Contents

MATH 452, Long-term Actuarial Math II

The course is a preparation for Long-Term Actuarial Mathematics Exam.

MWF 1:10 - 2:40 CW 206

A class on May 2 Tu for Friday classes.

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Office hours: M,T 7-8pm, through zoom.

https://binghamton.zoom.us/j/8265526594?pwd=d3l6OGx1cmZ4M3cxZEJwVGd1RGcrUT09

Meeting ID: 826 552 6594

Passcode: 031320

Textbook: Arcones' Manual For SOA Exam MLC (Second Volumn).

(Chapters to be covered: 7-12)

It is in my website

http://www2.math.binghamton.edu/p/people/qyu/qyu_personal

Course materials for 452, lecture note 2. **It will dispear next week**!

A pdf file with some tables needed in the homework can be downloaded from **my website**. http://www.math.binghamton.edu/qyu/qyu_personal
e.g, the Illustrative Life Table needed in some of the homework problems.

The lecture notes will be posted on my website.

Exams: 3 tests + final,

Feb. 20(M), Mar. 27(M), April 24(M),

Final May 11 12:50-2:50 pm FA 248 is changed to WH 329.

closed book,

You can bring a calculator **without the function of installing formulas**. I will check !!

Quizes: once a week, on Friday;

this week is on Friday, Formulas #1-9 for 450. (keys are in my website).

Homework: Due Wednesday before class, late homework will take 3 points off (out of 10).
Homework due this Friday: Do the final exam of 450 in my website **including Part A !**

Grading Policy:

1. 10% hw +10% quiz +45% tests +35% final

2. Correction: If you make correction and hand in with the old exam, **within 3 days** after I return the test **in class**, you can get 40% of the missing grades back. No partial credit for correction. Cannot ask me to help you in correction.

3. A- = 85 + and C = 60 +.

10+10+45*(0.3+0.4*0.7)+35*0.3=56

Student Attendance in Class:

The Bulletin states, “Students are expected to attend all scheduled classes, laboratories and discussions. Instructors may establish their own attendance criteria for a course. They may establish both the number of absences permitted to receive credit ....

CHAPTER 7

Benefit Reserves

7.1 Benefit reserves.

When doing the accounting of an entity, the balance sheet of this entity records its **assets** and **liabilities**. Assets are everything of value owned by this entity, including the value of payments to be received in the future. Liabilities include any obligation of this entity. The entity's **equity** is

$$\text{equity} = \text{assets} - \text{liabilities}.$$

When an insurer issues an insurance policy, it assumes a liability. The insurer becomes obligated to make a net cashflow of payments depending on a contingent event. This net cashflow includes both benefit payments and benefit premiums. In insurance parlance, liabilities are called **reserves**. The present value of an insurance liability depends on a contingent event. The **actuarial reserve** of an insurance liability is the expectation of the present value of the net cash flow generated by this insurance liability. An insurer must keep offsetting assets to pay off actuarial reserves. In actuarial parlance, equity is called **surplus**.

Definition 7.1. *The* **loss random variable** *(r.v.) at a certain time t for an insurance contract, which is in effect, is the difference between the present values of future benefit payments and future benefit premiums (payments−premiums). Its APV is called the* **benefit reserve** *at a certain time.*

The loss r.v. defined in Chapter 6 corresponds to the loss r.v. at $t = 0$.

In application, it is more useful to consider t as an integer. The actuarial reserve is further classified as **terminal ones and initial** ones as follows.

Definition 7.2. *The t-th* **terminal loss random variable** *is the loss r.v. for an insurance contract, which is in effect, t years after the issue of a policy immediately before funding is made. Its APV is called the t–***th terminal benefit reserve***. They are denoted by ${}_tL_x$ and ${}_tV_x$, respectively.*

Definition 7.3.

Definition 7.4. *The* $(t+1)$–**th initial benefit reserve** *of an insurance contract, which is in effect, is the benefit reserve t years after the issue of a policy immediately after funding is made. It is denoted by ${}_{t+1}I_x$.*

Main interests: For various insurance products, ${}_tV_x$, ${}_{t+1}I_x$, $Var({}_tL_x) = ?$

Theorem 7.1.

Definition 7.5.

7.2 Fully discrete insurance.

In this section, we assume that the benefit is paid as the end of the year of death and benefit premiums are received at the beginning of each year while the insurance is in effect. Benefit premiums for this insurance contract were discussed in Section 6.2.

7.2.1 Whole life insurance. In this section, we discuss the benefit reserves for a fully discrete whole life insurance. A death benefit is paid at the end of year of death. To fund a unit whole life insurance, the insuree makes a payment of P_x at the beginning of the year while he/she is alive. t years after the issue of the contract, if the contract is in effect (*i.e.*, $K_x > t$), the insurer has a liability on this contract. At this moment, the live's age is $x+t$. At time t, the future benefit payments are those of whole life insurance to $(x+t)$. At time t, the future premiums are $P_x\ddot{Y}_{x+t}$ $(= P_x \sum_{k=0}^{K_{x+t}-1} v^k)$.

t years after the issue of the contract, if the contract is in effect, the insurance company will pay a benefit of 1 (unit) at time K_{x+t}. The insurance company will receive the following cashflow of benefit premiums

Benefit premiums	P_x	P_x	P_x	$\cdots$	P_x
Time	t	$t+1$	$t+2$	$\cdots$	$K_{x+t}-1$

The difference between the present values at time t of future benefits (payments) and future benefit premiums is

$$ {}_tL_x = v^{K_{x+t}} - P_x \sum_{k\geq 0}^{K_{x+t}-1} v^k \qquad (= v^{K_{x+t}} - P_x \ddot{a}_{\overline{K_{x+t}}|} = Z_{x+t} - P_x\ddot{Y}_{x+t}.)$$

${}_0L_x = Z_x - P_x\ddot{Y}_x$ is the loss r.v. at issue of the policy, denoted by L_x in Chapter 6. Under equivalent principle $P_x = ?$ $A_x/\ddot{a}$?

Alternatively, we may define ${}_tL_x = (v^{K_x - t} - P_x\ddot{a}_{\overline{K_x-t}|})\big|(K_x > t)$.
The t-th terminal benefit reserve ${}_tV_x$ $(= E({}_tL_x) = A_{x+t} - P_x\ddot{a}_{x+t})$.
The $(t+1)$-th initial benefit reserve ${}_{t+1}I_x$ $(= {}_tV_x + P_x)$.

Definition 7.6.

Theorem 7.2.

Theorem 7.3.

Theorem 7.4.

Theorem 7.5.

Definition 7.7. *The* **net amount at risk**, *or the* **pure amount of protection**, *is the difference between the face value of a life insurance policy and its* <u>*cash*</u> *value. For a fully discrete* <u>*unit*</u> *whole life insurance, the net amount at risk during the t–th year is* $1 - {}_tV_x$.

Q: $1 - {}_0V_x = 1 - E(v^{K_x} - P_x \sum_{k=0}^{K_x-1} v^k) = ?$

If $t > 0$ then $1 - {}_tV_x < 1$? $\qquad$ $1 - {}_\infty I_x = ?$

The t–th terminal loss r.v.
${}_tL_x = v^{K_{x+t}} - P_x \sum_{k\geq 0}^{K_{x+t}-1} v^k \qquad = Z_{x+t} - P_x \ddot{Y}_{x+t} = v^{K_{x+t}} - P_x \ddot{a}_{\overline{K_{x+t}|}}$.
Or ${}_tL_x = (v^{K_x - t} - P_x \ddot{a}_{\overline{K_x - t|}}) \big| (K_x > t)$.
The t-th terminal benefit reserve ${}_tV_x$ $(= E({}_tL_x) = A_{x+t} - P_x \ddot{a}_{x+t})$.
The $(t+1)$-th initial benefit reserve ${}_{t+1}I_x$ $(= {}_tV_x + P_x)$. $\qquad$ =0 ? $\qquad$ < 1 ?

${}_tV_x = A_{x+t} - P_x \ddot{a}_{x+t}$, the t–th terminal benefit reserve is calculated using the APV of the future payments of the fully discrete whole life insurance, called the **prospective method**.

Some life insurance allows the insuree to cancel the policy a certain number of year after issue. The liability of having an open insurance policy is its terminal benefit reserve. If a policy is canceled t years after issue, before the benefit premium is paid, the amount to be returned to an insuree, before subtracting expenses, is the t–th terminal benefit reserve.

If (x) takes a whole life insurance and survives t years, (x) can cancel his policy at time t and get a payment of (at most) ${}_tV_x$. At time t, (x) has made annual payments of P_x at the beginning of the year for t years. Amount: tP_x, or including interests: $P_x \sum_{k=0}^{t-1} (1+i)^k$ $(> {}_tV_x)$ During these t years, (x) could have died and received a death benefit. The funding made by (x) is used to pay for this t–year term life insurance and for the t–th terminal benefit reserve. Hence, a t–year life annuity–due of P_x funds both a t–year term unit life insurance and a t–year pure endowment of (amount) ${}_tV_x$. We have that

$$P_x \ddot{a}_{x:\overline{t|}} = A^1_{x:\overline{t|}} + {}_tV_x \cdot {}_tE_x => {}_tV_x = \frac{P_x \ddot{a}_{x:\overline{t|}} - A^1_{x:\overline{t|}}}{{}_tE_x}). \qquad ({}_tE_x = v^t {}_tp_x)$$

Theorem 7.6. *(t–th retrospective terminal benefit reserve)*

$${}_tV_x = P_x \frac{\ddot{a}_{x:\overline{t|}}}{{}_tE_x} - \frac{A^1_{x:\overline{t|}}}{{}_tE_x} = P_x \ddot{s}_{x:\overline{t|}} - {}_tk_x, \text{ where } \ddot{s}_{x:\overline{t|}} = \frac{\ddot{a}_{x:\overline{t|}}}{{}_tE_x} \text{ and } {}_tk_x, = \frac{A^1_{x:\overline{t|}}}{{}_tE_x}$$

$P_x \ddot{s}_{x:\overline{t|}} \overset{def}{=}$ the accumulated value of the premiums received (in unit value insurance).

${}_tk_x \overset{def}{=}$ the **accumulated cost of insurance** over the age interval $(x, x+t]$ (in unit value insurance).

Two ways to compute the terminal benefit reserve ${}_tV_x$:
${}_tV_x = P_x \frac{\ddot{a}_{x:\overline{t|}}}{{}_tE_x} - \frac{A^1_{x:\overline{t|}}}{{}_tE_x}$ the retrospective method,
${}_tV_x = A_{x+t} - P_x \ddot{a}_{x+t}$, the prospective method.

Example 7.1. *For a fully discrete whole life insurance of \$50,000 on* (40)*, assume:*
(i) $i = 0.06$*;*
(ii) Mortality follows the life table for the US population in 2004 (p. 996-999).
The 10*–th terminal loss given that* (40) *dies in the 20–th year from issue = ?*

Solution: $B \cdot {}_{10}L_{40} = ?$ where $B = 50,000$, given $K_x = 20$ and $x = 40$.
Formulas: ${}_tL_x = Z_{x+t} - P_x\ddot{Y}_{x+t} = v^{K_{x+t}} - P_x\frac{1-v^{K_{x+t}}}{1-v}$ **(values of notations ?)**

$v = \frac{1}{1+i}$ and

$P_x = \frac{A_x}{\ddot{a}_x} = \frac{A_x}{\frac{1-A_x}{1-v}}$ by the equivalent principle.

From Table D.2 (p. 997) $A_{40} = 0.13264232$. Hence,

$$P_{40} = \frac{(1-v)A_{40}}{1-A_{40}}\Big|_{v=\frac{1}{1.06},A_{40}=0.13264232} = 0.008656239545.$$

$$B_{10}L_{40} = (50000)(v^{K_{40+10}} - P_{40}\frac{1-v^{K_{40+10}}}{1-v})\Big|_{K_{50}=10,v=1/1.06} \approx 24543.$$

Example 7.2. *Ten years ago, Joan entered a fully discrete whole life insurance contract with face value \$100,000. Joan was 40 years old when she entered this insurance. Suppose that $i = 5\%$, $A_{40} = 0.13$ and $A_{50} = 0.20$. This insurance allows the insuree to cancel this policy any time five years after issue. If the insuree cancels her insurance at time t, the insuree will receive the t–th terminal benefit reserve as compensation for canceling this insurance.*
(i) The annual benefit premium paid by Joan in each of these 10 years ?
(ii) The amount that Joan can receive if she cancels her insurance ?

Solution: (i) $BP_x = ?$ (ii) $B \cdot {}_tV_x = ?$ $B = ?$ $x = ?$ $t = ?$
(i) Equivalent principle: $E({}_0L_x) = 0$, where ${}_0L_x = Z_x - P_x\ddot{Y}_x = v^{K_x} - P_x\sum_{k=0}^{K_x-1} v^k$.
$=> A_x = P_x\ddot{a}_x$, where $\ddot{a}_x = E(\sum_{k=0}^{K_x-1} v^k) = E(\frac{1-v^{K_x}}{1-v}) = \frac{1-A_x}{1-v}$, as $E(v^{K_x}) = A_x$.

$P_{40} = \frac{A_x}{\ddot{a}_x} = \frac{A_{40}}{\frac{1-A_{40}}{1-v}} = \frac{(1-1/1.05))0.13}{1-0.13} = 0.007115489874.$

The annual benefit premium is $BP_x = 10^5(0.007115489874) \approx 711.55$.
(ii) $B \cdot {}_{10}V_{40} = ?$ 2 ways: ${}_tV_x = P_x\frac{\ddot{a}_{x:\overline{t}|}}{{}_tE_x} - \frac{A^1_{x:\overline{t}|}}{{}_tE_x} = A_{x+t} - P_x\ddot{a}_{x+t}$, **Which to choose ?**

$${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t} = A_{x+t} - P_x\frac{1-A_{x+t}}{1-v}\Big|_{A_{50}=0.2,v=1/1.05,P_{40}=?} = 0.08045977012.$$

The amount that Joan can receive is ${}_{10}V_{40} = (100000)(0.08045977012) = 8045.977$.
10 years payments: $10BP_x = 7115.5$ alone; with interests $\sum_{k=1}^{10} 711.55(1+0.05)^k = 9397.29$.

Example 7.3. *Suppose that $A^1_{x:\overline{5}|} = 0.28$ and $A_{x:\overline{5}|} = 0.45$. Find the accumulated cost of insurance in the first five years of a whole life insurance to (x) with face value \$10,000.*

Solution: $B \cdot {}_5k_x = ?$ $(B = 10^4)$.

Formulas: ${}_tk_x = \frac{A^1_{x:\overline{t}|}}{{}_tE_x}$, ${}_tE_x = A_{x:\overline{t}|}^{\ \ \ 1}$, $\overbrace{A_{x:\overline{n}}}^{=?} = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1} = \overbrace{A^1_{x:\overline{n}|}}^{=?} + {}_tE_x$.

${}_5E_x = 0.45 - 0.28 = 0.17$.

The accumulated cost of insurance $B \cdot {}_5k_x = (10000)\frac{A^1_{x:\overline{5}|}}{{}_tE_x} = (10000)\frac{0.28}{0.17} = 16470.59$.

Example 7.4. *Suppose that $P_x = 0.45$, $P^1_{x:\overline{n}|} = 0.30$, $P_{x:\overline{n}|}^{\ \ \ 1} = 0.2$, where P_x, $P^1_{x:\overline{n}|}$ and $P_{x:\overline{n}|}^{\ \ \ 1}$ are the corresponding benefit premiums due to the equivalent principle. Find ${}_nV_x$.*

Solution: Formulas: (1) ${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$. (2) ${}_tV_x = P_x\frac{\ddot{a}_{x:\bar{t}|}}{{}_tE_x} - \frac{A^1_{x:\bar{t}|}}{{}_tE_x}$ **Which to choose ?**
$A^1_{x:\bar{t}|} = P^1_{x:\bar{t}|}\ddot{a}_{x:\bar{t}|}$, ${}_tE_x = P_{x:\bar{t}|}^{\;\;1}\ddot{a}_{x:\bar{t}|}$, $A_x = P_x\ddot{a}_x$ **why ?** and $t = n$.
Loss r.v.s: $L_1 = Z^1_{x:\bar{t}|} - P^1_{x:\bar{t}|}\ddot{Y}_{x:\bar{t}|}$, $L_2 = Z_{x:\bar{t}|}^{\;\;1} - P_{x:\bar{t}|}^{\;\;1}\ddot{Y}_{x:\bar{t}|}$, $L_3 = Z_x - P_x\ddot{Y}_x$.

$\ddot{Y}_{x:\bar{t}|} = \sum_{k=0}^{(K_x\wedge t)-1} v^k = \frac{1-v^{K_x\wedge t}}{1-v}$, $E(Z^1_{x:\bar{t}|} - P^1_{x:\bar{t}|}\ddot{Y}_{x:\bar{t}|}) = 0$, $E(Z_{x:\bar{t}|}^{\;\;1} - P_{x:\bar{t}|}^{\;\;1}\ddot{Y}_{x:\bar{t}|}) = 0$.
Notice $\frac{A^1_{x:\bar{t}|}}{\ddot{a}_{x:\bar{t}|}} = P^1_{x:\bar{t}|} = ?$ and $\frac{{}_tE_x}{\ddot{a}_{x:\bar{t}|}} = P_{x:\bar{t}|}^{\;\;1} = ?$ Hence, (2) leads to

$$\underbrace{P_x}_{=?} = \frac{A^1_{x:\bar{t}|} + {}_tE_x \cdot {}_tV_x}{\ddot{a}_{x:\bar{t}|}} = \underbrace{P^1_{x:\bar{t}|}}_{=?} + \underbrace{P_{x:\bar{t}|}^{\;\;1}}_{=?}\cdot{}_tV_x \qquad (t = n). \tag{3}$$

Eq. (3) yields ${}_nV_x = {}_tV_x = \frac{P_x - P^1_{x:\bar{n}|}}{P_{x:\bar{n}|}^{\;\;1}} = \frac{0.45-0.3}{0.2} = 0.75$.
This actually leads to the theorem as follows.

Theorem 7.7. $P_x = P^1_{x:\bar{t}|} + P_{x:\bar{t}|}^{\;\;1} \cdot {}_tV_x$.

Example 7.5. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*
d_x	*33*	*56*	*54*	*45*	*34*	*28*	*0*

A whole life insurance to 80–year old individuals pays a death benefit of $B = \$50000$. The insuree will make level benefit premiums at the beginning of the year while he is alive. Let $i = 6.5\%$ and $P_{80} = 0.2710105645$. Suppose that 250 80–year old individuals enter this insurance contract and they die according with the deterministic group interpretation.
(iii) Compute ${}_3V_{80}$ using the prospective method ${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$.
(iv) Compute ${}_3V_{80}$ using the retrospective method $P_x\ddot{a}_{x:\bar{t}|} = A^1_{x:\bar{t}|} + {}_tE_x \cdot {}_tV_x$.
(v) Compute ${}_kI_x$, for $k = 1, 2, \ldots, 6$.

Solution: (iii) ${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$, $A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k)$, $\ddot{a}_x = \frac{1-A_x}{1-v}$, $f_{K_x}(k) = \frac{d_{x+k-1}}{\ell_x}$,

$$A_{83} = \sum_{k=1}^{?} v^k f_{K_{83}}(k) = \sum_{k=1}^{?} v^k \frac{d_{83+k-1}}{\ell_{83}} = (1.065)^{-1}\frac{45}{107} + \cdots + (1.065)^{-3}\frac{28}{107} + 0 = 0.891679545,$$
$$\ddot{a}_{83} = \frac{1-A_{83}}{1-v} = 1.774788994,$$
$${}_3V_{80} = A_{83} - P_{80}\ddot{a}_{83} = 0.891679545 - (0.2710105645)(1.774788994) = 0.4106929779.$$

Likewise,

k	0	1	2	3	4	5	6
${}_kV_{80}$	0	0.18044	0.30021	0.41069	0.52715	0.66796	0

Notice ${}_tV_x \geq 0$.

(iv) ${}_3V_{80} = P_{80}\frac{\ddot{a}_{80:\bar{3}|}}{{}_3E_{80}} - \frac{A^1_{80:\bar{3}|}}{{}_3E_{80}}$, $A^1_{x:\bar{n}|} = \sum_{k=1}^{n} v^k f_{K_x}(k)$, $\ddot{a}_{x:\bar{n}} = \sum_{k=0}^{n-1} v^k {}_kp_x$, ${}_kp_x = \ell_{x+k}/\ell_x$,

${}_nE_x = v^n {}_np_x$.

$$A^1_{80:\overline{3}|} = \sum_{k=1}^{?} v^k \frac{d_{80+k-1}}{\ell_{80}} = \frac{33}{250}(1.065)^{-1} + \frac{56}{250}(1.065)^{-2} + \frac{54}{250}(1.065)^{-3} = 0.5002507451,$$

$$\ddot{a}_{80:\overline{3}|} = \sum_{k=0}^{?} v^k \frac{\ell_{80+k}}{\ell_{80}} = 1 + \frac{217}{250}(1.065)^{-1} + \frac{161}{250}(1.065)^{-2} = 2.382812052,$$

$${}_3E_{80} = v^3 \frac{\ell_{83}}{\ell_{80}} = (1.065)^{-3}\frac{107}{250} = 0.3543194113,$$

$${}_3V_{80} = P_{80}\frac{\ddot{a}_{80:\overline{3}|}}{{}_3E_{80}} - \frac{A^1_{80:\overline{3}|}}{{}_3E_{80}} = (0.2710105645)\frac{2.382812052}{0.3543194113} - \frac{0.5002507451}{0.3543194113} = 0.4106929781.$$

(v) ${}_{t+1}I_x = {}_tV_x + P_x$ and $P_{80} = 0.2710105645$.

k	1	2	3	4	5	6
${}_kV_{80}$	0.18044	0.30021	0.41069	0.52715	0.66796	0
${}_kI_{80}$	0.27101	0.45146	0.57129	0.68170	0.79816	0.93897

Example 7.6.

Example 7.7. *Consider a fully discrete whole life insurance policy on* (35) *with face value of \$50000. Assume that* $i = 5.5\%$ *and death is modeled using the De Moivre model with terminal age 100.*
(i) Find the benefit annual premium for this policy.
(ii) Find 10*–th terminal benefit reserve using the prospective method.*
(iii) Find 10*–th terminal benefit reserve using the retrospective method.*
(iv) Find the variance of the 10*–th terminal loss random variable.*

Solution: (i) $BP_x = ?$ $B = 50000$. $P_{35} = \frac{A_{35}}{\ddot{a}_{35}} = \frac{A_{35}}{(1-A_{35})/(1-v)}$,
$A_x = \sum_{k=1}^{w-x} v^k f_{K_x}(k) = \sum_{k=1}^{w-x} v^k \frac{1}{\omega - x} = \frac{v(1-v^{w-x})}{1-v}\frac{1}{w-x}$.
$A_{35} = \frac{v(1-v^{w-x})}{1-v}\frac{1}{w-x}\Big|_{x=35,v=1.055^{-1},w=100} = 0.2711041133.$
$P_{35} = \frac{A_{35}}{(1-A_{35})/(1-v)} = 0.01939013521.$
$BP_{35} = 50000P_{35} = 969.5067605.$
(ii) $B{}_{10}V_{35} = B(A_{45} - P_{35}\ddot{a}_{45}) = BA_{45} - BP_{35}\frac{1-A_{45}}{1-v} = ?$
$A_{45} = \frac{v(1-v^{w-x})}{1-v}/(w-x)\Big|_{x=45,v=1.055^{-1},w=100} = 0.3131849179.$
$B{}_{10}V_{35} = BA_{45} - BP_{35}\frac{1-A_{45}}{1-v} = 2886.612842.$
(iii) $B{}_{10}V_{35} = B\frac{P_{35}\ddot{a}_{35:\overline{10}|} - A^1_{35:\overline{10}|}}{{}_{10}E_{35}} = ?$

$$A^1_{x:\overline{n}|} = E(v^{K_x} I(K_x \le n)) = \sum_{k=1}^{n} v^k f_{K_x}(k) = \frac{v(1-v^n)}{1-v}/(w-x)\Big|_{n=10,w=100,x=35,v=1/1.055}$$

$$A^1_{35:\overline{10}|} = 0.1159634743,$$

$$
\begin{aligned}
{}_{10}E_{35} &= v^{10}{}_{10}p_{35} = (1.055)^{-10}\frac{w-x-10}{w-x} = (1.055)^{-10}\frac{65-10}{65} = 0.4953643364,\\
\ddot{a}_{35:\overline{10}|} &= \frac{1-A_{35:\overline{10}|}}{1-v},\\
A_{35:\overline{10}|} &= A^1_{35:\overline{10}|} + {}_{10}E_{35} = 0.1159634743 + 0.4953643364 = 0.6113278107,\\
\ddot{a}_{35:\overline{10}|} &= \frac{1-A_{35:\overline{10}|}}{1-v} = \frac{1-0.6113278107}{0.055/1.055} = 7.455439267,\\
B_{10}V_{35} &= B\frac{P_{35}\ddot{a}_{35:\overline{10}|} - A^1_{35:\overline{10}|}}{{}_{10}E_{35}} = 2886.612842.
\end{aligned}
$$

(iv) $\mathrm{Var}({}_tL_x) = ?$ $\quad {}_tL_x = Z_{x+t} - P_x\ddot{Y}_{x+t} = Z_{x+t} - P_x\frac{1-Z_{x+t}}{d} = Z_{x+t}\left(1+\frac{P_x}{d}\right) - \frac{P_x}{d}$,

$$
\begin{aligned}
\mathrm{Var}({}_tL_x) &= \left(1+\frac{P_x}{d}\right)^2 ({}^2A_{x+t} - (A_{x+t})^2)\\
{}^2A_{45} = A_{45}(v^2) &= \frac{v(1-v^{w-x})}{1-v}\frac{1}{w-x}\Big|_{x=35, v=1.055^{-2}} = 0.1604201,\\
\mathrm{Var}(B\cdot{}_{10}L_{35}) &= B^2\left(1+\frac{P_x}{d}\right)^2 ({}^2A_{x+t} - (A_{x+t})^2) = 293321000.
\end{aligned}
$$

Theorem 7.8. *(***Uncovered benefit***, or* **paid–up insurance formula***)* ${}_tV_x = A_{x+t}\left(1-\frac{P_x}{P_{x+t}}\right)$.

Proof $\quad \ddot{a}_{x+t} = \frac{A_{x+t}}{P_{x+t}} \Rightarrow {}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t} = A_{x+t} - P_x\frac{A_{x+t}}{P_{x+t}} = A_{x+t}\left(1-\frac{P_x}{P_{x+t}}\right)$.

