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x} = 0.315237$.
$P = \frac{A_x}{\ddot{a}_x} = \frac{A_{50}}{(1-A_{50})/(1-v)} = 0.0260581$

The benefit annual premium is $BP = 2605.81$

$$<< BP_{0.25} \approx 4996.$$

(iii) Find $\mathbb{P}$(the loss random variable is positive) with P in (ii):

$$\mathbb{P}\left\{L_x = v^{K_{50}} - P\tfrac{1-v^{K_{50}}}{1-v} > 0\right\} = ? \quad (\text{P}\mathbf{= 4996 \ from \ (i) \ or \ 0.026 \ or} \ 2605.81 \ \mathbf{??})$$

$$\mathbb{P}\left\{L_x = v^{K_{50}} - P\frac{1-v^{K_{50}}}{1-v} > 0\right\} = \mathbb{P}\left\{K_{50} < \frac{\ln\frac{P/(1-v)}{1+P/(1-v)}}{\ln v}\Bigg|_{v=\frac{1}{1.06},P=0.026}\right\} \qquad by\ Ex.6.1$$

$$=\mathbb{P}\{K_{50} < 19.81210743\} \quad (= \mathbb{P}\{K_{50} \le 19\} \ \mathbf{or} \quad \mathbb{P}\{K_{50} \le 20\} \ \mathbf{??} \ )$$

$$=\mathbb{P}\{K_{50} \le 19\} = \mathbb{P}\{T_{50} \le 19\} = \frac{19}{50} = 0.38 \qquad\qquad > 0.25.$$

Example 6.7. *An insurance company offers a whole life insurance to lives aged* 20 *paying 75000 at the end of the year of death. Each insuree will make an annual premium of P at the beginning of year while he is alive to fund this insurance. Suppose that* 1000 *policyholders enter this insurance product. Use $i = 6\%$ and the life table D.2 (p.605) to calculate P so that the probability that the aggregate loss is positive is less than or equal to* 0.01.

Solution: Solve P based on $BL_x = BZ_x - P\sum_{k=0}^{K_x-1} v^k$, where B= ?
CLT: $\mathbb{P}(\frac{\overline{X}-\mu_X}{\sigma_{\overline{X}}} \le t) = \mathbb{P}(\frac{\sum_{i=1}^n \mathcal{L}_i - E(\sum_{i=1}^n \mathcal{L}_i)}{\sqrt{V(\sum_{i=1}^n \mathcal{L}_i)}} \le t) \approx \Phi(t) = 0.99$, where $t = 2.32$, $n = 10^3$,

$\mathcal{L}_1$, ..., $\mathcal{L}_n$ are i.i.d. from $\mathcal{L}$ ($= B\text{Ł}_x$). *i.e.*, $\mathbb{P}(\sum_{i=1}^n \mathcal{L}_i \le \underbrace{nE(\mathcal{L}) + \overbrace{2.32}^{=t}\sqrt{nV(\mathcal{L})}}_{=0}) \approx 0.99$.

Key : Solve for P, the 99th percentile of $\sum_{i=1}^n \mathcal{L}_i$, from

$$nE(\mathcal{L}) + 2.32\sqrt{nV(\mathcal{L})} = 0 \tag{1}$$

$$\mathcal{L} = 75000L_x = 75000Z_x - P\sum_{k=0}^{K_x-1} v^k = 75000Z_x - P\frac{1-Z_x}{1-v} \tag{2}$$

$$= \left(75000 + \frac{P}{1-v}\right)Z_x - \frac{P}{1-v} \qquad = aZ_x + b \tag{3}$$

$$nE[\mathcal{L}] = n[75000A_x - P\frac{1-A_x}{1-1/1.06}] \text{ (Table 7.2: } A_{20} = 0.05246 \text{ and } {}^2A_{20} = 0.01078\text{).} \qquad by\ (2)$$

$$= 3934500 - 16739.87333P,$$

$$n\text{V}(\mathcal{L}) = na^2V(Z_x) \qquad by\ (3)$$

$$= \underbrace{n}_{=?}\underbrace{(A_x(v^2) - (A_x(v))^2)}_{=?}a^2 \approx (8.03)\left(75000 + \frac{1.06P}{0.06}\right)^2.$$

Eq. (1) yields $0 = 3934500 - 16739.87P + (2.32)\sqrt{8.03}\left(75000 + \frac{1.06P}{0.06}\right) \qquad => P \approx 266.42.$

Example 6.8. *An insurer offers a fully discrete whole life insurances of \$10,000 on independent lives age* 30*, you are given:*

(i) $i = 0.06$
(ii) Mortality follows the life table C4.
(iii) Annual contract premium for a policy is $1.25P_x$.
Calculate the minimum number of policies the insurer must issue so that the probability that the aggregate loss for the issued policies is approximately less than 0.05.

Solution: CLT: $\mathbb{P}(\frac{\overline{X}-\mu_X}{\sigma_{\overline{X}}} \le t) = \mathbb{P}(\frac{\sum_{i=1}^n \mathcal{L}_i - E(\sum_{i=1}^n \mathcal{L}_i)}{\sigma_{(\sum_{i=1}^n \mathcal{L}_i)}} \le t) \approx \Phi(t) = 0.95$, where $\mathcal{L}_1$, ..., $\mathcal{L}_n$ are i.i.d. from the loss rv $\mathcal{L}$, $t = 1.645$, $n =$ **???** That is,

$$\mathbb{P}(\sum_{i=1}^n \mathcal{L}_i \le nE(\mathcal{L}) + t\sqrt{nV(\mathcal{L})}) \approx 0.95.$$

Key : Solve for n through the 95th percentile of $\sum_{i=1}^n \mathcal{L}_i$, from Eq. (1).

$$nE(\mathcal{L}) + 1.645\sqrt{nV(\mathcal{L})} = 0 \tag{1}$$

Now $\mathcal{L} = BL_x = 10^4 v^{K_x} - \pi \ddot{a}_{\overline{K_x}|}$, where $\pi = 1.25P_x$ is the annual contract premium,

$$\begin{aligned}
E(\mathcal{L}) =& 10^4 A_x - \pi \frac{1-A_x}{1-v} = (10^4 + \frac{\pi}{d})A_x - \frac{\pi}{d} \\
& \pi = 1.25P_x = 1.25B\frac{A_x}{\ddot{a}_x} && P_x = B\tfrac{A_x}{\ddot{a}_x},\ B = 10^4, \\
& A_x = 0.082295 \ \ and \ \ {}^2A_x = 0.0180 \ \ by \ \ Table 7.2 \\
& \ddot{a}_x = \frac{1-A_x}{1-v} = 16.213 \\
& \pi = (1.25)(10000)\frac{A_x}{\ddot{a}_x} = 63.44831308 \\
nE[\mathcal{L}] =& nE[(10000)L_x] = n(10000)A_x - \pi\ddot{a}_x \approx -205.7375n, \\
nV(\mathcal{L}) =& n\mathrm{V}((10000)L_x) = n\left(10000 + \frac{\pi}{d}\right)^2 ({}^2A_x - A_x^2) \approx n1384729.716,
\end{aligned}$$

Eq.(1) => $-205.7375n + (1.645)\sqrt{n(1384729.716)} = 0$, $\quad an + b\sqrt{n} = 0 \Rightarrow \sqrt{n} = (-b/a)$

$$n = \frac{(1.645)^2(1384729.716)}{(205.7375)^2} = 88.51504549? \ \ n = 88? \ \ n = 89?$$

Suppose that the funding scheme is limited to the first t years. The present value of the loss with unit payment (w.u.p.) is

$$L = v^{K_x} - P\ddot{a}_{\overline{\min(K_x,t)}|} = Z_x - P\ddot{Y}_{x:\bar{t}|},$$

and its APV is $A_x - P\ddot{a}_{x:\bar{t}|}$.

Definition 6.6. *The benefit premium for a fully discrete whole life insurance funded for the first t years that satisfies the equivalence principle is denoted by* ${}_tP_x$ $(= \frac{A_x}{\ddot{a}_{x:\bar{t}|}})$.

Example 6.9. *Ethan is 30 years old and purchases a whole life insurance policy with face value of \$50000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of the next 30 years while Ethan is alive. Assume that* $\delta = 0.05$ *and death is modeled using the constant force of mortality* $\mu = 0.03$*. Find the benefit annual premium for this policy.*

Solution: Find $50000P$ such that $A_{30} = P\ddot{a}_{x:\overline{30}|}$. => $50000P = 50000A_{30}/\ddot{a}_{30:\overline{30}|}$.

$A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k (S_{T_x}(k-1) - S_{T_x}(k))$
$= \sum_{k=1}^{\infty} v^k (e^{-(k-1)\mu} - e^{-k\mu})$
$= \sum_{k=1}^{\infty} v^k (e^{-k\mu+\mu} - e^{-k\mu})$
$= \sum_{k=1}^{\infty} v^k e^{-k\mu}(e^{\mu} - 1)$
$= (e^{\mu} - 1)\sum_{k=1}^{\infty}(t)^k = (e^{\mu} - 1)t\frac{1-t^{\infty}}{1-t}$ $\quad t = ??$
$= (e^{\mu} - 1)\frac{ve^{-\mu}}{1-ve^{-\mu}}\Big|_{v=e^{-0.05},\mu=0.03} \approx 0.366.$

$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{1-v} = \sum_{k=0}^{n-1} v^k {}_kp_x$. **Which is better here ?**

$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = \sum_{k=0}^{n-1} v^k e^{-\mu k} = \sum_{k=0}^{n-1} t^k = \frac{1-(ve^{-\mu})^n}{1-(ve^{-\mu})}\Big|_{v=e^{-0.05},\mu=0.03,n=30} \approx 11.827.$

Thus $50000P = 50000A_{30}/\ddot{a}_{30:\overline{30}|} \approx \frac{(50000)(0.366)}{11.827} \approx 1545.89.$

Plan	Loss
Whole life insurance	$Z_x - P\ddot{Y}_x$
t–year funded whole life insurance	$Z_x - P\ddot{Y}_{x:\overline{t}|}$
n–year term insurance	$Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}$
t–year funded n–year term insurance	$Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}$
n–year pure endowment insurance	$Z_{x:\overline{n}|}^{\ \ 1} - P\ddot{Y}_{x:\overline{n}|}$
t–year funded n–year pure endowment insurance	$Z_{x:\overline{n}|}^{\ \ 1} - P\ddot{Y}_{x:\overline{t}|}$
n–year endowment	$Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}$
t–year funded n–year endowment insurance	$Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}$
n–year deferred insurance	${}_{n}|Z_x - P\ddot{Y}_x$
t–year funded n–year deferred insurance	${}_{n}|Z_x - P\ddot{Y}_{x:\overline{t}|}$

Table 6.1: Loss in the fully discrete case

6.2.2 n–year term insurance.

An n–year term insurance paid at the end of the year of death funded at the beginning of the year while the insuree is alive is a called a fully discrete n–year term insurance.

