6.10 Premiums found including expenses.

When finding the annual premium expenses and commissions have to be taken into in account. Possible costs are underwriting (making the policy) and maintaining the policy. The annual premium which an insurance company charges is called the **gross annual premium**, the **contract premium**, the **loaded premium** and the expense–augmented premium. It often includes:

1. Issue cost.
2. Percentage of annual benefit premium.
3. Fixed amount per policy.
4. Percentage of (face value) contract amount.
5. Settlement cost.

Often the expenses related to the contract amount, are given as per thousand expenses, i.e. the per thousand expenses are the expenses made for each \$1,000 of the face value of the insurance. The loss is

$$L_e = \text{expenses-deposit (or } L_e = \text{expenses-total annual premium)}$$

Example 6.48. *A fully discrete whole life insurance policy with face value of \$50,000 is made to (x). The following costs are incurred:*
(i) \$800 for making the contract.
(ii) Percent of expense–loaded premium expenses are 6% in the first year and 2% thereafter.
(iii) Per thousand expenses are \$2 per year.
Assume $P_x = A_x/\ddot{a}_x = 0.11$ All expenses are paid at the beginning of the year. $d = 5\%$. Calculate the expense–augmented annual premium G using the equivalence principle.

Solution: Solve G from $E(L_e) = 0$, where

$$L_e = \underbrace{(50000)Z_x + 800 + \overbrace{(0.04)}^{or\ 0.06?} G + (0.02)G\ddot{Y}_x + (2)(50)\ddot{Y}_x}_{expenses} - \underbrace{G\ddot{Y}_x}_{deposit}.$$

$$E(L_e) = (50000)A_x + 800 + (0.04)G + (0.02)G\ddot{a}_x + (2)(50)\ddot{a}_x - G\ddot{a}_x = 0. \qquad A_x/\ddot{a}_x = 11$$

$$=> \quad G = \frac{(50000)A_x + 800 + (2)(50)\ddot{a}_x}{(1-0.02)\ddot{a}_x - 0.04} =? \ \ddot{a}_x =? \ \ A_x =? \qquad [17]\ \ddot{a}_x = \frac{1-A_x}{1-v}$$

$0.11 = P_x = \frac{A_x}{\ddot{a}_x} = \frac{1-d\ddot{a}_x}{\ddot{a}_x} = \frac{A_x}{(1-A_x)/d}$. $d = 1 - v = 0.05$.
$0.11 = \frac{1-d\ddot{a}_x}{\ddot{a}_x} => 0.11\ddot{a}_x = 1 - d\ddot{a}_x =>$

$\ddot{a}_x = \frac{1}{d}/(1 + \frac{0.11}{d}) = 1/(0.05 + 0.11) = 10/16$.

$A_x = P_x\ddot{a}_x = 11/16$.

$G = \frac{(50000)A_x+800+(2)(50)\ddot{a}_x}{(1-0.02)\ddot{a}_x-0.04} = 5883.319638$.

Example 6.49. *A whole life insurance policy with face value of \$40,000 payable at the end of the year of death is made to (45). Assume that $i = 4.5\%$ and death is modeled using the de Moivre model with terminal age 95. The annual benifit premium is paid at the beginning of the year, and the following costs are incurred and paid at the beginning of the year:*
(i) \$500 for making the contract.
(ii) Percent of expense–loaded premium expenses is 5% in the first year and 1% thereafter.
(iii) Per policy expenses are \$20 per year.
(iv) Per thousand expenses are \$1.2 per year.
(v) \$600 for settlement.
(a) Calculate the gross annual premium G using the equivalence principle.
(b) Calculate the expense–augmented loss for an insuree that dies 7 years, 5 months and 10 days after the issue of this policy.
(c) Calculate the variance of the expense–augmented loss.

Solution: (a) Solve G from $E(L_e) = 0$, where the loss

$$\begin{aligned} L_e =& 40000Z_{45} + 500 + 0.04G + 0.01G\ddot{Y}_{45} + 20\ddot{Y}_{45} + (1.2)(40)\ddot{Y}_{45} + 600Z_{45} - G\ddot{Y}_{45}, \quad (1)\\ => \quad 0 =& 40000A_{45} + 500 + 0.04G + 0.01G\ddot{a}_{45} + 20\ddot{a}_{45} + (1.2)(40)\ddot{a}_{45} + 600A_{45} - G\ddot{a}_{45},\\ => \quad G =& \frac{500 + 40600A_{45} + 68\ddot{a}_{45}}{0.99\ddot{a}_{45} - 0.04}. \quad \ddot{a}_{45}, A_{45} =? \quad T_{45} \sim U(0, 50). \end{aligned}$$