Suppose a whole life insurance contact **allows the insuree to modify the policy at time** t. The insuree can stop paying annual benefit premiums and receive a whole life insurance with a small face value of F. The APV of a whole life insurance to $(x+t)$ with face value F is FA_{x+t}, instead of 1 unit.

$$A_{x+t}\left(1-\frac{P_x}{P_{x+t}}\right) = {}_tV_x = FA_{x+t}. \tag{1}$$

Thus, $F = 1-\frac{P_x}{P_{x+t}}$ (<1). **See the example as follows.**

Example 7.8. *Ten years ago, Joan entered a fully discrete whole life insurance contract with face value \$100,000. Joan was 40 years old when she entered this insurance. Suppose that* $i = 5\%$, $A_{40} = 0.13$ *and* $A_{50} = 0.20$. *This insurance allows the insuree to stop paying benefit premiums payments and receive a whole life insurance with a face value of* F. *Calculate* F *if Joan decides to stop paying benefit premiums.*

Solution: $F = B(1-\frac{P_x}{P_{x+t}}) = ?$ not the F in Eq. (1) above ! $P_x = \frac{A_x}{\frac{1-A_x}{1-v}}$.

$$F = (100000)\left(1-\frac{P_{40}}{P_{50}}\right) = (100000)\left(1-\frac{\frac{A_{50}}{\frac{1-A_{50}}{1-v}}}{\frac{A_{40}}{\frac{1-A_{40}}{1-v}}}\right)\Bigg|_{A_{40}=0.13, A_{50}=0.2, v=\frac{1}{1.05}} 40229.88503.$$

That is, the face value is ≈ 40230, instead of 10^5, originally.

Example 7.9. *Suppose that* $\ddot{a}_{40} = 13$ *and* $\ddot{a}_{55} = 9$. *Find* ${}_{15}V_{40}$.

Solution: Formulas: $\begin{cases} (1)\ {}_tV_x = P_x \frac{\ddot{a}_{x:\overline{t}|}}{{}_tE_x} - \frac{A^1_{x:\overline{t}|}}{{}_tE_x} \\ (2)\ {}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}. \end{cases}$ **Which is more useful here ?**

${}_tV_x = {}_{15}V_{40} = A_{55} - P_{40}\underbrace{\ddot{a}_{55}}_{=9} = ?$ with $(x,t) = (40, 15)$. Convert (A_{55}, P_{40}) to $(\ddot{a}_{40}, \ddot{a}_{50})$.

$$\ddot{a}_{55} = \frac{1 - A_{55}}{1 - v} \Rightarrow A_{55} = 1 - (1-v)\ddot{a}_{55} = 1 - d\ddot{a}_{55},\ d = ?$$

$$A_x = P_x\ddot{a}_x \quad \Rightarrow \quad P_{40} = A_{40}/\ddot{a}_{40} = \frac{1 - (1-v)\ddot{a}_{40}}{\ddot{a}_{40}} = \frac{1}{\ddot{a}_{40}} - d$$

$$\begin{aligned} {}_{10}V_{40} &= A_{55} - P_{40}\ddot{a}_{55} = (1 - d\ddot{a}_{55}) - (\frac{1}{\ddot{a}_{40}} - d)\ddot{a}_{55} = 1 - d\ddot{a}_{55} - \frac{\ddot{a}_{55}}{\ddot{a}_{40}} + d\ddot{a}_{55} \\ &= 1 - \frac{\ddot{a}_{55}}{\ddot{a}_{40}} = 1 - \frac{9}{13} = 0.3076923077. \end{aligned}$$

Theorem 7.9. *(***Ratio of annuities formula***)* ${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$.

Example 7.10. *Suppose that* ${}_{10}V_{20} = 0.2$ *and* ${}_5V_{30} = 0.25$. *Calculate* ${}_{15}V_{20}$.

Solution: In view of the previous theorem, ${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$.
${}_{10}V_{20} = 1 - \frac{\ddot{a}_{20+10}}{\ddot{a}_{20}} = 1 - \frac{\ddot{a}_{30}}{\ddot{a}_{20}}$.
${}_5V_{30} = 1 - \frac{\ddot{a}_{30+5}}{\ddot{a}_{30}} = 1 - \frac{\ddot{a}_{35}}{\ddot{a}_{30}}$.

$${}_{15}V_{20} = 1 - \frac{\ddot{a}_{20+15}}{\ddot{a}_{20}} = 1 - \frac{\ddot{a}_{35}?}{\ddot{a}_{20}?}$$

$(1 - {}_tV_x)(1 - {}_sV_{x+t}) = \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\frac{\ddot{a}_{x+t+s}}{\ddot{a}_{x+t}} = \frac{\ddot{a}_{x+t+s}}{\ddot{a}_x} = 1 - {}_{t+s}V_x$.
$\Rightarrow 1 - {}_{15}V_{20} = (1 - {}_{10}V_{20})(1 - {}_5V_{30}) = (1 - 0.2)(1 - 0.25) = 0.6 \Rightarrow {}_{15}V_{20} = 0.4$.

Theorem 7.10. *For each* $s, t > 0$, $(1 - {}_tV_x)(1 - {}_sV_{x+t}) = 1 - {}_{t+s}V_x$.

Theorem 7.11. *(***Benefit premiums formula***) We have that*

$${}_tV_x = 1 - \frac{P_x + d}{P_{x+t} + d}.$$

Theorem 7.12. *(***Life insurance formula***)* ${}_tV_x = 1 - \frac{1 - A_{x+t}}{1 - A_x}$.

Theorem 7.13. *(***Iterative formula for the** t**–th reserve benefit***)*
${}_tV_x + P_x = vq_{x+t} + v \cdot {}_{t+1}V_x \cdot p_{x+t}$.

Proof. Formulas: $A_x = vq_x + vp_xA_{x+1}$ [14] and $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$ [18]

$$\begin{aligned} {}_tV_x &= A_{x+t} - P_x\ddot{a}_{x+t} = vq_{x+t} + vp_{x+t}A_{x+t+1} - P_x(1 + vp_{x+t}\ddot{a}_{x+t+1}) \\ &= -P_x + vq_{x+t} + vp_{x+t}(A_{x+t+1} - P_x\ddot{a}_{x+t+1}) = -P_x + vq_{x+t} + vp_{x+t} \cdot {}_{t+1}V_x. \end{aligned}$$

The theorem then follows. ∎

Example 7.11. *Consider a fully discrete whole life insurance of* 1000 *on* (x)*. Assume:*
(i) $q_{x+5} = 0.04$.
(ii) $i = 0.07$.
(iii) The initial benefit reserve for policy year 6 is 540. $=_{5+1} I_x$
Calculate the net amount at risk for policy year 6.

Solution: $1000(1 - {}_6V_x) = ?$
By formula: ${}_tV_x + P_x = vq_{x+t} + v \cdot {}_{t+1}V_x \cdot p_{x+t}$ (see Th7.13).

$${}_{5+1}I_x = 540 = (1000)({}_5V_x + P_x) = 1000(vq_{x+5} + vp_{x+5} \cdot {}_6V_x).$$

Hence,

$$(1000) \cdot {}_6V_x = \frac{\frac{540}{v} - q_{x+5}1000}{p_{x+5}} = \frac{(540)(1.07) - (0.04)(1000)}{1 - 0.04} = 560.2083333.$$

The net amount at risk for policy year 6 is $1000 - 560.2083333 = 439.7916667$.

Example 7.12. *Consider a fully discrete whole life insurance of \$10,000 on* (x)*. Assume:*
(i) $q_{x+10} = 0.04$.
(ii) $q_{x+11} = 0.03$.
(iii) $i = 0.07$.
(iv) The annual net benefit premium is 450 ($= BP_x$*).*
(v) The terminal benefit reserve for policy year 10 is 4000 ($= B({}_{10}V_x)$*).*
Calculate the terminal benefit reserve for policy year 11 and for policy year 12. ($B({}_tV_x)$ *??).*

Solution: Using that ${}_{10}V_x + P_x = vq_{x+10} + vp_{x+10} \cdot {}_{11}V_x$, we get that

$$\begin{aligned}(10^4)_{11}V_x =& (10^4)\frac{({}_{10}V_x + P_x)(1+i) - q_{x+10}}{1 - q_{x+10}} \\ =& \frac{(4000 + 450)(1.07) - (10000)(0.04)}{0.96} = 4543.229167.\end{aligned}$$

Similarly, we get that

$$\begin{aligned}(10000)_{12}V_x =& \frac{(10000)({}_{11}V_x + P_x)(1+i) - (10000)q_{x+11}}{1 - q_{x+11}} \\ =& \frac{(4543.229167 + 450)(1.07) - (10000)(0.03)}{0.97} = 5198.71671.\end{aligned}$$

(**Iteative formula for the t–th reserve benefit for fully discrete whole life insurance**)
${}_tV_x + P_x = vq_{x+t} + v \cdot {}_{t+1}V_x \cdot p_{x+t}$.

Theorem 7.14. $P_x = (v \cdot {}_{t+1}V_x - {}_tV_x)p_{x+t} + (v - {}_tV_x)q_{x+t}$. $\quad$ *(*$= v \cdot {}_{t+1}V_x p_{x+t} + vq_{x+t} - {}_tV_x$*.)*

Theorem 7.15. ${}_{t+1}V_x = ({}_tV_x + P_x)(1+i) - q_{x+t}(1 - {}_{t+1}V_x)$.

Theorem 7.16. *For a whole life insurance which is funded for h years,* ${}^h_tL_x = v^{K_{x+t}} - {}_hP_x\sum_{j=0}^{K_{x+t}\wedge(h-t)-1} v^j I(0 \le t < h)$ *and*

the t-th terminal benefit reserve $${}^h_tV_x = \begin{cases} A_{x+t} - {}_hP_x\ddot{a}_{x+t:\overline{h-t}|} & \text{if } 0 \le t < h, \\ A_{x+t} & \text{if } t \ge h. \end{cases}$$

Example 7.13. *For a fully discrete 10–payment whole life insurance of 1000 on (x), you are given:*
(i) The annual benefit premium is 200.
(ii) The terminal reserve at the end of year 9 is 150.
Calculate $1000A_{x+9}$.

Solution: ${}^h_tV_x = A_{x+t} - {}_hP_x\ddot{a}_{x+t:\overline{h-t}|}$ and $t = 9 < 10 = h$,
$1000\,{}^h_tV_x = \underbrace{1000A_{x+t}}_{=?} - 1000{}_hP_x\ddot{a}_{x+t:\overline{h-t}|}$, and $\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k{}_kp_x$.
$150 = (1000)A_{x+9} - (200)\ddot{a}_{x+t:\overline{1}|} = (1000)A_{x+9} - 200.$
So, $(1000)A_{x+9} = 350$.

7.2.2 n–year term insurance. $L = Z^1_{x:\overline{n}|} - P^1_{x:\overline{n}|}\ddot{Y}_{x:\overline{n}|}$ [19].

Theorem 7.17. *For an n–year term insurance, the t–th terminal loss r.v. is*
${}_tL^1_{x:\overline{n}|} = Z^1_{x+t:\overline{n-t}|} - P^1_{x:\overline{n}|}\ddot{Y}_{x+t:\overline{n-t}|} = v^{K_{x+t}}I(K_{x+t} \le n-t) - P^1_{x:\overline{n}|}\sum_{j=0}^{K_{x+t}\wedge(n-t)-1} v^j.$

Alternatively, ${}_tL^1_{x:\overline{n}|} = [v^{K_x-t}I(K_x \le n) - P^1_{x:\overline{n}|}\ddot{a}_{\overline{\min(K_x-t,n-t)}|}]|(K_x > t)$.

Theorem 7.18. *The t–th terminal benefit reserve is*
${}_tV^1_{x:\overline{n}|} = [A^1_{x+t:\overline{n-t}|} - P^1_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|}]I(0 \le t < n).$

Formula [17]: $\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k{}_kp_x$.

Example 7.14. *Consider an n–year term life insurance to (x) with face value $55,000$. Suppose $i = 0.06$, $P^1_{x:\overline{n}|} = 0.17$, $p_{x+n-2} = 0.02$ and $p_{x+n-1} = 0.03$. Find the mean and the standard deviation (SD) of the $(n-2)$–th terminal loss random variable for this insurance if $K_x > n-2$ (i.e., $\mathbb{P}(K_{x+n-2} > 0) = 1$).*

Solution: Two approaches: (1) Compute $[\overbrace{A^1_{x+t:\overline{n-t}|}}^{=?} - P^1_{x:\overline{n}|}\overbrace{\ddot{a}_{x+t:\overline{n-t}|}}^{=\sum_{k=0}^{n-t-1} v^k{}_kp_{x+t}}]$.
(2) Directly as follows. Let ${}_{n-2}L$ be the $(n-2)$–th terminal loss r.v. for this insurance.
${}_{n-2}L = B(v^{K_{x+n-2}}I(K_{x+n-2} \le 2) - P^1_{x:\overline{n}|}\sum_{j=0}^{K_{x+n-2}\wedge 2-1} v^j) = g(K_{x+n-2}).$
$E[{}_{n-2}L] = E(g(K_{x+n-2})) = \sum_k g(k)f_{K_{x+n-2}}(k) = ?$ $\quad E[({}_{n-2}L)^2] = ?$

$${}_{n-2}L = 55000((1.06)^{-1} - (0.17)(1)) = 42536.79245, \text{ if } K_{x+n-2} = 1,$$
$${}_{n-2}L = 55000((1.06)^{-2} - (0.17)(1 + (1.06)^{-1})) = 30779.04948, \text{ if } K_{x+n-2} = 2,$$
$${}_{n-2}L = 55000(0 - (0.17)(1 + (1.06)^{-1})) = -18170.75472 \text{ ?? if } K_{x+n-2} > 2.$$

[8] $\underline{K_x} = \lceil T(x) \rceil \geq 1$, and $\underline{f_{K_x}(k)} = {}_{k-1}|q_x \;= {}_{k-1}p_x \cdot q_{x+k-1}$.

$$\begin{aligned} &f_{K_{x+n-2}}(1) = {}_0|_1 q_{x+n-2} = q_{x+n-2} = 0.98,\\ &f_{K_{x+n-2}}(2) = p_{x+n-2}q_{x+n-1} = (0.02)(0.97) = 0.0194,\\ &\mathbb{P}(K_{x+n-2} > 2) = 1 - f_{K_{x+n-2}}(1) - f_{K_{x+n-2}}(2) = 0.0006. \end{aligned} \tag{14.1}$$

Hence,

$$\begin{aligned} &E[{}_{n-2}L] = E(g(K_{x+n-2})) = \sum_k g(k) f_{K_{x+n-2}}(k)\\ =&g(1)f_{K_{x+n-2}}(1) + g(2)f_{K_{x+n-2}}(2) + (-18170.75472)\mathbb{P}(K_{x+n-2} > 2)\\ =&(42536.79245)(0.98) + (30779.04948)(0.0194) + (-18170.75472)(0.0006) = 42272.26771,\\ &E[({}_{n-2}L)^2]\\ =&(42536.79245)^2(0.98) + (30779.04948)^2(0.0194) + (-18170.75472)^2(0.0006) = 1791767831. \end{aligned}$$

The SD of ${}_{n-2}L = \sqrt{1791767831 - (42272.26771)^2} = 2196.181608$.

7.2.3 n–year pure endowment insurance. $L = Z_{x:\overset{\;1}{\overline{n}|}} - P_{x:\overset{\;1}{\overline{n}|}}\ddot{Y}_{x:\overline{n}|}$ [19].

Theorem 7.19. *For an n–year pure endowment insurance, the t–th terminal loss random variable is ${}_tL_{x:\overset{\;1}{\overline{n}|}} = v^{n-t}I(K_{x+t} > n-t) - P_{x:\overset{\;1}{\overline{n}|}}\sum_{j=0}^{K_{x+t}\wedge(n-t)-1} v^j$ if $t \leq n$.*

Theorem 7.20. *For an n–year pure endowment insurance, the t–th terminal benefit reserve is* ${}_tV_{x:\overset{\;1}{\overline{n}|}} = \begin{cases} A_{x+t:\overset{\quad 1}{\overline{n-t}|}} - P_{x:\overset{\;1}{\overline{n}|}}\ddot{a}_{x+t:\overline{n-t}|} & \text{if } 0 < t < n,\\ 1 & \text{if } t = n. \end{cases}$

Example 7.15. *Consider an n–year pure endowment insurance to (x) with face value $55,000$. Suppose $i = 0.06$, $P_{x:\overset{\;1}{\overline{n}|}} = 0.23$, $p_{x+n-2} = 0.02$ and $p_{x+n-1} = 0.03$. Find the mean and the standard deviation of the $(n-2)$–th terminal loss random variable for this insurance if $K_x > n-2$.*

Solution: Assumptions are the same as Ex.7.14, except the loss r.v.. There are two approches, we use the 2nd one. Let ${}_{n-2}L$ be the $(n-2)$–th terminal loss rv. $E({}_{n-2}L)$, $\sigma_{{}_{n-2}L} = ?$
$E(g(K_t)) = \sum_{j=0}^{K_t-1} g(k)f_{K_t}(k)$, ... Write ${}_{n-2}L = g(K_{x+n-2})$.
${}_{n-2}L = B[v^2 I(K_{x+n-2} > 2) - P_{x:\overset{\;1}{\overline{n}|}}\sum_{j=0}^{K_{x+n-2}\wedge 2-1} v^j] = ?$ and B=?

$$\begin{aligned} &{}_{n-2}L = (55000)[(1.06)^{-2}\cdot 0 - (0.23)\sum_{j=0}^{0} v^j] = -12650, \text{ if } K_{x+n-2} = 1\ (\leq 2),\\ &{}_{n-2}L = (55000)[0 - (0.23)(1 + (1.06)^{-1})] = -24583.96226, \text{ if } K_{x+n-2} = 2\ (\leq 2),\\ &{}_{n-2}L = (55000)[(1.06)^{-2} - (0.23)(1 + (1.06)^{-1})] = 24365.84194, \text{ if } K_{x+n-2} > 2, \end{aligned}$$

[8] $\underline{f_{K_x}(k)} = {}_{k-1}|q_x \;= {}_{k-1}p_x \cdot q_{x+k-1}$ and [4] ${}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m}$.

$\mathbb{P}\{K_{x+n-2} = 1\}$, $= 2$, > 2 are the same as in Eq.(14.1) in Ex.7.14.

$$\begin{aligned}
&E[{}_{n-2}L] = \sum g(i) f_{K_{x+n-2}}(i)\\
=&(-12650)(0.98) + (-24583.96226)(0.0194) + (24365.84194)(0.0006) = -12859.30936,\\
&E[({}_{n-2}L)^2] = \sum (g(i))^2 f_{K_{x+n-2}})(i)\\
=&(-12650)^2(0.98) + (-24583.96226)^2(0.0194) + (24365.84194)^2(0.0006) = 168903067.8,\\
&\sqrt{\mathrm{Var}({}_{n-2}L)} = \sqrt{168903067.8 - (-12859.30936)^2} = 1881.815768.
\end{aligned}$$

7.2.4 n–year endowment insurance. $L = Z_{x:\overline{n}|} - P_{x:\overline{n}|}\ddot{Y}_{x:\overline{n}|}$ [19].

Theorem 7.21. *For an n–year endowment insurance, the t–th terminal random loss is $v^{K_{x+t}\wedge(n-t)} - P_{x:\overline{n}|}\sum_{j=0}^{K_{x+t}\wedge(n-t)-1} v^j$ $(= Z_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ddot{Y}_{x+t:\overline{n-t}|})$ if $t \le n$.*

Theorem 7.22. *For an n–year endowment insurance, the t–th terminal benefit reserve is*

$${}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|} = \begin{cases} A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|} & \text{if } t < n,\\ 1 & \text{if } t = n.\end{cases}$$

Example 7.16. *Consider a 3–year endowment to (x) with face value $80,000$. Suppose $i = 0.06$, $p_x = 0.98$ and $p_{x+1} = 0.95$. Find the mean and the standard deviation of the first terminal loss random variable for this insurance.*

Solution: Let ${}_1L = B \cdot {}_tL_{x:\overline{n}|} \stackrel{def}{=} g(K_{x+1} \wedge 2)$. $\sigma_{{}_1L} = ?$
${}_tL = B(v^{K_{x+1}\wedge 2} - P\sum_{k=0}^{K_{x+1}\wedge 2-1} v^k)$ (see Th 7.21). $K_{x+1} \wedge 2 \in \{1, 2\}$! !
by [19], [17] & [14] $P = P_{x:\overline{3}|} = \frac{A_{x:\overline{3}|}}{\ddot{a}_{x:\overline{3}|}} = \frac{A_{x:\overline{3}|}}{(1-A_{x:\overline{3}|})/(1-v)}$, $A_{x:\overline{n}|} = E(v^{K_x\wedge n})$ and
[8] $f_{K_x}(k) = {}_{k-1|}q_x = \prod_{1\le j<k} p_{x+j-1}q_{x+k-1}$.
$A_{x:\overline{3}|} = \sum_{k=1}^3 v^k f_{K_x}(k) = (1.06)^{-1}(0.02) + (1.06)^{-2}(0.98)(0.05) + (1.06)^{-3}(0.98)(0.95) = 0.8441633.$

${}_1L = B(v^{K_{x+1}\wedge 2} - P\sum_{k=0}^{K_{x+1}\wedge 2-1} v^k) = 80000((1.06)^{-1} - \frac{A_{x:\overline{3}|}}{\frac{1-A_{x:\overline{3}|}}{1-\frac{1}{1.06}}}) = 50942.00556$, if $K_{x+1} = 1$,
${}_1L = 80000((1.06)^{-2} - \frac{A_{x:\overline{3}|}}{\frac{1-A_{x:\overline{3}|}}{1-\frac{1}{1.06}}}(1 + (1.06)^{-1})) = 23528.80326$, if $K_{x+1} \ge 2$, and
$\mathbb{P}\{K_{x+1} = 1\} = q_{x+1} = 0.05$ and $\mathbb{P}\{K_{x+1} \ge 2\} = p_{x+1} = 0.95$. Hence,

$$\begin{aligned}
&E[{}_1L] = E(g(K_{x+1})) = g(1)q_{x+1} + g(2)p_{x+1} = (50942.00)(0.05) + (23528.80)(0.95) = 24899.46,\\
&E[({}_1L)^2] = (50942.00556)^2(0.05) + (23528.80326)^2(0.95) = 655678750.2,\\
&\mathrm{Var}({}_1L) = 655678750.2 - (24899.46)^2 = 35695473.59,\\
&\sqrt{\mathrm{Var}({}_1L)} = \sqrt{35695473.59} = 5974.568904.
\end{aligned}$$

Theorem 7.23. *If $t < n$,* $\quad (L = Z_{x:\overline{n}|} - P_{x:\overline{n}|}\ddot{Y}_{x:\overline{n}|} \quad [19])$,

$$
{}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|} = \frac{P_{x:\overline{n}|}\ddot{a}_{x:\overline{t}|} - A^{1}_{x:\overline{t}|}}{{}_tE_x} = (P_{x+t:\overline{n-t}|} - P_{x:\overline{n}|})\ddot{a}_{x+t:\overline{n-t}|}
$$

$$
= \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+t:\overline{n-t}|}}\right) A_{x+t:\overline{n-t}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} = 1 - \frac{1 - A_{x+t:\overline{n-t}|}}{1 - A_{x:\overline{n}|}} = 1 - \frac{P_{x:\overline{n}|} + d}{P_{x+t:\overline{n-t}|} + d}.
$$

Theorem 7.24. *For a h–payment n–year endowment insurance, the prospective t–th terminal benefit reserve is* $ {}^h_tV_{x:\overline{n}|} = \begin{cases} A_{x+t:\overline{n-t}|} - {}_hP_{x:\overline{n}|}\ddot{a}_{x+t:\overline{h-t}|} & \text{if } 0 \le t < h < n, \\ A_{x+t:\overline{n-t}|} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$ *and*

the retrospective one is $ {}^h_tV_{x:\overline{n}|} = \begin{cases} {}_hP_{x:\overline{n}|}\ddot{s}_{x:\overline{t}|} - \frac{A^1_{x:\overline{t}|}}{{}_tE_x} & \text{if } 0 \le t < h < n, \\ \frac{{}_hP_{x:\overline{n}|}\ddot{a}_{x:\overline{h}|}}{{}_tE_x} - \frac{A^1_{x:\overline{t}|}}{{}_tE_x} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$ *(recall $\ddot{s}_{x:\overline{t}|} = \frac{\ddot{a}_{x:\overline{t}|}}{{}_tE_x}$)*

7.2.5 n–year deferred insurance.

If an n–year deferred insurance is funded during the first n years, the benefit premium is ${}_nP({}_n|A_x) = \frac{{}_n|A_x}{\ddot{a}_{x:\overline{n}|}}$. $\quad (L = {}_n|Z_x - P\ddot{Y}_{x:\overline{n}|} \quad [19])$.

Theorem 7.25. *For an n–year deferred insurance, funded during the first n years, the t–th terminal benefit reserve is* ${}^n_tV({}_n|A_x) = \begin{cases} {}_{n-t}|A_{x+t} - {}_nP({}_n|A_x)\ddot{a}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n, \\ A_{x+t} & \text{if } t \ge n. \end{cases}$

7.2.6 n–year deferred annuity.

$(L = \sum_{k\ge n}^{K_x-1} v^k - P\sum_{k=0}^{(K_x\wedge n)-1} v^k)$.