Definition 6.7. *The loss r.v. for a fully discrete n–year term insurance w.u.p. is denoted by* $L^1_{x:\overline{n}|}$ $(= Z^1_{x:\overline{n}} - P\ddot{Y}_{x:\overline{n}|} = v^{K_x} I(K_x \le n) - P\sum_{k=0}^{K_x \wedge n-1} v^k)$.

Definition 6.8. *The benefit premium for a fully discrete n–year term insurance obtained using the equivalence principle is denoted by* $P^1_{x:\overline{n}|}$ *or* $P(A^1_{x:\overline{n}|})$ $(= \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}})$.

Example 6.10. Skip the example.

Example 6.11. *An insurer offers a four–year life insurance of 25 years old. Mortality is given by the table:*

x	25	26	27	28
q_x	0.01	0.02	0.03	0.04

This life insurance is funded by benefit payments at the beginning of the year. The benefit payment is 10000. $i = 5\%$. Calculate the annual benefit premium using the equivalence principle.

Solution: Solve $BP^1_{25:\overline{4}|}$. $B = ?$

Equivalent principle => $P^1_{25:\overline{4}|} = \frac{A^1_{25:\overline{4}|}}{\ddot{a}_{25:\overline{4}|}}$.

$$\begin{aligned}
A^1_{25:\overline{4}|} =& E(v^{K_x}\mathbf{1}(K_x \le n)) = \sum_{k=1}^{4} v^k f_{K_x}(k) = \sum_{k=1}^{4} v^k \prod_{j\ge 0}^{k-2} p_{x+j}q_{x+k-1} \qquad \text{by [8]}\\
=& v^1 q_{x+1-1}\prod_{j\ge 0}^{1-2} p_{x+j} + v^2 q_{x+2-1}\prod_{j\ge 0}^{2-2} p_{x+j} + v^3 q_{x+3-1}\prod_{j\ge 0}^{3-2} p_{x+j} + v^4 q_{x+4-1}\prod_{j\ge 0}^{4-2} p_{x+j}\\
=& (1.05)^{-1}(0.01)(1) + (1.05)^{-2}(0.02)(0.99) + (1.05)^{-3}(0.03)(0.99)(0.98)\\
&+ (1.05)^{-4}(0.04)(0.99)(0.98)(0.97) = 0.08359546485.
\end{aligned}$$

Now two ways for $\ddot{a}_{25:\overline{4}|}$:

(1) by [17] $\ddot{a}_{25:\overline{4}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$, where ${}_1p_x = 1 - q_x$, ${}_2p_x = p_x p_{x+1}$, ...

(2) $\ddot{a}_{25:\overline{4}|} = \frac{1-A_{25:\overline{4}|}}{d}$ due to $\ddot{Y}_{x:\overline{n}|} = \frac{1-Z_{x:\overline{n}|}}{d}$ [17]. **Which is better ??**

$$\begin{aligned}
A_{25:\overline{4}|} =& A^1_{25:\overline{4}|} + v^4 {}_4p_{25} \qquad \text{by [14]}\\
=& A^1_{25:\overline{4}|} + v^4 p_{25}p_{26}p_{27}p_{28} \qquad \text{by [4]}\\
=& A^1_{25:\overline{4}|} + (1.05)^{-4}(0.99)(0.98)(0.97)(0.96) = 0.8268662132,\\
\ddot{a}_{25:\overline{4}|} =& \frac{1 - A_{25:\overline{4}|}}{d} = 3.635809523,\\
BP^1_{25:\overline{4}|} =& (10000)\frac{A^1_{25:\overline{4}|}}{\ddot{a}_{25:\overline{4}|}} = (10000)\frac{0.08359546485}{3.635809523} = 229.9225641.
\end{aligned}$$

Theorem 6.5. $L^1_{x:\overline{n}|} \equiv v^{K_x} I(K_x \le n) - P\ddot{a}_{\overline{K_x \wedge n}|}$

$L^1_{x:\overline{n}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|} = \left(1 + \frac{P}{d}\right) Z^1_{x:\overline{n}|} + \frac{P}{d} Z_{x:\overline{n}|}^{\ \ 1} - \frac{P}{d}$ *(for computing variance)*

.

Example 6.12.

Example 6.13.

Example 6.14.

Example 6.15.

Monday lecture starts from this page. Announcement:
A. Quiz on Wednesday: 450 formulae [13], [17]-[19]
B. In final, Part A is all formulae [1]-[19] for 450.
C. 2 out of 3 problems in Final will be similar to the homeworks assigned on Friday.

Example 6.12. William is 40 years old and purchases a 25–year term life insurance policy with face value of \$150000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of the next 25 years while William is alive. Assume that $i = 6.5\%$ and death is modeled using de Moivre's model with terminal age 90.
(i) Calculate the benefit annual premium for this policy using the equivalence principle.
(ii) Calculate the standard deviation of the loss for this policy.

Solution: (i) Solve $BP^1_{40:\overline{25}|} = B\frac{A^1_{40:\overline{25}|}}{\ddot{a}_{40:\overline{25}|}}$ due to $L = Z^1_{40:\overline{25}|} - P\ddot{Y}_{40:\overline{25}|}$. $B = ?$
Formula: $\ddot{a}_{40:\overline{25}|} = \frac{1-A_{40:\overline{25}|}}{d} = \sum_{k=0}^{n-1} v^k {}_kp_x$ **???** [14] $A_{40:\overline{25}|} = E(v^{K_x \wedge n})$. $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\;\;1}$.

$$A^1_{40:\overline{25}|} = E(v^{K_x} I(K_x \le n)) = \sum_{k=1}^{n} v^k \frac{1}{w-x} = \frac{v}{50}\frac{1-v^{25}}{1-v} = 0.2439575345. \qquad (2)$$

$$A_{40:\overline{25}|} = E(Z^1_{40:\overline{25}|} + Z_{x:\overline{25}|}^{\;\;1}) = A^1_{40:\overline{25}|} + v^n {}_np_x = A^1_{40:\overline{25}|} + v^n \frac{50-n}{50} = 0.3475265409, \quad (1)$$

$$\ddot{a}_{40:\overline{25}|} = \frac{1 - A_{40:\overline{25}|}}{d} = \frac{1 - 0.3475265409}{1 - (1/1.065)} = 10.69052668$$

The benefit annual premium for this policy is $BP^1_{40:\overline{25}|} = 150000\frac{A^1_{40:\overline{25}|}}{\ddot{a}_{40:\overline{25}|}} = 3422.996011$.
(ii) $\sigma_L = ?$ where $L = BL^1_{x:\overline{n}|}$, [19]: $L^1_{x:\overline{n}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|} = v^{K_x} I(K_x \le n) - P\sum_{i=0}^{(K_x \wedge n)-1} v^i$.
$\sigma_L = B\sigma_{L^1_{x:\overline{n}|}}$. $\sigma_{L^1_{x:\overline{n}|}} = ?$
$L^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n) - P\frac{1-v^{K_x \wedge n}}{1-v}$ $\qquad$ $P = 3423$ or $\frac{3423}{150000}$?
$= v^{K_x} I(K_x \le n) - P\frac{1-(v^{K_x} I(K_x \le n) + v^n I(K_x > n))}{1-v}$
$= v^{K_x} I(K_x \le n)(1 + \frac{P}{1-v}) + \frac{P}{1-v} v^n I(K_x > n) - \frac{P}{1-v}$

$V(L^1_{x:\overline{n}|}) = V(aX + bY + c) = a^2 V(X) + b^2 V(Y) + 2abCov(X, Y)$, $\qquad (X, Y, a, b, c) = ?$
$Cov(X, Y) = E(XY) - E(X)E(Y) = -E(X)E(Y)$, as $E(Z^1_{x:\overline{n}|} Z_{x:\overline{n}|}^{\;\;1}) = 0$ (see [14]),

$$\mathrm{V}(L^1_{40:\overline{25}|}) = \left(1 + \frac{P}{d}\right)^2 [A^1_{x:\overline{n}|}(v^2) - (A^1_{x:\overline{n}|})^2] + (\frac{Pv^n}{d})^2 {}_np_x(1 - {}_np_x) + 0 - 2(1 + \frac{P}{d})\frac{P}{d} A^1_{x:\overline{n}|} A_{x:\overline{n}|}^{\;\;1}$$

$${}^2A^1_{40:\overline{25}|} = A^1_{40:\overline{25}|}(v^2) = \frac{v}{50}\frac{1-v^{25}}{1-v}\Big|_{v=1/1.06^2} = 0.1426103697, \qquad (by\ (2))$$

$$A_{x:\overline{n}|}^{\;\;1} = v^n {}_np_x, \quad P = \frac{3423}{150000} \text{ and } d = 1 - 1/1.065.$$

The standard deviation of the loss for this policy is $\sigma_L = B\sqrt{V((L^1_{40:\overline{25}|})} = 54578.29029$.

Theorem 6.6. *(skip)*.$\mathrm{V}(L^1_{x:\overline{n}|}) = (1+\frac{P}{d})^2 \cdot {}^2A^1_{x:\overline{n}|} + \frac{P^2}{d^2} \cdot {}^2A_{\underset{x:\overline{n}|}{1}} - (E[L^1_{x:\overline{n}|}] + \frac{P}{d})^2.$

The present value of the loss for a t–year $(1 \le t \le n)$ funded n–year term insurance w.u.p. is

$v^{K_x} I(K_x \le n) - P\ddot{a}_{\overline{\min(K_x,t)}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}$, with its APV $A^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{t}|}$.

The benefit premium which satisfies the equivalence principle is ${}_tP^1_{x:\overline{n}|} = P({}_tA^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}$.

Example 6.16.

Example 6.17. *A 20–year term life insurance policy to (x) with face value of \$10000 payable at the end of the year of death is funded by a level benefit annual premium at the beginning of the next 10 years while (x) is alive. Assume that $\delta = 0.05$ and death is modeled using the constant force of mortality $\mu = 0.02$. Find the benefit annual premium for this policy.*

Solution: Solve P, where $L = 10^5 Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}$.

$$P = (10000)\frac{A^1_{x:\overline{20}|}}{\ddot{a}_{x:\overline{10}|}} \qquad \textit{by } E(L) = 0$$

$$\begin{aligned}
A^1_{x:\overline{20}|} &= \sum_{k=1}^{n} v^k f_{K_x}(k) = \sum_{k=1}^{n} v^k (p_x)^{k-1} q_x \quad (or\ (= \sum_{k=1}^{n} v^k (e^{-\mu(k-1)} - e^{-\mu k})) \\
&= \sum_{k=1}^{n} v^k (p_x)^k \frac{q_x}{p_x} = vp_x \frac{1-(vp_x)^n}{1-vp_x}\frac{q_x}{p_x} = 0.2099039122, \\
\ddot{a}_{x:\overline{10}|} = \frac{1-A_{x:\overline{10}|}}{1-v} &= \sum_{k=0}^{n-1} v^k {}_kp_x \quad \textbf{(compare to U(0,b) !!)} \qquad {}_kp_x = e^{-\mu k} = p_x^k \\
&= \sum_{k=0}^{n-1} v^k p_x^k = \frac{1-(vp_x)^n}{1-vp_x} = \frac{1-e^{-(10)(0.05+0.02)}}{1-e^{-(0.05+0.02)}} = 7.446282211, \\
P =& (10000)\frac{A^1_{x:\overline{20}|}}{\ddot{a}_{x:\overline{10}|}} = 281.8908903.
\end{aligned}$$

Example 6.18.

