Announcement: The homework assigned this weekend due next Wednesday.

Example 6.38. *Consider a whole life insurance policy to* (40) *with face value of \$250000 payable at the end of the year of death. This policy will be paid by benefit annual premiums paid at the beginning of each year while* (40) *is alive. Suppose that the premiums increase by 6% each year. Assume that $i = 6\%$ and death is modeled using the de Moivre model with terminal age 100. Find the amount of the first benefit annual premium for this policy.*

Solution: Let π be the amount of the first benefit premium (in unit value).
Solve $B\pi$ from $E(L) = 0$, where $B = 250000$,
$L = v^{K_x} - \sum_{k=0}^{K_x-1} \pi_k v^k$, and $\pi_k = \pi(1.06)^k$, $k = 0, 1, 2, \ldots$
$L = v^{K_x} - \sum_{k=0}^{K_x-1} \pi(1.06)^k v^k = v^{K_x} - \sum_{k=0}^{K_x-1} \pi(1.06v)^k$.
$0 = A_x - \pi E \sum_{k=0}^{K_x-1}(1.06v)^k)$. =>

$$\pi = A_x/E(\sum_{k=0}^{K_x-1}(1.06v)^k)$$

$$A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(x) = \sum_{k=1}^{w-x} v^k \frac{1}{w-x} = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x} = 0.2693571284.$$

$$E(\sum_{k=0}^{K_x-1}(1.06v)^k) = E(\sum_{k=0}^{K_x-1} 1) = E(K_x) = \sum_{k=1}^{\infty} k f_{K_x}(k) = \sum_{k=1}^{60} k\frac{1}{60} \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$= \frac{(60)(61)}{2}\frac{1}{60} = 30.5.$$

$$\pi = A_x/E\sum_{k=0}^{K_x-1}(1.06v)^k) = \frac{0.2693571284}{30.5}$$

$$B\pi = (250000)\frac{0.2693571284}{30.5} = 2207.845$$

250000*0.2693571284/30.5

Additonal. Redo #20 with two changes: (1) aged 20 − > aged 30.

(2) the last probability 0.01 being changed to 0.02.

6.20. An insurance company offers a whole life insurance to lives aged 20 paying 75000 at the end of the year of death. Each insuree will make an annual premium of P at the beginning of year while he is alive to fund this insurance. Suppose that $i = 6\%$ and that mortality follows the life table for the USA population in 2004 (see pages **??**–**??**). P is 15% higher than the value obtained using the equivalence principle. Using the central limit theorem, calculate the smallest number of policies that the insurance company must sell so that the probability the aggregate loss is positive is less or equal than 0.01. **Answer:** 795.

Sol. It suffices to assume unit payment: $L = Z_x - P\ddot{Y}_x$.

$P(L > 0) \approx 0.02 <=> \frac{L-E(L)}{\sigma_L} \sim N(0,1)$.

qnorm$(1 - 0.02)$ [1] 2.053749

$\frac{0-E(L)}{\sigma_L} \approx 2.05$ or 2.06. **(From Normal table)**

$L = Z_x - P\ddot{Y}_x = Z_x - P\frac{1-Z_x}{1-v} = Z_x(1 + \frac{P}{1-v}) - \frac{P}{1-v}$

$E(L) = A_x(1 + \frac{P}{1-v}) - \frac{P}{1-v}$

$A_x = 0.08229543$

${}^2A_x$=0.01796859

$P = 1.15\frac{A_x}{\ddot{a}_x} = 1.15\frac{A_x}{(1-A_x)/(1-v)}$.

$V(L) = (A_x(v^2) - (A_x(v))^2)(1 + \frac{P}{1-v})^2$

$-nE(L) + 2.053\sqrt{nV(L)} \approx 0$

$-\sqrt{n}E(L) + 2.053\sqrt{V(L)} \approx 0$

$n \approx 2.053^2 V(L)/(E(L))^2$

R codes:

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v=1/1.06
A=0.08229543
B= 0.01796859 # replacing A =0.052456 and B= 0.010781
P=1.15*A/(1-A)*(1-v)
E=A*(1+P/(1-v))-P/(1-v)
V=(B-A*A)*(1+P/(1-v))**2
n=2.053**2*V/E**2
n [1] 376.8422
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Ans: n=377.

Example 6.39.

Example 6.40.

Example 6.41.

Example 6.42.

Theorem 6.11.

Theorem 6.12.

Theorem 6.13.

Theorem 6.14.

Theorem 6.15.

Theorem 6.16.

Theorem 6.17.

Theorem 6.18.

Theorem 6.19.

Theorem 6.20.

Theorem 6.21.

Theorem 6.22.

Theorem 6.23.

Theorem 6.24.

Example 6.43.

6.9 Computing benefit premiums from a life table

In this section, we disucuss how to obtain the benefit rates for different insurance products.