Theorem 7.26. *The t–th terminal benefit reserve for an n–year deferred annuity–due funded at the beginning of the year over the deferral period is*

$$
{}_tV({}_n|\ddot{a}_x) = \begin{cases} {}_{n-t}|\ddot{a}_{x+t} - P({}_n|\ddot{a}_x)\ddot{a}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n, \\ \ddot{a}_{x+t} & \text{if } t \ge n. \end{cases}
$$

Theorem 7.27. *If $0 \le t < n$,* ${}_tV({}_n|\ddot{a}_x) = P({}_n|\ddot{a}_x)\ddot{a}_{x:\overline{t}|}/{}_tE_x$

$$
= (P({}_{n-t}|\ddot{a}_{x+t}) - P({}_n|\ddot{a}_x))\ddot{a}_{x+t:\overline{n-t}|} = \left(1 - \frac{P({}_n|\ddot{a}_x)}{P({}_{n-t}|\ddot{a}_{x+t})}\right){}_{n-t}E_{x+t}\cdot\ddot{a}_{x+n}.
$$

The t–th terminal benefit reserve for an n–year deferred annuity–immediate funded at the beginning of the year over the deferral period is

$$
{}_tV({}_n|a_x) = \begin{cases} {}_{n-t}|a_{x+t} - P({}_n|a_x)\ddot{a}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n, \\ a_{x+t} & \text{if } t \ge n. \end{cases} \quad a_{x+t} = \ddot{a}_{x+t} - 1.
$$

$$
L_{im} = \sum_{k>n}^{K_x} v^k - P\sum_{k=0}^{(K_x\wedge n)-1} v^k \text{ versus } \quad L_{due} = \sum_{k\ge n}^{K_x-1} v^k - P\sum_{k=0}^{(K_x\wedge n)-1} v^k.
$$

Insurance type	t–th terminal loss r.v.
Whole life insurance	${}_tL_x = Z_{x+t} - P_x\ddot{Y}_{x+t}$
h–payment whole life insurance	${}^h_tL_x = \begin{cases} Z_{x+t} - {}_hP_x\ddot{Y}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h, \\ Z_{x+t} & \text{if } t \ge h. \end{cases}$
n–year term insurance	${}_tL^1_{x:\overline{n}\rvert} = Z^1_{x+t:\overline{n-t}\rvert} - P^1_{x:\overline{n}\rvert}\ddot{Y}_{x+t:\overline{n-t}\rvert}$ if $0 \le t < n$.
n–year pure endowment	${}_tL_{x:\overline{n}\rvert}^{\ \ 1} = \begin{cases} Z_{x+t:\overline{n-t}\rvert}^{\ \ \ \ 1} - P_{x:\overline{n}\rvert}^{\ \ 1}\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
h–payment n–year endowment	${}^h_tL_{x:\overline{n}\rvert} = \begin{cases} Z_{x+t:\overline{n-t}\rvert} - {}_hP_{x:\overline{n}\rvert}\ddot{Y}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h < n, \\ Z_{x+t:\overline{n-t}\rvert} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year endowment	${}_tL_{x:\overline{n}\rvert} = \begin{cases} Z_{x+t:\overline{n-t}\rvert} - P_{x:\overline{n}\rvert}\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year deferred insurance	${}^n_tL({}_n\lvert Z_x) = \begin{cases} {}_{n-t}\lvert Z_{x+t} - {}_nP({}_n\lvert Z_x)\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ Z_{x+t} & \text{if } t \ge n. \end{cases}$
n–year deferred annuity–due	${}_tL({}_n\lvert \ddot{Y}_x) = \begin{cases} {}_{n-t}\lvert \ddot{Y}_{x+t} - P({}_n\lvert \ddot{Y}_x)\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ \ddot{Y}_{x+t} & \text{if } t \ge n. \end{cases}$

Insurance type	t–th terminal benefit reserve
Whole life insurance	${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$
h–payment whole life insurance	${}^h_tV_x = \begin{cases} A_{x+t} - {}_hP_x\ddot{a}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h, \\ A_{x+t} & \text{if } t \ge h. \end{cases}$
n–year term insurance	${}_tV^1_{x:\overline{n}\rvert} = A^1_{x+t:\overline{n-t}\rvert} - P^1_{x:\overline{n}\rvert}\ddot{a}_{x+t:\overline{n-t}\rvert}$ if $0 \le t < n$.
n–year pure endowment	${}_tV_{x:\overline{n}\rvert}^{\ \ 1} = \begin{cases} A_{x+t:\overline{n-t}\rvert}^{\ \ \ \ 1} - P_{x:\overline{n}\rvert}^{\ \ 1}\ddot{a}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
h–payment n–year endowment	${}^h_tV_{x:\overline{n}\rvert} = \begin{cases} A_{x+t:\overline{n-t}\rvert} - {}_hP_{x:\overline{n}\rvert}\ddot{a}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h < n, \\ A_{x+t:\overline{n-t}\rvert} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year endowment	${}_tV_{x:\overline{n}\rvert} = \begin{cases} A_{x+t:\overline{n-t}\rvert} - P_{x:\overline{n}\rvert}\ddot{a}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year deferred insurance	${}^n_tV({}_n\lvert A_x) = \begin{cases} {}_{n-t}\lvert A_{x+t} - {}_nP({}_n\lvert A_x)\ddot{a}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ A_{x+t} & \text{if } t \ge n. \end{cases}$
n–year deferred annuity–due	${}_tV({}_n\lvert \ddot{a}_x) = \begin{cases} {}_{n-t}\lvert \ddot{a}_{x+t} - P({}_n\lvert \ddot{a}_x)\ddot{a}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ \ddot{a}_{x+t} & \text{if } t \ge n. \end{cases}$

Table 7.1: t–th terminal benefit reserve for some fully discrete contracts.

7.3 Fully continuous insurance

7.3.1 Whole life insurance. Recall that $T_x = (X - x)\lvert(X > x)$ is the interval of time when death occurs. For a whole life unity insurance paid at the time of death and funded continuously a benefit of 1 is paid at T_x and benefit premiums are paid at an annual rate of $P = \overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x}$.

$$L = \overline{Z}_x - P\overline{Y}_x = v^{T_x} - P\int_0^{T_x} v^t dt = v^{T_x} - P\frac{1-v^{T_x}}{-\ln v} = v^{T_x}(1+\frac{P}{-\ln v}) + P/\ln v = e^{-\delta T_x}(1+\frac{P}{\delta}) - \frac{P}{\delta}$$

Definition 7.8. *The loss r.v. at time t for a whole life unity insurance paid at the time of* *death and funded continuously is denoted by* ${}_t\overline{L}(\overline{A}_x)$.

Theorem 7.28. *For a whole life unity insurance paid at the time of death and funded continuously, the loss random variable at time t is*

$$ {}_t\overline{L}(\overline{A}_x) = v^{T_{x+t}} - \overline{P}(\overline{A}_x)\overline{a}_{\overline{T_{x+t}}|} = \overline{Z}_{x+t} - \overline{P}(\overline{A}_x)\overline{Y}_{x+t} = \overline{Z}_{x+t}\left(1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right) - \frac{\overline{P}(\overline{A}_x)}{\delta}. $$

[17] $\overline{Y}_x = \int_0^{T_x} v^t dt = \frac{1-\overline{Z}_x}{\delta}$, $\overline{Z}_x = ?$

Theorem 7.29. *The t–th terminal benefit reserve of a whole life insurance paid at the time of death and funded continuously is*

$$ {}_t\overline{V}(\overline{A}_x) = E[{}_t\overline{L}(\overline{A}_x)] = \overline{A}_{x+t} - \overline{P}(\overline{A}_x)\overline{a}_{x+t}. $$

For a fully continuous whole life unity insurance, the **net amount at risk** is $1 - {}_t\overline{V}(\overline{A}_x)$. $1 - {}_t\overline{V}(\overline{A}_x)$ is the amount of the unit death benefit that cannot be paid using the t–th terminal benefit reserve. Suppose that, at time t, an insurer has funds of ${}_t\overline{V}(\overline{A}_x)$ to pay future benefits, and death happens at time t. Then, the insurer is unprotected by $1 - {}_t\overline{V}(\overline{A}_x)$.

Example 7.17. *A fully cts whole life insurance to* (45) *provides a death benefit of 40,000. Assume that* $\delta = 0.07$ *and death is modeled using De Moivre's model with terminal age 100. Calculate the* 15*–th terminal benefit reserve and the variance of the* 15*–th terminal loss random variable.*

Solution: Let ${}_{15}L = B(v^{T_{x+15}} - P\int_0^{T_{x+15}} v^t dt) = B(v^{T_{x+15}} - P\frac{1-v^{T_{x+15}}}{\delta})$. $B = 40000$.
(1) $E({}_{15}L) = B \cdot {}_{15}\overline{V}(\overline{A}_{45}) = ?$ (2) $\mathrm{Var}({}_{15}\overline{L}) = ?$

${}_{15}\overline{V}(\overline{A}_{45}) = \overline{A}_{60} - \overline{P}(\overline{A}_{45})\overline{a}_{60}$. [17] $\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x}$, $\overline{a}_x = (1-\overline{A}_{45})/\delta$.

$$ \overline{A}_x = E(v^{T_x}) = \int_0^{w-x} \frac{v^t}{w-x}dt = \frac{v^t}{(w-x)\ln v}\Big|_0^{w-x} = \frac{1-v^{w-x}}{-(w-x)\ln v} = \frac{1-\exp(-\delta(w-x))}{\delta(w-x)} \text{ and } v = e^{-\delta}. $$

$$ \overline{A}_{45} = \frac{1-e^{-(0.07)(55)}}{(0.07)(55)} = 0.2542130555, $$

$$ \overline{P}(\overline{A}_{45}) = \frac{\overline{A}_x}{\overline{a}_x} = \frac{\overline{A}_{45}}{(1-\overline{A}_{45})/\delta} = \frac{(0.07)(0.2542130555)}{1-0.2542130555} = 0.02386058648, $$

$$ \overline{A}_{60} = \frac{1-e^{-(0.07)(40)}}{(0.07)(40)} = 0.3354249776, $$

$$ \overline{a}_{60} = \frac{1-\overline{A}_{60}}{\delta} = \frac{1-0.3354249776}{0.07} = 9.493928891, $$

${}_{15}\overline{V}(\overline{A}_{45}) = \overline{A}_{60} - \overline{P}(\overline{A}_{45})\overline{a}_{60} = 0.3354249776 - (0.02386058648)(9.493928891) = 0.1088942663.$

Ans (1): $E({}_{15}L) = B \cdot {}_{15}\overline{V}(\overline{A}_{45}) = (40000)(0.1088942663) = 4355.770652.$

$\mathrm{Var}({}_{15}\overline{L}) = B^2\left(1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right)^2 \left({}^2\overline{A}_{x+t} - (\overline{A}_{x+t})^2\right)\Big|_{t=15,x=45,\delta=0.07} = 0.1175861125B^2,$
as ${}^2\overline{A}_{60} = \overline{A}_x(v^2) = \frac{1-e^{-(2)(0.07)(40)}}{(2)(0.07)(40)} = 0.1779110958.$
Ans (2): $\mathrm{Var}({}_{15}\overline{L}) = (40000)^2(0.1175861125) = 188137780.$

Theorem 7.30.

Theorem 7.31.

The prospective method: ${}_t\overline{V}_x = \overline{A}_{x+t} - P \cdot \overline{a}_{x+t}$.
The retrospective method: ${}_t\overline{V}(\overline{A}_x) = P_x \overline{s}_{x:\overline{t}|} - {}_t\overline{k}_x \quad \overset{def}{=} \frac{\overline{P}(\overline{A}_x)\overline{a}_{x:\overline{t}|} - \overline{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x}$. $P = P_x = \overline{P}(\overline{A}_x)$!!

In the fully continuous case, **accumulated cost of insurance** over the age interval $(x, x+t]$ is defined as ${}_t\overline{k}_x = \frac{\overline{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x}$.

Theorem 7.32. *We have that*

$$\begin{aligned}{}_t\overline{V}(\overline{A}_x) &= \overline{A}_{x+t} - \overline{P}(\overline{A}_x)\overline{a}_{x+t} = \frac{\overline{P}(\overline{A}_x)\overline{a}_{x:\overline{t}|} - \overline{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x} = (\overline{P}(\overline{A}_{x+t}) - \overline{P}(\overline{A}_x))\overline{a}_{x+t} \\ &= \overline{A}_{x+t}\left(1 - \frac{\overline{P}(\overline{A}_x)}{\overline{P}(\overline{A}_{x+t})}\right) = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x} = 1 - \frac{1-\overline{A}_{x+t}}{1-\overline{A}_x} = 1 - \frac{\overline{P}(\overline{A}_x)+\delta}{\overline{P}(\overline{A}_{x+t})+\delta}.\end{aligned}$$

Theorem 7.32 is similar to the results in Fully discrete cases, *e.g.*, if $0 \le t < n$,

$$\begin{aligned}{}_tV_{x:\overline{n}|} &= A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|} = \frac{P_{x:\overline{n}|}\ddot{a}_{x:\overline{t}|} - A^{\,1}_{x:\overline{t}|}}{{}_tE_x} = (P_{x+t:\overline{n-t}|} - P_{x:\overline{n}|})\ddot{a}_{x+t:\overline{n-t}|} \\ &= \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+t:\overline{n-t}|}}\right)A_{x+t:\overline{n-t}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} = 1 - \frac{1-A_{x+t:\overline{n-t}|}}{1-A_{x:\overline{n}|}} = 1 - \frac{P_{x:\overline{n}|}+d}{P_{x+t:\overline{n-t}|}+d}.\end{aligned}$$

Theorem 7.33. *The cdf. of* ${}_t\overline{L}(\overline{A}_x)$ *is*

$$\begin{aligned}F_{{}_t\overline{L}(\overline{A}_x)}(u) &= \begin{cases} 1 - F_{T_{x+t}}\left(-\frac{1}{\delta}\log\left(\frac{\delta u + \overline{P}(\overline{A}_x)}{\delta + \overline{P}(\overline{A}_x)}\right)\right) \ \textbf{Why ? (1)} & \text{if } -\frac{\overline{P}(\overline{A}_x)}{\delta} \le u \le 1, \ \textbf{Why ? (2)} \\ 1 & \text{if } 1 < u. \end{cases} \\ &= \begin{cases} \frac{1 - F_{T_x}\left(t - \frac{1}{\delta}\log\left(\frac{\delta u + \overline{P}(\overline{A}_x)}{\delta + \overline{P}(\overline{A}_x)}\right)\right)}{1 - F_{T_x}(t)} \ \textbf{Why ? (3)} & \text{if } -\frac{\overline{P}(\overline{A}_x)}{\delta} \le u \le 1, \\ 1 & \text{if } 1 < u. \end{cases}\end{aligned}$$

Reasons: $\quad L = e^{-\delta T_{x+t}}\left(1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right) - \frac{\overline{P}(\overline{A}_x)}{\delta}$.

(1) $F_{{}_t\overline{L}}(u) = \mathbb{P}({}_t\overline{L} \le u) = \mathbb{P}(e^{-\delta T_{x+t}}a + b \le u) = \mathbb{P}(e^{-\delta T_{x+t}} \le \frac{u-b}{a}) \qquad ((a,b) = \textbf{???})$
$= \mathbb{P}(-\delta T_{x+t} \le \ln\frac{u-b}{a}) = \mathbb{P}(T_{x+t} \ge -\frac{\ln\frac{u-b}{a}}{\delta}) = 1 - \mathbb{P}(T_{x+t} < -\frac{1}{\delta}\ln\frac{u-b}{a}) = ...$

(2) $T_{x+t} > 0 \Rightarrow -\frac{\ln\frac{u-b}{a}}{\delta} > 0 \Rightarrow \ln\frac{u-b}{a} < 0 \Rightarrow \frac{u-b}{a} < e^0 = 1 \Rightarrow u < a + b = 1$
if $u < -P/\delta$, then $F_{{}_t\overline{L}}(u) < \mathbb{P}({}_t\overline{L} \le -P/\delta) = \mathbb{P}(e^{-\delta T_{x+t}}a - P/\delta \le -P/\delta) = 0$.

(3) $1 - F_{T_{x+t}}(u) = S_{T_{x+t}}(u) = \frac{S_{T_x}(t+u)}{S_{T_x}(t)}$.

Theorem 7.34. *The probability density function of* ${}_t\overline{L}(\overline{A}_x)$ *is*

$$\begin{aligned} f_{{}_t\overline{L}(\overline{A}_x)}(u) =& \frac{f_{T_{x+t}}\left(-\frac{1}{\delta}\log\left(\frac{\delta u+\overline{P}(\overline{A}_x)}{\delta+\overline{P}(\overline{A}_x)}\right)\right)}{\delta u+\overline{P}(\overline{A}_x)} && \text{if } -\frac{\overline{P}(\overline{A}_x)}{\delta} < u < 1,\\ =& \frac{f_{T_x}\left(t-\frac{1}{\delta}\log\left(\frac{\delta u+\overline{P}(\overline{A}_x)}{\delta+P}\right)\right)}{(\delta u+\overline{P}(\overline{A}_x))(1-F_{T_x}(t))} && \text{if } -\frac{\overline{P}(\overline{A}_x)}{\delta} < u < 1, \end{aligned}$$

$F_L(u) = F_T(-\frac{1}{\delta}(\log(\delta u + P) - (\ln\delta + P)))$, $f_L(u) = F_L'(u)$, $g(h(u))' = g'(h(u))h'(u)$.

Corollary 7.1. *Under De Moivre's model with terminal age* ω, ${}_t\overline{L}(\overline{A}_x)$ *is a continuous r.v. with cumulative distribution function*

$$F_{{}_t\overline{L}(\overline{A}_x)}(u) = \frac{\delta(\omega-x-t)+\log\left(\frac{\delta u+\overline{P}(\overline{A}_x)}{\delta+\overline{P}(\overline{A}_x)}\right)}{\delta(\omega-x-t)}, \quad \text{if } e^{-\delta(\omega-x-t)} - \overline{P}(\overline{A}_x)\overline{a}_{\overline{\omega-x-t}|} < u < 1.$$

and density function

$$f_{{}_t\overline{L}(\overline{A}_x)}(u) = \frac{1}{(\omega-x-t)(\delta u+\overline{P}(\overline{A}_x))}, \quad \text{if } e^{-\delta(\omega-x-t)} - \overline{P}(\overline{A}_x)\overline{a}_{\overline{\omega-x-t}|} < u \le 1,$$

where $\overline{P}(\overline{A}_x) = \frac{\delta\overline{A}_x}{1-\overline{A}_x} = \frac{\delta\frac{\overline{a}_{\overline{\omega-x}|}}{\omega-x}}{1-\frac{\overline{a}_{\overline{\omega-x}|}}{\omega-x}} = \frac{\delta\overline{a}_{\overline{\omega-x}|}}{\omega-x-\overline{a}_{\overline{\omega-x}|}}$.

Corollary 7.2. *Under constant force of mortality* μ, ${}_t\overline{L}(\overline{A}_x)$ *is a continuous r.v. with cumulative distribution function*

$$F_{{}_t\overline{L}(\overline{A}_x)}(u) = \left(\frac{\delta u+\mu}{\delta+\mu}\right)^{\frac{\mu}{\delta}}, \quad -\frac{\mu}{\delta} < u < 1$$

and density function

$$f_{{}_t\overline{L}(\overline{A}_x)}(u) = \mu\frac{(\delta u+\mu)^{\frac{\mu}{\delta}-1}}{(\delta+\mu)^{\frac{\mu}{\delta}}}, \quad -\frac{\mu}{\delta} < u < 1.$$

Proof. Write ${}_t\overline{L} = {}_t\overline{L}(\overline{A}_x)$ and $P = \overline{P}(\overline{A}_x)$. Under constant force of mortality μ, $P = \frac{\overline{A}_x}{\frac{1-\overline{A}_x}{\delta}} = \frac{\mu}{\mu+\delta}/\frac{1-\frac{\mu}{\mu+\delta}}{\delta} = \mu$, as $v = e^{-\delta}$ and $\overline{A} = \int_0^\infty v^t e^{-\mu t}\mu dt = \int_0^\infty e^{-(\mu+\delta)t}\mu dt = \frac{\mu}{\mu+\delta}$. Then ${}_t\overline{L} = e^{-\delta T_{x+t}}\left(1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right) - \frac{\overline{P}(\overline{A}_x)}{\delta} = (1+\frac{\mu}{\delta})e^{-\delta T_{x+t}} - \mu/\delta$. Since $F_{T_x}(t) = 1-e^{-\mu t}$, if $0 \le t < \infty$; we have that for $-\frac{\mu}{\delta} < u < 1$, $F_{{}_t\overline{L}(\overline{A}_x)}(u) = P({}_t\overline{L} \le u) = P(T_{x+t} \ge (-\frac{1}{\delta}\log\frac{u+\mu/\delta}{1+\frac{\mu}{\delta}})) = \exp\left(\frac{\mu}{\delta}\ln\left(\frac{\delta u+\mu}{\delta+\mu}\right)\right) = \left(\frac{\delta u+\mu}{\delta+\mu}\right)^{\frac{\mu}{\delta}}$.

$f_{{}_t\overline{L}(\overline{A}_x)}(u) = F'_{{}_t\overline{L}(\overline{A}_x)}(u) = \mu\frac{(\delta u+\mu)^{\frac{\mu}{\delta}-1}}{(\delta+\mu)^{\frac{\mu}{\delta}}}, \quad -\frac{\mu}{\delta} < u < 1.$ ∎

Quiz on Friday: Upto [16].

Example 7.18. Skip it! *Under the assumptions in Example 7.17,*
(i) find $\mathbb{P}($ *the* 15*–th terminal loss r.v.*$>$ *its mean),*
(ii) find the first quartile of the 15*–th terminal loss r.v..*

Solution. (i) ${}_{15}L = B(v^{T_{x+t}} - P\frac{1-v^{T_{x+t}}}{\delta})$, $B = 40000$, $P \approx 0.024$ and ${}_{15}V = E({}_{15}L) \approx 0.109B \approx 4356$.
$\mathbb{P}(B(v^{T_{x+t}} - P\frac{1-v^{T_{x+t}}}{\delta}) > E({}_{15}L)) = ?$

$\approx \mathbb{P}(v^{T_{x+t}} - P\frac{1-v^{T_{x+t}}}{\delta} > 0.109) = \mathbb{P}(v^{T_{x+t}}(1+\frac{P}{\delta}) - \frac{P}{\delta} > 0.109)$
$= \mathbb{P}(v^{T_{x+t}} > \frac{\frac{P}{\delta}+0.109)}{(1+\frac{P}{\delta})}) = \mathbb{P}(T_{x+t}\ln v > \ln\frac{\frac{P}{\delta}+0.109)}{(1+\frac{P}{\delta})}) = \mathbb{P}(T_{x+t} < \frac{-1}{\delta}\ln\frac{\frac{P}{\delta}+0.109)}{(1+\frac{P}{\delta})})$
$= \mathbb{P}(T_{x+t} < a) = \frac{a-0}{w-(x+t)} \approx 0.389$

(ii) $\mathbb{P}(v^{T_{x+t}} - P\frac{1-v^{T_{x+t}}}{\delta} \le \eta_{1/4}) = \mathbb{P}(v^{T_{x+t}}(1+P/\delta) - P/\delta \le \eta_{1/4}) = 1/4$. $\eta_{0.25} = ?$
Two ways: (1) $c/40 = \mathbb{P}(T_{x+t} \le c) = 1/4$ where $c = 10$.

$\eta_{1/4} = v^{10}(1+P/\delta) - P/\delta = 0.324$.

The 1st quartile of ${}_tL$ is $B * 0.324 = 12960$.

(2) $1/4 = \mathbb{P}(v^{T_{x+t}}(1+P/\delta) - P/\delta \le \eta_{1/4}) = \mathbb{P}(T_{x+t} < \frac{-1}{\delta}\ln\frac{\frac{P}{\delta}+\eta_{1/4})}{(1+\frac{P}{\delta})}) = \mathbb{P}(T_{x+t} < 10)$. =>
$\frac{-1}{\delta}\ln\frac{\frac{P}{\delta}+\eta_{1/4})}{(1+\frac{P}{\delta})}) = 10$.
Solving the equation yields $\eta_{1/4} = 0.324$.

For a fully continuous whole life insurance which is funded for h years,

$$ {}^h_t\overline{V}(\overline{A}_x) = \begin{cases} \overline{A}_{x+t} - {}_h\overline{P}(\overline{A}_x)\overline{a}_{x+t:\overline{h-t}|} & \text{if } 0 \le t < h, \\ \overline{A}_{x+t} & \text{if } t \ge h. \end{cases} $$

Theorem 7.35. *If* $0 \le t < h$*, then*

$$ {}^h_t\overline{V}(\overline{A}_x) = \overline{A}_{x+t} - {}_h\overline{P}(\overline{A}_x)\overline{a}_{x+t:\overline{h-t}|} = \frac{{}_h\overline{P}(\overline{A}_x)\overline{a}_{x:\overline{t}|} - \overline{A}^{1}_{x:\overline{t}|}}{{}_tE_x} \qquad {}_h\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_{x:\overline{h}|}} $$
$$ = ({}_{h-t}\overline{P}(\overline{A}_{x+t}) - {}_h\overline{P}(\overline{A}_{x+t}))\overline{a}_{x+t:\overline{h-t}|} = \left(1 - \frac{{}_h\overline{P}(\overline{A}_x)}{{}_{h-t}\overline{P}(\overline{A}_{x+t})}\right)\overline{A}_{x+t}. \quad {}_{h-t}\overline{P}(\overline{A}_{x+t}) = \frac{\overline{A}_{x+t}}{\overline{a}_{x+t:\overline{h-t}|}} $$

7.3.2 n–year term insurance.

Theorem 7.36. *For an* n*–year term insurance, the* t*–th terminal loss r.v. is*
$$ {}_t\overline{L}(\overline{A}^{1}_{x:\overline{n}|}) = \begin{cases} \overline{Z}^{1}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}^{1}_{x:\overline{n}|})\overline{Y}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n, \\ 0 & \text{if } t = n \end{cases}, $$

$\overline{Z}^{1}_{x+t:\overline{n-t}|} = v^{T_{x+t}} I(T_{x+t} \le (n-t))$ and $\overline{Y}_{x+t:\overline{n-t}|} = \int_0^{T_{x+t}\wedge(n-t)} v^u du = \frac{1-v^{T_{x+t}\wedge(n-t)}}{\delta}$.

Theorem 7.37. *If $0 \le t < n$,*

$$
\begin{aligned}
&{}_t\overline{V}(\overline{A}^{\,1}_{x:\overline{n}|}) = \overline{A}^{\,1}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}^{\,1}_{x:\overline{n}|})\overline{a}_{x+t:\overline{n-t}|} = \frac{\overline{P}(\overline{A}^{\,1}_{x:\overline{n}|})\overline{a}_{x:\overline{t}|} - \overline{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x} \qquad \overline{P}(\overline{A}^{\,1}_{x:\overline{n}|}) = \frac{\overline{A}^{\,1}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}}\\
=&(\overline{P}(\overline{A}^{\,1}_{x+t:\overline{n-t}|}) - \overline{P}(\overline{A}^{\,1}_{x:\overline{n}|}))\overline{a}_{x+t:\overline{n-t}|} = \left(1 - \frac{\overline{P}(\overline{A}^{\,1}_{x:\overline{n}|})}{\overline{P}(\overline{A}^{\,1}_{x+t:\overline{n-t}|})}\right)\overline{A}^{\,1}_{x+t:\overline{n-t}|}.
\end{aligned}
$$

7.3.3 n–year pure endowment insurance.

Theorem 7.38. *For an n–year pure endowment insurance, the t–th terminal loss r.v. is*

$$
{}_t\overline{L}(\overline{A}_{x:\overline{n}|}^{\ \ 1}) = \begin{cases} \overline{Z}_{x+t:\overline{n-t}|}^{\ \ 1} - \overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1})\overline{Y}_{x+t:\overline{n-t}|} = \\ v^{n-t}I(n-t < T_{x+t}) - P\frac{1-v^{T_{x+t}\wedge(n-t)}}{\delta} & \text{if } 0 \le t < n,\\ 1 & \text{if } t = n. \end{cases}
$$

Theorem 7.39. *If $t < n$,*

$$
\begin{aligned}
&{}_t\overline{V}(\overline{A}_{x:\overline{n}|}^{\ \ 1}) = \overline{A}_{x+t:\overline{n-t}|}^{\ \ 1} - \overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1})\overline{a}_{x+t:\overline{n-t}|} = \frac{\overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1})\overline{a}_{x:\overline{t}|} - \overline{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x} \qquad \overline{a}_{x:\overline{n}|} = \int_0^n v^t\,{}_tp_x dt\\
=&(\overline{P}(\overline{A}_{x+t:\overline{n-t}|}^{\ \ 1}) - \overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1}))\overline{a}_{x+t:\overline{n-t}|} = \left(1 - \frac{\overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1})}{\overline{P}(\overline{A}_{x+t:\overline{n-t}|}^{\ \ 1})}\right)\overline{A}_{x+t:\overline{n-t}|}^{\ \ 1}).
\end{aligned}
$$

7.3.4 n–year endowment insurance.