6.3 Benefits paid annually funded continuously.

6.3.1 Whole life insurance.

Example 6.19. *Rita is 52 years old and purchases a whole life insurance policy with face value of \$70000 paid at the end of the year of death. This policy will be paid with a continuous lifetime payment. Assume that $i = 0.065$ and death is modeled using the de Moivre model with terminal age 95. Find the net single premium and the benefit annual premium for this policy.*

Plan	Loss
Whole life insurance	$Z_x - P\overline{Y}_x$
t–year funded whole life insurance	$Z_x - P\overline{Y}_{x:\overline{t}\|}$
n–year term insurance	$Z^1_{x:\overline{n}\|} - P\overline{Y}_{x:\overline{n}\|}$
t–year funded n–year term insurance	$Z^1_{x:\overline{n}\|} - P\overline{Y}_{x:\overline{t}\|}$
n–year pure endowment insurance	$Z_{x:\overline{n}\|}^{\ \ 1} - P\overline{Y}_{x:\overline{n}\|}$
t–year funded n–year pure endowment insurance	$Z_{x:\overline{n}\|}^{\ \ 1} - P\overline{Y}_{x:\overline{t}\|}$
n–year endowment	$Z_{x:\overline{n}\|} - P\overline{Y}_{x:\overline{n}\|}$
t–year funded n–year endowment insurance	$Z_{x:\overline{t}\|} - P\overline{Y}_{x:\overline{t}\|}$
n–year deferred insurance	${}_n\|Z_x - P\overline{Y}_x$
t–year funded n–year deferred insurance	${}_n\|Z_x - P\overline{Y}_{x:\overline{t}\|}$

Table 6.2: Loss in the case of annually funded continuously

Solution: (1) Solve BA_x, $B = ?$

$$A_{52} = E(v^{K_x}) = \sum_{k=1}^{w-x} v^k f_{K_x}(k) = \frac{1}{w-x}\sum_{k=1}^{w-x} v^k = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}\Bigg|_{v=\frac{1}{1.065},w-x=43} \approx 0.3339.$$

The net single premuim is $BA_x \approx (70000)(0.3339) \approx 23374.85$.
(2) Solve BP, through $E(L_x) = 0$, $\quad L_x = Z_x - P\ddot{Y}_x$??? $\quad L_x = Z_x - P\overline{Y}_x$???
$A_x - P\overline{a}_x = 0 => P = A_x/\overline{a}_x$. [17]: $\overline{a}_x = E(\frac{1-\overline{Z}_x}{\delta}) = \int_0^\infty v^t {}_tp_x dt = \int_0^{w-x} v^t(1-\frac{t}{w-x})dt$.

$$\overline{A}_{52} = E(v^{T_x}) = \int_0^\infty v^t f_{T_x}(t)dt = (\int_0^{w-x} v^t dt)\frac{1}{w-x} = \frac{1}{w-x}\frac{v^t}{\ln v}\Bigg|_0^{w-x} = 0.344665,$$

$$\overline{a}_{52} = \frac{1-\overline{A}_{52}}{\delta} = \frac{1-0.344665}{\ln(1.065)} = 10.406. \qquad (\delta = -\ln v)$$

The benefit annual premiums is $BP = BA_x/\overline{a}_x = (70000)\frac{0.3339.}{10.406} = 2246.220478$.

6.3.2 n–year term insurance.

Example 6.20. *Angela is 47 years old and purchases a 20–year term insurance policy with face value of \$120,000 paid at the end of the year of death. This policy will be paid continuously for the next 20 years while Angela is alive. Assume that $i = 7.5\%$ and death is modeled using the de Moivre model with terminal age 95. Find the benefit annual premium for this policy.*

Solution: Solve for BP or P ? $\quad$ Need L so that $E(L) = 0$ to solve P.

$B = ?$ [19]: $L = Z^1_{x:\overline{20}|} - P\ddot{Y}_{x:\overline{20}|}$? or $L = Z^1_{x:\overline{20}|} - P\overline{Y}_{x:\overline{20}|}$? $=> P = \frac{A^1_{x:\overline{20}|}}{\overline{a}_{x:\overline{20}|}}$

$$A^1_{47:\overline{20}|} = E(v^{K_x} I(K_x \le n)) = \sum_{k=1}^{20} v^k f_{K_x}(k) = (\sum_{k=1}^{20} v^k)\frac{1}{95-47} = v\frac{1-v^{20}}{1-v}\frac{1}{48} \approx 0.212385.$$

$$2 \text{ ways } \overline{a}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x dt = E(\frac{1-\overline{Z}_{x:\overline{n}|}}{\delta}) = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta} \qquad \textit{which is better?}$$

$$\begin{aligned}
\overline{A}_{47:\overline{20}|} =& E(v^{T_x \wedge n}) = \int_0^\infty v^{t\wedge n} f_{T_x}(t)dt = \int_0^{20} v^t \frac{1}{48}dt + \int_{20}^{48} v^{20}\frac{1}{48}dt \qquad \textit{easier} \\
=& \frac{v^t}{48\ln v}\Big|_0^{20} + (1.075)^{-20}\frac{t}{48}\Big|_{20}^{48} \approx 0.357578, \\
\overline{a}_{47:\overline{20}|} =& \frac{1-\overline{A}_{47:\overline{20}|}}{\delta} = \frac{1-0.357578}{-\ln v} \approx 8.882965907 \\
BP =& B\frac{A^1_{x:\overline{20}|}}{\overline{a}_{x:\overline{20}|}} = 120000\frac{0.212385}{8.882965907} \approx 2869.11.
\end{aligned}$$

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Example 6.21.

Example 6.22.

Example 6.23.

Theorem 6.7.

Theorem 6.8.

Corollary 6.1.

Corollary 6.2.

Theorem 6.9.

Theorem 6.10.

6.4 Benefit premiums for fully continuous insurance.

In this section, we will consider the funding of insurance products paid at the time of death and funded continuously. This type of insurance is called **fully continuous**.

6.4.1 Whole life insurance. Suppose that an insurance company funds a continuous whole life insurance w.u.p. with payments at a continuous rate of P while the individual is alive. The

Plan	Loss
Whole life insurance	$\overline{Z}_x - P\overline{Y}_x$
t–year funded whole life insurance	$\overline{Z}_x - P\overline{Y}_{x:\overline{t}\vert}$
n–year term insurance	$\overline{Z}^1_{x:\overline{n}\vert} - P\overline{Y}_{x:\overline{n}\vert}$
t–year funded n–year term insurance	$\overline{Z}^1_{x:\overline{n}\vert} - P\overline{Y}_{x:\overline{t}\vert}$
n–year pure endowment insurance	$\overline{Z}_{x:\overset{1}{\overline{n}\vert}} - P\overline{Y}_{x:\overline{n}\vert}$
t–year funded n–year pure endowment insurance	$\overline{Z}_{x:\overset{1}{\overline{n}\vert}} - P\overline{Y}_{x:\overline{t}\vert}$
n–year endowment	$\overline{Z}_{x:\overline{n}\vert} - P\overline{Y}_{x:\overline{n}\vert}$
t–year funded n–year endowment insurance	$\overline{Z}_{x:\overline{n}\vert} - P\overline{Y}_{x:\overline{t}\vert}$
n–year deferred insurance	${}_{n\vert}\overline{Z}_x - P\overline{Y}_x$
t–year funded n–year deferred insurance	${}_{n\vert}\overline{Z}_x - P\overline{Y}_{x:\overline{t}\vert}$

Table 6.3: Loss in the fully continuous case

loss random variable is $\overline{L} = v^{T_x} - P\overline{a}_{\overline{T_x}\vert} = \overline{Z}_x - P\overline{Y}_x$, where $\overline{a}_{\overline{n}\vert} = \int_0^n v^t dt$ and $\overline{Y}_x = \frac{1-\overline{Z}_x}{\delta}$.

Example 6.24. *An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.*
(i) Calculate the expected loss at issue.
(ii) Calculate the variance of the loss at issue random variable.
(iii) What is the loss if (x) die at age x+50 ?
(iv) Calculate the probability that the loss at issue is positive.

Solution: (i) Solve $E(B\overline{L})$ with $B =$?? $\overline{L} = Z_x - P\ddot{Y}_x$? or $Z_x - P\overline{Y}_x$? or $\overline{Z}_x - P\overline{Y}_x$?
$\overline{L} = \overline{Z}_x - P\overline{Y}_x$ **Q:** $P = 125$ **? or** $BP = 125$ **?**
$= v^{T_x} - P\frac{1-v^{T_x}}{\delta}$ (see [17])
$= (1+\frac{P}{\delta})v^{T_x} - \frac{P}{\delta}$ **Why do this ?**

$E(\overline{L}) = aE(\overline{Z}_x) + b = a\overline{A}_x + b \qquad (a,b)$**=?**

$$\overline{A}_x = \int_0^\infty v^t f_{T_x}(t)dt = \int_0^\infty v^t \mu e^{-\mu t}dt = \mu\int_0^\infty e^{-(\mu+\delta)t}dt$$
$$= \frac{\mu}{\mu+\delta}\int_0^\infty (\mu+\delta)e^{-(\mu+\delta)t}dt = \frac{\mu}{\mu+\delta} = \frac{\mu}{\mu-\ln v} \quad \textbf{why in } v \ ? \quad (1)$$

$$E(B\overline{L}) = B[(1+\frac{P}{\delta})\frac{\mu}{\mu+\delta} - \frac{P}{\delta}]\Big|_{P=0.125,\delta=0.06,\mu=?} = -35.71428571.$$

(ii) Solve $V(B\overline{L}) = B^2V(\overline{L}) \qquad \overline{L} = (1+\frac{P}{\delta})v^{T_x} - \frac{P}{\delta}.$

$$\mathrm{V}((1000)\overline{L}_x) = (1000)^2\left(1+\frac{P}{\delta}\right)^2\left(\overline{A}_x(v^2) - (\overline{A}_x(v))^2\right) = 82515.79 \qquad (\textit{by Eq. } (1))$$

(iii) The loss is $1000\overline{L} = [(1000)v^{T_x} - (125)\frac{1-v^{T_x}}{\delta}]\Big|_{T_x=??} = -1149.826$

(iv) The probability that the loss is positive is

$$\begin{aligned}
\mathbb{P}\{1000\overline{L} =& (1000)v^{T_x} - (125)\frac{1-v^{T_x}}{\delta} > 0\} \\
=& \mathbb{P}((1+\frac{P}{\delta})v^{T_x} - \frac{P}{\delta} > 0) && P = 125?\ or\ \frac{125}{1000}? \\
=& P(av^{T_x} - b > 0) && (a,b) =? \\
=& P(v^{T_x} > b/a) \\
=& P(T_x < \frac{\ln(b/a)}{\ln v}) \\
=& \mathbb{P}\left\{T_x < \frac{\ln\frac{0.125/\delta}{1+0.125/\delta}}{-\delta}\right\} = \mathbb{P}\left\{T_x < \frac{-\ln(1.48)}{-0.06}\right\} \\
=& 1 - e^{-(0.01)\frac{\ln(1.48)}{0.06}} = 1 - e^{\frac{-1}{6}\ln(1.48)} = 1 - (1.48)^{-1/6} \approx 0.06.
\end{aligned}$$

Example 6.25.