$$\begin{aligned} [14] \to \quad A_{45} =& E(v^{K_x}) = \sum_{k=1}^{50} v^k \frac{1}{50} = v\frac{1-v^{50}}{1-v}\frac{1}{50}\bigg|_{v=1/1.045} = 0.3952401556, \qquad (2)\\ [17] \to \quad \ddot{a}_{45} =& \frac{1-A_{45}}{1-v} = \frac{1-0.3952401556}{1-1/1.045} = 14.0438675,\\ G =& \frac{500 + 40600A_{45} + 68\ddot{a}_{45}}{0.99\ddot{a}_{45} - 0.04} = 1262.439006. \end{aligned}$$

(b) Solve $L_e(K_x)$. Where is K_x in L_e ? Insuree die between 7 and 8 years, $K_x =$??

$$\begin{aligned} L_e =& [40000Z_{45} + 500 + 0.04G + 0.01G\ddot{Y}_{45} + 20\ddot{Y}_{45} + 40 \times 1.2\ddot{Y}_{45} + 600Z_{45} - G\ddot{Y}_{45}] \quad \textit{by } (1)\\ =& (40600)Z_{45} + 550.4975602 - 1181.814616\ddot{Y}_{45}\\ =& (40600)v^{K_{45}} + 550.4975602 - 1181.814616\frac{1-v^{K_{45}}}{1-v} \quad (\textit{see } [14] \textit{ and } [17])\\ =& \underbrace{68044.36v^{K_{45}} - 26893.86}_{\textit{why do this step??}} = 18244.28524 \qquad \textit{if } K_{45} = 8. \end{aligned}$$

$$\begin{aligned} (c) \quad \mathrm{V}(L_e) =& a^2\mathrm{V}(Z_{45}) = (68044.36)^2(A_{45}(v^2) - (A_{45})^2) \qquad (\textit{see } (2))\\ =& (68044.36)^2(v\frac{1-v^{50}}{(1-v)50}\bigg|_{v=1/1.045^2} - (A_{45})^2)\\ =& (68044.36)^2(0.2146684865 - (0.3952401556)^2) = 270642713.1. \end{aligned}$$

Skip to Page 205 Example 6.49.

Theorem 6.28. *Suppose that we have a whole life insurance on (x), with a death benefit of b paid at the end of the year of death. The fixed annual cost has an amount of e. In the first year, there exists an additional cost of e_0^*. The percentage of the expense–augmented premium paid in expenses each year is r. During the first year, it is paid an additional percentage of the expense–augmented premium of r_0^*. The settlement cost is s. All cost except the settlement cost are paid at the beginning of the year. The insurance is funded by an expense–augmented premium of G paid at the beginning of the year while (x) is alive. If the equivalence principle is used, then*

$$G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}. \tag{1}$$

Using that $P_x = \frac{A_x}{\ddot{a}_x}$ and $P_x + d = \frac{1}{\ddot{a}_x}$ (easy to verify), we get that the expense–augmented annual benefit premium using the equivalence principle is

$$G = \frac{e_0^*(P_x + d) + (b+s)P_x + e}{1 - r - r_0^*(P_x + d)}. \tag{2}$$

Q: Why Eq. (2) ?

Eq. (1) needs to specify f_{T_x}, but Eq. (2) does not.

The expense–augmented loss at issue random variable is the present value of expenses plus the present value of benefit minus the present value of premiums, i.e. it is

$$\begin{aligned} {}_0L_e =& e_0^* + r_0^*G + (b+s)Z_x + (rG+e)\ddot{Y}_x - G\ddot{Y}_x, \\ =& e_0^* + r_0^*G + (b+s)Z_x - ((1-r)G - e)\frac{1-Z_x}{d} = aZ_x + b. \end{aligned}$$

Theorem 6.29. *Under the conditions in the previous theorem,*

(i) The expense–augmented loss at issue random variable is

$$\begin{aligned} {}_0L_e =& (e_0^* + r_0^*G + b + s)(Z_x - P_x\ddot{Y}_x) \\ =& (e_0^* + r_0^*G + b + s){}_0L_x. \end{aligned}$$