6.9.1 Fully discrete insurance. We assume that the death benefits are paid at the end of the year of death. But, the benefit premiums are at the beginning of each m–thly period. The annual benefit premium in this situation is higher than the regular situation. Benefit premiums, instead of being received at the beginning of the year, they are received later on. During a year when an insuree dies, benefit premiums may not be received during the whole year. From a life table, we can find $\ddot{a}_x$, then we can estimate $\ddot{a}_x^{(m)}$.

Example 6.44. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	46	54	45	62	28	0

Assume that $i = 6.5\%$ and uniform distribution of deaths over each year of death. Find $P_{80}^{(12)}$, using that $A_{80} = 0.8161901166$ (see Example 4.9 in page 87).

Solution: There are two appoaches for $P_{80}^{(m)} = \frac{A_{80}}{\ddot{a}_{80}^{(m)}}$:

(1) Based on basic formulas: $v \to v^{1/m}$ and ${}_{j-1}|q_x \to {}_{\frac{j-1}{m}}|_{\frac{1}{m}}q_x = f_{J_x}(j)$ $\quad ({}_0|q_x = q_x \to {}_{\frac{1}{m}}q_x)$.

$$\begin{aligned}
A_x^{(m)} =& A_{80}^{(12)} = E(v^{J_x}) = \sum_{j=1}^{\infty}(v^{1/12})^j f_{J_x}(j) \\
=& (v^{1/12})^1 f_{J_x}(1) + \cdots + (v^{1/12})^{12} f_{J_x}(12) \\
&+ (v^{1/12})^{12+1} f_{J_x}(12+1) + \cdots + (v^{1/12})^{12+12} f_{J_x}(12+12) \\
&+ \cdots \\
=& \sum_{k=0}^{\infty}\sum_{j=1}^{m}(v^{1/12})^{km+j}\frac{1}{m}\frac{d_{x+k}}{\ell_x} \\
=& \frac{1}{m}\sum_{k=0}^{\infty}(v^{1/12})^{km}\frac{d_{x+k}}{\ell_x}\sum_{j=1}^{m}(v^{1/12})^j \\
=& \frac{1}{m}\sum_{j=1}^{m}(v^{1/12})^j\sum_{k=0}^{\infty}v^k\frac{d_{x+k}}{\ell_x} \\
=& \frac{1}{m}\frac{v^{1/12}(1-v^{m/12})}{1-v^{1/12}}A_x = 0.8402293189,
\end{aligned}$$

$\ddot{a}_x^{(12)} = E(\sum_{j=0}^{J_x-1}(v^{1/12})^j) = E(\frac{1-v^{J_x/12}}{1-v^{1/12}}) = \frac{1-A_x^{(12)}}{1-v^{1/12}} = 2.543720348.$
$P_{80}^{(12)} = \frac{A_{80}}{\ddot{a}_{80}^{(12)}} = \frac{0.8161901166}{2.543720348} = 0.3208647198,$

(2) Using formulas: $\ddot{a}_{80}^{(m)} = \dfrac{1-A_{80}^{(m)}}{d^{(m)}}, \qquad d^{(m)} = m(1-(1+i)^{-\frac{1}{m}})$

$A_{80}^{(m)} = \dfrac{i}{i^{(m)}}A_{80}, \qquad i^{(m)} = m((1+i)^{\frac{1}{m}}-1).$

Thus

$i^{(m)} = m((1+i)^{\frac{1}{m}}-1) \approx 6.314\% \qquad A_{80}^{(12)} = \dfrac{i}{i^{(12)}}A_{80} = \dfrac{0.065}{0.06314033132}(0.8161901166) \approx 0.840,$

$d^{(m)} = m(1-(1+i)^{-\frac{1}{m}}) \approx 6.281\% \qquad \ddot{a}_{80}^{(12)} = \dfrac{1-A_{80}^{(12)}}{d^{(12)}} = \dfrac{1-0.8402293189}{0.06281} \approx 2.544,$

$$P_{80}^{(12)} = \frac{A_{80}}{\ddot{a}_{80}^{(12)}} = \frac{0.8161901166}{2.544} \approx 0.321.$$

For a period of length $\frac{1}{m}$:

(i) the interest factor is $(1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$.

(ii) the effective rate of interest is $(1+i)^{1/m} - 1 = \frac{i^{(m)}}{m}$.

(iii) the discount factor is $(1+i)^{-1/m} = v^{1/m} = (1-d)^{1/m} = 1 - \frac{d^{(m)}}{m}$.

(iv) the effective rate of discount is $1 - v^{1/m} = \frac{d^{(m)}}{m}$.

Theorem 6.25. Skip the theorem.

6.9.2 Semicontinuous insurance.

For a semicontinuous insurance, the death benefit is paid at the time of the death and benefit premiums are paid at the beginning of the year.