Example 6.26.

Example 6.27.

Example 6.28.

Example 6.29.

Example 6.30. *Kayla is 35 years old and purchases a 10–year deferred life insurance policy with face value of \$250,000 paid at the time of her death. This policy will be paid continuously for the next 10 years while Kayla is alive. Assume that $d = 6\%$ and death is modeled using De Moivre's model with terminal age 95. Find (i) the benefit annual premium for this policy and (ii) the standard deviation of the loss r.v..*

Solution: (i) Solve BP where $B = 250,000$, $E(\overline{L}) = 0$, and
$\overline{L} = {}_n|\overline{Z}_x - P\overline{Y}_{x:\overline{n}|} = v^{T_x}I(T_x > n) - P\int_0^{T_x\wedge n} v^t dt.$
$E(\overline{L}) = 0$ yields $P = {}_n|\overline{A}_x/\overline{a}_{x:\overline{n}|}$, where $\overline{a}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x dt = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$ (latter is easier).

$${}_{10}|\overline{A}_{35} = E(v^{T_x}I(T_x > n)) = \int_{10}^{60} v^t\frac{1}{60}dt = \left.\frac{v^t}{60\ln v}\right|_{10}^{60} = \frac{v^{10}-v^{60}}{-60\ln v}, \qquad (d = 1 - v = 0.06) \tag{1}$$

$$\overline{A}_{35:\overline{10}|} = E(v^{T_x\wedge n}) = \int_0^\infty v^{t\wedge 10} f_{T_x}(t)dt = \int_0^{10} v^t\frac{1}{60}dt + \int_{10}^{60} v^{10}\frac{1}{60}dt = \frac{1-v^{10}}{60(-\ln v)} + \frac{50}{60}v^{10}$$

$$P = {}_n|\overline{A}_x/\overline{a}_{x:\overline{n}|} = \frac{\frac{v^{10}-v^{60}}{-60\ln v}}{\frac{1-(\frac{1-v^{10}}{60(-\ln v)}+\frac{50}{60}v^{10})}{\delta}} \approx 0.009786197.$$

The benefit annual premium for this policy is $BP = 250000P = 2446.549$.

(ii) Solve $V(B\overline{L})$ first, where $\overline{L} = {}_n|\overline{Z}_x - P\overline{Y}_{x:\overline{n}|} = v^{T_x} I(T_x > n) - P\frac{1-v^{T_x \wedge n}}{\delta}$.
Write $\overline{L} = v^{T_x} I(T_x > n) - P\frac{1-v^{T_x} I(T_x \le n) - v^n I(T_x > n)}{\delta}$ by [14]. Note $I(T_x \le n)I(T_x > n) = ?$

$$\overline{L} = (v^{T_x} + P\frac{v^n}{\delta}) I(T_x > n) + P\frac{v^{T_x} I(T_x \le n)}{\delta} - \frac{P}{\delta} = W + U + c \qquad W = ?$$

$$V(\overline{L}) = V(W) + V(U) + 2Cov(W, U) = V(W) + V(U) + 0 - 2E(W)E(U)$$

$$E(W) = {}_n|\overline{A}_x + P\frac{v^n}{\delta}\mathbb{P}(T_x > n) \;\; (see\ Eq.(1)) \qquad E(W(v))^{\,2} = E(W(v^2))??$$

$$E(U) = \frac{P}{\delta}\overline{A}^{1}_{x:\overline{n}|} \qquad U = \frac{P}{\delta} v^{T_x} I(T_x \le n), \qquad E(U^2) = ??$$

$$\overline{A}^{1}_{x:\overline{n}|} = E(v^{T_x} I(T_x \le n) = \int_0^n v^t \frac{1}{60} dt = \frac{1 - v^n}{60(-\ln v)} = \cdots \tag{2}$$

$$E(W^2) = E(((v^{T_x} + P\frac{v^n}{\delta}) I(T_x > n))^2) = E((v^{2T_x} + 2\frac{Pv^n v^{T_x}}{\delta} + P^2 \frac{v^{2n}}{\delta^2}) I(T_x > n))$$

$$= {}_n|\overline{A}_x(v^2) + 2\frac{Pv^n}{\delta}{}_n|\overline{A}_x + \frac{P^2 v^{2n}}{\delta^2}{}_n p_x = \cdots \text{ (see Eq. (1))}$$

$$V(W) = {}_n|\overline{A}_x(v^2) + 2\frac{Pv^n}{\delta}{}_n|\overline{A}_x + \frac{P^2 v^{2n}}{\delta^2}{}_n p_x - ({}_n|\overline{A}_x + \frac{Pv^n {}_n p_x}{\delta})^2$$

$$V(U) = V(P\frac{v^{T_x} I(T_x \le n)}{\delta}) = \frac{P^2}{\delta^2} V(\overline{Z}^{1}_{x:\overline{n}|}) = \frac{P^2}{\delta^2}(\overline{A}^{1}_{x:\overline{n}|}(v^2) - (\overline{A}^{1}_{x:\overline{n}|})^2) = \cdots \text{ (see Eq. (2))}$$

$$V(\overline{L}) = V(W) + V(U) - 2E(W)E(U) \approx 0.01851821.$$

The SD of the loss of the policy $\approx B * \sqrt{0.01851821} \approx 34020.41$.

6.5 Benefit premiums for semicontinuous insurance.

This section discusses the funding of insurance products paid at the time of death and funded at the beginning of the year. This type of insurance is called **semicontinuous insurance**. We skip this secction.

Plan	Loss
Whole life insurance	$\overline{Z}_x - P\ddot{Y}_x$
t–year funded whole life insurance	$\overline{Z}_x - P\ddot{Y}_{x:\overline{t}\|}$
n–year term insurance	$\overline{Z}^{1}_{x:\overline{n}\|} - P\ddot{Y}_{x:\overline{n}\|}$
t–year funded n–year term insurance	$\overline{Z}^{1}_{x:\overline{n}\|} - P\ddot{Y}_{x:\overline{t}\|}$
n–year pure endowment insurance	$\overline{Z}_{x:\overline{n}\|}^{\;\;\,1} - P\ddot{Y}_{x:\overline{n}\|}$
t–year funded n–year pure endowment insurance	$\overline{Z}_{x:\overline{n}\|}^{\;\;\,1} - P\ddot{Y}_{x:\overline{t}\|}$
n–year endowment	$\overline{Z}_{x:\overline{n}\|} - P\ddot{Y}_{x:\overline{n}\|}$
t–year funded n–year endowment insurance	$\overline{Z}_{x:\overline{n}\|} - P\ddot{Y}_{x:\overline{t}}\|$
n–year deferred insurance	${}_n\|\overline{Z}_x - P\ddot{Y}_x$
t–year funded n–year deferred insurance	${}_n\|\overline{Z}_x - P\ddot{Y}_{x:\overline{t}\|}$

Table 6.4: Loss in the semicontinuous case

6.6 Benefit premium for an n–year deferred annuity due.

6.6.1 n–year deferred annuity due funded discretely.

$$L = {}_n|\ddot{Y}_x - P\ddot{Y}_{x:\overline{n}|} = \sum_{k\geq n}^{K_x-1} v^k - P\sum_{k=0}^{(n\wedge K_x)-1} v^k = \frac{v^n - v^{K_x}}{1-v} I(K_x > n) - P\frac{1-v^{n\wedge K_x}}{1-v}. \qquad (1)$$

Example 6.31. *Jasmine is 45 years old and purchases a 20–year deferred contingent annuity with a face value of \$40000 paid at the beginning of year while she is alive. This policy will be paid by level payments made at the beginning of the next 20 years while Jasmine is alive. Assume that $\delta = 0.05$ and constant force of mortality is 0.02. Find the annual benefit premium for this policy using the equivalent principle. Derive the variance of the loss (in unit payment of insurance).*

Solution: (a) Solve BP, $B = 40,000$ and by $E(L) = 0$, where

$$L = {}_n|\ddot{Z}_x - P\ddot{Y}_{x:\overline{n}|} \ ? \qquad L = {}_n|Z_x - P\ddot{Y}_{x:\overline{n}|} \ ? \qquad L = {}_n|\ddot{Y}_x - P\ddot{Y}_{x:\overline{n}|} \ ?$$

$=> P = \frac{{}_n|\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}$. Many formulas below, which is easier ?

$$\ddot{a}_{45:\overline{20}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = E(\frac{1 - Z_{x:\overline{n}|}}{d}) \qquad need\ A_{x:\overline{n}|}$$

$${}_{20}|\ddot{a}_{45} = \sum_{k=n}^{\infty} v^k {}_kp_x = {}_nE_x \ddot{a}_{x+n} = vp_x \cdot {}_{n-1}|\ddot{a}_{x+1},$$

$${}_n|\ddot{Y}_x = \frac{v^n - v^{K_x}}{1-v} I(K_x > n) \text{ and } \ddot{Y}_x = \ddot{Y}_{x:\overline{n}|} + {}_n|\ddot{Y}_x \qquad [17],\ [18]$$