(ii) The variance of the expense–augmented loss is

$$\begin{aligned} \mathrm{V}({}_0L_e) =& (e_0^* + r_0^*G + b + s)^2\mathrm{V}(L_x) \\ =& (e_0^* + r_0^*G + b + s)^2\left(1 + \frac{P_x}{d}\right)^2\mathrm{V}(Z_x) \\ =& (e_0^* + r_0^*G + b + s)^2\frac{{}^2A_x - A_x{}^2}{(1-A_x)^2} \\ =& (e_0^* + r_0^*G + b + s)^2\frac{{}^2A_x - A_x{}^2}{(d\ddot{a}_x)^2}. \end{aligned}$$

When we compute the expense–augmented loss we get an expression of the type $c_1 + c_2Z_x - c_2\ddot{Y}_x$. The proof of the previous theorem gives that

$${}_0L_e = c_1 + c_2Z_x - c_2\ddot{Y}_x = (c_1 + c_2)L_x \text{ and}$$

$$\mathrm{V}({}_0L_e) = (c_1 + c_2)^2 \mathrm{V}(L_x).$$

The loss without including expenses for a whole life insurance with death benefit b is ${}_0L = bL_x = b(Z_x - P_x\ddot{Y}_x)$. The variance of this loss is

$$\mathrm{V}(bL_x) = b^2\mathrm{V}(L_x) = b^2\left(1 + \frac{P_x}{d}\right)^2 \mathrm{V}(Z_x).$$

Hence, if $e_0^* + r_0^*G + s > 0$ and $\mathrm{V}(Z_x) > 0$,

$$\mathrm{V}({}_0L_e) > \mathrm{V}(b \cdot {}_0L_x).$$

The increase in the loss by including expenses is

$${}_0L_e - {}_0L = (e_0^* + r_0^*G + s) \cdot {}_0L_x.$$

Many variations of this model are possible.

In the fully continuous case, the expense–augmented loss and the expense–augmented premium have expressions similar to the fully discrete case. Let b be the death benefit death paid at the time of the death. The fixed issue cost is e_0^*. The percentage of the expense–augmented premium paid in expenses at issue is r_0^*. There is an annual rate of contract expenses of e paid continuously while (x) is alive. The percentage of the expense–augmented premium paid continuously in expenses while (x) is alive is r. The settlement cost is s.

Let G be the expense–augmented premium rate using the equivalence principle. We have that

$$\begin{aligned} G\overline{a}_x &= b\overline{A}_x + e_0^* + r_0^*G + e\overline{a}_x + rG\overline{a}_x + s\overline{A}_x \\ &= e_0^* + r_0^*G + (b+s)\overline{A}_x + (rG + e)\overline{a}_x. \end{aligned}$$

So,

$$G = \frac{e_0^* + (b+s)\overline{A}_x + e\overline{a}_x}{(1-r)\overline{a}_x - r_0^*}.$$

In this situation the expense–augmented loss at issue random variable is

$${}_0\overline{L}_e = e_0^* + r_0^*G + (b+s)\overline{Z}_x - ((1-r)G - e)\overline{Y}_x.$$

Computations identical to the ones done in the fully discrete case give that

$${}_0\overline{L}_e = (e_0^* + r_0^*G + b + s)\overline{L}_x = (e_0^* + r_0^*G + b + s)(\overline{A}_x - \overline{P}_x\overline{a}_x)$$

$$\begin{aligned} \textit{and} \quad \mathrm{V}({}_0\overline{L}_e) &= (e_0^* + r_0^*G + b + s)^2\left(1 + \frac{\overline{P}_x}{\delta}\right)^2 \mathrm{V}(\overline{Z}_x) \\ &= (e_0^* + r_0^*G + b + s)^2 \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(1 - \overline{A}_x)^2} \\ &= (e_0^* + r_0^*G + b + s)^2 \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(\delta\overline{a}_x)^2}. \end{aligned}$$

The loss for whole life insurance paying a death benefit of b at the time of the death without including expenses is ${}_0\overline{L} = b\overline{L}_x = b(\overline{Z}_x - \overline{P}_x\overline{Y}_x)$. The increase in the loss by including expenses:

$$ {}_0\overline{L}_e - b \cdot {}_0\overline{L}_x = (e_0 + r_0G + s) \cdot {}_0\overline{L}_x = (e_0 + r_0G + s)(\overline{Z}_x - \overline{P}_x\overline{Y}_x). $$

Connecting Ex. 6.48.