Theorem 7.40. *For an n–year endowment insurance, the t–th terminal loss r.v. is*

$$
{}_t\overline{L}(\overline{A}_{x:\overline{n}|}) = \begin{cases} \overline{Z}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{Y}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n,\\ 1 & \text{if } t = n. \end{cases} \qquad (\overline{Z}_{x:\overline{n}|} = v^{T_x \wedge n})
$$

Theorem 7.41. *If $0 \le t < n$,*

$$
\begin{aligned}
&{}_t\overline{V}(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+t:\overline{n-t}|} = \frac{\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x:\overline{t}|} - \overline{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x}\\
=&(\overline{P}(\overline{A}_{x+t:\overline{n-t}|}) - \overline{P}(\overline{A}_{x:\overline{n}|}))\overline{a}_{x+t:\overline{n-t}|} = \left(1 - \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\overline{P}(\overline{A}_{x+t:\overline{n-t}|})}\right)\overline{A}_{x+t:\overline{n-t}|} = 1 - \frac{\overline{a}_{x+t:\overline{n-t}|}}{\overline{a}_{x:\overline{n}|}}\\
=&1 - \frac{1 - \overline{A}_{x+t:\overline{n-t}|}}{1 - \overline{A}_{x:\overline{n}|}} = 1 - \frac{\overline{P}(\overline{A}_{x:\overline{n}|}) + \delta}{\overline{P}(\overline{A}_{x+t:\overline{n-t}|}) + \delta}.
\end{aligned}
$$

Theorem 7.42. *For an n–year endowment insurance funded for h years, the t–th terminal* *loss r.v. is*

$$ {}^{h}_{t}\overline{L}(\overline{A}_{x:\overline{n}|}) = \begin{cases} \overline{Z}_{x+t:\overline{n-t}|} - {}_{h}\overline{P}(\overline{A}_{x:\overline{n}|})\overline{Y}_{x+t:\overline{h-t}|} & \text{if } t \le h < n, \\ \overline{Z}_{x+t:\overline{n-t}|} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases} $$

Theorem 7.43. *If $t \le h < n$,*

$$ {}^{h}_{t}\overline{V}(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+t:\overline{n-t}|} - {}_{h}\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+t:\overline{h-t}|} = \frac{{}_{h}\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x:\overline{t}|} - \overline{A}^{1}_{x:\overline{t}|}}{{}_{t}E_x} $$

$$ = ({}_{h-t}\overline{P}(\overline{A}_{x+t:\overline{n-t}|}) - {}_{h}\overline{P}(\overline{A}_{x:\overline{n}|}))\overline{a}_{x+t:\overline{h-t}|} = \left(1 - \frac{{}_{h}\overline{P}(\overline{A}_{x:\overline{n}|})}{{}_{h-t}\overline{P}(\overline{A}_{x+t:\overline{n-t}|})}\right)\overline{A}_{x+t:\overline{n-t}|}. $$

7.3.5 n–year deferred insurance.

If an n–year deferred insurance is funded during the first n years, the benefit premium is ${}_{n}P({}_{n|}\overline{A}_x) = \frac{{}_{n|}\overline{A}_x}{\overline{a}_{x:\overline{n}|}}$.

Theorem 7.44. *For a fully continuous n–year deferred insurance, funded during the first n years, the t–th terminal loss r.v. is*

$$ {}^{n}_{t}\overline{L}({}_{n|}\overline{A}_x) = \begin{cases} {}_{n-t|}\overline{Z}_{x+t} - {}_{n}P({}_{n|}\overline{A}_x)\overline{Y}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n, \\ \overline{Z}_{x+t} & \text{if } t \ge n. \end{cases} \qquad ({}_{n|}\overline{Z}_x = v^{T_x} I(T_x \ge n)) $$

Insurance type	t–th terminal benefit reserve
Whole life insurance	${}_{t}\overline{V}(\overline{A}_x) = \overline{A}_{x+t} - \overline{P}(\overline{A}_x)\overline{a}_{x+t}$
h–payment whole life insurance	${}^{h}_{t}\overline{V}(\overline{A}_x) = \begin{cases} \overline{A}_{x+t} - {}_{h}\overline{P}(\overline{A}_x)\overline{a}_{x+t:\overline{h-t}|} & \text{if } 0 \le t < h, \\ \overline{A}_{x+t} & \text{if } t \ge h. \end{cases}$
n–year term insurance	${}_{t}\overline{V}(\overline{A}^{1}_{x:\overline{n}|}) = \overline{A}^{1}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}^{1}_{x:\overline{n}|})\overline{a}_{x+t:\overline{n-t}|}$ if $0 \le t < n$.
n–year pure endowment	${}_{t}\overline{V}(\overline{A}_{x:\overset{1}{\overline{n}|}}) = \begin{cases} \overline{A}_{x+t:\overset{1}{\overline{n-t}|}} - \overline{P}(\overline{A}_{x:\overset{1}{\overline{n}|}})\overline{a}_{x+t:\overline{n-t}|}, & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year endowment	${}_{t}\overline{V}(\overline{A}_{x:\overline{n}|}) = \begin{cases} \overline{A}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+t:\overline{n-t}|} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
h–payment n–year endowment	${}^{h}_{t}\overline{V}(\overline{A}_{x:\overline{n}|}) = \begin{cases} \overline{A}_{x+t:\overline{n-t}|} - {}_{h}\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+t:\overline{h-t}|} & \text{if } t \le h < n, \\ \overline{A}_{x+t:\overline{n-t}|} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year deferred insurance	${}^{n}_{t}\overline{V}({}_{n\|}\overline{A}_x) = \begin{cases} {}_{n-t\|}\overline{A}_{x+t} - {}_{n}P({}_{n\|}\overline{A}_x)\overline{a}_{x+t:\overline{n-t}\|} & \text{if } 0 \le t < n, \\ \overline{A}_{x+t} & \text{if } t \ge n. \end{cases}$
n–year deferred life annuity	${}_{t}\overline{V}({}_{n\|}\overline{a}_x) = \begin{cases} {}_{n-t\|}\overline{a}_{x+t} - \overline{P}({}_{n\|}\overline{a}_x)\overline{a}_{x+t:\overline{n-t}\|} & \text{if } 0 \le t < n, \\ \overline{a}_{x+t} & \text{if } t \ge n. \end{cases}$

Table 7.2: t–th terminal benefit reserve for some fully continuous contracts.

7.3.6 n–year deferred life annuity.

Theorem 7.45. *The t–th loss r.v. for a n–year deferred continuous annuity is*

$$ {}_t\overline{L}({}_{n|}\overline{a}_x) = \begin{cases} {}_{n-t|}\overline{Y}_{x+t} - \overline{P}({}_{n|}\overline{a}_x)\overline{Y}_{x+t:\overline{n-t}|} & \textit{if } 0 \le t < n, \\ \overline{Y}_{x+t} & \textit{if } t \ge n. \end{cases} \qquad \left({}_{n|}\overline{Y}_x = \int_n^{T_x} v^t dt I(T_x > n)\right) $$

Theorem 7.46. *If $t < n$,*

$$ {}_t\overline{V}({}_{n|}\overline{a}_x) = {}_{n-t|}\overline{a}_{x+t} - \overline{P}({}_{n|}\overline{a}_x)\overline{a}_{x+t:\overline{n-t}|} = \frac{\overline{P}({}_{n|}\overline{a}_x)\overline{a}_{x:\overline{t}|}}{{}_tE_x} $$

$$ =(\overline{P}({}_{n-t|}\overline{a}_x) - \overline{P}({}_{n|}\overline{a}_x))\overline{a}_{x+t:\overline{n-t}|} = \left(1 - \frac{\overline{P}({}_{n|}\overline{a}_x)}{\overline{P}({}_{n-t|}\overline{a}_x)}\right){}_{n-t}E_{x+t}\overline{a}_{x+n} = \frac{\overline{a}_{x:\overline{t}|}}{\overline{a}_{x:\overline{n}|}}\left(\overline{a}_x - \overline{a}_{x:\overline{t}|}\right). $$

7.4 Reserves for insurance paid discretely and funded continuously.

In this section, we assume that benefits are paid at the end of the year of death and the benefit premium is continuous.

Insurance type	t–th terminal loss r.v.
Whole life insurance	${}_tL = Z_{x+t} - P\overline{Y}_{x+t}$
h–payment whole life insurance	${}^h_tL = \begin{cases} Z_{x+t} - P\overline{Y}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h, \\ Z_{x+t} & \text{if } t \ge h. \end{cases}$
n–year term insurance	${}_tL = Z^1_{x+t:\overline{n-t}\rvert} - P\overline{Y}_{x+t:\overline{n-t}\rvert}$ if $0 \le t < n$,
n–year pure endowment	${}_tL = \begin{cases} Z_{x+t:\overset{\ 1}{\overline{n-t}}\rvert} - P\overline{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
h–payment n–year endowment	${}^h_tL = \begin{cases} Z_{x+t:\overline{n-t}\rvert} - P\overline{Y}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h < n, \\ Z_{x+t:\overline{n-t}\rvert} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year endowment	${}_tL = \begin{cases} Z_{x+t:\overline{n-t}\rvert} - P\overline{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year deferred insurance	${}^n_tL = \begin{cases} {}_{n-t\rvert}Z_{x+t} - P\overline{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ Z_{x+t} & \text{if } t \ge n. \end{cases}$
n–year deferred annuity–due	${}_tL = \begin{cases} {}_{n-t\rvert}\ddot{Y}_{x+t} - P\overline{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ \ddot{Y}_{x+t} & \text{if } t \ge n. \end{cases}$

Table 7.3: t–th terminal loss r.v. paid discretely and funded continuously.

We have study two typical types for $t-$th teminal benifit reserves:

Fully discrete Insurance, *e.g.*, $L = v^{K_{x+t}} - P\sum_{k=0}^{K_{x+t}-1} v^k$;

Fully continuous Insurance, *e.g.*, $L = v^{T_{x+t}} - P\int_0^{T_{x+t}} v^y dy$;

In Section 7.4 and 7.5, two more types are discussed.

Insurance paid discretely, and funded continuously *e.g.*, $L = v^{K_{x+t}} - P\int_0^{T_{x+t}} v^y dy$;

Insurance paid immediately and funded discretely *e.g.*, $L = v^{T_{x+t}} - P\sum_{k=0}^{K_{x+t}-1} v^k$.

Insurance type	t–th terminal loss r.v. with unit benefit
Whole life insurance	${}_tL = \overline{Z}_{x+t} - P\ddot{Y}_{x+t}$
h–payment whole life insurance	${}^h_tL = \begin{cases} \overline{Z}_{x+t} - P\ddot{Y}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h, \\ \overline{Z}_{x+t} & \text{if } t \ge h. \end{cases}$
n–year term insurance	${}_tL = \overline{Z}^{1}_{x+t:\overline{n-t}\rvert} - P\ddot{Y}_{x+t:\overline{n-t}\rvert}$ if $0 \le t < n$,
n–year pure endowment	${}_tL = \begin{cases} \overline{Z}_{x+t:\overset{1}{\overline{n-t}\rvert}} - P\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
h–payment n–year endowment	${}^h_tL = \begin{cases} \overline{Z}_{x+t:\overline{n-t}\rvert} - P\ddot{Y}_{x+t:\overline{h-t}\rvert} & \text{if } 0 \le t < h < n, \\ \overline{Z}_{x+t:\overline{n-t}\rvert} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year endowment	${}_tL = \begin{cases} \overline{Z}_{x+t:\overline{n-t}\rvert} - P\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year deferred insurance	${}^n_tL = \begin{cases} {}_{n-t}\rvert\overline{Z}_{x+t} - P\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ \overline{Z}_{x+t} & \text{if } t \ge n. \end{cases}$
n–year deferred annuity–due	${}_tL = \begin{cases} {}_{n-t}\rvert\ddot{Y}_{x+t} - P\ddot{Y}_{x+t:\overline{n-t}\rvert} & \text{if } 0 \le t < n, \\ \ddot{Y}_{x+t} & \text{if } t \ge n. \end{cases}$

Table 7.4: t–th terminal loss r.v. paid immediatly and funded discretely.

7.5 Reserves for insurance paid immediately and funded discretely.

In this section, we consider the benefit reserves for the insurance paid immediately and funded at the beginning of the year.

7.6 Benefit reserves for general fully discrete insurance.

In this subsection, we consider a general insurance contract, with non–level benefits paid at the end of the year and non–level benefit premiums paid at the beginning of the year. This set–up applies to all fully discrete insurances considered before that only offer death benefit.

$L = b_{K_x}v^{K_x} - \sum_{k=0}^{K_x-1}\pi_k v^k$, where
b_k is the benefit paid by an insurance company at the end of year k, and π_{k-1} is the benefit premium received by an insurance company at the beginning of year k, $k = 1, 2, \ldots$.

Hence, the APV of the contingent benefit $b_{K_x}v^{K_x}$ is

$$E(b_{K_x}v^{K_x}) = \sum_{k=1}^{\infty} b_k v^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1}\rvert q_x. \qquad (\textstyle\sum_{k=1}^{\infty} Bv^k \cdot {}_{k-1}\rvert q_x)$$

The contingent cashflow of benefit premiums is

benefit premiums	π_0	π_1	π_2	π_3	$\cdots$
Time after issue	0	1	2	3	$\cdots$

The APV of the contingent benefit premiums is

$$E(\sum_{k=0}^{K_x-1}\pi_k v^k) = \sum_{k=0}^{\infty}\pi_k v^k \mathbb{P}\{K_x > k\} = \sum_{k=0}^{\infty}\pi_k v^k \cdot {}_kp_x. \qquad (\textstyle\sum_{k=0}^{\infty} Pv^k \cdot {}_kp_x)$$

Under the equivalence principle,

$$\sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1}|q_x = \sum_{k=0}^{\infty} \pi_k v^k \cdot {}_k p_x.$$

The general insurance defined above includes endowments and annuities.

Example 7.19. *Let $b_k = b_1(1+r_b)^{k-1}$, $k \geq 1$, and $\pi_k = \pi_0(1+r_\pi)^k$, $k \geq 0$. Derive a formula for π_0.*

Sol. $\sum_{k=1}^{\infty} b_1(1+r_b)^{k-1} v^k \cdot {}_{k-1}|q_x = \sum_{k=0}^{\infty} \pi_0(1+r_\pi)^k v^k \cdot {}_k p_x$ *and*

$$\pi_0 = \frac{\sum_{k=1}^{\infty} b_1(1+r_b)^{k-1} v^k \cdot {}_{k-1}|q_x}{\sum_{k=0}^{\infty}(1+r_\pi)^k v^k \cdot {}_k p_x}.$$

Definition 7.9. $L = b_{K_x} v^{K_x} - \sum_{k=0}^{K_x-1} \pi_k v^k$.

If a policy is in effect at time t, the t–th terminal loss random variable is

$$_tL = b_{K_x} v^{K_{x+t}} - \sum_{k=0}^{K_{x+t}-1} \pi_{t+k} v^k = \sum_{k=1}^{\infty} b_{t+k} v^k I(K_{x+t} = k) - \sum_{k=0}^{\infty} \pi_{t+k} v^k I(K_{x+t} > k). \quad (1)$$

Reason : *2nd Expression*

$$\begin{aligned}
&= \sum_{k=1}^{\infty} b_{t+k} v^k I(K_x = k+t, K_{x+t} = k) - \sum_{i=1}^{\infty}\sum_{k=0}^{i-1} \pi_{t+k} v^k I(K_{x+t} = i) \\
&= \sum_{k=1}^{\infty} b_{t+k} v^k I(K_{x+t} = k) - \sum_{k=0}^{\infty} \pi_{t+k} v^k \sum_{i>k}^{\infty} I(K_{x+t} = i) \\
&= \sum_{k=1}^{\infty} b_{t+k} v^k I(K_x = k+t) - \sum_{k=0}^{\infty} \pi_{t+k} v^k \sum_{i>k}^{\infty} I(K_x = i+t) \\
&= \sum_{k=1}^{\infty} b_{t+k} v^k I(K_x = k+t) - \sum_{k=0}^{\infty} \pi_{t+k} v^k I(K_x > k+t) \\
&= \sum_{k=1}^{\infty} b_{t+k} v^k I(K_{x+t} = k) - \sum_{k=0}^{\infty} \pi_{t+k} v^k I(K_{x+t} > k) \quad (3rd\ expression)
\end{aligned}$$

Alternatively, $_tL = (b_{K_x} v^{K_x - t} - \sum_{k=0}^{K_x-t-1} \pi_{t+k} v^k)|(K_x > t)$. That is, if $K_x > t$,

$$_tL = b_{K_x} v^{K_x-t} - \sum_{k=0}^{K_x-t-1} \pi_{t+k} v^k = \sum_{k=1}^{\infty} b_{t+k} v^k I(K_x = t+k) - \sum_{k=0}^{\infty} \pi_{t+k} v^k I(K_x > t+k). \quad (2) v.s. (1)$$

Definition 7.10. *We denote the t–th terminal benefit reserve of a general life insurance assuming that the policy is in effect by $_tV$.*

$$_tV = E[_tL] = \sum_{j=1}^{\infty} b_{t+j} v^j \cdot {}_{j-1}|q_{x+t} - \sum_{j=0}^{\infty} \pi_{t+j} v^j \cdot {}_j p_{x+t}.$$

${}_tL = \sum_{j=1}^{\infty} b_{t+j} v^j I(K_x = t+j) - \sum_{j=0}^{\infty} \pi_{t+j} v^j I(K_x > t+j)$.

${}_0L = \sum_{i=1}^{\infty} b_i v^i I(K_x = i) - \sum_{j=0}^{\infty} \pi_j v^j I(K_x > j)$.

${}_0L = \sum_{j+1=1}^{\infty} b_{j+1} v^{j+1} I(K_x = j+1) - \sum_{j=0}^{\infty} \pi_j v^j I(K_x > j)$, $i = j+1$.

${}_0L = \sum_{j=0}^{\infty} [b_{j+1} v^{j+1} I(K_x = j+1) - \pi_j v^j I(K_x > j)]$.

${}_0L = \sum_{j=0}^{\infty} v^j \underbrace{[b_{j+1} v I(K_x = j+1) - \pi_j I(K_x > j)]}_{\overset{def}{=} C_j}$. ${}_0V = E({}_0L) = 0$.

Definition 7.11. $C_j = vb_{j+1} I(K_x = j+1) - \pi_j I(K_x > j)$ *denotes the present value at time* j *of the net cash loss during the* $(j+1)$*–th year.*

The cashflow in the reserves during the $(j+1)$–th year is:

(i) benefit of b_{j+1} paid at time $j+1$ if $K_x = j+1$.

(ii) benefit premium of π_j received at time j if $K_x > j$.

Examples: For a fully discrete unity whole life insurance to (x),

$C_j = vI(K_x = j+1) - P_x I(K_x > j)$, if $j \geq 0$ v.s. $C_j = vb_{j+1} I(K_x = j+1) - \pi_j I(K_x > j)$

For an n–year unit endowment insurance to (x), ${}_0L{=}vv^{K_x \wedge n-1} - P_{x:\overline{n}|} \sum_{j=0}^{K_x \wedge n-1} v^j$.

$$C_j = \begin{cases} vI(K_x = j+1) - P_{x:\overline{n}|} I(K_x > j) & \text{if } 0 \leq j \leq n-1 \\ I(K_x > n) & \text{if } j = n, \end{cases}$$

Theorem 7.47. ${}_tL = \sum_{j=0}^{\infty} v^j C_{t+j}$. *(*$C_j = vb_{j+1} I(K_x = j+1) - \pi_j I(K_x > j)$*)*

Proof. $\vdash$: $\sum_{j=0}^{\infty} v^j C_{t+j} = {}_tL$ $(\overset{def}{=} b_{K_x} v^{K_{x+t}} - \sum_{k=0}^{K_{x+t}-1} \pi_{t+k} v^k)$.

$$\sum_{j=0}^{\infty} v^j C_{t+j} = \sum_{j=0}^{\infty} v^j (vb_{t+j+1} I(K_x = t+j+1) - \pi_{t+j} I(K_x > t+j))$$
$$= \sum_{j=0}^{\infty} v^{j+1} b_{t+j+1} I(K_x = t+j+1) - \sum_{j=0}^{\infty} v^j \pi_{t+j} I(K_x > t+j)$$
$$= \sum_{j=1}^{\infty} v^j b_{t+j} I(K_x = t+j) - \sum_{j=0}^{\infty} v^j \pi_{t+j} I(K_x > t+j) = {}_tL \text{ (by Eq.(1) after Def. 7.9).}$$

Theorem 7.48. ${}_tL = C_t + v \cdot {}_{t+1}L$.

Proof. ${}_tL = \sum_{j=0}^{\infty} v^j C_{t+j}$ (by Th.7.47)

$= C_t + \sum_{j=1}^{\infty} v^j C_{t+j} = C_t + \sum_{i=0}^{\infty} v^{i+1} C_{t+i+1}$ $(j = i+1)$

$= C_t + v \sum_{j=0}^{\infty} v^j C_{t+j+1} = C_t + v \cdot {}_{t+1}L$ (by Th. 7.47).

Theorem 7.49. ${}_{t+1}L = ({}_tL + \pi_t)(1+i) I(K_x > t) - b_{t+1} I(K_x = t+1)$

$= ({}_tL + \pi_t)(1+i) I(K_x > t+1)$.

Proof. Definitions for ${}_tL$ and C_j:

(1) ${}_tL = b_{K_x} v^{K_{x+t}} - \sum_{k=0}^{K_{x+t}-1} \pi_{t+k} v^k$;
(2) $C_j = v b_{j+1} I(K_x = j+1) - \pi_j I(K_x > j)$.
Using that ${}_tL = {}_tL[I(K_x \le t) + I(K_x > t)]$,

$$\begin{aligned} {}_{t+1}L &= \frac{1}{v}{}_tL - \frac{1}{v}C_t \qquad (by\ Th.\ 7.48:\ {}_tL = C_t + v \cdot {}_{t+1}L) \\ &=(1+i){}_tL I(K_x > t) - b_{t+1} I(K_x = t+1) + (1+i)\pi_t I(K_x > t) \qquad (by\ (2)\ , v = \frac{1}{i+1}) \\ &=({}_tL + \pi_t)(1+i) I(K_x > t) - b_{t+1} I(K_x = t+1).\ \text{Done for 1st equation.} \end{aligned}$$

If $I(K_x = t+1) = 1$, then ${}_tL = v b_{t+1} - \pi_t$ (by (1))
$=> b_{t+1} = ({}_tL + \pi_t)(1+i)$. So,

$$\begin{aligned} &({}_tL + \pi_t)(1+i) I(K_x > t) - b_{t+1} I(K_x = t+1) \\ =\ & b_{t+1}[I(K_x > t) - I(K_x = t+1)] \\ =\ & ({}_tL + \pi_t)(1+i) I(K_x > t+1). \end{aligned}$$

Theorem 7.50. *$E[C_t | K_x > t] = v b_{t+1} q_{x+t} - \pi_t$ and*

$$\mathrm{Var}(C_t | K_x > t) = v^2 b_{t+1}^2 p_{x+t} q_{x+t}.$$

Proof. Conditional on $K_x > t$, $C_t = v b_{t+1} I(K_x = t+1) - \pi_t$ by (2) above. Hence,

$$\begin{aligned} E[C_t | K_x > t] =& v b_{t+1} \mathbb{P}\{K_x = t+1 | K_x > t\} - \pi_t = v b_{t+1} q_{x+t} - \pi_t, \\ \mathrm{Var}(C_t | K_x > t) =& \mathrm{Var}(v b_{t+1} I(K_x = t+1) | K_x > t) \\ =& v^2 b_{t+1}^2 \mathbb{P}\{K_{x+t} = 1 | K_{x+t} > 0\}(1 - \mathbb{P}\{K_{x+t} = 1 | K_{x+t} > 0\}) \\ =& v^2 b_{t+1}^2 \cdot q_{x+t} \cdot p_{x+t}. \end{aligned}$$

Theorem 7.51. *$E[{}_{t+1}L | K_x > t] = {}_{t+1}V p_{x+t}$.*

Proof. Since ${}_{t+1}L_x = 0$ if $K_x \le t+1$,

$$\begin{aligned} E[{}_{t+1}L | K_x > t] &= E[{}_{t+1}L \cdot (I(K_x = t+1) + I(K_x > t+1)) | K_x > t] \\ =& 0 \frac{\mathbb{P}\{K_x = t+1\}}{\mathbb{P}\{K_x > t\}} + \frac{E[{}_{t+1}L I(K_x > t+1)] \mathbb{P}\{K_x > t+1\}}{\mathbb{P}\{K_x > t\}} = {}_{t+1}V p_{x+t}. \end{aligned}$$

Theorem 7.52. *(***Iterative formula for the terminal benefit reserve using the initial reserve***)*

$${}_tV + \pi_t = v b_{t+1} q_{x+t} + v \cdot {}_{t+1}V p_{x+t}.$$

Previous theorem states that the initial reserve is used to paid death benefit to the deceased and to fund the benefit reserve for the survivors.

It follows from Theorem 7.52 that:

Theorem 7.53. *(***Iterative formula for the terminal benefit reserve using the net amount at risk***)*

$${}_{t+1}V = ({}_tV + \pi_t)(1+i) - q_{x+t}(b_{t+1} - {}_{t+1}V).$$

The **net amount at risk in the** $(t+1)$**–th year** is $b_{t+1} - {}_{t+1}V$. The decrease in the benefit reserves depends on the net amount at risk and the mortality during the year $x+t$.

It also follows from Theorem 7.52 that:

Theorem 7.54. *(***Iterative formula for the terminal benefit reserve using the benefit premium***)*

$$\pi_t = (v \cdot {}_{t+1}V - {}_tV)p_{x+t} + (vb_{t+1} - {}_tV)q_{x+t} = vb_{t+1}q_{x+t} + (v \cdot {}_{t+1}Vp_{x+t} - {}_tV)\,. \tag{1}$$

Two interpretations of Eq.(1):

In $\pi_t = (v \cdot {}_{t+1}V - {}_tV)p_{x+t} + (vb_{t+1} - {}_tV)q_{x+t}$ the benefit premium is decomposed into survivors and deceased. The terminal benefit reserve for survivors is ${}_tV$ at time t and ${}_{t+1}V$ at time $t+1$. The present value at time t of the adjustment in the terminal benefit reserves for survivors is $(v \cdot {}_{t+1}V - {}_tV)$. Deceased have a terminal benefit reserve of ${}_tV$ at time t and received a death benefit of b_{t+1} at time $t+1$. The present value at time t of the adjustment in the terminal benefit reserves for deceased is $vb_{t+1} - {}_tV$.

In $\pi_t = vb_{t+1}q_{x+t} + (v \cdot {}_{t+1}Vp_{x+t} - {}_tV)$ the benefit premium is decomposed into the part that goes to pay benefit and the part that goes to adjust benefit reserves.

Example 7.20. *A special four–year term insurance pays non level benefits to* (x). *Benefits and annual mortality rates are given by:*

k	b_k	q_{x+k-1}
1	*10000*	*0.01*
2	*5000*	*0.02*
3	*4000*	*0.01*
4	*1000*	*0.02*

Assume that $i = 0.06$.