Due to $S_{T_x} \sim \text{Exp}(\mu)$, it is simpler to use

$$\ddot{a}_{45:\overline{20}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = \sum_{k=0}^{n-1} v^k e^{-\mu k} = \sum_{k=0}^{n-1} (ve^{-\mu})^k = \frac{1-(ve^{-\mu})^n}{1-ve^{-\mu}}\Bigg|_{v=e^{-0.05},\mu=0.02} \approx 11.14,$$

$${}_{20}|\ddot{a}_{45} = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{\infty}|} - \ddot{a}_{x:\overline{n}|} = [\frac{1-(ve^{-\mu})^\infty}{1-ve^{-\mu}} - \frac{1-(ve^{-\mu})^n}{1-ve^{-\mu}}\Bigg|_{v=e^{-0.05},\mu=0.02} \approx 3.65,$$

$$or\ = \sum_{k=n}^{\infty} v^k {}_kp_x = \sum_{k=n}^{\infty} v^k e^{-\mu k} = \sum_{k=n}^{\infty} (ve^{-\mu})^k = (ve^{-\mu})^n \sum_{j=0}^{\infty} (ve^{-\mu})^j \approx 3.65, \quad (j = k-n)$$

$$BP = (40000)P \approx \frac{(40000)(3.65)}{11.14} \approx 13092.43.$$

(2) Solve $V(L)$

$$\begin{aligned}
L =& \sum_{k=n}^{K_x-1} v^k - P \sum_{k=0}^{(K_x\wedge n)-1} v^k = v^n \sum_{j=0}^{K_x-1-n} v^j - P \sum_{k=0}^{(K_x\wedge n)-1} v^k \quad j=k-n \\
=& \frac{v^n - v^{K_x}}{1-v} I(K_x > n) - P\frac{1-v^{K_x\wedge n}}{1-v} \\
=& \frac{v^n - v^{K_x}}{1-v} I(K_x > n) - P\frac{1-v^{K_x} I(K_x \le n) - v^n I(K_x > n)}{1-v} \\
=& \frac{v^n(1+P) - v^{K_x}}{1-v} I(K_x > n) + P\frac{v^{K_x}}{1-v} I(K_x \le n) - P/d \quad (see\ [14]) \\
V(L) =& V(X) + V(Y) + 2Cov(X,Y) \qquad (X,Y) = ? \\
=& V(X) + V(Y) - 2E(X)E(Y) \qquad\qquad Cov(X,Y) = E(XY) - E(X)E(Y) \\
=& V(X) + V(Y) - 2(\frac{1+P}{1-v} v^n {}_np_x - \frac{1}{1-v} {}_{n|}A_x)(P \cdot \frac{A^1_{x:\overline{n}|}}{1-v})
\end{aligned}$$

$$V(Y) = V(P\frac{v^{K_x} I(K_x \le n)}{1-v}) = \frac{P^2}{(1-v)^2}[A^1_{x:\overline{n}|}(v^2) - (A^1_{x:\overline{n}|})^2]$$

$$V(X) = V(\frac{v^n(1+P) - v^{K_x}}{1-v} I(K_x > n)) = \frac{V(U)}{(1-v)^2} = \frac{E(U^2) - (E(U))^2}{(1-v)^2} \qquad U = ?$$

$$\begin{aligned}
E(U^2) =& E((v^n(1+P) - v^{K_x})^2 I(K_x > n)) \\
=& (v^{2n}(1+P)^2 {}_np_x - 2(1+P)v^n \underbrace{E(v^{K_x} I(K_x > n))}_{{}_{n|}A_x} + \underbrace{E(v^{2K_x} I(K_x > n))}_{{}_{n|}A_x(v^2)} \\
E(U) =& E((v^n(1+P) - v^{K_x}) I(K_x > n)) = v^n(1+P) {}_np_x - \underbrace{E(v^{K_x} I(K_x > n))}_{{}_{n|}A_x}
\end{aligned}$$

$$\begin{aligned}
E(v^{K_x}) =& \sum_{k=1}^{\infty} v^k (e^{-(k-1)\rho} - e^{-k\rho}) = \sum_{k=1}^{\infty} v^k ({}_{k-1}p_x - {}_{k-1}p_x p_x) = \sum_{k=1}^{\infty} v^k {}_{k-1}p_x q_x = \sum_{k=1}^{\infty} v^k p_x^k q_x / p_x \\
=& vp_x \frac{1-(vp_x)^\infty}{1-vp_x} q_x/p_x \\
A^1_{x:\overline{n}|} =& E(v^{K_x} I(K_x \le n)) = \sum_{k=1}^{n} v^k (e^{-(k-1)\rho} - e^{-k\rho}) \\
{}_{n|}A_x =& E(v^{K_x} I(K_x > n)) = E(v^{K_x}) - E(v^{K_x} I(K_x \le n)) \\
V(L) =& V(X) + V(Y) + 2Cov(X,Y) = \cdots \approx 16.92 + 4.51 - 14.96 \approx = 6.47
\end{aligned}$$

6.7 Premiums paid m times a year.

Example 6.32.

Example 6.33.

Example 6.34.

Example 6.35.

Example 6.36.

6.8 Non–level premiums and/or benefits.

Insured products can be funded with non–level premiums. Premiums may increase according to the inflation rate.

Let b_k be the benefit **(insurance, not annuity)** paid by an insurance company at the end of year k, $k = 1, 2, \dots$. The contingent cashflow of benefits is

benefits	0	b_1	b_2	b_3	$\cdots$
Time after issue	0	1	2	3	$\cdots$

Hence, the APV of the contingent benefit is $\sum_{k=1}^{\infty} b_k v^k \mathbb{P}\{K_x = k\}$
Let π_{k-1} be the benefit premium received by an insurance company at the beginning of year k, $k = 1, 2, \dots$. The contingent cashflow of benefit premiums is

benefit premiums	π_0	π_1	π_2	π_3	$\cdots$
Time after issue	0	1	2	3	$\cdots$

Hence, the APV of the contingent benefit premiums is

$$\sum_{j=1}^{\infty}\sum_{k=0}^{j-1} \pi_k v^k f_{K_x}(j) = \sum_{k=0}^{\infty} \pi_k v^k \mathbb{P}\{K_x > k\} = \sum_{k=0}^{\infty} \pi_k v^k \cdot {}_k p_x.$$

The loss is $L = b_{K_x} v^{K_x} - \sum_{k=0}^{K_x - 1} \pi_k v^k$,

$L =$ benefit of insurance – cumulative payment for buying the insurance
Under the equivalence principle $E(L) = 0$, *i.e.*,

$$\sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1|}q_x = \sum_{k=0}^{\infty} \pi_k v^k \cdot {}_k p_x = \sum_{j=1}^{\infty}\sum_{k=0}^{j-1} \pi_k v^k f_{K_x}(j) \qquad {}_{k-1|}q_x = f_{K_x}(j)?$$

Example 6.37. *For a special fully discrete* <u>15–payment</u> <u>*whole*</u> *life insurance on (20):*
(i) The death benefit is 1000 for the first 10 years and is 6000 thereafter.
(ii) The benefit premium paid during each of the first 5 years is half of the benefit premium paid during the subsequent years.
(iii) Mortality is given by the life table for the USA population in 2004 (see pages 603)
(iv) $i = 0.06$.
Calculate the initial annual benefit premium.

Solution: Find π, which is the initial annual benefit premium and sastisfies $E(L)=0$ where

$$L=[(1000)v^{K_x}I(K_x\le 10)+(6000)v^{K_x}I(K_x>10)]-[\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k+\sum_{k\ge 5}^{(K_x\wedge 15)-1}2\pi v^k].$$

$$\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k+\sum_{k\ge 5}^{(K_x\wedge 15)-1}2\pi v^k=\begin{cases}\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k & \text{if } K_x\le 5\\ \sum_{k=0}^{5-1}\pi v^k+\sum_{k\ge 5}^{(K_x\wedge 15)-1}2\pi v^k & \text{if } K_x>5\end{cases}$$
$$=\begin{cases}\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k & \text{if } K_x\le 5\\ -\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k+\sum_{k\ge 0}^{(K_x\wedge 15)-1}2\pi v^k & \text{if } K_x>5\end{cases} (easier)$$

$$E(L)=0 \text{ yields} \qquad (1000)A^1_{20:\overline{10}|}+(6000)\cdot {}_{10}|A_{20}=E(\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k+\sum_{k\ge 5}^{(K_x\wedge 15)-1}2\pi v^k)$$
$$=(1000)A_{20}+(5000)\cdot {}_{10}|A_{20}=E(-\sum_{k=0}^{(K_x\wedge 5)-1}\pi v^k+\sum_{k=0}^{(K_x\wedge 15)-1}2\pi v^k)$$
$$(by[14])\quad (1000)A_{20}+(5000)\cdot {}_{10}E_{20}A_{20+10}=-\pi\ddot{a}_{20:\overline{5}|}+2\pi\ddot{a}_{20:\overline{15}|} \qquad \text{by [17]},$$
$$(1000)A_{20}+(5000)\cdot {}_{10}E_{20}A_{20+10}=\pi(2\ddot{a}_{20:\overline{15}|}-\ddot{a}_{20:\overline{5}|})$$
$$\pi=\frac{(1000)A_{20}+(5000)\cdot {}_{10}E_{20}A_{20+10}}{2\ddot{a}_{20:\overline{15}|}-\ddot{a}_{20:\overline{5}|}}=\frac{easy}{tricky} \tag{1}$$

$2\ddot{a}_{20:\overline{15}|}-\ddot{a}_{20:\overline{5}|}=?$ tables only give ${}_5E_x,\ {}_{10}E_x,\ {}_{20}E_x,\ \ell_x,\ \ddot{a}_x,$

$$\ddot{a}_{x:\overline{n}|}=\underbrace{\ddot{a}_x-{}_n|\ddot{a}_x}_{by\ [17]}=\underbrace{\ddot{a}_x-{}_nE_x\cdot\ddot{a}_{x+n}}_{by\ [17]},\ and\ \ {}_{15}E_x=v^{15}{}_{15}p_x={}_{m+n}E_x={}_mE_x\cdot{}_nE_{x+m}$$

$$\ddot{a}_{20:\overline{15}|}=\ddot{a}_{20}-{}_{15}|\ddot{a}_{20}=\ddot{a}_{20}-{}_{15}E_{20}\cdot\ddot{a}_{35}=\ddot{a}_{20}-v^{15}\frac{\ell_{35}}{\ell_{20}}\cdot\ddot{a}_{35}$$
$$=16.739946-(1.06)^{-15}\frac{97250}{98709}(15.817689)=10.23733295,$$
$$\ddot{a}_{20:\overline{5}|}=\ddot{a}_{20}-{}_5E_{20}\cdot\ddot{a}_{25}=16.739946-(0.743753117)(16.514250)=4.457421088,$$
$$2\ddot{a}_{20:\overline{15}|}-\ddot{a}_{20:\overline{5}|}=((2)(10.23733295)-4.457421088)=16.01724481.$$

$(1000)A_{20}+(5000)\cdot{}_{10}E_{20}\cdot A_{30}=52.45587+(5)(0.553116815)(82.29543)=280.0508007.$
Hence, $\pi=\frac{280.0508007}{16.01724481}=17.48433042.$

Announcement: The homework assigned this weekend due next Wednesday.

Example 6.38. *Consider a whole life insurance policy to (40) with face value of \$250000 payable at the end of the year of death. This policy will be paid by benefit annual premiums paid at the beginning of each year while (40) is alive. Suppose that the premiums increase by 6% each year. Assume that $i=6\%$ and death is modeled using the de Moivre model with terminal age 100. Find the amount of the first benefit annual premium for this policy.*

Solution: Let π be the amount of the first benefit premium (in unit value). Solve $B\pi$ from $E(L) = 0$, where $B = 250000$,

$L = v^{K_x} - \sum_{k=0}^{K_x-1} \pi_k v^k$, and $\pi_k = \pi(1.06)^k$, $k = 0, 1, 2, \dots$

$L = v^{K_x} - \sum_{k=0}^{K_x-1} \pi(1.06)^k v^k = v^{K_x} - \sum_{k=0}^{K_x-1} \pi(1.06v)^k$.