Example 6.50. *For a fully continuous whole life insurance of \$50,000 on* (x)*, suppose:*
(i) The issuing expenses are \$1000 and 5% of the expense–augmented annual premium rate.
(ii) The annual rate of continuous maintenance expense is \$250.
(iii) There exists a continuous rate of expenses which is 10% of the benefit premium rate.
(iv) $\delta = 0.06$.
(v) $\overline{a}_x = 12$.
(vi) $\mathrm{V}(\overline{Z}_x) = 0.15$.
(a) Calculate the expense–augmented annual premium rate G using the equivalence principle.
(b) Calculate the variance of the expense-augmented loss random variable.

Solution: (a) Find G, the expense–augmented annual premium rate, such that $E(L_e) = 0$.

What is the benefit premium rate in (iii) ?

Face value of death benefit B= ?

$$ L_e = \underbrace{(50000)\overline{Z}_x + 1000 + (0.05)G + 250\overline{Y}_x + (0.10)G\overline{Y}_x}_{expenses} - \underbrace{G\overline{Y}_x}_{deposit}. $$

$$ E(L_e) = 0 = (50000)\overline{A}_x + 1000 + (0.05)G + 250\overline{a}_x + (0.10)G\overline{a}_x - G\overline{a}_x, \quad \overline{a}_x = 12 \qquad (1) $$

$(G, \overline{A}_x) = ?$

By [17], $12 = \overline{a}_x = \frac{1-\overline{A}_x}{\delta}$,

$\overline{A}_x = 1 - (0.06)\overline{a}_x = 1 - (0.06)(12) = 0.28$ and

$$ \begin{aligned} Eq.(1) => \quad 0 =& (50000)(0.28) + 1000 + (0.05)G + (250)(12) + (0.10)(12)G - (12)G \\ =& 18000 + 1.25G - (12)G \\ G =& \frac{18000}{12 - 1.25} = 1674.418605. \end{aligned} $$

(b) Solve $V(L_e)$ with given $V(\overline{Z}_x) = 0.15$.

$$ \begin{aligned} L_e =& (50000)\overline{Z}_x + 1000 + (0.05)G + 250\overline{Y}_x + (0.10)G\overline{Y}_x - G\overline{Y}_x \\ =& (50000)\overline{Z}_x + 1000 + (0.05)G + (250 - 0.90G)\frac{1 - \overline{Z}_x}{\delta} \quad (see\ [17]) \\ =& (50000 - \frac{250 - 0.9G}{\delta})\overline{Z}_x + b \\ =& a\overline{Z}_x + b \quad (a = 50000 - \frac{250 - 0.9G}{\delta}). \\ \mathrm{V}(L_e) =& a^2 V(\overline{Z}_x) = (50000 - \frac{250 - 0.9G}{\delta})^2 \times 0.15 = 755077125. \end{aligned} $$

Different types of expenses can be paid during different length of times. This happens if benefits premiums and the policy are in hold for different periods.

Example 6.51. *A 10–payment, fully discrete, 20–year term insurance policy with face value of \$90000 payable at the end of the year of death is made to (45). The costs are:*
(i) \$275 at the beginning of each year which the policy is active.
(ii) Per thousand expenses are \$2.5 at the beginning of each year which the policy is active.
(iii) 1% for each annual premium received.
Assume that $i = 6\%$ and death follows the life table for the USA population in 2004 (see page 602). Find the gross annual premium using the equivalence principle.

Solution: Find G, where

$$L = \underbrace{(90000)Z^1_{45:\overline{20}|} + 275\ddot{Y}_{45:\overline{20}|} + (2.5)(90)\ddot{Y}_{45:\overline{20}|} + (0.01)G\ddot{Y}_{45:\overline{10}|}}_{expenses} - \underbrace{G\ddot{Y}_{45:\overline{10}|}}_{deposit}$$

Why sometime 20 and sometime 10 in $\ddot{Y}_{45:\overline{20}|}$ and $\ddot{Y}_{45:\overline{10}|}$?