(i) Find the level net annual benefit premium which follows the equivalence principle.
(ii) Calculate ${}_kV$ *for* $k = 1, 2, 3, 4$.
(iii) Calculate C_j, $j = 0, 1, 2, 3$, *given that* (x) *dies in the third year. (See Def. 7.11).*

Solution: (i) $\sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1|}q_x = \sum_{k=0}^{\infty} \pi_k v^k \cdot {}_kp_x$ by $E(L) = 0$.
$\sum_{k=1}^{4} b_k v^k \cdot {}_{k-1|}q_x = \sum_{k=0}^{4} \pi v^k \cdot {}_kp_x$ by (i) and 4 year term insurance.

$$\begin{aligned}
&\sum_{k=1}^{4} b_k v^k \cdot {}_{k-1|}q_x \qquad\qquad ({}_{k-1|}q_x = \prod_{j\geq 0}^{k-2} p_{x+j}q_{x+k-1})\\
=&(10000)(1.06)^{-1}(0.01) + (5000)(1.06)^{-2}(0.99)(0.02) + (4000)(1.06)^{-3}(0.99)(0.98)(0.01)\\
&+ (1000)(1.06)^{-4}(0.99)(0.98)(0.99)(0.02) = 230.2493029\\
=&\sum_{k=0}^{4} \pi v^k \cdot {}_kp_x \qquad\qquad ({}_kp_x = \prod_{j\geq 0}^{k-1} p_{x+j})\\
=&\pi + \pi(1.06)^{-1}(0.99) + \pi(1.06)^{-2}(0.99)(0.98) + \pi(1.06)^{-3}(0.99)(0.98)(0.99)\\
=&3.603889452\pi.
\end{aligned}$$

Hence, $\pi = \frac{230.2493029}{3.603889452} = 63.88911368$.

(ii) Two ways for ${}_tV$: (1) Definition, (2) Th. 7.52.
(1) ${}_tV = E({}_tL) = \sum_{k=1}^{\infty} b_{t+k} v^k f_{K_{x+t}}(k) - \sum_{k=0}^{\infty} \pi_{t+k} v^k {}_kp_{x+t}$, due to
${}_tL = b_{K_x} v^{K_{x+t}} - \sum_{k=0}^{K_{x+t}-1} \pi_{t+k} v^k = \sum_{k=1}^{\infty} b_{t+k} v^k I(K_{x+t} = k) - \sum_{k=0}^{\infty} \pi_{t+k} v^k I(K_{x+t} > k)$.
(2) ${}_0V = E({}_0L) = 0$ and ${}_{t+1}V = \frac{(\pi_t + {}_tV)(1+i) - b_{t+1} q_{x+t}}{p_{x+t}}$, due to ${}_tV + \pi_t = v b_{t+1} q_{x+t} + v_{t+1} V p_{x+t}$.
Which to choose ?

$$
\begin{aligned}
{}_1V &= \frac{(\pi + {}_0V)(1+i) - b_1 q_x}{p_x} = \frac{(63.88911368)(1.06) - (10000)(0.01)}{0.99} = -32.60357525, \\
{}_2V &= \frac{(\pi + {}_1V)(1+i) - b_2 q_{x+1}}{p_{x+1}} = \frac{(63.88911368 - 32.60357525)(1.06) - (5000)(0.02)}{0.98} = -68.20135639, \\
{}_3V &= \frac{(63.88911368 - 68.20135639)(1.06) - (4000)(0.01)}{0.99} = -45.02118916, \\
{}_4V &= \frac{(63.88911368 - 45.02118916)(1.06) - (1000)(0.02)}{0.98} = 0.
\end{aligned}
$$

(iii) If $K_x = 3$, $C_j = v b_{j+1} I(K_x = j+1) - \pi_j I(K_x > j)$=? for $j = 0, 1, 2, 3$,

$$C_0 = C_1 = -\pi \approx -63.9, \quad C_2 = (1.06)^{-1}(4000) - 63.88911368 = 3709.695792, \quad C_3 = 0.$$

Example 7.21. *For a fully discrete* 10*–year term life insurance of* 50000 *on* (40)*, you are given:*

t	$(50000) \cdot {}_tV_{40}$	q_{40+t}
5	1345.00	0.02
6	1206.08	0.025
7	806.96	0.03

Calculate i and the annual benefit premium P.

Solution: $(i, P) = ?$ Th. 7.52 or formula sheet: ${}_tV + \pi_t = v b_{t+1} q_{x+t} + v \cdot {}_{t+1}V p_{x+t}$, $v = \frac{1}{1+i}$,
${}_tV = 50000 {}_tV_{40}$, $b_t = 50000$. $\pi_t = P$.
We need 2 equations. $({}_tV + P)(i+1) = 50000 q_{x+t} + {}_{t+1}V p_{x+t}$, $t \in \{5, 6\}$ from above.

$$
\begin{aligned}
&(1)\ (1345 + P)(1+i) = (50000)(0.02) + (1206.08)(1 - 0.02) = 2181.9584, \\
&(2)\ (1206.08 + P)(1+i) = (50000)(0.025) + (806.96)(1 - 0.025) = 2036.786.
\end{aligned}
$$

Eq. (1)-Eq. (2) yields
$(1345 - 1206.08)(1+i) = 2181.9584 - 2036.786 \Rightarrow i = 0.045007198$.
Eq. (1) yields $P = \frac{2181.9584}{1.045007198} - 1345 = 742.9840868$.

Example 7.22. *For a fully discrete 25–payment whole life insurance of 1000 on* (x)*, you are given:*
(i) $i = 6\%$.
(ii) The annual benefit premium is 60.
(iii) The terminal reserve at the end of year 19 is 520.
(iv) $q_{x+19} = 0.02$.
Calculate the terminal reserve at the end of year 20.

Solution: ${}_{20}V = ?$ Formula: ${}_tV + \pi_t = v b_{t+1} q_{x+t} + v \cdot {}_{t+1}V p_{x+t}$,
$t = 19$, ${}_{19}V = 520$, $\pi_t = P = 60$.

That is, ${}_{19}V + P = v \cdot 1000 q_{x+19} + v \cdot {}_{20}V p_{x+19}$. So,
$(520 + 60)(1.06) = (1000)(0.02) + {}_{20}V(0.98)$ =>

$${}_{20}V = \frac{(520 + 60)(1.06) - (1000)(0.02)}{0.98} = 606.9387755.$$

Example 7.23. *For a special fully discrete whole life insurance on* (40)*:*
(i) The death benefit for year k is \$1000 plus the benefit reserve at the end of year k, $k = 1, \ldots$.
(ii) $i = 0.08$
(iii) Mortality has constant force 0.01*.*
Calculate the level annual benefit premium.

Solution: $10^3\pi = ?$ Formula: ${}_tV + \pi = vb_{t+1}q_{x+t} + vp_{x+t} \cdot {}_{t+1}V$ and (i) $b_t = 1 + {}_tV$.

$$\begin{aligned}
{}_tV + \pi =& vb_{t+1}q_{x+t} + vp_{x+t} \cdot {}_{t+1}V \qquad (b_{t+1} = 1 + {}_{t+1}V)\\
=& vq_{x+t} + v_{t+1}Vq_{x+t} + vp_{x+t} \cdot {}_{t+1}V\\
=& vq_{x+t} + v \cdot {}_{t+1}V\\
=> \quad & {}_tV - v \cdot {}_{t+1}V = v^1 q_{x+t} - \pi.\\
=> \quad & v^t \cdot {}_tV - v^{t+1} \cdot {}_{t+1}V = v^{t+1}q_{x+t} - v^t\pi \qquad (1)\\
& ({}_0V - v_1V) + (v_1V - v^2{}_2V) + \cdots = vq_x - v\pi + v^2q_{x+1} - v^2\pi + \cdot
\end{aligned}$$

$$\begin{aligned}
&\underbrace{0 = {}_0V - v^\infty{}_\infty V}_{why?} && {}_0V =? \quad v^\infty =?\\
=&\sum_{t=0}^{\infty}\left(v^t{}_tV - v^{t+1} \cdot {}_{t+1}V\right) = \sum_{t=0}^{\infty}\left(v^{t+1}q_{x+t} - v^t\pi\right) && \textit{by Eq. } (1)\\
=&v\sum_{t=0}^{\infty}v^tq_{x+t} - \sum_{t=0}^{\infty}v^t\pi. && \sum_{t=0}^{n-1}v^t = \ddot{a}_{\overline{n}|}
\end{aligned}$$

=> $\pi = \frac{v\sum_{t=0}^{\infty}v^tq_{x+t}}{\sum_{t=0}^{\infty}v^t}$ and $q_{x+t} = P(T_{x+t} \le 1) = 1 - e^{-\mu}$. Thus, the level annual benefit premium is $10^3\pi = 10^3 v\frac{\sum_{t=0}^{\infty}v^tq_{40+t}}{\ddot{a}_{\overline{\infty}|}} = 10^3\frac{v\sum_{t=0}^{\infty}v^t(1-e^{-0.01})}{\ddot{a}_{\overline{\infty}|}} = 10^3\frac{1}{1.08}\frac{\ddot{a}_{\overline{\infty}|}}{\ddot{a}_{\overline{\infty}|}}(1 - e^{-0.01}) = 9.21$.

Example 7.24. *For a special fully discrete n–year term life insurance on (x): (i) $i = 0.08$*
(ii) Mortality has constant force 0.02*.*
(iii) The death benefit for year k is \$$10^3$+the benefit reserve at the end of year k, $k = 1, \ldots, n$.
Calculate the level annual benefit premium.

Solution: $10^3\pi = ?$ where ${}_tV + \pi = vb_{t+1}q_{x+t} + vp_{x+t} \cdot {}_{t+1}V$.

$b_t = (1 + {}_tV)$ if $t \in \{1, ..., n\}$.

$$\begin{aligned}
{}_tV + \pi =& vq_{x+t}(1 + {}_{t+1}V) + vp_{x+t} \cdot {}_{t+1}V \quad \text{similiar to deriving (1)} \\
=& vq_{x+t} + v \cdot {}_{t+1}V \qquad \text{if } t = 0, \dots, n-1. \\
=> \ & v^t \cdot {}_tV - v^{t+1} \cdot {}_{t+1}V = v^{t+1}q_{x+t} - v^t\pi \qquad (2)
\end{aligned}$$

$$\underbrace{0 = {}_0V - v^n \cdot {}_nV}_{why?} = \sum_{t=0}^{n-1} \left(v^t {}_tV - v^{t+1} \cdot {}_{t+1}V\right) = \sum_{t=0}^{n-1} \left(v^{t+1}q_{x+t} - v^t\pi\right) \qquad (by\ (2))$$

$$= v\sum_{t=0}^{n-1} v^t q_{x+t} - \sum_{t=0}^{n-1} v^t\pi.$$

$$=> \ \pi = \frac{v\sum_{t=0}^{n-1} v^t q_{x+t}}{\sum_{t=0}^{n-1} v^t} = \frac{v\sum_{t=0}^{n-1} v^t(1-e^{-0.02})}{\sum_{t=0}^{n-1} v^t} = v(1-e^{-0.02}) \qquad \sum_{t=0}^{n-1} v^t = \ddot{a}_{\overline{n}|}$$

$$10^3\pi = 10^3 v(1-e^{-0.02}) \approx 18.68.$$

Theorem 7.55. *A special fully discrete whole life insurance on (x) pays a unit benefit plus the terminal benefit reserve. This insurance is funded by a level annual premium of π. Then, $\pi = \frac{\sum_{t=0}^{\infty} v^{t+1}q_{x+t}}{\ddot{a}_{\overline{\infty}|}}$.*

Theorem 7.56. *A special fully discrete n–year term life insurance on (x) pays a unit death benefit plus the terminal benefit reserve. This insurance is funded by a level annual premium π. Then, $\pi = \frac{\sum_{t=0}^{n-1} v^{t+1}q_{x+t}}{\ddot{a}_{\overline{n}|}}$.* *if $n = \infty$, it becomes Th.7.55 $\pi = \frac{\sum_{t=0}^{\infty} v^{t+1}q_{x+t}}{\ddot{a}_{\overline{\infty}|}}$.*

Theorem 7.57. *A special fully discrete n–year endowment insurance on (x) pays a unit benefit plus the terminal benefit reserve. This insurance is funded by a level annual premium π. Then, $\pi = \frac{v^n + \sum_{t=0}^{n-1} v^{t+1}q_{x+t}}{\ddot{a}_{\overline{n}|}}$.*

Theorem 7.58. *A special n–year deferred discrete life unit annuity–due on (x) makes an* **extra death benefit** *equal to the terminal benefit reserve if death happens during the deferral period. This insurance is funded by a level annual premium π paid at the beginning of the year during the deferral period. Then, $\pi = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}} = \frac{\ddot{a}_{x+n}}{\ddot{a}_{\overline{n}|}/v^n}$.*

Proof. Assumption: $b_t = {}_tV$ for $t = 1, \dots, n$. For $t = 0, \dots, n-1$,

$$\begin{aligned}
{}_tV + \pi =& vb_{t+1}q_{x+t} + vp_{x+t} \cdot {}_{t+1}V = v \cdot {}_{t+1}V q_{x+t} + vp_{x+t} \cdot {}_{t+1}V \\
=& v \cdot {}_{t+1}V. \\
v \cdot {}_{t+1}V - {}_tV =& \pi \\
v^{t+1} \cdot {}_{t+1}V - v^t \cdot {}_tV =& v^t\pi
\end{aligned}$$

$$v^n \cdot {}_nV = \sum_{t=0}^{n-1} \left(v^{t+1} \cdot {}_{t+1}V - v^t \cdot {}_tV\right) = \sum_{t=0}^{n-1} v^t\pi = \ddot{a}_{\overline{n}|}\pi.$$

Hence, $v^n \cdot {}_nV = \pi\ddot{a}_{\overline{n}|}$. By the prospective method in standard case

$${}_tV(standard) = \begin{cases} {}_{n-t}|\ddot{a}_{x+t} - P\ddot{a}_{x+t:\overline{n-1}|} & if\ 0 \le t < n \\ \ddot{a}_{x+t} & if\ \ t \ge n. \end{cases}$$

Thus the modified one is ${}_tV = \begin{cases} \sum_{j=1}^{n-t} b_{t+j} v^j f_{K_{x+t}}(j) + \left({}_{n-t}|\ddot{a}_{x+t} - P\ddot{a}_{x+t:\overline{n-1}|}\right) & if\ 0 \le t < n \\ \ddot{a}_{x+t} & if\ \ t \ge n. \end{cases}$

So, ${}_nV = \ddot{a}_{x+n}$ and $v^n \cdot {}_nV = \pi \ddot{a}_{\overline{n}|}$. Hence, $v^n \ddot{a}_{x+n} = \pi \ddot{a}_{\overline{n}|}$ and $\pi = \frac{\ddot{a}_{x+n}}{\ddot{a}_{\overline{n}|}/v^n} \overset{def}{=} \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}$. ■

Example 7.25. *For a special fully discrete 4–year endowment insurance on* (50)*, assume:*
(i) $i = 6\%$.
(ii) The maturity value is $\$10^4$ *(i.e., benefit is* $\$10^4$ *if (50) is alive at 54).*
(iii) The death benefit is $\$10^4$ *plus the benefit reserve at the end of the year of death.*
(iv) Mortality follows the life table for the USA population in 2004.
Calculate the level benefit premium for this insurance.

Solution: $10^4\pi = ?$ Note that for the level-payment, the t-th terminal loss is

$$\begin{aligned} {}_tL^* =& 10^4[v^{K_{x+t}\wedge(n-t)} - \sum_{k=0}^{K_{x+t}\wedge(n-t)-1} \pi v^k] \qquad (L = B[v^{K_x \wedge n} - \sum_{k=0}^{K_x\wedge n-1} \pi v^k]) \\ =& 10^4[\sum_{j=1}^{n-t} v^j I(K_{x+t} = j) + v^{n-t} I(K_{x+t} > n-t) - \sum_{k=0}^{n-t-1} \pi v^k I(K_{x+t} > k)] \\ =& 10^4[\sum_{j=1}^{n-t} v^j I(K_{x+t} = j) - \sum_{k=0}^{n-t} \pi_k v^k I(K_{x+t} > k)] \quad (\pi_k = \begin{cases} \pi & if\ k < n-t \\ -1 & if\ k = n-t \end{cases}) \end{aligned}$$

$$\begin{aligned} \text{Here } {}_tL =& 10^4[b_{K_{x+t}} v^{K_{x+t}\wedge(n-t)} - \sum_{k=0}^{n-t-1} \pi v^k I(K_{x+t} > k)] \\ =& 10^4[\sum_{j=1}^{n-t} \underbrace{(1 + {}_jV)}_{=b_{t+j}} v^j I(K_{x+t} = j) - \sum_{k=0}^{n-t} \pi_k v^k I(K_{x+t} > k)]. \end{aligned}$$

${}_tV + \pi_t = v b_{t+1} q_{x+t} + v p_{x+t} \cdot {}_{t+1}V$ (i.e., Th. 7.54), $q_x + p_x = 1$ and $b_{n+1} = {}_{n+1}V = 0$ yield

$$\begin{cases} {}_tV + \pi = v(1 + {}_{t+1}V) q_{x+t} + v p_{x+t} \cdot {}_{t+1}V = v q_{x+t} + v \cdot {}_{t+1}V & \text{if } t = 0, ..., n-1 \\ {}_nV - 1 = v b_{n+1} q_{x+n} + v p_{x+n} \cdot {}_{n+1}V = 0 & \text{if } t = n \end{cases}$$

Hence, for $t = 0, ..., n-1$, $\quad {}_tV - v \cdot {}_{t+1}V = v q_{x+t} - \pi,$

$$=> \quad v^t \cdot {}_tV - v^{t+1} \cdot {}_{t+1}V = v^{t+1} q_{x+t} - v^t \pi.$$

$$=> \quad -v^n = {}_0V - v^n \cdot {}_nV = \sum_{t=0}^{n-1} \left(v^t {}_tV - v^{t+1} \cdot {}_{t+1}V\right) = \sum_{t=0}^{n-1} \left(v^{t+1} q_{x+t} - v^t \pi\right)$$

$$-v^n \quad = \sum_{t=0}^{n-1} v^{t+1} q_{x+t} - \pi \sum_{t=0}^{n-1} v^t => \ \pi = \frac{v^n + \sum_{t=0}^{n-1} v^{t+1} q_{x+t}}{\ddot{a}_{\overline{n}|}}. \qquad (n = 4)$$

Ans: $10^4\pi = (10^4)\frac{v^4+\sum_{t=0}^{3} v^{t+1}q_{x+t}}{\ddot{a}_{\overline{4}|}} = 10^4\frac{0.8091146}{3.673011949} \approx 2202.86,$ *as*

$$v^4 + \sum_{t=0}^{3} v^{t+1}q_{x+t} = (1.06)^{-4} + (1.06)^{-1}(0.00439537) + (1.06)^{-2}(0.00475767) + (1.06)^{-3}(0.00511418) + (1.06)^{-4}(0.00548678) = 0.8091146,$$

$$\ddot{a}_{\overline{4}|} = \sum_{k=0}^{3} 1.06^{-k} = 3.673011949.$$

The cash loss C_j does not take in account the change in benefit reserves.

Definition 7.12. *The* **yearly accrued loss** *at time j, denoted by Λ_j, is the present value of the net cash loss during the year $[j, j+1]$ (i.e., C_j), plus the increase in the APV of liability during the year $[j, j+1]$ (i.e., $v \cdot {}_{j+1}VI(K_x > j+1) - {}_jVI(K_x > j)$).*

Theorem 7.59. $\Lambda_j \overset{def}{=} C_j + v \cdot {}_{j+1}VI(K_x > j+1) - {}_jVI(K_x > j)$.

Th. 7.59 is in Formulas sheet (up-dated). **Quiz this week: All 450**

Definition 7.13.

The sum of the present value of all accrued losses is equal to the difference of the loss random variable and the benefit reserve. Notice that the benefit reserve goes to zero as $t \to \infty$. Precisely, we have that:

Theorem 7.60. ${}_tL = {}_tVI(K_x > t) + \sum_{j=0}^{\infty} v^j \Lambda_{t+j}$.

Theorem 7.61. *If $j \ge t$, $E[({}_jL - {}_jV)I(K_x > j)] = E[({}_jL - {}_jV)I(K_x > j)|K_x > t] = 0$.*

Proof. (1) $E[({}_jL - {}_jV)I(K_x > j)] = {}_jp_x E[({}_jL - {}_jV)|I(K_x > j)] = 0$.
(2) $E[({}_jL - {}_jV)I(K_x > j)|K_x > t] = \frac{E[({}_jL - {}_jV)I(K_x > j)]}{{}_tp_x} = 0$. ∎

Theorem 7.62. *If $j \ge t$, $E[\Lambda_j|K_x > t] = 0$. (see formulas of Λ_j, ${}_tV_x$ and C_j)*

Theorem 7.63. $\mathrm{Var}(\Lambda_j|K_x > j) = v^2(b_{j+1} - {}_{j+1}V)^2 p_{x+j}q_{x+j}$. *(see above and Th.7.59)*

Theorem 7.64. *If $j \ge t$,*

$\mathrm{Var}(\Lambda_j|K_x > t) = {}_{j-t}p_{x+t}\mathrm{Var}(\Lambda_j|K_x > j) = {}_{j-t}p_{x+t}v^2(b_{j+1} - {}_{j+1}V_x)^2 p_{x+j}q_{x+j}$.

Theorem 7.65. *If $k > j \ge t$, $\mathrm{Cov}(\Lambda_j, \Lambda_k|K_x > t) = 0$. ($\Lambda_j$ and $C_j = g(I(A))$)*

Theorem 7.66. (**Hattendorf's theorem**) *We have that*

$$\mathrm{Var}({}_tL|K_x > t) = \sum_{j=0}^{\infty} \mathrm{Var}(v^j \Lambda_{t+j}|K_x > t) = \sum_{j=0}^{\infty} {}_jp_{x+t} v^{2j+2}(b_{t+j+1} - {}_{t+j+1}V)^2 p_{x+t+j} q_{x+t+j}.$$

Example 7.26. *For a fully discrete* 15*-year term life insurance of \$1,000 on* (x)*, let*
(i) $i = 7\%$,
(ii) $1000 \cdot {}_{14}V^1_{x:\overline{15}|} = 10.23$,
(iii) $q_{x+13} = 0.05$,
(iv) $q_{x+14} = 0.06$.
Calculate $\mathrm{Var}({}_{13}L|K_x > 13)$ *(or simply* $\mathrm{Var}({}_{13}L)$*), where* ${}_{13}L = 10^3{}_{13}L^1_{x:\overline{15}|}$.

Solution: Let ${}_tV = 10^3 E({}_tL^1_{x:\overline{15}|}) = 10^3{}_tV^1_{x:\overline{15}|}$.

$Var({}_tL) =$? Two ways.
Basic way: ${}_{13}L^1_{x:\overline{15}|}$ takes only 3 values. $Var({}_tL) = E(({}_tL)^2) - (E({}_tL))^2$.

$$ {}_tL^1_{x:\overline{n}|} = v^{K_{x+t}} I(K_{x+t} \le n-t) - P10^{-3} \sum_{k\ge 0}^{[K_{x+t}\wedge(n-t)]-1} v^k. \tag{1}$$

Formula: ${}_tV + \pi_t = vb_{t+1}q_{x+t} + v \cdot {}_{t+1}V p_{x+t}$. (Th7.53) and formula sheet.
Assumption (ii) & above => $10.23 = {}_{14}V = vb_{15}q_{x+14} + v \cdot \underbrace{{}_{15}V}_{?} p_{x+14} - \underbrace{P}_{?}$ (as $\pi_t = P$).

${}_{15}V = E(10^3 v^{K_{x+15}} I(K_{x+15} \le 15-15) - P \sum_{k\ge 0}^{[K_{x+15}\wedge(15-15)]-1} v^k) = E(0-0) = 0.$
That is, $10.23 = {}_{14}V = 10^3(1.07)^{-1}q_{x+14} + v \cdot 0p_{x+14} - P$.
=> $P = 10^3(1.07)^{-1}(0.06) - 10.23 = 45.84476636$. ${}_{13}L = 1000{}_tL^1_{x:\overline{n}|} = g(K_{x+13})$ by Eq. (1),

$${}_{13}L = \begin{cases} (1000)(1.07)^{-1} - 45.84476636 \sum_{j=0}^{1\wedge(15-13)-1} v^j = 888.7346729 & \text{if } K_{x+13} = 1 \\ (1000)(1.07)^{-2} - 45.84476636(1 + (1.07)^{-1}) = 784.7483859 & \text{if } K_{x+13} = 2 \\ -45.84476636(1 + (1.07)^{-1}) = -88.6903424 & \text{if } K_{x+13} > 2, \end{cases}$$

$\mathbb{P}\{K_{x+13} = 1\} = f_{K_{x+13}}(1) = q_{x+13} = 0.05,$
$\mathbb{P}\{K_{x+13} = 2\} = f_{K_{x+13}}(2) = p_{x+13}q_{x+14} = (0.95)(0.06) = 0.057,$
$\mathbb{P}\{K_{x+13} > 2\} = \sum_{j>2} f_{K_{x+13}}(j) = {}_2p_{x+13} = p_{x+13}p_{x+14} = (0.95)(0.94) = 0.893.$
${}_{13}V = E({}_{13}L) = \sum g(j) f_{K_{x+13}}(j)$
$=(888.7346729)(0.05) + (784.7483859)(0.057) + (-88.690342)(0.893) = 9.966915878,$
$E[({}_{13}L)^2|K_x > 13] = E[({}_{13}L)^2] = \sum (g(j))^2 f_{K_{x+13}}(j)$
$=(888.7346729)^2(0.05) + (784.7483859)^2(0.057) + (-88.690342)^2(0.893) = 81619.09492,$
$\mathrm{Var}({}_{13}L|K_x > 13) = 81619.09492 - (9.966915878)^2 = 81519.75551.$

2nd Way: Use **Hattendorf's theorem**

$$\mathrm{Var}({}_{13}L|K_x > 13) = \sum_{j=0}^{\infty} \mathrm{Var}(\Lambda_{13+j}|K_x > 13) = \sum_{j=0}^{1} \mathrm{Var}(\Lambda_{13+j}|K_x > 13) = \cdots, \qquad (2)$$

as by formulas: $\Lambda_{15} = C_{15} + v \cdot {}_{16}VI(K_x > 16) - {}_{15}VI(K_x > 15)$ $(= C_{15})$ and

$C_{15} = b_{16}vI(K_x = 16) - \pi_{15}I(K_x > 15) = 0.$
Now compute RHS of (2):

$$\begin{aligned}
\mathrm{Var}(\Lambda_{13}|K_x > 13) =& {}_{13-13}p_{x+13}v^2(b_{14} - {}_{14}V)^2 p_{x+13}q_{x+13} && (\textit{see formula sheet}) \\
=& (1.07)^{-2}(1000 - 10.23)^2(0.95)(0.05) = 40643.83004, \\
\mathrm{Var}(\Lambda_{14}|K_x > 13) =& {}_{14-13}p_{x+13}v^2(b_{15} - {}_{15}V)^2 p_{x+14}q_{x+14} && {}_{15}V = 0 \\
=& (0.95)(1.07)^{-2}(1000)^2(0.94)(0.06) = 46798.84706, \\
\mathrm{Var}({}_{13}L|K_x > 13) =& \mathrm{Var}(\Lambda_{13}|K_x > 13) + v^2\mathrm{Var}(\Lambda_{14}|K_x > 13) \\
=& (40643.83004) + (1.07)^{-2}(46798.84706) = 81519.7555.
\end{aligned}$$

Example 7.27. *For a fully discrete ten–year endowment insurance of \$1,000 on* (x)*, let (i)* $i = 0.06$*. (ii)* $p_{x+7} = 0.95$*. (iii)* $p_{x+8} = 0.9$*. (iv)* ${}_8V = 359.62455$ *(v)* ${}_9V = 653.6745$*. Calculate* $\mathrm{Var}({}_7L)$*.*

Solution: $Var({}_7L) = ?$ where
${}_tL = 10^3(v^{K_{x+t}\wedge(n-t)} - P\sum_{k\geq 0}^{[K_{x+t}\wedge(n-t)]-1} v^k)$ takes 3 values, due to $K_{x+7} = 1, 2$ or > 2.
2 steps: (a) find $P = \pi/10^3 = \pi_t/10^3$? (b) find ${}_7L$ and f_{K_x} at $K_{x+7} = 1, 2$ or > 2 for $E({}_7L)$.
(a) Formula: (1) ${}_tV + \pi_t = vb_{t+1}q_{x+t} + v \cdot {}_{t+1}Vp_{x+t}$; (Th7.53 and formula sheet).

$$(2)\ {}_tV_{x+t:\overline{n-t|}} = A_{x+t:\overline{n-t|}} - P\ddot{a}_{x+t:\overline{n-t|}}.$$

Both work. Use (2).