$0 = A_x - \pi E \sum_{k=0}^{K_x-1}(1.06v)^k)$. =>

$$\pi = A_x/E(\sum_{k=0}^{K_x-1}(1.06v)^k)$$

$$A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(x) = \sum_{k=1}^{w-x} v^k \frac{1}{w-x} = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x} = 0.2693571284.$$

$$E(\sum_{k=0}^{K_x-1}(1.06v)^k) = E(\sum_{k=0}^{K_x-1} 1) = E(K_x) = \sum_{k=1}^{\infty} k f_{K_x}(k) = \sum_{k=1}^{60} k\frac{1}{60} \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$= \frac{(60)(61)}{2}\frac{1}{60} = 30.5.$$

$$\pi = A_x/E\sum_{k=0}^{K_x-1}(1.06v)^k) = \frac{0.2693571284}{30.5}$$

$$B\pi = (250000)\frac{0.2693571284}{30.5} = 2207.845$$

250000*0.2693571284/30.5

Example 6.39.

Example 6.40.

Example 6.41.

Example 6.42.

Theorem 6.11.

Theorem 6.12.

Theorem 6.13.

Theorem 6.14.

Theorem 6.15.

Theorem 6.16.

Theorem 6.17.

Theorem 6.18.

Theorem 6.19.

Theorem 6.20.

Theorem 6.21.

Theorem 6.22.

Theorem 6.23.

Theorem 6.24.

Example 6.43.

6.9 Computing benefit premiums from a life table

In this section, we disucuss how to obtain the benefit rates for different insurance products.

6.9.1 Fully discrete insurance. We assume that the death benefits are paid at the end of the year of death. But, the benefit premiums are at the beginning of each m–thly period. The annual benefit premium in this situation is higher than the regular situation. Benefit premiums, instead of being received at the beginning of the year, they are received later on. During a year when an insuree dies, benefit premiums may not be received during the whole year. From a life table, we can find $\ddot{a}_x$, then we can estimate $\ddot{a}_x^{(m)}$.

Example 6.44. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*
d_x	*33*	*46*	*54*	*45*	*62*	*28*	*0*

Assume that $i = 6.5\%$ and uniform distribution of deaths over each year of death. Find $P_{80}^{(12)}$, using that $A_{80} = 0.8161901166$ (see Example 4.9 in page 87).

Solution: There are two appoaches for $P_{80}^{(m)} = \frac{A_{80}}{\ddot{a}_{80}^{(m)}}$:

(1) Based on basic formulas: $v \to v^{1/m}$ and ${}_{j-1|}q_x \to {}_{\frac{j-1}{m}|\frac{1}{m}}q_x = f_{J_x}(j)$ $({}_{0|}q_x = q_x \to {}_{\frac{1}{m}}q_x)$.

$$
\begin{aligned}
A_x^{(m)} =& A_{80}^{(12)} = E(v^{J_x}) = \sum_{j=1}^{\infty}(v^{1/12})^j f_{J_x}(j) \\
=&(v^{1/12})^1 f_{J_x}(1) + \cdots + (v^{1/12})^{12} f_{J_x}(12) \\
&+ (v^{1/12})^{12+1} f_{J_x}(12+1) + \cdots + (v^{1/12})^{12+12} f_{J_x}(12+12) \\
&+ \cdots \\
=&\sum_{k=0}^{\infty}\sum_{j=1}^{m}(v^{1/12})^{km+j}\frac{1}{m}\frac{d_{x+k}}{\ell_x} \\
=&\frac{1}{m}\sum_{k=0}^{\infty}(v^{1/12})^{km}\frac{d_{x+k}}{\ell_x}\sum_{j=1}^{m}(v^{1/12})^j \\
=&\frac{1}{m}\sum_{j=1}^{m}(v^{1/12})^j\sum_{k=0}^{\infty}v^k\frac{d_{x+k}}{\ell_x} \\
=&\frac{1}{m}\frac{v^{1/12}(1-v^{m/12})}{1-v^{1/12}}A_x = 0.8402293189,
\end{aligned}
$$

$\ddot{a}_x^{(12)} = E(\sum_{j=0}^{J_x-1}(v^{1/12})^j) = E(\frac{1-v^{J_x/12}}{1-v^{1/12}}) = \frac{1-A_x^{(12)}}{1-v^{1/12}} = 2.543720348.$

$P_{80}^{(12)} = \frac{A_{80}}{\ddot{a}_{80}^{(12)}} = \frac{0.8161901166}{2.543720348} = 0.3208647198,$

$$
\text{(2) Using formulas: } \ddot{a}_{80}^{(m)} = \frac{1-A_{80}^{(m)}}{d^{(m)}}, \qquad d^{(m)} = m(1-(1+i)^{-\frac{1}{m}})
$$

$$
A_{80}^{(m)} = \frac{i}{i^{(m)}}A_{80}, \qquad i^{(m)} = m((1+i)^{\frac{1}{m}}-1).
$$

Thus

$$
i^{(m)} = m((1+i)^{\frac{1}{m}}-1) \approx 6.314\% \qquad A_{80}^{(12)} = \frac{i}{i^{(12)}}A_{80} = \frac{0.065}{0.06314033132}(0.8161901166) \approx 0.840,
$$

$$
d^{(m)} = m(1-(1+i)^{-\frac{1}{m}}) \approx 6.281\% \qquad \ddot{a}_{80}^{(12)} = \frac{1-A_{80}^{(12)}}{d^{(12)}} = \frac{1-0.8402293189}{0.06281} \approx 2.544,
$$

$$
P_{80}^{(12)} = \frac{A_{80}}{\ddot{a}_{80}^{(12)}} = \frac{0.8161901166}{2.544} \approx 0.321.
$$

For a period of length $\frac{1}{m}$:

(i) the interest factor is $(1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$.

(ii) the effective rate of interest is $(1+i)^{1/m} - 1 = \frac{i^{(m)}}{m}$.

(iii) the discount factor is $(1+i)^{-1/m} = v^{1/m} = (1-d)^{1/m} = 1 - \frac{d^{(m)}}{m}$.

(iv) the effective rate of discount is $1 - v^{1/m} = \frac{d^{(m)}}{m}$.

Theorem 6.25. Skip the theorem.

6.9.2 Semicontinuous insurance.

For a semicontinuous insurance, the death benefit is paid at the time of the death and benefit premiums are paid at the beginning of the year.

Example 6.45. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	46	54	45	62	28	0

Assume that $i = 6.5\%$ and uniform distribution of deaths over each year of death. The death benefit is paid at the time of the death and benefit premiums P are paid at the beginning of the year. Find P using equivalent principle.

Solution: $L = v^{T_x} - P\sum_{k=0}^{K_x-1} v^k = v^{T_x} - P\frac{1-v^{K_x}}{1-v}$. $P = \overline{A}_x/\ddot{a}_x$ and $\ddot{a}_x = \frac{1-A_x}{1-v}$ by [17].
$A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k d_{x+k-1}/\ell_x = 0.8161901166$,
$\ddot{a}_{80} = \frac{1-A_{80}}{d} = \frac{1-0.8161901166}{1-(1/1.065)} = 3.011654243$.
$\overline{A}_x = \frac{i}{\delta}A_x$ (**true only for** $\overline{A}_x$**,** $\overline{A}^{\,1}_{x:\overline{n}|}$ **and** ${}_n|\overline{A}_x$ **!**), or
$\overline{A}_x = \sum_{i=1}^{6}\int_{i-1}^{i} v^t f_{T_x}(t)dt$ (**always true**). Use this approach here.
$f_{T_x}(t) = -\frac{d}{dt}{}_tp_x$ (by (10) of 447).
${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, (by (11))
$\ell_{x+t} =$

$$\begin{cases} \ell_x + t(\ell_{x+1}-\ell_x) & if\ t \in (x, x+1] \\ \ell_{x+1} + t(\ell_{x+2}-\ell_{x+1}) & if\ t \in (x+1, x+2] \\ \ell_{x+2} + t(\ell_{x+3}-\ell_{x+2}) & if\ t \in (x+2, x+3] \\ \ell_{x+3} + t(\ell_{x+4}-\ell_{x+3}) & if\ t \in (x+3, x+4] \\ \ell_{x+4} + t(\ell_{x+5}-\ell_{x+4}) & if\ t \in (x+4, x+5] \\ \ell_{x+5} + t(\ell_{x+6}-\ell_{x+5}) & if\ t \in (x+5, x+6] \end{cases} \quad f_{T_x}(t) = -{}_tp'_x = \begin{cases} \frac{d_x}{\ell_x} & if\ t \in (x, x+1] \\ \frac{d_{x+1}}{\ell_x} & if\ t \in (x+1, x+2] \\ \frac{d_{x+2}}{\ell_x} & if\ t \in (x+2, x+3] \\ \frac{d_{x+3}}{\ell_x} & if\ t \in (x+3, x+4] \\ \frac{d_{x+4}}{\ell_x} & if\ t \in (x+4, x+5] \\ \frac{d_{x+5}}{\ell_x} & if\ t \in (x+5, x+6]. \end{cases}$$

$$f_{T_x}(t) = f_{K_x}(t)?? \qquad f_{T_x}(t) = f_{K_x}(\lceil t \rceil)??$$

$$f_{T_x}(t) = f_{K_x}(\lceil t \rceil) \text{ if } t \text{ is not an integer ??}$$

$$\overline{A}_{80} = \int_0^{\infty} v^t f_{T_x}(t)dt = \sum_{i=1}^{6}\int_{i-1}^{i} v^t \frac{d_{x+i-1}}{l_x}dt = \sum_{i=1}^{6}\left(\frac{v^t}{\ln v}\bigg|_{i-1}^{i}\right)\frac{d_{x+i-1}}{l_x} = 0.8424379003.$$

$$P = \frac{\overline{A}_{80}}{\ddot{a}_{80}} = \frac{0.8424379003}{3.011654243} = 0.2797259686.$$

Theorem 6.26.

Theorem 6.27.

6.9.3 Fully continuous insurance.

For a fully continuous insurance, we need to know $\overline{A}_x$ and $\overline{a}_x$. From a life table, we can find A_x and a_x. Then, we need to estimate $\overline{A}_x$ and $\overline{a}_x$.

Example 6.46. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

Assume that $i = 6.5\%$ *and uniform distribution of deaths over each year of death. Find* $\overline{P}(\overline{A}_{80})$*, using that* $A_{80} = 0.8161901166$ *(see Example 4.9 in page 87).*

Solution: We have that

$$\overline{P} = \frac{\overline{A}_x}{\overline{a}_x} = \frac{\overline{A}_x}{\frac{1-\overline{A}_x}{\delta}}; \quad \overline{A}_{80} = \frac{i}{\delta} A_{80} = \frac{0.065}{\ln(1.065)}(0.8161901166) = 0.8424379003,$$

$$\overline{P}(\overline{A}_{80}) = \frac{\delta \overline{A}_{80}}{1 - \overline{A}_{80}} = \frac{\ln(1.065)(0.8424379003)}{(1 - 0.8424379003)} = 0.3367076072.$$

Example 6.47.