Tables give A_x, ${}_{20}A_x$, $\ddot{a}_x$ etc.

$$E(L) = 0 \quad => \quad (90000)A^1_{45:\overline{20}|} + 275\ddot{a}_{45:\overline{20}|} + (2.5)(90)\ddot{a}_{45:\overline{20}|} + (0.01)G\ddot{a}_{45:\overline{10}|} - G\ddot{a}_{45:\overline{10}|} = 0$$

$$G = \frac{90000A^1_{45:\overline{20}|} + 500\ddot{a}_{45:\overline{20}|}}{0.99\ddot{a}_{45:\overline{10}|}}. \quad A^1_{45:\overline{20}|} = ? \quad \ddot{a}_{45:\overline{20}|} = ? \quad \ddot{a}_{45:\overline{10}|} = ??$$

$$\begin{aligned}
A^1_{45:\overline{20}|} &= A_{45} - {}_{20|}A_{45} = A_{45} - {}_{20}E_{45}A_{65} \quad (\textit{see } [14]) \\
&= 0.16656845 - (0.271632162)(0.37609614) = 0.06440864237, \\
\ddot{a}_{45:\overline{10}|} &= \ddot{a}_{45} - {}_{10|}\ddot{a}_{45} = \ddot{a}_{45} - {}_{10}E_{45}\ddot{a}_{55} \quad (\textit{see } [18]) \\
&= 14.723957 - (0.534696682)(13.160819) = 7.686910748, \\
\ddot{a}_{45:\overline{20}|} &= \ddot{a}_{45} - {}_{20|}\ddot{a}_{45} = \ddot{a}_{45} - {}_{20}E_{45}\ddot{a}_{65} \quad (\textit{see } [18]) \\
&= 14.723957 - (0.271632162)(11.022302) = 11.72994528.
\end{aligned}$$

$$G = \frac{(90000)(0.06440864237) + (500)(11.72994528)}{(0.99)(7.686910748)} = 1532.416116.$$

Often the first year expenses are different from the rest of the years. Usually, it is easier to express expenses as a level expense for all years plus an extra first year expense.

Example 6.52. *For a 5–payment 20–year endowment insurance of \$100,000 on (25), you are given the following:*
(i) Percent of expense–loaded premium expenses are 10% in the first year and 2% thereafter.
(ii) Per active policy expenses are \$200 in the first year and \$80 in each year thereafter.
(iii) Expenses are paid at the beginning of each policy year.
(iv) Death benefits are payable at the end of the year of death.
(v) $i = 6\%$.
(vi) Mortality follows the life table for the USA population in 2004 (see page 603).
Calculate the expense-loaded premium using the equivalence principle.

Solution: Solve G, where the loss is

$$\begin{aligned} L =&(100000)Z_{25:\overline{20}|} + (0.08)G + (0.02)G\ddot{Y}_{25:\overline{5}|} + 120 + 80\ddot{Y}_{25:\overline{20}|} - G\ddot{Y}_{25:\overline{5}|}. \\ E(L) = 0 =&(100000)A_{25:\overline{20}|} + (0.08)G + (0.02)G\ddot{a}_{25:\overline{5}|} + 120 + 80\ddot{a}_{25:\overline{20}|} - G\ddot{a}_{25:\overline{5}|} \\ G =&\frac{(100000)A_{25:\overline{20}|} + 120 + 80\ddot{a}_{25:\overline{20}|}}{(1-0.02)\ddot{a}_{25:\overline{5}|} - 0.08}. \quad \textit{need } A_{25:\overline{20}|}, \quad \ddot{a}_{25:\overline{20}|}, \quad \ddot{a}_{25:\overline{5}|} \end{aligned}$$

We have that $A_{25:\overline{20}|} = A^{1}_{25:\overline{20}|} + {}_{20}E_{25}$ (*see* [14])

$$\begin{aligned} \textit{but } A^{1}_{25:\overline{20}|} &= A_{25} - {}_{20|}A_{25} = A_{25} - {}_{20}E_{25}A_{45} \quad (\textit{see } [14]) \\ &= 0.065231113 - (0.302791379)(0.16656845) = 0.01479562233, \\ A_{25:\overline{20}|} &= 0.01479562233 + 0.302791379 = 0.3175870013, \\ \ddot{a}_{25:\overline{5}|} &= \ddot{a}_{25} - {}_{5|}\ddot{a}_{25} = \ddot{a}_{25} - {}_{5}E_{25} \cdot \ddot{a}_{30} \quad (\textit{see } [18]) \\ &= 16.51425 - (0.743683357)(16.212781) = 4.4570746, \\ \ddot{a}_{25:\overline{20}|} &= \ddot{a}_{25} - {}_{20}E_{25}\ddot{a}_{45} \\ &= 16.51425 - (0.302791379)(14.723957) = 12.05596276. \end{aligned}$$

Hence,

$$G = \frac{(100000)(0.3175870013) + 120 + 80(12.05596276)}{(4.4570746)(0.98) - 0.08} = 7659.442515.$$