$653.6745 = {}_9V = 10^3[E(v^{K_{x+9}\wedge(10-9)}) - P \cdot \sum_{k=0}^{(10-9)-1} v^k {}_kp_{x+9}]$

$=> 653.7 = 10^3(1.06)^{-1} - 10^3P.$
So, $1000P = 1000(1.06)^{-1} - 653.6745 = 289.7217264.$
(b) ${}_7L(k) = 10^3(v^{K_{x+t}\wedge(n-t)} - P\sum_{i\geq 0}^{[K_{x+t}\wedge(n-t)]-1} v^i)$ $\qquad (t = 7, K_{x+t} = k)$

$$= \begin{cases} (1000)(1.06)^{-1} - 289.7217264 = 653.675 & \text{if } K_{x+7} = k = 1 \\ (1000)(1.06)^{-2} - 289.722(1 + (1.06)^{-1}) = 326.952 & \text{if } K_{x+7} = k = 2 \\ (1000)(1.06)^{-3} - 289.7217264(1 + (1.06)^{-1} + (1.06)^{-2}) = 18.724 & \text{if } K_{x+7} = k > 2 \end{cases}$$

$$f_{K_{x+7}}(1) = \mathbb{P}\{K_x = 8 | K_x > 7\} = q_{x+7} = 0.05,$$
$$f_{K_{x+7}}(2) = \mathbb{P}\{K_x = 9 | K_x > 7\} = {}_1|_1 q_{x+7} = p_{x+7}q_{x+8} = (0.95)(0.1) = 0.095,$$
$$\mathbb{P}(K_{x+7} > 2) = \mathbb{P}\{K_x > 9 | K_x > 7\} = {}_2p_{x+7} = p_{x+7}p_{x+8} = (0.95)(0.9) = 0.855.$$

$$\begin{aligned} {}_7V =& E({}_7L) = \sum_{k=1}^{2} {}_7L(k) f_{K_{x+7}}(k) + \sum_{k>2} {}_7L(k) P(K_{x+7} = k) \\ =& (653.675)(0.05) + (326.952)(0.095) + (18.724)(0.855) = 79.75310364, \\ E[({}_7L)^2] =& (653.675)^2(0.05) + (326.952)^2(0.095) + (18.724)^2(0.855) = 31819.55976, \\ \mathrm{Var}({}_7L) =& 31819.55976 - (79.75310364)^2 = 25459.00222. \end{aligned}$$

Condition (iv) is not needed here.

Theorem 7.67. *A special fully discrete whole life insurance on (x) pays a unit benefit plus the terminal benefit reserve. This insurance is funded by a level annual premium π. Then,*
$\mathrm{Var}({}_tL) = \sum_{j=0}^{\infty} {}_jp_{x+t} v^{2j+2} p_{x+t+j} q_{x+t+j}$.

Proof Hattendorf's Th. => $\mathrm{Var}({}_tL | K_x > t) = \sum_{j=0}^{\infty} {}_jp_{x+t} v^{2j+2} (b_{t+j+1} - {}_{t+j+1}V)^2 p_{x+t+j} q_{x+t+j}$ $= \sum_{j=0}^{\infty} {}_jp_{x+t} v^{2j+2} (1)^2 p_{x+t+j} q_{x+t+j}$, as $(b_t - {}_tV)^2 = (1 + {}_tV - {}_tV)^2 = 1$.

Theorem 7.68. *A special fully discrete n–year endowment insurance on (x) pays a death benefit equals to one plus the terminal benefit reserve. This insurance is funded by a level annual premium π. Then,* $\mathrm{Var}({}_tL) = \sum_{j=0}^{n-1} {}_jp_{x+t} v^{2j+2} p_{x+t+j} q_{x+t+j}$.

Proof. It follows from Hattendorf's theorem noting that for each $1 \le t \le n$, $b_t = 1 + {}_tV$ and each $t \ge n+1$, $b_t - {}_tV = 0$. ■

7.7 Benefit reserves for general fully cts insurances

Suppose that a life insurance policy pays to (x) b_{K_x} at the time of death, say and the benefit premiums are paid at the continuous rate π_t. Then the loss random variable

$$\overline{L}_x = b_{T_x} v^{T_x} - \int_0^{T_x} \pi_s v^s ds.$$

The t–th terminal loss random variable is ${}_t\overline{L} = b_{T_{x+t}+t} v^{T_{x+t}} - \int_0^{T_{x+t}} \pi_{s+t} \cdot v^s \, ds$

$$(= b_{T_x} v^{T_x - t} - \int_t^{T_x} \pi_s \cdot v^{s-t} \, ds = b_{T_x} v^{T_x - t} - \int_0^{T_x - t} \pi_{s+t} \cdot v^s \, ds, \text{ if } T_x > t).. \quad \textbf{Why } b_{T_x} = b_{T_{x+t}+t}\ ?$$

If $0 < t < 4$ and (x) dies at the 4−the year, then T_x= ? $T_{x+t} + t = (4 - t) + t =$?
$P(T_x > 0) = 1$? $P(T_{x+t} > 0) = 1$? $P(T_x - t > 0) = 1$?

$P(T_x > 0|T_x > t) = 1$? $\quad P(T_{x+t} > 0|T_x > t) = 1$? $\quad P(T_x - t > 0|T_x > t) = 1$?

$E[b_{T_x} v^{T_x}] = \int_0^\infty b_s v^s f_{T_x}(s)\,ds = \int_0^\infty b_s v^s \mu_x(s)_s p_x\,ds$ (for compute $(\overline{L}_x) = 0$)

$E\left[\int_0^{T_x} v^s \pi_s\,ds\right] = E\left[\int_0^\infty I(s < T_x) v^s \pi_s\,ds\right] = \int_0^\infty E[I(s < T_x)] v^s \pi_s\,ds = \int_0^\infty \pi_s \cdot v^s \cdot {}_s p_x\,ds.$

Under the equivalence principle $E(\overline{L}) = 0$, so

$$\int_0^\infty b_s v^s \mu_x(s)_s p_x\,ds = \int_0^\infty \pi_s \cdot v^s \cdot {}_s p_x\,ds.$$

The t–th terminal benefit reserve assuming that the policy is in effect is

$$\begin{aligned} {}_t\overline{V} =& E[{}_t\overline{L}] = E[b_{T_{x+t}+t} v^{T_{x+t}} - \int_0^{T_{x+t}} \pi_{s+t} \cdot v^s\,ds] \\ =& \int_0^\infty b_{s+t} v^s f_{T_{x+t}}(s)\,ds - \int_0^\infty \pi_{s+t} v^s \cdot {}_s p_{x+t}\,ds \end{aligned}$$

The retrospective method, $\underbrace{\int_0^t \pi_s v^s {}_s p_x\,ds}_{total\ payments} = \underbrace{\int_0^t b_s v^s f_{T_x}(s)\,ds}_{insurance} + \underbrace{{}_t E_x \cdot {}_t\overline{V}}_{refund} =>$

$${}_t\overline{V} = \frac{\int_0^t \pi_s v^s {}_s p_x\,ds - \int_0^t b_s v^s f_{T_x}(s)\,ds}{{}_t E_x}.$$

Example 7.28. *For a special fully continuous whole life insurance on (x):*
(i) The death benefit at time t is $b_t = 1000e^{0.03t}$, $t \ge 0$.
(ii) The benefit premium rate at time t is $\pi_t = \pi e^{0.02t}$, $t \ge 0$.
(iii) $\mu_x(t) = 0.01$, $t \ge 0$
(iv) $\delta = 0.06$.
Calculate ${}_{10}\overline{V}$, the benefit reserve at the end of year 10.

Solution: Formula: ${}_t\overline{V} = E[{}_t\overline{L}] = E[b_{T_x} v^{T_{x+t}} - \int_0^{T_{x+t}} \pi_{s+t} \cdot v^s\,ds]$
$= \int_0^\infty b_{t+s} v^s f_{T_{x+t}}(s)\,ds - \int_0^\infty v^s \pi_{t+s} \cdot {}_s p_{x+t}\,ds$=? where $\pi_t = \pi e^{0.02t}$.
$\pi =$? π is from the equivalent principle $E(b_{T_x} v^{T_x}) = E(\int_0^{T_x} \pi_s v^s ds)$ and $f_{T_x}(s) = \mu_x(s)_s p_x$.

$$\begin{aligned} E(b_{T_x} v^{T_x}) =& \int_0^\infty b_s v^s \mu_x(s)_s p_x\,ds = \int_0^\infty 1000e^{0.03t} e^{-0.06t}(0.01)e^{-0.01t}\,dt \\ =& \frac{(1000)(0.01)}{0.04} \int_0^\infty 0.04e^{-0.04t}\,dt = \frac{1000(0.01)}{0.04} = 250 \end{aligned}$$

$$= E(\int_0^{T_x} \pi_s v^s ds) = \int_0^\infty \pi_s \cdot v^s \cdot {}_s p_x\,ds = \int_0^\infty \pi e^{0.02t} e^{-0.06t} e^{-0.01t}\,dt = 20\pi \quad => \quad \pi = 12.5.$$

At the end of year 10, ${}_t\overline{V} = \int_0^\infty b_{t+s} v^s f_{T_{x+t}}(s)\,ds - \int_0^\infty v^s \pi_{t+s} \cdot {}_s p_{x+t}\,ds$ $(t = 10)$.

$$\int_0^\infty b_{t+s} v^s f_{T_{x+t}}(s)\,ds = \int_0^\infty 10^3 e^{0.03(10+s)} e^{-0.06s}(0.01)e^{-0.01s}\,ds = e^{0.3} \int_0^\infty 10e^{-0.04s}\,ds \approx 337.46.$$

$$\int_0^\infty v^s \pi_{t+s} \cdot {}_s p_{x+t}\,ds = \int_0^\infty (12.5)e^{0.02(10+s)} e^{-0.06s} e^{-0.01s}\,ds = 12.5e^{0.2} \int_0^\infty e^{-0.05s}\,ds \approx 305.35$$

Hence, ${}_{10}\overline{V} \approx 337.46 - 305.35 = 32.11$.

Theorem 7.69. $\frac{d}{dt}{}_t\overline{V} = \pi_t - b_t\mu_x(t) + (\delta + \mu_x(t)) \cdot {}_t\overline{V}$.

Proof. Formulas: ${}_t\overline{V} = \frac{\int_0^t \pi_s v^s {}_sp_x\,ds - \int_0^t b_s v^s {}_sp_x \mu_x(s)\,ds}{{}_tE_x} = \frac{\int_0^t (\pi_s - b_s\mu_x(s)) v^s {}_sp_x\,ds}{v^t \cdot {}_tp_x}$,
$\frac{d\int_a^s g(t)dt}{ds} = g(s)$ and quotient rule.

$$\frac{d}{dt}{}_t\overline{V} = \frac{(\pi_t - b_t\mu_x(t))v^t{}_tp_x \cdot v^t{}_tp_x - \int_0^t (\pi_s - b_s\mu_x(s))v^s{}_sp_x\,ds \cdot \frac{d}{dt}(v^t \cdot {}_tp_x)}{(v^t \cdot {}_tp_x)^2}.$$

$$\frac{\frac{d}{dt}(v^t \cdot {}_tp_x)}{v^t \cdot {}_tp_x} = \frac{d}{dt}\log(v^t \cdot {}_tp_x) = \frac{d}{dt}\log(v^t) + \frac{d}{dt}\log({}_tp_x) = -\delta - \mu_x(t).$$

$$\begin{aligned}\frac{d}{dt}{}_t\overline{V} =& \frac{(\pi_t - b_t\mu_x(t))v^t{}_tp_x \cdot v^t{}_tp_x - \int_0^t (\pi_s - b_s\mu_x(s))v^s{}_sp_x\,ds \cdot (-\delta - \mu_x(t))v^t \cdot {}_tp_x}{(v^t \cdot {}_tp_x)^2} \\ =& \pi_t - b_t\mu_x(t) + (\delta + \mu_x(t)) \cdot {}_t\overline{V}. \quad \square\end{aligned}$$

Previous theorem states the rate of the t–th terminal benefit reserve depends on three terms:
π_t, the benefit premium rate,
$b_t\mu_x(t)$, the rate of paid benefit,
$(\delta + \mu_x(t)) \cdot {}_t\overline{V}$, the rate of increase of the t–th terminal benefit reserve due to interest and mortality.

7.8 Benefit reserves for m–thly payed premiums.

Here, we consider the benefit reserves for insurance contracts with benefit premiums paid m times a year. The benefit premiums of these contracts was discussed in Section 6.7. In this section, we suppose that the benefit premiums are level payments made at the beginning of period of length $\frac{1}{m}$ years while the individual is alive. In this section, we give the formulas for the t–th terminal benefit reserve for an insurance contract, where t is a nonnegative integer.

The t–th terminal benefit reserve for a whole life insurance to (x) paid at the end of the year of death and funded m times a year is

$${}_tV_x^{(m)} = A_{x+t} - P_x^{(m)}\ddot{a}_{x+t}^{(m)} \qquad where \ \ \ddot{a}_x^{(m)} = \frac{1}{m}\sum_{j=0}^{\infty} v^{j/m}{}_{j/m}p_{x+t}.$$

The t–th terminal benefit reserve for a n–year term insurance to (x) paid at the end of the year of death and funded m times a year is

$${}_tV^{1\,(m)}_{x:\overline{n}|} = A^1_{x+t:\overline{n-t}|} - P^{1\,(m)}_{x:\overline{n}|}\ddot{a}^{(m)}_{x+t:\overline{n-t}|} \qquad where \ \ \ddot{a}^{(m)}_{x+t:\overline{n-t}|} = \frac{1}{m}\sum_{j=0}^{(n-t)m-1} v^{j/m}{}_{j/m}p_{x+t}.$$

The t–th terminal benefit reserve for an n–year pure endowment to (x) and funded m times a year is

$${}_tV_{x:\overline{n}|}^{\ \ 1\,(m)} = A_{x+t:\overline{n-t}|}^{\ \ \ \ 1} - P_{x:\overline{n}|}^{\ \ 1\,(m)} \cdot \ddot{a}^{(m)}_{x+t:\overline{n-t}|}.$$

The t–th terminal benefit reserve for an n–year endowment to (x) paid at the end of the year of death funded m times a year is

$$ {}_tV^{(m)}_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P^{(m)}_{x:\overline{n}|}\ddot{a}^{(m)}_{x+t:\overline{n-t}|}. $$

The t–th terminal benefit reserve for an n–year deferred insurance to (x) paid at the end of the year of death and funded n years is

$$ {}_tV^{(m)}_x = {}_{n-t}|A_{x+t:\overline{n-t}|} - P^{(m)}({}_n|A_x)\ddot{a}^{(m)}_{x+t:\overline{n-t}|}. $$

The t–th terminal benefit reserve for an n–year deferred annuity–due to (x) and funded n years m times a year is

$$ {}_tV^{(m)}({}_n|\ddot{a}_x) = {}_{n-t}|\ddot{a}_{x+t} - P^{(m)}({}_n|\ddot{a}_x)\ddot{a}^{(m)}_{x+t:\overline{n-t}|}. $$

Next, we present the formulas when the benefit is immediately paid.

The t–th terminal benefit reserve for a whole life insurance to (x) immediately paid and funded m times a year is

$$ {}_tV^{(m)}(\overline{A}_x) = \overline{A}_{x+t} - P(\overline{A}^{(m)}_x)\ddot{a}^{(m)}_{x+t}. $$

The t–th terminal benefit reserve for an n–year term insurance to (x) immediately paid and funded m times a year is

$$ {}_tV^{(m)}(\overline{A}^{1}_{x:\overline{n}|}) = \overline{A}^{1}_{x+t:\overline{n-t}|} - P^{(m)}(\overline{A}^{1}_{x:\overline{n}|})\ddot{a}^{(m)}_{x+t:\overline{n-t}|}. $$

The t–th terminal benefit reserve for an n–year endowment to (x) immediately paid and funded m times a year is

$$ {}_tV^{(m)}(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+t:\overline{n-t}|} - P^{(m)}(\overline{A}_{x:\overline{n}|})\ddot{a}^{(m)}_{x+t:\overline{n-t}|}. $$

The t–th terminal benefit reserve for an n–year deferred insurance immediately paid to (x) and funded n years m times a year is

$$ {}_tV^{(m)}_x({}_n|\overline{A}_x) = {}_{n-t}|\overline{A}_{x+t} - P^{(m)}({}_n|\overline{A}_x)\ddot{a}^{(m)}_{x+t:\overline{n-t}|}. $$

7.9 Benefit reserves including expenses.

The benefit reserve including expenses is the excess of
the APV of future benefits and expenses over the APV of the future gross premiums;
called **expense–augmented reserves** and **gross premium reserves**.
Consider a fully discrete whole life insurance with expenses.

b – the death benefit.

e – the fixed annual cost.

e^*_0 – the first year additional cost.

r – percentage of the expense–augmented premium paid in expanses each year.

r^*_0 – the 1st year additional percentage of the expense–augmented premium.

s – the settlement cost.

G – the expense–augmented premium using the equivalence principle.

Then

$$\begin{aligned} 0 &= bA_x + e\ddot{a}_x + e_0^* + rG\ddot{a}_x + r_0^*G + sA_x - G\ddot{a}_x \\ &= e_0^* + r_0^*G + (b+s)A_x + (rG+e)\ddot{a}_x - G\ddot{a}_x. \qquad (1) \\ &=> \quad G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}. \end{aligned}$$

The k–th **terminal expense–augmented reserve** (for $k \geq 0$) is

$$\begin{aligned} {}_kV_e =& (e_0^* + r_0^*G)\mathbf{1}(k=0) + (b+s)A_{x+k} + (rG+e)\ddot{a}_{x+k} - G\ddot{a}_{x+k} \qquad (see\ (1)) \\ =& (b+s)A_{x+k} - ((1-r)G - e)\ddot{a}_{x+k} + (e_0^* + r_0^*G)\mathbf{1}(k=0). \qquad (1.11) \end{aligned}$$

The k–th **terminal benefit reserve** is

$${}_kV = b(A_{x+k} - P_x\ddot{a}_{x+k}).$$

The excess of the k–th terminal expense–augmented reserve over the k–th terminal benefit reserve is called the k–th **terminal expense reserve**, denoted by ${}_kV^e$. Hence,

$${}_kV^e = {}_kV_e - {}_kV = sA_{x+k} - (bP_x + (1-r)G - e)\ddot{a}_{x+k} + (e_0^* + r_0^*G)\mathbf{1}(k=0).$$

Example 7.29. *Using the information in Example 6.48, for:*
(i) Calculate the 10–th terminal expense–augmented reserve.
(ii) Calculate the 10–th terminal expense reserve.

Example 6.48. A whole life insurance policy with face value of \$40,000 payable at the end of the year of death is made to (45). Assume that $i = 4.5\%$ and death is modeled using the de Moivre model with terminal age 95.
The following costs are incurred and paid at the beginning of the year:
(i) \$500 for making the contract.
(ii) Percent of expense–loaded premium expenses is 5% in the first year and 1% thereafter.
(iii) Per policy expenses are \$20 per year.
(iv) Per thousand expenses are \$1.2 per year.
(v) \$600 for settlement.
Calculate the gross annual premium G using the equivalence principle.

Results: The equivalent principle $0 = {}_0V_e$ *i.e.*,

$$0 = {}_kV_e = (b+s)A_{x+k} - ((1-r)G - e)\ddot{a}_{x+k} + (e_0^* + r_0^*G)\mathbf{1}(k=0) \text{ with } k=0 \text{ yields}$$

$$0 = \underbrace{(40000+600)}_{b+s}A_{45} - \underbrace{((1-0.01)G - 20 - 1.2 \times 40)}_{(1-r)G-e}\ddot{a}_{45} + 500 + 0.04G$$

$$A_x = \textstyle\sum_{t=1}^{95-x} v^t \frac{1}{95-x} = v\frac{1-v^{95-x}}{1-v}\frac{1}{95-x},\ \ddot{a}_x = \frac{1-A_x}{1-v},\ G = 1262.439006.$$

Solution: (i) ${}_kV_e = (b+s)A_{x+k} - ((1-r)G - e)\ddot{a}_{x+k} = ?\ (k=10,\ x=45)$
The 10–th terminal expense–augmented reserve is

$${}_{10}V_e = 40600 * 0.4600396105 - ((1-0.01)1262.439006 - 68)12.53908016 \approx 3858.7.$$

(ii) ${}_{10}V^e = {}_{10}V_e - {}_{10}V = ?$

$${}_{10}V = (40000)(A_{55} - P_{45}\ddot{a}_{55}) = ? \text{ and } P_{45} = ?$$

$$\begin{aligned} A_{45} &= v\frac{1-v^{50}}{1-v}/50 = 0.3952401556, \\ P_{45} &= \frac{A_{45}}{\ddot{a}_{45}} = \frac{A_{45}}{\frac{1-A_{45}}{1-v}} = 0.02814325581, \\ {}_{10}V &= (40000)(0.4600396105 - (0.02814325581)(12.53908016)) = 4285.962797. \end{aligned}$$

The 10–th terminal expense reserve is ${}_{10}V^e = 3858.739983 - 4285.962797 = -427.222814$.

Example 7.30. *Find the expense–augmented reserve for the 5–th year, using the information in Example 6.50.*

Example 6.50. A 10–payment, fully discrete, 20–year term insurance policy with face value of \$90000 payable at the end of the year of death is made to (45). The costs are :
(i) 275 at the beginning of each year which the policy is active.
(ii) Per thousand expenses are \$2.5 at the beginning of each year which the policy is active.
(iii) 1% for each annual premium received.
Assume that $i = 6\%$ and death follows the life table for the USA population in 2004.
Find the gross annual premium using the equivalence principle.
Copy from results: $b = 90000$, $s = 0$, $e = 275 + 90 * 2.5$, $r = 0.01$.

$$L = \underbrace{\overbrace{(90000)}^{b+s} Z^1_{45:\overline{20}|} + \overbrace{(275 + 90 * 2.5)}^{e} \ddot{Y}_{45:\overline{20}|} + \overbrace{0.01}^{r} G\ddot{Y}_{45:\overline{10}|}}_{expenses} - \underbrace{G\ddot{Y}_{45:\overline{10}|}}_{premium} \qquad (1)$$

Using the equivalence principle, $G = 1532.416116$.

Solution: Recall (1.11): ${}_kV_e = (b+s)A_{x+k} - ((1-r)G - e)\ddot{a}_{x+k} + (e_0^* + r_0^*G)\mathbf{1}(k=0)$.
${}_5V_e = (b+s)A_{x+5} - ((1-r)G - e)\ddot{a}_{x+5} + (e_0^* + r_0^*G)\mathbf{1}(5=0)$, **is it right ?**
Eq. (1) and (1.11) lead to **(no** $(e_0^* + r_0^*G$ **!)**
${}_kV_e = (b+s)A^1_{x+k:\overline{n-k}|} + e\ddot{a}_{x+k:\overline{n-k}|} + rG\ddot{a}_{x+k:\overline{h-k}|} - G\ddot{a}_{x+k:\overline{h-k}|}$, where $(n,k,x,h) = (20,5,45,10)$.

$$\begin{aligned} {}_5V_e &= (90000)A^1_{50:\overline{15}|} + (275 + 90(2.5))\ddot{a}_{50:\overline{15}|} - G(1-0.01)\ddot{a}_{50:\overline{5}|} \\ &= (90000)A^1_{50:\overline{15}|} + 500\ddot{a}_{50:\overline{15}|} - G(0.99)\ddot{a}_{50:\overline{5}|}. \end{aligned}$$

To solve $A^{1}_{50:\overline{15}|}$, $\ddot{a}_{50:\overline{15}|}$ and $\ddot{a}_{50:\overline{5}|}$, make use of the Life Table and Formulas in 450 [14],

$$
\begin{aligned}
&A^{1}_{50:\overline{15}|} = A_{50} - {}_{15}E_{50}A_{65}, \quad \ddot{a}_{50:\overline{15}|} = \frac{1 - A_{50:\overline{15}|}}{1-v}, \quad \ddot{a}_{50:\overline{5}|} = \frac{1 - A_{50:\overline{5}|}}{1-v} \\
&{}_{15}E_{50} = v^{15}{}_{15}p_x = v^{15}{}_{10}p_{x5}p_{x+10} = {}_{10}E_{50} \cdot {}_{5}E_{60} = (0.524456813)(0.705463729) = 0.369985259, \\
&A^{1}_{50:\overline{15}|} = A_{50} - {}_{15}E_{50}A_{65} = 0.20695786 - (0.369985259)(0.37609614) = 0.06780783223, \\
&A_{50:\overline{15}|} = A^{1}_{50:\overline{15}|} + {}_{15}E_{50} = 0.06780783223 + 0.369985259 = 0.4377930912, \\
&A_{50:\overline{5}|} = A^{1}_{x:\overline{5}|} + {}_{5}E_x = A_x - {}_{5}E_xA_{x+5} + {}_{5}E_x = A_{50} + {}_{5}E_{50}(1 - A_{55}) \\
&\qquad\qquad = 0.20695786 + (0.728300687)(1 - 0.25504797) = 0.7495069352, \\
&\ddot{a}_{50:\overline{5}|} = \frac{1 - A_{50:\overline{5}|}}{1-v} = \frac{1 - 0.7495069352}{(0.06/1.06)} = 4.425377478, \\
&\ddot{a}_{50:\overline{15}|} = \frac{1 - A_{50:\overline{15}|}}{1-v} = \frac{1 - 0.4377930912}{(0.06/1.06)} = 9.932322055.
\end{aligned}
$$

Hence, the 5–th year expense–augmented reserve is

$$
{}_{5}V_e \approx (90000)(0.0678) + (500)(9.9323) - (1532.4161)(0.99)(4.4254) \approx 4355.161359
$$

In general, we may assume that payments depend on the year. Then notations become
b_k – the benefit payable at the end of the k–th year;
G_{k-1} – expense–augmented benefit paid at the beginning of the k–th year;
r_k – the proportion of the k–th expense–augmented premium that is used to pay expenses;
e_k – the fixed contract expense for the k–th year.