Example 6.45. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	46	54	45	62	28	0

Assume that $i = 6.5\%$ and uniform distribution of deaths over each year of death. The death benefit is paid at the time of the death and benefit premiums P are paid at the beginning of the year. Find P using equivalent principle.

Solution: $L = v^{T_x} - P\sum_{k=0}^{K_x-1} v^k = v^{T_x} - P\frac{1-v^{K_x}}{1-v}$. $P = \overline{A}_x/\ddot{a}_x$ and $\ddot{a}_x = \frac{1-A_x}{1-v}$ by [17].
$A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k d_{x+k-1}/\ell_x = 0.8161901166$,
$\ddot{a}_{80} = \frac{1-A_{80}}{d} = \frac{1-0.8161901166}{1-(1/1.065)} = 3.011654243$.
$\overline{A}_x = \frac{i}{\delta}A_x$ (**true only for $\overline{A}_x$, $\overline{A}^{\,1}_{x:\overline{n}|}$ and ${}_{n|}\overline{A}_x$!**), or
$\overline{A}_x = \sum_{i=1}^{6}\int_{i-1}^{i} v^t f_{T_x}(t)dt$ (**always true**). Use this approach here.
$f_{T_x}(t) = -\frac{d}{dt}{}_tp_x$ (by (10) of 447).
${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, (by (11))
$\ell_{x+t} =$

$$\begin{cases} \ell_x + t(\ell_{x+1}-\ell_x) & if\ t \in (x, x+1] \\ \ell_{x+1} + t(\ell_{x+2}-\ell_{x+1}) & if\ t \in (x+1, x+2] \\ \ell_{x+2} + t(\ell_{x+3}-\ell_{x+2}) & if\ t \in (x+2, x+3] \\ \ell_{x+3} + t(\ell_{x+4}-\ell_{x+3}) & if\ t \in (x+3, x+4] \\ \ell_{x+4} + t(\ell_{x+5}-\ell_{x+4}) & if\ t \in (x+4, x+5] \\ \ell_{x+5} + t(\ell_{x+6}-\ell_{x+5}) & if\ t \in (x+5, x+6] \end{cases} \quad f_{T_x}(t) = -{}_tp'_x = \begin{cases} \frac{d_x}{\ell_x} & if\ t \in (x, x+1] \\ \frac{d_{x+1}}{\ell_x} & if\ t \in (x+1, x+2] \\ \frac{d_{x+2}}{\ell_x} & if\ t \in (x+2, x+3] \\ \frac{d_{x+3}}{\ell_x} & if\ t \in (x+3, x+4] \\ \frac{d_{x+4}}{\ell_x} & if\ t \in (x+4, x+5] \\ \frac{d_{x+5}}{\ell_x} & if\ t \in (x+5, x+6]. \end{cases}$$

$$f_{T_x}(t) = f_{K_x}(t)?? \qquad f_{T_x}(t) = f_{K_x}(\lceil t \rceil)??$$

$$f_{T_x}(t) = f_{K_x}(\lceil t \rceil) \text{ if } t \text{ is not an integer ??}$$

$$\overline{A}_{80} = \int_0^{\infty} v^t f_{T_x}(t)dt = \sum_{i=1}^{6}\int_{i-1}^{i} v^t \frac{d_{x+i-1}}{l_x}dt = \sum_{i=1}^{6}\left(\frac{v^t}{\ln v}\Big|_{i-1}^{i}\right)\frac{d_{x+i-1}}{l_x} = 0.8424379003.$$

$$P = \frac{\overline{A}_{80}}{\ddot{a}_{80}} = \frac{0.8424379003}{3.011654243} = 0.2797259686.$$

Theorem 6.26.

Theorem 6.27.

6.9.3 Fully continuous insurance.

For a fully continuous insurance, we need to know $\overline{A}_x$ and $\overline{a}_x$. From a life table, we can find A_x and a_x. Then, we need to estimate $\overline{A}_x$ and $\overline{a}_x$.

Example 6.46. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assume that $i = 6.5\%$ and uniform distribution of deaths over each year of death. Find $\overline{P}(\overline{A}_{80})$, using that $A_{80} = 0.8161901166$ (see Example 4.9 in page 87).

Solution: We have that

$$\overline{P} = \frac{\overline{A}_x}{\overline{a}_x} = \frac{\overline{A}_x}{\frac{1-\overline{A}_x}{\delta}}; \quad \overline{A}_{80} = \frac{i}{\delta}A_{80} = \frac{0.065}{\ln(1.065)}(0.8161901166) = 0.8424379003,$$

$$\overline{P}(\overline{A}_{80}) = \frac{\delta\overline{A}_{80}}{1-\overline{A}_{80}} = \frac{\ln(1.065)(0.8424379003)}{(1-0.8424379003)} = 0.3367076072.$$

Example 6.47.