6.10 Premiums found including expenses.

When finding the annual premium expenses and commissions have to be taken into in account. Possible costs are underwriting (making the policy) and maintaining the policy. The annual premium which an insurance company charges is called the **gross annual premium**, the **contract premium**, the **loaded premium** and the expense–augmented premium. It often includes:

1. Issue cost.
2. Percentage of annual benefit premium.
3. Fixed amount per policy.
4. Percentage of (face value) contract amount.
5. Settlement cost.

Often the expenses related to the contract amount, are given as per thousand expenses, i.e. the per thousand expenses are the expenses made for each \$1,000 of the face value of the insurance. The loss is

$$L_e = \text{expenses-deposit (or } L_e = \text{expenses-total annual premium)}$$

Example 6.48. *A fully discrete whole life insurance policy with face value of \$50,000 is made to (x). The following costs are incurred:*
(i) \$800 for making the contract.
(ii) Percent of expense–loaded premium expenses are 6% in the first year and 2% thereafter.
(iii) Per thousand expenses are \$2 per year.
Assume $P_x = A_x/\ddot{a}_x = 0.11$ All expenses are paid at the beginning of the year. $d = 5\%$. Calculate the expense–augmented annual premium G using the equivalence principle.

Solution: Solve G from $E(L_e) = 0$, where

$$L_e = \underbrace{(50000)Z_x + 800 + \overbrace{(0.04)}^{or\ 0.06?} G + (0.02)G\ddot{Y}_x + (2)(50)\ddot{Y}_x}_{expenses} - \underbrace{G\ddot{Y}_x}_{deposit}.$$

$$E(L_e) = (50000)A_x + 800 + (0.04)G + (0.02)G\ddot{a}_x + (2)(50)\ddot{a}_x - G\ddot{a}_x = 0. \qquad A_x/\ddot{a}_x = 11$$

$$=> \quad G = \frac{(50000)A_x + 800 + (2)(50)\ddot{a}_x}{(1-0.02)\ddot{a}_x - 0.04} =?\ \ddot{a}_x =?\ \ A_x =? \qquad [17]\ \ddot{a}_x = \frac{1-A_x}{1-v}$$

$0.11 = P_x = \frac{A_x}{\ddot{a}_x} = \frac{1-d\ddot{a}_x}{\ddot{a}_x} = \frac{A_x}{(1-A_x)/d}$. $d = 1 - v = 0.05$.
$0.11 = \frac{1-d\ddot{a}_x}{\ddot{a}_x} => 0.11\ddot{a}_x = 1 - d\ddot{a}_x =>$
$\ddot{a}_x = \frac{1}{d}/(1 + \frac{0.11}{d}) = 1/(0.05 + 0.11) = 10/16$.
$A_x = P_x\ddot{a}_x = 11/16$.
$G = \frac{(50000)A_x+800+(2)(50)\ddot{a}_x}{(1-0.02)\ddot{a}_x-0.04} = 5883.319638$.

Example 6.49. *A whole life insurance policy with face value of \$40,000 payable at the end of the year of death is made to (45). Assume that $i = 4.5\%$ and death is modeled using the de Moivre model with terminal age 95. The annual benifit premium is paid at the beginning of the year, and the following costs are incurred and paid at the beginning of the year:*
(i) \$500 for making the contract.
(ii) Percent of expense–loaded premium expenses is 5% in the first year and 1% thereafter.
(iii) Per policy expenses are \$20 per year.
(iv) Per thousand expenses are \$1.2 per year.
(v) \$600 for settlement.
(a) Calculate the gross annual premium G using the equivalence principle.
(b) Calculate the expense–augmented loss for an insuree that dies 7 years, 5 months and 10 days after the issue of this policy.
(c) Calculate the variance of the expense–augmented loss.

Solution: (a) Solve G from $E(L_e) = 0$, where the loss

$$\begin{aligned} L_e =&40000Z_{45} + 500 + 0.04G + 0.01G\ddot{Y}_{45} + 20\ddot{Y}_{45} + (1.2)(40)\ddot{Y}_{45} + 600Z_{45} - G\ddot{Y}_{45}, \quad (1)\\ => \quad 0 =&40000A_{45} + 500 + 0.04G + 0.01G\ddot{a}_{45} + 20\ddot{a}_{45} + (1.2)(40)\ddot{a}_{45} + 600A_{45} - G\ddot{a}_{45},\\ => \quad G =&\frac{500 + 40600A_{45} + 68\ddot{a}_{45}}{0.99\ddot{a}_{45} - 0.04}. \quad \ddot{a}_{45}, A_{45} =? \quad T_{45} \sim U(0, 50). \end{aligned}$$

$$\begin{aligned} [14] \to \quad A_{45} =&E(v^{K_x}) = \sum_{k=1}^{50} v^k \frac{1}{50} = v\frac{1-v^{50}}{1-v}\frac{1}{50}\bigg|_{v=1/1.045} = 0.3952401556, \quad (2)\\ [17] \to \quad \ddot{a}_{45} =&\frac{1-A_{45}}{1-v} = \frac{1-0.3952401556}{1-1/1.045} = 14.0438675,\\ G =&\frac{500 + 40600A_{45} + 68\ddot{a}_{45}}{0.99\ddot{a}_{45} - 0.04} = 1262.439006. \end{aligned}$$

(b) Solve $L_e(K_x)$. Where is K_x in L_e ? Insuree die between 7 and 8 years, $K_x =??$

$$\begin{aligned} L_e =&[40000Z_{45} + 500 + 0.04G + 0.01G\ddot{Y}_{45} + 20\ddot{Y}_{45} + 40 \times 1.2\ddot{Y}_{45} + 600Z_{45} - G\ddot{Y}_{45}] \quad \textit{by } (1)\\ =&(40600)Z_{45} + 550.4975602 - 1181.814616\ddot{Y}_{45}\\ =&(40600)v^{K_{45}} + 550.4975602 - 1181.814616\frac{1-v^{K_{45}}}{1-v} \quad (\textit{see } [14] \textit{ and } [17])\\ =&\underbrace{68044.36v^{K_{45}} - 26893.86}_{\textit{why do this step??}} = 18244.28524 \qquad \textit{if } K_{45} = 8. \end{aligned}$$

$$\begin{aligned} (c) \quad \mathrm{V}(L_e) =&a^2\mathrm{V}(Z_{45}) = (68044.36)^2(A_{45}(v^2) - (A_{45})^2) \qquad (\textit{see } (2))\\ =&(68044.36)^2\Big(v\frac{1-v^{50}}{(1-v)50}\bigg|_{v=1/1.045^2} - (A_{45})^2\Big)\\ =&(68044.36)^2(0.2146684865 - (0.3952401556)^2) = 270642713.1. \end{aligned}$$

Skip to Page 205 Example 6.49.

Theorem 6.28. *Suppose that we have a whole life insurance on (x), with a death benefit of b paid at the end of the year of death. The fixed annual cost has an amount of e. In the first year, there exists an additional cost of e_0^*. The percentage of the expense–augmented premium paid in expenses each year is r. During the first year, it is paid an additional percentage of the expense–augmented premium of r_0^*. The settlement cost is s. All cost except the settlement cost are paid at the beginning of the year. The insurance is funded by an expense–augmented premium of G paid at the beginning of the year while (x) is alive. If the equivalence principle is used, then*

$$G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}. \tag{1}$$

Using that $P_x = \frac{A_x}{\ddot{a}_x}$ and $P_x + d = \frac{1}{\ddot{a}_x}$ (easy to verify), we get that the expense–augmented annual benefit premium using the equivalence principle is

$$G = \frac{e_0^*(P_x+d) + (b+s)P_x + e}{1 - r - r_0^*(P_x+d)}. \tag{2}$$

Q: Why Eq. (2) ?

Eq. (1) needs to specify f_{T_x}, but Eq. (2) does not.

The expense–augmented loss at issue random variable is the present value of expenses plus the present value of benefit minus the present value of premiums, i.e. it is

$$\begin{aligned} {}_0L_e =& e_0^* + r_0^*G + (b+s)Z_x + (rG+e)\ddot{Y}_x - G\ddot{Y}_x, \\ =& e_0^* + r_0^*G + (b+s)Z_x - ((1-r)G - e)\frac{1-Z_x}{d} = aZ_x + b. \end{aligned}$$

Theorem 6.29. *Under the conditions in the previous theorem,*

(i) The expense–augmented loss at issue random variable is

$$\begin{aligned} {}_0L_e =& (e_0^* + r_0^*G + b + s)(Z_x - P_x\ddot{Y}_x) \\ =& (e_0^* + r_0^*G + b + s){}_0L_x. \end{aligned}$$

(ii) The variance of the expense–augmented loss is

$$\begin{aligned} \mathrm{V}({}_0L_e) =& (e_0^* + r_0^*G + b + s)^2 \mathrm{V}(L_x) \\ =& (e_0^* + r_0^*G + b + s)^2 \left(1 + \frac{P_x}{d}\right)^2 \mathrm{V}(Z_x) \\ =& (e_0^* + r_0^*G + b + s)^2 \frac{{}^2A_x - A_x{}^2}{(1-A_x)^2} \\ =& (e_0^* + r_0^*G + b + s)^2 \frac{{}^2A_x - A_x{}^2}{(d\ddot{a}_x)^2}. \end{aligned}$$

When we compute the expense–augmented loss we get an expression of the type $c_1 + c_2Z_x - c_2\ddot{Y}_x$. The proof of the previous theorem gives that

$${}_0L_e = c_1 + c_2Z_x - c_2\ddot{Y}_x = (c_1 + c_2)L_x \text{ and}$$

$$\mathrm{V}({}_0L_e) = (c_1 + c_2)^2 \mathrm{V}(L_x).$$

The loss without including expenses for a whole life insurance with death benefit b is ${}_0L = bL_x = b(Z_x - P_x \ddot{Y}_x)$. The variance of this loss is

$$\mathrm{V}(bL_x) = b^2 \mathrm{V}(L_x) = b^2 \left(1 + \frac{P_x}{d}\right)^2 \mathrm{V}(Z_x).$$

Hence, if $e_0^* + r_0^* G + s > 0$ and $\mathrm{V}(Z_x) > 0$,

$$\mathrm{V}({}_0L_e) > \mathrm{V}(b \cdot {}_0L_x).$$

The increase in the loss by including expenses is

$${}_0L_e - {}_0L = (e_0^* + r_0^* G + s) \cdot {}_0L_x.$$

Many variations of this model are possible.

In the fully continuous case, the expense–augmented loss and the expense–augmented premium have expressions similar to the fully discrete case. Let b be the death benefit death paid at the time of the death. The fixed issue cost is e_0^*. The percentage of the expense–augmented premium paid in expenses at issue is r_0^*. There is an annual rate of contract expenses of e paid continuously while (x) is alive. The percentage of the expense–augmented premium paid continuously in expenses while (x) is alive is r. The settlement cost is s.