The iterative formulas yield, for $k = 0, 1, 2, \ldots$

$$
\begin{aligned}
{}_{k}V_e + G_k(1 - r_k) - e_k &= b_{k+1}vq_{x+k} + v \cdot {}_{k+1}V_e p_{x+k}, && (1) \\
{}_{k}V + G_k &= b_{k+1}vq_{x+k} + v \cdot {}_{k+1}V p_{x+k}, \qquad \text{(the k–th terminal benefit reserve)} && (2) \\
(1) - (2) \quad => \quad {}_{k}V^e - G_kr_k - e_k &= p_{x+k}v \cdot {}_{k+1}V^e, \quad k = 0, 1, 2, \ldots, && (3)
\end{aligned}
$$

the iterative formula for k–th **terminal expense–augmented reserve**.

Example 7.31. *A whole life insurance policy with face value of \$50,000 payable at the end of the year of death is made to* (30). *The cost are :*
(i) Percents of expense–loaded premium expenses are 5% in Year 1 and 1% thereafter.
(ii) Per policy expenses are \$700 in the first year and \$1000 thereafter.
(iii) $i = 6\%$.
(iv) Death follows the life table for the USA population in 2004.
Calculate the expense–augmented reserve for the first two years.

Solution 1: Note $G_k = G$ and $b_k = 50000$. It is not the general case emphasized above. But Eq. (1), (2), (3) hold. Equivalent equation yields

$$
(50000)A_{30} + (0.05 - 0.01)G + (0.01)G\ddot{a}_{30} + 700 - 1000 + 1000\ddot{a}_{30} = G\ddot{a}_{30}.
$$

$$=> \quad G = \frac{(50000)A_{30} - 300 + 1000\ddot{a}_{30}}{(1-0.01)\ddot{a}_{30} - 0.04} \qquad \ddot{a}_x = \frac{1-A_x}{1-v}$$
$$= \frac{(50000)(0.08229543) - 300 + 1000(16.212781)}{(1-0.01)(16.212781) - 0.04} = 1250.889159.$$

${}_0V_e = 0$ and Eq. (1) (iterative formula) with $k = 0$ =>

$$\begin{aligned} LHS =& 0 + 1250.889159(1-0.05) - 700 = 488.344701 \\ = RHS =& (50000)(1.06)^{-1}(0.00099206) + (1.06)^{-1}{}_1V_e(1-0.00099206) \\ =& 46.79528302 + (1.06)^{-1}(1-0.00099206){}_1V_e, \\ => \quad {}_1V_e =& \frac{488.344701 - 46.79528302}{(1.06)^{-1}(1-0.00099206)} = 468.5071703, \end{aligned}$$

Eq. (1) with k=1 yields

$$\begin{aligned} LHS =& 468.5071703 + 1250.889159(1-0.01) - 1000 = 706.8874377 \\ = RHS =& (50000)(1.06)^{-1}(0.00102376) + (1.06)^{-1} \cdot {}_2V_e(1-0.00102376) \\ =& 48.29056604 + (1.06)^{-1}(1-0.00102376) \cdot {}_2V_e, \\ => \quad {}_2V_e =& \frac{706.8874377 - 48.29056604}{(1.06)^{-1}(1-0.00102376)} = 698.8059414. \end{aligned}$$

Solution 2: Same as the beginning of the Solution 1, find $G \approx 1251$ first. then use ${}_tV_e = (b+s)A_{x+t} - ((1-r)G - e)\ddot{a}_{x+t}$, $s = 0$ (settlement cost).

$$\begin{aligned} {}_1V_e =& (50000)A_{31} - ((1-0.01)G - 1000)\ddot{a}_{31} \\ =& (50000)(0.08632673) - ((1-0.01)G - 1000)(16.141561) = 468.5068724, \\ {}_2V_e =& (50000)A_{32} - ((1-0.01)G - 1000)\ddot{a}_{32} \\ =& (50000)(0.0905753) - ((1-0.01)G - 1000)(16.066503) = 698.8277185. \end{aligned}$$

The **asset share** at certain time is the accumulated financial value of an insurance policy, *i.e.* the actuarial accumulated value of (the premiums$-(benefits+expenses)$.

In Example 7.31 the APV of the premiums minus benefits and expenses is

$$c_0+r_0G+G\ddot{a}_{x:\overline{k}|}-[bA^1_{x:\overline{k}|}+sA^1_{x:\overline{k}|}+(rG+e)\ddot{a}_{x:\overline{k}|}] \quad (\stackrel{def}{=}d,)$$

the **asset share at the end of year** k is

$$_kAS=\frac{c_0+r_0G+G\ddot{a}_{x:\overline{k}|}-(b+s)A^1_{x:\overline{k}|}-(rG+e)\ddot{a}_{x:\overline{k}|}}{_kE_x} \quad (>d \text{ if } \; d>0), \tag{1}$$

where c_0 and r_0 refer to that insurance companies often put a surplus to fund a benefit reserve. In this case $_0AS$ is not zero. The following recurrence relation is useful,

$$\begin{aligned}&_0AS+G-rG-e-r_0G-c_0=(b+s)vq_x+p_xv\cdot {}_1AS,\\ &_kAS+G-rG-e=(b+s)vq_{x+k}+p_{x+k}v\cdot {}_{k+1}AS, \quad k=1,2,\ldots.\end{aligned}$$

That is, at the beginning of the $k+1$–th year, we have an asset share of $_kAS$ and $G-Gr-e$ (the premiums minus expenses made at the beginning of year k). The future value of this cashflow at the end of year $k+1$ is $(_kAS+G-Gr-c_1)(1+i)$. At the end of year k, deceased received a death benefit of $b+s$ and survivors have an average liability of $_{k+1}AS$.

$$\underbrace{c_0+r_0G+G\ddot{a}_{x:\overline{k}|}-(rG+c_1)\ddot{a}_{x:\overline{k}|}}_{total\ balance}=\underbrace{(b+s)A^1_{x:\overline{k}|}}_{insurance}+\underbrace{_kE_x{}_kAS}_{asset\ share} \quad \text{(compare to Eq.s (1) and (3)).} \tag{2}$$

$$\underbrace{\sum_{i=0}^{k-1}\pi_iv^i{}_ip_x}_{total\ payments}=\underbrace{\sum_{i=1}^{k}b_iv^if_{K_x}(i)}_{insurance}+\underbrace{_kE_x\cdot {}_kV}_{refund}. \tag{3}$$

7.10 Benefit reserves at fractional durations

Recall $_tV_x=A_{x+t}-P_x\ddot{a}_{x+t}, \quad t\in\{0,1,2,...\}$.

Hereafter, let t be a positive integer and let $0<s<1$. If q_x is given in integers assuming UDD, how to compute $_{t+s}V$ by modifying the iterative formula for the fully discrete case ?

$$\begin{aligned}_tV+\pi_t=&v^1[b_{t+1}\cdot q_{x+t}+{}_{t+1}V\cdot p_{x+t}], \quad t\in\{0,1,2,...\}; \qquad (formula\ [1.7])\\ =&v^1\cdot[b_{t+1}\mathbb{P}\{T_{x+t}\le 1\}+{}_{t+1}V\mathbb{P}\{T_{x+t}>1\}]\\ _{t+s}V=&v^{1-s}\cdot[b_{t+1}\mathbb{P}\{T_{x+t+s}\le 1-s\}+{}_{t+1}V\mathbb{P}\{T_{x+t+s}>1-s\}]\\ =&v^{1-s}[b_{t+1}\cdot {}_{1-s}q_{x+t+s}+{}_{t+1}V\cdot {}_{1-s}p_{x+t+s}].\end{aligned}$$

Theorem 7.70. $_{t+s}V=v^{1-s}\cdot b_{t+1}\cdot {}_{1-s}q_{x+t+s}+v^{1-s}\cdot {}_{1-s}p_{x+t+s}\cdot {}_{t+1}V$.

Since $x+t+s$ in $_{1-s}p_{x+t+s}$ is not an integer, revise Th. 7.70 as follows.

Theorem 7.71. $v^s{}_sp_{x+t}\cdot {}_{t+s}V=vb_{t+1}\cdot {}_{s|1-s}q_{x+t}+vp_{x+t}\cdot {}_{t+1}V$.

Proof. Multipying $v^s{}_sp_{x+t}$ on both sides of the equation in Th. 7.70 yields

$$\begin{aligned} v^s{}_sp_{x+t}\cdot{}_{t+s}V =& v^s{}_sp_{x+t}(v^{1-s}b_{t+1}\cdot{}_{1-s}q_{x+t+s}+v^{1-s}{}_{1-s}p_{x+t+s}\cdot{}_{t+1}V) \\ =& vb_{t+1}\cdot{}_sp_{x+t}\cdot{}_{1-s}q_{x+t+s}+v\cdot{}_sp_{x+t}\cdot{}_{1-s}p_{x+t+s}\cdot{}_{t+1}V \\ =& vb_{t+1}\cdot{}_{s|1-s}q_{x+t}+vp_{x+t}\cdot{}_{t+1}V \qquad \text{(by Formula [4] } {}_{m+n}p_x={}_mp_x\cdot{}_np_{x+m} \end{aligned} \tag{7.1}$$

and [3] ${}_{s|t}q_x=P(s<T_x\le s+t)={}_sp_x\cdot{}_tq_{x+s}$).

Assuming uniform distribution of deaths in $[x+t,x+t+1)$,
the density of T_{x+t} is $f_{T_{x+t}}(u)=q_{x+t}$, for $0\le u<1$. Hence,

$$ {}_{s|1-s}q_{x+t}=\mathbb{P}\{s<T_{x+t}<1\}=\int_s^1 q_{x+t}\,du=(1-s)q_{x+t}, \tag{7.2}$$

$$ {}_sp_{x+t}=1-{}_sq_{x+t}=1-\int_0^s q_{x+t}\,du=1-sq_{x+t} \quad \textit{and (7.1) =>}$$

Theorem 7.72. *Under uniform distribution of death between years of death,*

$$v^s(1-sq_{x+t})\cdot{}_{t+s}V=vb_{t+1}(1-s)q_{x+t}+vp_{x+t}\cdot{}_{t+1}V.$$

Example 7.32. *For an insurance contract on (x), you are given that:*
(i) ${}_tV=450$.
(ii) $b_{t+1}=1000$.
(iii) $\pi_t=250$.
(iv) $i=0.06$.
(v) $p_{x+t}=0.95$.
(vi) The distribution of death in the year $[t,t+1]$ is uniform.
Compute ${}_{t+0.2k}V$, for $k=1,2,3,4,5$.

Solution: Theorem 7.72 yields

$$ {}_{t+s}V=\frac{v^{1-s}b_{t+1}(1-s)q_{x+t}+v^{1-s}p_{x+t}\cdot{}_{t+1}V}{1-sq_{x+t}}, \quad s=? \quad {}_{t+1}V=? \tag{1}$$

Formula [1.7] ${}_tV+\pi_t=b_{t+1}vq_{x+t}+{}_{t+1}Vvp_{x+t}$ leads to
$450+250=(1000)(1.06)^{-1}(1-0.95)+{}_{t+1}V\cdot(1.06)^{-1}(0.95)$.
=> ${}_{t+0.2(5)}V={}_{t+1}V=\frac{(450+250)(1.06)-(1000)(0.05)}{0.95}=728.4210526$.
By Eq. (1), ${}_{t+0.2(1)}V=705.71$, ${}_{t+0.2(2)}V=711.42$, ${}_{t+0.2(3)}V=711.42$, ${}_{t+0.2(4)}V=722.78$.

Theorem 7.73. *Under the assumption in Th7.72,*
$v^s{}_sp_{x+t} \cdot {}_{t+s}V = ({}_tV + \pi_t)\frac{{}_{s|1-s}q_{x+t}}{q_{x+t}} + vp_{x+t} \cdot {}_{t+1}V \cdot \frac{{}_sq_{x+t}}{q_{x+t}}$.

Proof. Formula: ${}_tV + \pi_t = vb_{t+1}q_{x+t} + v \cdot {}_{t+1}Vp_{x+t}$.

$$=> \ vb_{t+1} = \frac{{}_tV + \pi_t - vp_{x+t} \cdot {}_{t+1}V}{q_{x+t}}.$$

$$v^s{}_sp_{x+t} \cdot {}_{t+s}V = vb_{t+1} \cdot {}_{s|1-s}q_{x+t} + vp_{x+t} \cdot {}_{t+1}V \qquad \text{by Th. 7.71}$$

$$\begin{aligned}
&= \frac{{}_tV + \pi_t - vp_{x+t} \cdot {}_{t+1}V}{q_{x+t}} \cdot {}_{s|1-s}q_{x+t} + vp_{x+t} \cdot {}_{t+1}V \\
&= ({}_tV + \pi_t)\underbrace{\frac{{}_{s|1-s}q_{x+t}}{q_{x+t}}}_{w} + vp_{x+t} \cdot {}_{t+1}V \underbrace{\frac{q_{x+t} - {}_{s|1-s}q_{x+t}}{q_{x+t}}}_{1-w} \quad \text{weighted average} \qquad (7.3) \\
&= ({}_tV + \pi_t)\frac{{}_{s|1-s}q_{x+t}}{q_{x+t}} + vp_{x+t} \cdot {}_{t+1}V \cdot \frac{{}_sq_{x+t}}{q_{x+t}}
\end{aligned}$$

as $q_{x+t} - {}_{s|1-s}q_{x+t} = \mathbb{P}\{T_{x+t} < 1\} - \mathbb{P}\{s < T_{x+t} < 1\} = \mathbb{P}\{T_{x+t} < s\} = {}_sq_{x+t}$. □

Recall Eq. (7.3) in the previous page,

$${}_{s|1-s}q_{x+t} = (1-s)q_{x+t}, \text{ assuming uniform distribution of deaths in } [x+t, x+t+1].$$

Hence,

$$w = \frac{{}_{s|1-s}q_{x+t}}{q_{x+t}} = 1 - s \text{ and } 1 - w = \frac{{}_sq_{x+t}}{q_{x+t}} = s.$$

It together with Eq. (7.3) yields

Theorem 7.74. *Assuming uniform distribution of deaths in* $[x+t, x+t+1]$,

$$v^s{}_sp_{x+t} \cdot {}_{t+s}V = ({}_tV + \pi_t)(1-s) + vp_{x+t} \cdot {}_{t+1}Vs.$$

Previous theorem states that the APV at time t of the $(t+s)$–th terminal reserve is linear interpolation of the $(t+1)$–th initial reserve and the APV at time t of the $(t+1)$–th terminal reserve. An approximation to the previous formula is

$${}_{t+s}V = ({}_tV + \pi_t)(1-s) + {}_{t+1}Vs, \ if \ s \approx 0+, \ e.g, \ v = 1/1.06 \approx 1 \text{ and } p_{x+t} = 0.95 \approx 1,$$

which approaches the $(t+s)$–th benefit reserve by linear interpolation of the $(t+1)$–th initial reserve and the $t+1$–th terminal reserve. We have that

$$({}_tV + \pi_t)(1-s) + {}_{t+1}Vs = {}_tV(1-s) + {}_{t+1}Vs + \pi_t(1-s).$$

${}_tV(1-s) + {}_{t+1}Vs$ is called the **mid–terminal reserve factor**. $\pi_t(1-s)$ is called the **unearned premium reserve**. The previous formula says that the $(t+s)$–th benefit reserve is the linear combination of the terminal reserved plus a fraction of the premium benefit.

Review: ${}_0L_x = Z_x - P_x\ddot{Y}_x$.

$$[1.2]\ {}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}.$$

$$[1.10] \Rightarrow\ {}_tV = \sum_{k=1}^{\infty} b_{t+k}v^k \cdot f_{K_{x+t}}(k) - \sum_{k=0}^{\infty} \pi_{t+k}v^k \cdot {}_kp_{x+t}.$$

$$[1.11]\ {}_tV_e = (b+s)A_{x+t} - ((1-r)G - e)\ddot{a}_{x+t} + \mathbf{1}(t=0)(e_0 + c_0G),$$

$${}_tV_x + P_x = vq_{x+t} + v \cdot {}_{t+1}V_x \cdot p_{x+t}.$$

$$[1.7]\ {}_tV + \pi_t = vb_{t+1} \cdot q_{x+t} + v \cdot {}_{t+1}V \cdot p_{x+t}.$$

$${}_tV_e + G_t(1-r_t) - e_t = vb_{t+1}q_{x+t} + v \cdot {}_{t+1}V_e p_{x+t}.$$

$$[12]\ {}_tV^e = {}_tV_e - {}_tV$$

$${}_tAS + G - rG - c_1 - \mathbf{1}(t=0)(r_0G + c_0) = v(b+s)q_{x+t} + v \cdot {}_{t+1}ASp_{x+t}.$$

$$[1.4]\ P_x\ddot{a}_{x:\bar{t}|} = A^1_{x:\bar{t}|} + {}_tE_x \cdot {}_tV_x.$$

$$\underbrace{\sum_{k=0}^{t-1} \pi_k v^k {}_kp_x}_{total\ payments} = \underbrace{\sum_{k=1}^{t} b_k v^k f_{K_x}(k)}_{insurance} + \underbrace{{}_tE_x \cdot {}_tV}_{refund}$$

$$\underbrace{c_0 + r_0G + G\ddot{a}_{x:\bar{k}|} - (rG + c_1)\ddot{a}_{x:\bar{k}|}}_{total\ balance} = \underbrace{(b+s)A^1_{x:\bar{k}|}}_{insurance} + \underbrace{{}_kAS_kE_x}_{asset\ share}.$$

Formulas in chapter 7:

1.1. ${}_tL_x = Z_{x+t} - P_x\ddot{Y}_{x+t}$.
1.2. ${}_tV_x = E({}_tL_x)\ (= A_{x+t} - P_x\ddot{a}_{x+t})$.
1.3. ${}_{t+1}I_x = {}_tV_x + P_x$.
1.4. $P_x\ddot{a}_{x:\bar{t}|} = A^1_{x:\bar{t}|} + {}_tE_x \cdot {}_tV_x$.
1.5. $F = 1 - \frac{P_x}{P_{x+t}}$.
1.6. ${}_tV_x = P_x\frac{\ddot{a}_{x:\bar{t}|}}{{}_tE_x} - \frac{A^1_{x:\bar{t}|}}{{}_tE_x} = P_x\ddot{s}_{x:\bar{t}|} - {}_tk_x$ (whole life insurance).
1.7. ${}_tV + \pi_t = vb_{t+1}q_{x+t} + v \cdot {}_{t+1}Vp_{x+t}$. (general ${}_tV_x$)
1.8. $C_j = vb_{j+1}I(K_x = j+1) - \pi_jI(K_x > j)$.
1.9. $\Lambda_j = C_j + v \cdot {}_{j+1}VI(K_x > j+1) - {}_jVI(K_x > j)$.
1.10. ${}_tL = b_{K_x}v^{K_{x+t}} - \sum_{k=0}^{K_{x+t}-1} \pi_{t+k}v^k = \sum_{k=1}^{\infty} b_{t+k}v^kI(K_{x+t} = k) - \sum_{k=0}^{\infty} \pi_{t+k}v^kI(K_{x+t} > k)$.
1.11. ${}_tV_e = (b+s)A_{x+t} - ((1-r)G - e)\ddot{a}_{x+t} + (e_0^* + r_0^*G)\mathbf{1}(t=0)$
1.12. ${}_tV^e = {}_tV_e - {}_tV$.

Insurance type	t–th terminal loss r.v.
Whole life insurance	${}_tL_x = Z_{x+t} - P_x\ddot{Y}_{x+t}$
h–payment whole life insurance	${}^h_tL_x = \begin{cases} Z_{x+t} - {}_hP_x\ddot{Y}_{x+t:\overline{h-t}\vert} & \text{if } 0 \le t < h, \\ Z_{x+t} & \text{if } t \ge h. \end{cases}$
n–year term insurance	${}_tL^1_{x:\overline{n}\vert} = \begin{cases} Z^1_{x+t:\overline{n-t}\vert} - P^1_{x:\overline{n}\vert}\ddot{Y}_{x+t:\overline{n-t}\vert} & \text{if } 0 \le t < n, \\ 0 & \text{if } t = n. \end{cases}$
n–year pure endowment	${}_tL_{x:\overset{1}{\overline{n}\vert}} = \begin{cases} Z_{x+t:\overset{1}{\overline{n-t}\vert}} - P_{x:\overset{1}{\overline{n}\vert}}\ddot{Y}_{x+t:\overline{n-t}\vert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
h–payment n–year endowment	${}^h_tL_{x:\overline{n}\vert} = \begin{cases} Z_{x+t:\overline{n-t}\vert} - {}_hP_{x:\overline{n}\vert}\ddot{Y}_{x+t:\overline{h-t}\vert} & \text{if } 0 \le t < h < n, \\ Z_{x+t:\overline{n-t}\vert} & \text{if } h \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year endowment	${}_tL_{x:\overline{n}\vert} = \begin{cases} Z_{x+t:\overline{n-t}\vert} - P_{x:\overline{n}\vert}\ddot{Y}_{x+t:\overline{n-t}\vert} & \text{if } 0 \le t < n, \\ 1 & \text{if } t = n. \end{cases}$
n–year deferred insurance	${}^n_tL({}_n\vert Z_x) = \begin{cases} {}_{n-t}\vert Z_{x+t} - {}_nP({}_n\vert Z_x)\ddot{Y}_{x+t:\overline{n-t}\vert} & \text{if } 0 \le t < n, \\ Z_{x+t} & \text{if } t \ge n. \end{cases}$
n–year deferred annuity–due	${}_tL({}_n\vert\ddot{Y}_x) = \begin{cases} {}_{n-t}\vert\ddot{Y}_{x+t} - P({}_n\vert\ddot{Y}_x)\ddot{Y}_{x+t:\overline{n-t}\vert} & \text{if } 0 \le t < n, \\ \ddot{Y}_{x+t} & \text{if } t \ge n. \end{cases}$

Table 7.5: t–th terminal loss random variables for some fully discrete contracts.

CHAPTER 8
Multiple Life Functions

In this chapter, we consider life insurance applied to several lives.

8.1 Multivariate random variables.

In this section, we need consider several random variables at the same time. Suppose that we have a sample space Ω with a probability $\mathbb{P}$ defined on Ω. If $X_1, \dots, X_d$ are random variables defined on Ω, then, $(X_1, \dots, X_d)$ is a random vector (r.v.) with values on $\mathbb{R}^d$. $(X_1, \dots, X_d)$ is also called a multivariate random variable.

Definition 8.1. *The (joint) cdf of a $\mathbb{R}^d$–valued r.v. $(X_1, \dots, X_d)$ is the function F from $\mathbb{R}^d$ into $\mathbb{R}$ defined by*

$$F_{(X_1,\dots,X_d)}(x_1, \dots, x_d) = \mathbb{P}\{X_1 \le x_1, \dots, X_d \le x_d\}, x_1, \dots, x_d \in \mathbb{R}.$$

For each $1 \le j \le d$, the c.d.f. F_{X_j} of X_j is called the marginal c.d.f. of X_j. It is easy to see that the marginal c.d.f. of X_j satisfies:

$$F_{X_j}(x_j) = \lim_{x_1,\dots,x_{j-1},x_{j+1},\dots,x_d \to \infty} F_{X_1,\dots,X_d}(x_1, \dots, x_d), x_j \in \mathbb{R}.$$

Definition 8.2. *The* **survival function** *of a $\mathbb{R}^d$–valued r.v. $(X_1, \dots, X_d)$ is the function S from $\mathbb{R}^d$ into $\mathbb{R}$ defined by*

$$S_{(X_1,\dots,X_d)}(x_1, \dots, x_d) = \mathbb{P}\{X_1 > x_1, \dots, X_d > x_d\}, x_1, \dots, x_d \in \mathbb{R}.$$

In this section, we assume $(X_1, ..., X_n)$ is cts. If $(X_1, ..., X_n)$ is cts, then the joint density of $(X_1, \dots, X_d)$ is

$$f_{(X_1,\dots,X_d)}(x_1, \dots, x_d) = \frac{\partial^d}{\partial x_1 \dots \partial x_d} F(x_1, \dots, x_d).$$

For each $1 \le j \le d$, the density f_{X_j} of X_j is called a **marginal density** of X_j. We have that

$$f_{X_j}(x_j) = \int_{\mathbb{R}^{d-1}} f_{(X_1,\dots,X_d)}(x_1, \dots, x_d)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_d.$$

See also (17-32) in formulas of 447.

Example 8.1.

Definition 8.3. *The r.v.'s $X_1, \dots, X_n$ are independent if*

$$\mathbb{P}\{X_1 \in A_1, \dots, X_n \in A_n\} = \mathbb{P}\{X_1 \in A_1\} \cdots \mathbb{P}\{X_n \in A_n\} \ \forall\ A_1, \dots, A_n \subset \mathbb{R}.$$

Example 8.2.

8.2 Joint life status.

In this section, we consider **joint life statuses**. A joint life status is obtained by combining several individual lives and defining the status as alive if and only if all the individual lives in the status are alive. We will mainly consider the case of two lives. The case of more than two lives is developed similarly. Consider two lives aged x and y, a two life joint status consisting by x and y is denoted by (xy) or $(x:y)$ (two people whose ages are x and y).

The future lifetime random variable of (xy) is denoted by $T_{xy} \stackrel{def}{=} \min(T_x, T_y)$. Often, we will assume that T_x and T_y are independent random variables, but not always.

Theorem 8.1. *Suppose that T_x and T_y have joint survival function S_{T_x,T_y}. Then, T_{xy} has survival function $S_{T_{xy}}(t) = S_{T_x,T_y}(t,t)$.*

Proof. The survival function of T_{xy} is
$S_{T_{xy}}(t) = \mathbb{P}\{T_{xy} > t\} = \mathbb{P}\{T_x \wedge T_y > t\} = \mathbb{P}\{T_x > t, T_y > t\} = S_{T_x,T_y}(t,t)$. ■

Different joint survival functions S_{T_x,T_y} can give rise to the same survival function of T_{xy}.

Example 8.3. *Actuary 1 believes that T_x and T_y have joint survival function*

$$S_{T_x,T_y}(s,t) = \begin{cases} 1 & \textit{if } s<0, t<0, \\ \frac{(10-s)^3(10-t)}{10^4} & \textit{if } 0 \le s,t \le 10, \\ \frac{(10-s)^3}{10^3} & \textit{if } 0 \le s \le 10, t<0, \\ \frac{(10-t)}{10} & \textit{if } s<0, 0 \le t \le 10,, \\ 0 & \textit{if } s>10, \ \textit{or } t>10, \end{cases}$$

Actuary 2 believes that T_x and T_y have joint survival function

$$S_{T_x,T_y}(s,t) = \begin{cases} 1 & \textit{if } s<0, t<0, \\ \frac{(10-s)^2(10-t)^2}{10^4} & \textit{if } 0 \le s,t \le 10, \\ \frac{(10-s)^2}{10^2} & \textit{if } 0 \le s \le 10, t<0, \\ \frac{(10-t)^2}{10^2} & \textit{if } s<0, 0 \le t \le 10,, \\ 0 & \textit{if } s>10, \ \textit{or } t>10, \end{cases}$$

Calculate the survival function of T_{xy} for each of the survival functions given.