Let G be the expense–augmented premium rate using the equivalence principle. We have that

$$\begin{aligned} G\overline{a}_x &= b\overline{A}_x + e_0^* + r_0^* G + e\overline{a}_x + rG\overline{a}_x + s\overline{A}_x \\ &= e_0^* + r_0^* G + (b+s)\overline{A}_x + (rG + e)\overline{a}_x. \end{aligned}$$

So,

$$G = \frac{e_0^* + (b+s)\overline{A}_x + e\overline{a}_x}{(1-r)\overline{a}_x - r_0^*}.$$

In this situation the expense–augmented loss at issue random variable is

$${}_0\overline{L}_e = e_0^* + r_0^* G + (b+s)\overline{Z}_x - ((1-r)G - e)\overline{Y}_x.$$

Computations identical to the ones done in the fully discrete case give that

$$\begin{aligned} {}_0\overline{L}_e &= (e_0^* + r_0^* G + b + s)\overline{L}_x = (e_0^* + r_0^* G + b + s)(\overline{A}_x - \overline{P}_x \overline{a}_x) \\ \textit{and} \quad \mathrm{V}({}_0\overline{L}_e) &= (e_0^* + r_0^* G + b + s)^2 \left(1 + \frac{\overline{P}_x}{\delta}\right)^2 \mathrm{V}(\overline{Z}_x) \\ &= (e_0^* + r_0^* G + b + s)^2 \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(1 - \overline{A}_x)^2} \\ &= (e_0^* + r_0^* G + b + s)^2 \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(\delta \overline{a}_x)^2}. \end{aligned}$$

The loss for whole life insurance paying a death benefit of b at the time of the death without including expenses is ${}_0\overline{L} = b\overline{L}_x = b(\overline{Z}_x - \overline{P}_x\overline{Y}_x)$. The increase in the loss by including expenses:

$$ {}_0\overline{L}_e - b \cdot {}_0\overline{L}_x = (e_0 + r_0G + s) \cdot {}_0\overline{L}_x = (e_0 + r_0G + s)(\overline{Z}_x - \overline{P}_x\overline{Y}_x). $$

Connecting Ex. 6.48.

Example 6.50. *For a fully continuous whole life insurance of \$50,000 on (x), suppose:*
(i) The issuing expenses are \$1000 and 5% of the expense–augmented annual premium rate.
(ii) The annual rate of continuous maintenance expense is \$250.
(iii) There exists a continuous rate of expenses which is 10% of the benefit premium rate.
(iv) $\delta = 0.06$.
(v) $\overline{a}_x = 12$.
(vi) $\mathrm{V}(\overline{Z}_x) = 0.15$.
(a) Calculate the expense–augmented annual premium rate G using the equivalence principle.
(b) Calculate the variance of the expense-augmented loss random variable.

Solution: (a) Find G, the expense–augmented annual premium rate, such that $E(L_e) = 0$.

What is the benefit premium rate in (iii) ?

Face value of death benefit B= ?

$$ L_e = \underbrace{(50000)\overline{Z}_x + 1000 + (0.05)G + 250\overline{Y}_x + (0.10)G\overline{Y}_x}_{expenses} - \underbrace{G\overline{Y}_x}_{deposit}. $$

$$ E(L_e) = 0 = (50000)\overline{A}_x + 1000 + (0.05)G + 250\overline{a}_x + (0.10)G\overline{a}_x - G\overline{a}_x, \quad \overline{a}_x = 12 \qquad (1) $$

$(G, \overline{A}_x) = ?$

By [17], $12 = \overline{a}_x = \frac{1-\overline{A}_x}{\delta}$,

$\overline{A}_x = 1 - (0.06)\overline{a}_x = 1 - (0.06)(12) = 0.28$ and

$$ \begin{aligned} Eq.(1) => \quad 0 =& (50000)(0.28) + 1000 + (0.05)G + (250)(12) + (0.10)(12)G - (12)G \\ =& 18000 + 1.25G - (12)G \\ G =& \frac{18000}{12 - 1.25} = 1674.418605. \end{aligned} $$

(b) Solve $V(L_e)$ with given $\qquad V(\overline{Z}_x) = 0.15.$

$$ \begin{aligned} L_e =& (50000)\overline{Z}_x + 1000 + (0.05)G + 250\overline{Y}_x + (0.10)G\overline{Y}_x - G\overline{Y}_x \\ =& (50000)\overline{Z}_x + 1000 + (0.05)G + (250 - 0.90G)\frac{1 - \overline{Z}_x}{\delta} \quad (see\ [17]) \\ =& (50000 - \frac{250 - 0.9G}{\delta})\overline{Z}_x + b \\ =& a\overline{Z}_x + b \quad (a = 50000 - \frac{250 - 0.9G}{\delta}). \\ \mathrm{V}(L_e) =& a^2V(\overline{Z}_x) = (50000 - \frac{250 - 0.9G}{\delta})^2 \times 0.15 = 755077125. \end{aligned} $$

Different types of expenses can be paid during different length of times. This happens if benefits premiums and the policy are in hold for different periods.

Example 6.51. *A 10–payment, fully discrete, 20–year term insurance policy with face value of \$90000 payable at the end of the year of death is made to (45). The costs are:*
(i) \$275 at the beginning of each year which the policy is active.
(ii) Per thousand expenses are \$2.5 at the beginning of each year which the policy is active.
(iii) 1% for each annual premium received.
Assume that $i = 6\%$ and death follows the life table for the USA population in 2004 (see page 602). Find the gross annual premium using the equivalence principle.

Solution: Find G, where

$$L = \underbrace{(90000)Z^1_{45:\overline{20}|} + 275\ddot{Y}_{45:\overline{20}|} + (2.5)(90)\ddot{Y}_{45:\overline{20}|} + (0.01)G\ddot{Y}_{45:\overline{10}|}}_{expenses} - \underbrace{G\ddot{Y}_{45:\overline{10}|}}_{deposit}$$

Why sometime 20 and sometime 10 in $\ddot{Y}_{45:\overline{20}|}$ and $\ddot{Y}_{45:\overline{10}|}$?

Tables give A_x, ${}_{20}A_x$, $\ddot{a}_x$ etc.

$$E(L) = 0 \quad => \quad (90000)A^1_{45:\overline{20}|} + 275\ddot{a}_{45:\overline{20}|} + (2.5)(90)\ddot{a}_{45:\overline{20}|} + (0.01)G\ddot{a}_{45:\overline{10}|} - G\ddot{a}_{45:\overline{10}|} = 0$$

$$G = \frac{90000A^1_{45:\overline{20}|} + 500\ddot{a}_{45:\overline{20}|}}{0.99\ddot{a}_{45:\overline{10}|}}. \quad A^1_{45:\overline{20}|} = ? \quad \ddot{a}_{45:\overline{20}|} = ? \quad \ddot{a}_{45:\overline{10}|} = ??$$

$$\begin{aligned}
A^1_{45:\overline{20}|} &= A_{45} - {}_{20|}A_{45} = A_{45} - {}_{20}E_{45}A_{65} \quad (\textit{see } [14]) \\
&= 0.16656845 - (0.271632162)(0.37609614) = 0.06440864237, \\
\ddot{a}_{45:\overline{10}|} &= \ddot{a}_{45} - {}_{10|}\ddot{a}_{45} = \ddot{a}_{45} - {}_{10}E_{45}\ddot{a}_{55} \quad (\textit{see } [18]) \\
&= 14.723957 - (0.534696682)(13.160819) = 7.686910748, \\
\ddot{a}_{45:\overline{20}|} &= \ddot{a}_{45} - {}_{20|}\ddot{a}_{45} = \ddot{a}_{45} - {}_{20}E_{45}\ddot{a}_{65} \quad (\textit{see } [18]) \\
&= 14.723957 - (0.271632162)(11.022302) = 11.72994528.
\end{aligned}$$

$$G = \frac{(90000)(0.06440864237) + (500)(11.72994528)}{(0.99)(7.686910748)} = 1532.416116.$$

Often the first year expenses are different from the rest of the years. Usually, it is easier to express expenses as a level expense for all years plus an extra first year expense.

Example 6.52. *For a 5–payment 20–year endowment insurance of \$100,000 on (25), you are given the following:*
(i) Percent of expense–loaded premium expenses are 10% in the first year and 2% thereafter.
(ii) Per active policy expenses are \$200 in the first year and \$80 in each year thereafter.
(iii) Expenses are paid at the beginning of each policy year.
(iv) Death benefits are payable at the end of the year of death.
(v) $i = 6\%$.
(vi) Mortality follows the life table for the USA population in 2004 (see page 603).
Calculate the expense-loaded premium using the equivalence principle.

Solution: Solve G, where the loss is

$$\begin{aligned} L =&(100000)Z_{25:\overline{20}|} + (0.08)G + (0.02)G\ddot{Y}_{25:\overline{5}|} + 120 + 80\ddot{Y}_{25:\overline{20}|} - G\ddot{Y}_{25:\overline{5}|}. \\ E(L) = 0 =&(100000)A_{25:\overline{20}|} + (0.08)G + (0.02)G\ddot{a}_{25:\overline{5}|} + 120 + 80\ddot{a}_{25:\overline{20}|} - G\ddot{a}_{25:\overline{5}|} \\ G =&\frac{(100000)A_{25:\overline{20}|} + 120 + 80\ddot{a}_{25:\overline{20}|}}{(1-0.02)\ddot{a}_{25:\overline{5}|} - 0.08}. \quad \textit{need } A_{25:\overline{20}|}, \quad \ddot{a}_{25:\overline{20}|}, \quad \ddot{a}_{25:\overline{5}|} \end{aligned}$$

We have that $A_{25:\overline{20}|} = A^{1}_{25:\overline{20}|} + {}_{20}E_{25}$ (*see* [14])

$$\begin{aligned} \textit{but } A^{1}_{25:\overline{20}|} &= A_{25} - {}_{20|}A_{25} = A_{25} - {}_{20}E_{25}A_{45} \quad (\textit{see } [14]) \\ &= 0.065231113 - (0.302791379)(0.16656845) = 0.01479562233, \\ A_{25:\overline{20}|} &= 0.01479562233 + 0.302791379 = 0.3175870013, \\ \ddot{a}_{25:\overline{5}|} &= \ddot{a}_{25} - {}_{5|}\ddot{a}_{25} = \ddot{a}_{25} - {}_{5}E_{25} \cdot \ddot{a}_{30} \quad (\textit{see } [18]) \\ &= 16.51425 - (0.743683357)(16.212781) = 4.4570746, \\ \ddot{a}_{25:\overline{20}|} &= \ddot{a}_{25} - {}_{20}E_{25}\ddot{a}_{45} \\ &= 16.51425 - (0.302791379)(14.723957) = 12.05596276. \end{aligned}$$

Hence,

$$G = \frac{(100000)(0.3175870013) + 120 + 80(12.05596276)}{(4.4570746)(0.98) - 0.08} = 7659.442515.$$