Solution: For the first joint survival function, is it correct to have

$$S_{T_x,T_y}(t,t) = \begin{cases} 1 & \text{if } s<0, t<0, \quad ? \\ \frac{(10-t)^3(10-t)}{10^4} & \text{if } 0 \le s,t \le 10, \quad ? \\ \frac{(10-t)^3}{10^3} & \text{if } 0 \le s \le 10, t<0, \quad ? \\ \frac{(10-t)}{10} & \text{if } s<0, 0 \le t \le 10, \quad ? \\ 0 & \text{if } s>10, \ \text{or } t>10, \quad ? \end{cases}$$

For both joint survival functions, we have that

$$S_{T_{xy}}(t) = S_{T_x,T_y}(t,t) = \begin{cases} 1 & \text{if } t < 0, \\ \frac{(10-t)^4}{10^4} & \text{if } 0 \le t \le 10, \\ 0 & \text{if } t > 10. \end{cases}$$

Or $S_{T_{xy}}(t) = \frac{(10-s)^2}{10^2}$ if $0 \le t \le 10$.

Theorem 8.2.

Theorem 8.3.

Example 8.4. *Suppose that T_x and T_y have joint density $f_{T_x,T_y}(s,t) = \frac{12}{(1+2s+3t)^3}$, $s,t,\ge 0$. Find the survival function and the density of T_{xy}*

Solution: The survival function of T_{xy} is

$$\begin{aligned} &S_{T_{xy}}(t) = \mathbb{P}\{T_{xy} > t\} = \mathbb{P}\{\min(T_x,T_y) > t\} = \mathbb{P}\{T_x > t, T_y > t\} \\ &= \int_t^\infty \int_t^\infty f_{T_x,T_y}(u,v)\,dv\,du = \int_t^\infty \int_t^\infty \frac{12}{(1+2u+3v)^3}\,dv\,du \\ &= \int_t^\infty \int_t^\infty \frac{12}{3^3}((1/3+2u/3)+v)^{-3}\,dv\,du \qquad \int (a+v)^n dv = \frac{(a+v)^{n+1}}{n+1} \\ &= \int_t^\infty \frac{12}{3^3\cdot(-3+1)}((1/3+2u/3)+v)^{-2}\Big|_t^\infty\,du \\ &= \int_t^\infty \frac{2}{(1+2u+3t)^2}\,du \\ &= \int_t^\infty \frac{2}{2^2(1/2+3/2t+u)^2}\,du = \frac{1}{1+5t},\ t > 0. \end{aligned}$$

The density function of T_{xy} is

$$f_{T_{xy}}(t) = -\frac{d}{dt}S_{T_{xy}}(t) = -\frac{d}{dt}\frac{1}{1+5t} = \frac{5}{(1+5t)^2},\ \textbf{Done ??}$$

Theorem 8.4. *Suppose that T_x and T_y are independent,*
(i) The survival function of T_{xy} is ${}_tp_{xy} = {}_tp_x \cdot {}_tp_y$, $t \ge 0$.
(ii) The cumulative distribution function of T_{xy} is

$${}_tq_{xy} = 1 - {}_tp_{xy} = 1 - (1 - {}_tq_x)(1 - {}_tq_y) = {}_tq_x + {}_tq_y - {}_tq_x \cdot {}_tq_y,\ t \ge 0.$$

(iii) T_{xy} has density $f_{T_{xy}}(t) = f_{T_x}(t)S_{T_y}(t) + f_{T_y}(t)S_{T_x}(t)$, $t > 0$.

Proof (i) $S_{T_{xy}}(t) = {}_tp_{xy} = \mathbb{P}\{T_{xy} > t\} = \mathbb{P}\{\min(T_x,T_y) > t\}$
$= \mathbb{P}\{T_x > t, T_y > t\} = \mathbb{P}\{T_x > t\}\mathbb{P}\{T_y > t\} = {}_tp_x \cdot {}_tp_y,\ t \ge 0.$
(ii) $F_{T_{xy}}(t) = {}_tq_{xy} = 1 - {}_tp_{xy} = 1 - {}_tp_x \cdot {}_tp_y = 1 - (1 - {}_tq_x)(1 - {}_tq_y) = \cdots,\ t \ge 0.$
(iii) $f_{T_{xy}}(t) = -\frac{d}{dt}{}_tp_{xy} = -\frac{d}{dt}({}_tp_x \cdot {}_tp_y),\ t \ge 0.$

$f_{T_{xy}}(t) = f_{T_x}(t)S_{T_y}(t) + S_{T_x}(t)f_{T_y}(t)$.

We will abbreviate $p_{xy} = {}_1p_{xy}$ and $q_{xy} = {}_1q_{xy}$.

Example 8.5. *Assume that T_{30} and T_{40} are independent, ${}_{10}p_{30:40} = 0.3$. Calculate ${}_{20}p_{30}$.*

Solution: ${}_tp_{xy} = {}_tp_x \cdot {}_tp_y$ and ${}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m}$.

$0.3 = {}_{10}p_{30:40} = {}_{10}p_{30} \cdot {}_{10}p_{30+10} = {}_{10+10}p_{30}$.

Example 8.6. *Suppose that:*
(i) T_{40} and T_{50} are independent.
(ii) T_{40} and T_{50} follow De Moivre's law with terminal age 100.
Find the survival function of $T_{40:50}$.

Solution: Formula: ${}_tp_{xy} = {}_tp_x{}_tp_y$ if $T_x \perp T_y$.

The survival function of De Moivre's law with terminal age 100 is $s(t) = \frac{100-t}{100}$, $0 \le t \le 100$.

The survival function of T_x is ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{100-x-t}{100-x}$, if $0 \le t \le 100 - x$, which implies that

$${}_tp_x = 1 \text{ if } t < 0 \text{ and } {}_tp_x = 0 \text{ if } t > 100 - x.$$

The survival function of $T_{40:50}$ is

$${}_tp_{40:50} = {}_tp_{40} \cdot {}_tp_{50} = \frac{60-t}{60}\frac{50-t}{50} = \frac{(60-t)(50-t)}{(60)(50)}, \text{ if } \ 0 \le t \le 50.$$

Theorem 8.5. ${}_{n+t}p_{xy} = {}_np_{xy} \cdot {}_tp_{x+n:y+n}$ *(recall [4]* ${}_{n+t}p_x = {}_np_x \cdot {}_tp_{x+n}$*)*

Example 8.7. *The probability that* $(30:25)$ *survives 10 years is* 0.9*. The probability that* $(30:25)$ *survives 15 years is* 0.8*. Calculate the probability that* $(40:35)$ *survives 5 years.*

Solution: ${}_{n+t}p_{xy} = {}_np_{xy} \cdot {}_tp_{x+n:y+n}$ with $n = 10$, $t = 5$, $(x, y) = (30, 25)$.

$0.8 = {}_{10+5}p_{xy} = {}_{10}p_{xy} \cdot {}_5p_{x+10:y+10} = 0.9 \cdot {}_5p_{x+10:y+10}$.

The probability that $(40:35)$ survives 5 years is ${}_5p_{40:35} = \frac{{}_{15}p_{30:25}}{{}_{10}p_{30:25}} = \frac{8}{9}$.

Theorem 8.6. *The pdf of T_{xy} is $f_{T_{xy}}(t) = -\frac{d}{dt}\,{}_tp_{xy}$, $t \ge 0$.*

Theorem 8.7. *The mortality rate of T_{xy} is $\mu_{x:y}(t) = -\frac{d}{dt}\log({}_tp_{xy}) = \frac{f_{T_{xy}}(t)}{{}_tp_{xy}}$, $t \ge 0$.*

Recall $\mu_{T_{x+t}}(0) = \mu_{x+t}(0) = \mu_{x+t} = \mu_x(t) = \mu_{T_x}(t)$.

Theorem 8.8. $\mu_{T_{x+t:y+t}}(0) = \mu_{x+t:y+t}(0) = \underbrace{\mu_{x+t:y+t} = \mu_{x:y}(t)}_{main\ eq.} = \mu_{T_{xy}}(t)$.

Theorem 8.9. *Suppose that T_x and T_y are independent, then:*

(i) The mortality rate of T_{xy} is $\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$.

(ii) The density function of T_{xy} is $f_{T_{xy}}(t) = (\mu_{x+t} + \mu_{y+t}) \cdot {}_tp_x \cdot {}_tp_y$.

Proof. (i) $\mu_{x+t:y+t} = -\frac{d}{dt}\log({}_tp_{xy}) = -\frac{d}{dt}\log({}_tp_x \cdot {}_tp_y)$
$= -\frac{d}{dt}\log({}_tp_x) - \frac{d}{dt}\log({}_tp_y) = \mu_{x+t} + \mu_{y+t}$.

(ii) We have that $f_{T_{xy}}(t) = \mu_{x+t:y+t} \cdot {}_tp_{xy} = (\mu_{x+t} + \mu_{y+t}){}_tp_x \cdot {}_tp_y$. ■

Theorem 8.10.

Future life of joint life status (xy): $T_{xy} = T_x \wedge T_y$.
${}_tp_{xy} = S_{T_{xy}}(t) = S_{T_x,T_y}(t,t)$,
${}_tp_{xy} = {}_tp_x {}_tp_y$ if $T_x \perp T_y$.
$f_{T_{xy}}(t) = -\frac{d}{dt}\,{}_tp_{xy}$.
$\mu_{x:y}(t) = -\frac{d}{dt}\log({}_tp_{xy}) = \frac{f_{T_{xy}}(t)}{{}_tp_{xy}}$.

Example 8.8. *T_{40} and T_{50} are independent. T_{40} and T_{50} follow De Moivre's law with terminal age 100.*
(i) Find the density function of $T_{40:50}$.
(ii) Find the force of mortality of $T_{40:50}$.
(iii) Find the force of mortality of $T_{50:60}$.
(iv) Find ${}_tp_{50:60}$.

Solution: (i) By Example 8.6, the survival function of $T_{40:50}$ is

$${}_tp_{40:50} = {}_tp_{40} \cdot {}_tp_{50} = \frac{(60-t)(50-t)}{(60)(50)} = \frac{3000 - 110t + t^2}{3000},\ 0 \le t \le 50.$$

The density function of $T_{40:50}$ is

$$f_{T_{40:50}}(t) = -\frac{d}{dt}{}_tp_{40:50} = \frac{110-2t}{3000},\ 0 < t < 50.$$

(ii) The force of mortality of $T_{40:50}$ is

$$\mu_{40:50}(t) = \frac{f}{S} = \frac{110-2t}{3000-110t+t^2},\ 0 \le t \le 50.$$

(iii) The force of mortality of $T_{50:60}$ is
$\mu_{50:60}(t) = \mu_{40:50}(t+10) = \frac{110-2(t+10)}{3000-110(t+10)+(t+10)^2},\ 0 \le t,\ (t+10) \le 50.$

(iv) Two ways: (1) ${}_tp_{50:60} = {}_tp_{50} \cdot {}_tp_{60} = \frac{(50-t)(40-t)}{50*40}$;

(2) ${}_tp_{50:60} = \exp(-\int_0^t \mu_{50:60}(s)ds) = ...$ **How to continue ?**

$$\int \frac{\alpha x + \beta}{x^2 + bx + c}dx = \int \frac{\alpha_1}{x-a_1} + \frac{\alpha_2}{x-a_2}dx, \text{ where } a_1, a_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

The complete expectation of the future lifetime of the entity (xy) is

$$\overset{\circ}{e}_{xy} \overset{def}{=} E(T_{xy}) = \int_0^\infty t f_{T_{xy}}(t)\,dt = \int_0^\infty {}_tp_{xy}\,dt = \int_0^\infty \int_0^\infty (t \wedge s) f_{T_x,T_y}(s,t)\,dsdt. \qquad (1)$$

$$E[{T_{xy}}^2] = \int_0^\infty t^2 f_{T_{xy}}(t)\,dt = \int_0^\infty 2t \cdot {}_tp_{xy}\,dt = \int_0^\infty \int_0^\infty (t \wedge s)^2 f_{T_x,T_y}(s,t)\,dsdt. \quad (\textit{see } [1]\&[2])$$

$$\mathrm{Var}(T_{xy}) = E[{T_{xy}}^2] - (\mathring{e}_{xy})^2.$$

Example 8.9. *Suppose that the joint density function of T_x and T_y is $f_{T_x,T_y}(s,t) = \frac{6}{(1+s+t)^4}$, $s,t \geq 0$. Find $\mathring{e}_{xy}$.*

Solution: 3 ways (see Eq. (1)). **Q:** $T_x \perp T_y$ **?**.

Method (3). Notice that $\min(s,t) = sI(s \leq t) + tI(t < s)$.

$$\mathring{e}_{xy} = \int_0^\infty \int_0^\infty (sI(s \leq t) + tI(t < s)) f_{T_x,T_y}(s,t)\,ds\,dt$$

$$= \int_0^\infty \int_0^t s f_{T_x,T_y}(s,t)\,ds\,dt + \int_0^\infty \int_t^\infty t f_{T_x,T_y}(s,t)\,ds\,dt. \qquad \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int_0^\infty \int_0^t s f_{T_x,T_y}(s,t)\,ds\,dt = \int_0^\infty \int_0^t \frac{s6}{(1+s+t)^4}\,ds\,dt \overset{??}{=} \int_0^\infty \int_s^\infty \frac{6s}{(1+s+t)^4}\,dt\,ds$$

$$\overset{n=?}{=} \int_0^\infty \frac{6s(1+s+t)^{n+1}}{n+1}\Big|_s^\infty ds = \int_0^\infty \frac{2s}{(1+2s)^3}\,ds = \int_0^\infty \frac{2s+1-1}{(1+2s)^3}\,ds$$

$$= \int_0^\infty \left((1+2s)^{-2} - (1+2s)^{-3}\right)\,ds = \left(-\frac{1}{2(1+2s)} + \frac{1}{4(1+2s)^2}\right)\Big|_0^\infty = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

$$\int_0^\infty \int_t^\infty t f_{T_x,T_y}(s,t)\,ds\,dt = \int_0^\infty \int_t^\infty t\frac{6}{(1+s+t)^4}\,ds\,dt = \frac{1}{4} \text{ (by symmetry).}$$

Hence, $\mathring{e}_{xy} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Method (2). Notice that ${}_tp_{xy} = \mathbb{P}\{T_{xy} > t\} = \mathbb{P}\{T_x > t, T_y > t\} = \int_t^\infty \int_t^\infty f_{T_x,T_y}(u,v)\,du\,dv$

$$= \int_t^\infty \int_t^\infty \frac{6}{(1+u+v)^4}\,du\,dv = \int_t^\infty \frac{-2}{(1+u+v)^3}\Big|_t^\infty dv = \int_t^\infty \frac{2}{(1+u+t)^3}\,dv$$

$$= \frac{-1}{(1+u+t)^2}\Big|_t^\infty = \frac{1}{(1+2t)^2}.$$

$\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy}\,dt = \int_0^\infty \frac{1}{(1+2t)^2}\,dt = -\frac{1}{1+2t}\Big|_0^\infty = \frac{1}{2}$.

Method (1) $\mathring{e}_{xy} = \int_0^\infty t f_{T_{xy}}(t)\,dt = \int_0^\infty t\frac{-\partial_t p_{xy}}{\partial t}\,dt$ ...... (doable, but not preferred).

Example 8.10. *Suppose that: (i) T_x and T_y are independent. (ii) T_x and T_y have constant mortality forces, say, μ_1 and μ_2. Find $\mathring{e}_{xy}$ and $\mathrm{Var}(T_{xy})$.*

Solution: $\mu_{x+t:y+t} = -\frac{d}{dt}\ln {}_tp_{xy} = -\frac{d}{dt}\ln({}_tp_x \cdot {}_tp_y) = \mu_{x+t} + \mu_{y+t} = \mu_1 + \mu_2$.
Hence, $S_{T_{xy}} = e^{-t(\mu_1+\mu_2)} = e^{-t\mu}$, $t > 0$. So, $E(T_{xy}) = \frac{1}{\mu}$ and $\sigma^2_{T_{xy}} = \frac{1}{\mu^2}$. Or
$\mathring{e}_{xy} = \frac{1}{\mu_1+\mu_2}$ and $\mathrm{Var}(T_{xy}) = \frac{1}{(\mu_1+\mu_2)^2}$.

Let $K_{xy} = \lceil T_{xy} \rceil$. Then
$\mathbb{P}\{K_{xy} = k\} = \mathbb{P}\{k-1 < T_{xy} \leq k\} = {}_{k-1|}q_{xy} = {}_{k-1}p_{xy} - {}_kp_{xy} = {}_{k-1}p_{xy} \cdot q_{x+k-1:y+k-1}$.
If $T_x \perp T_y$ then $\mathbb{P}\{K_{xy} = k\} = {}_{k-1}p_{xy} - {}_kp_{xy} = {}_{k-1}p_x \cdot {}_{k-1}p_y - {}_kp_x \cdot {}_kp_y$.

Example 8.11. *Suppose that $T_x \perp T_y$ and*

$q_x = 0.01$	$q_{x+1} = 0.02$	$q_{x+2} = 0.02$
$q_y = 0.01$	$q_{y+1} = 0.03$	$q_{y+2} = 0.03$

Find $\mathbb{P}\{K_{xy} = k\}$, for $k = 1, 2, 3$.

Solution: $\mathbb{P}\{K_{xy} = k\} = {}_{k-1}p_{xy} - {}_k p_{xy}$, ${}_k p_x = p_x \cdots p_{x+k-1}$ and ${}_k p_{xy} = {}_k p_x \cdot {}_k p_y$.

$$\begin{aligned}
{}_1p_{xy} &= p_x p_y = (0.99)(0.99) \approx 0.98,\\
{}_2p_{xy} &= {}_2p_x \cdot {}_2p_y = p_x p_{x+1} p_y p_{y+1} \approx (0.99)(0.98)(0.99)(0.97) \approx 0.93,\\
{}_3p_{xy} &= {}_3p_x \cdot {}_3p_y = p_x p_{x+1} p_{x+2} p_y p_{y+1} p_{y+2} = (0.99)(0.98)(0.98)(0.99)(0.97)(0.97) \approx 0.89.
\end{aligned}$$

$$\text{Hence,} \quad \begin{aligned}
\mathbb{P}\{K_{xy} = 1\} &= 1 - p_{xy} \approx 0.02,\\
\mathbb{P}\{K_{xy} = 2\} &= {}_1p_{xy} - {}_2p_{xy} \approx 0.05,\\
\mathbb{P}\{K_{xy} = 3\} &= {}_2p_{xy} - {}_3p_{xy} \approx 0.06.
\end{aligned}$$

The curtate expectation of future lifetime for the joint status (xy) is defined as
$e_{xy} = E[K_{xy}] = \sum_{k=1}^{\infty} {}_k p_{xy}$. Formula [2].
The temporary curtate expectation of future lifetime for the joint status (xy) is defined as
$e_{xy:\overline{n}|} = E(K_{xy} \wedge n) = \sum_{k=1}^{n} {}_k p_{xy}$. Formula [2].

Example 8.12. *Suppose that: (i) $T_x \perp T_y$. (ii) T_x and T_y have constant mortality forces, say μ_1 and μ_2. Find e_{xy}.*

Solution: Since $\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t} = \mu_1 + \mu_2$,

$$e_{xy} = \sum_{k=1}^{\infty} {}_k p_{xy} = \sum_{k=1}^{\infty} e^{-(\mu_1+\mu_2)k} = p \sum_{i=0}^{\infty} p^i = p \frac{1 - p^{\infty}}{1 - p}\Big|_{p=?} = \frac{e^{-(\mu_1+\mu_2)}}{1 - e^{-(\mu_1+\mu_2)}} = \frac{1}{e^{\mu_1+\mu_2} - 1}.$$

8.3 Last–survivor status.

Definition 8.4. *The* **last survivor status** *is obtained from several lives by making the age–at–death of the status the last of the individuals deaths forming the status.*

Definition 8.5. *The last survivor status consisting of (x) and (y) is denoted by $\overline{xy}$ or $\overline{x : y}$. The future life r.v. of $\overline{xy}$ is denoted $T_{\overline{xy}}$. It is easy to see that $T_{\overline{xy}} = \max(T_x, T_y) = T_x \vee T_y$.*

Notations $S_{T_{\overline{xy}}}(t) = {}_t p_{\overline{xy}}$ and $F_{T_{\overline{xy}}}(t) = {}_t q_{\overline{xy}}$.

Theorem 8.11. $F_{T_{\overline{xy}}}(t) = F_{T_x,T_y}(t,t) = \underbrace{F_{T_x}(t) F_{T_y}(t)}_{if\ T_x \perp T_y}.$

Proof. $F_{T_{\overline{xy}}}(t) = \mathbb{P}\{T_{\overline{xy}} \le t\} = \mathbb{P}\{\max(T_x, T_y) \le t\} = \mathbb{P}\{T_x \le t, T_y \le t\} = F_{T_x,T_y}(t,t)$.
If $T_x \perp T_y$ then $F_{T_{\overline{xy}}}(t) = \mathbb{P}\{T_x \le t, T_y \le t\} = \mathbb{P}\{T_x \le t)\mathbb{P}\{T_y \le t\} = F_{T_x}(t) F_{T_y}(t)$. ∎

${}_t q_{\overline{xy}} = {}_t q_x {}_t q_y$ if $T_x \perp T_y$, $f_{T_{\overline{xy}}}(t) = \frac{d}{dt} {}_t q_{\overline{xy}}$ and $\mu_{T_{\overline{xy}}}(t) = \frac{f_{T_{\overline{xy}}}(t)}{{}_t p_{\overline{xy}}}$.

Example 8.13. *There are two cdfs:*

$$F_{T_x,T_y}(s,t)=\begin{cases}0 & \text{if } s<0, \text{ or } t<0,\\ \frac{s^3t}{10^4} & \text{if } 0\le s,t\le 10,\\ \frac{s^3}{10^3} & \text{if } 0\le s\le 10, 10<t,\\ \frac{t}{10} & \text{if } 10<s, 0\le t\le 10,\\ 1 & \text{if } s>10, t>10,\end{cases}\qquad F_{T_x,T_y}(s,t)=\begin{cases}0 & \text{if } s<0, t<0,\\ \frac{s^2t^2}{10^4} & \text{if } 0\le s,t\le 10,\\ \frac{s^2}{10^2} & \text{if } 0\le s\le 10, 10<t,\\ \frac{t^2}{10^2} & \text{if } 10<s, 0\le t\le 10,\\ 1 & \text{if } s>10, t>10,\end{cases}$$

${}_tq_{\overline{xy}} =$?

Solution: For both of them, ${}_tq_{\overline{xy}} = F_{T_{\overline{xy}}}(t) = F_{T_x,T_y}(t,t) = \begin{cases}1 & \text{if } t<0,\\ \frac{t^4}{10^4} & \text{if } 0\le t\le 10,\\ 0 & \text{if } t>10.\end{cases}$

Theorem 8.12. $f_{T_{\overline{xy}}}(t) = \int_0^t f_{T_x,T_y}(t,v)\,dv + \int_0^t f_{T_x,T_y}(u,t)\,du.$

$F_{T_x,T_y}(t,t) = \int_0^t\int_0^t f_{T_x,T_y}(u,v)dudv,$

$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} g(u,c(t))du = g(b(t),c(t))b'(t) - g(a(t),c(t))a'(t) + \int_{a(t)}^{b(t)}\frac{\partial}{\partial t}g(u,c(t))du$

$= \int_0^t f_{T_x,T_y}(t,v)\,dv - 0 + \int_0^t \frac{\partial}{\partial t}\int_0^t f_{T_x,T_y}(u,t)\,dudv =$ ......

Theorem 8.13. *If $T_x \perp T_y$, then $f_{T_{\overline{xy}}}(t) = f_{T_x}(t)F_{T_y}(t) + F_{T_x}(t)f_{T_y}(t)$, $(\neq f_{T_x}(t)f_{T_y}(t).)$*

Example 8.14. *Suppose that: (a) $T_{40}\perp T_{50}$.*
(b) T_{40} and T_{50} follow De Moivre's law with terminal age 100.
Find the c.d.f and the density function of $T_{\overline{40:50}}$.

Solution: Formulas: $F_{T_{\overline{xy}}}(t) = F_{T_x}(t)F_{T_y}(t)$ and $F'(t)=f(t)$. Thus $F_{T_{40}}(t) = \begin{cases}\frac{t}{60} & \text{if } 0\le t\le 60,\\ 1 & \text{if } 60<t.\end{cases}$

$F_{T_{50}}(t) = \begin{cases}\frac{t}{50} & \text{if } 0\le t\le 50,\\ 1 & \text{if } 50<t.\end{cases}$ The c.d.f. of $T_{\overline{40:50}}$ is $F_{T_{\overline{40:50}}}(t) = F_{T_{40}}(t)F_{T_{50}}(t) = \begin{cases}\frac{t}{60}\frac{t}{50} & \text{if } 0\le t\le 50,\\ \frac{t}{60} & \text{if } 50<t\le 60,\\ 1 & \text{if } 60<t.\end{cases}$

Hence, the density of $T_{\overline{40:50}}$ is $f_{T_{\overline{40:50}}}(t) = \frac{d}{dt}F_{T_{\overline{40:50}}}(t) = \begin{cases}\frac{t}{1500} & \text{if } 0<t<50,\\ \frac{1}{60} & \text{if } 50<t<60,\end{cases}$

Theorem 8.14. *(i) ${}_tq_{\overline{xy}} + {}_tq_{xy} = {}_tq_x + {}_tq_y$. (ii) ${}_tp_{\overline{xy}} + {}_tp_{xy} = {}_tp_x + {}_tp_y$.*

Proof. (1) $\vdash$: $I(t>T_x) + I(t>T_y) = I(t>T_{xy}) + I(t>T_{\overline{xy}})$.
Notice that if $T_x = T_y$, then $T_x = T_y = T_{xy} = T_{\overline{xy}}$ and equality (1) holds.
If $T_x \neq T_y$, then T_{xy} is one of T_x and T_y, and $T_{\overline{xy}}$ is the other one of them.
Hence, equality (1) holds for each $t\ge 0$.

Taking expectations of (1) yields

$$\begin{aligned}
0 =& P(T_x < t) + P(T_y < t) - P(T_{xy} < t) - P(T_{\overline{xy}} < t) \\
=& P(T_x \le t) + P(T_y \le t) - P(T_{xy} \le t) - P(T_{\overline{xy}} \le t) \qquad \textbf{Why ?} \\
=& {}_tq_x + {}_tq_y - {}_tq_{\overline{xy}} - {}_tq_{xy} \qquad => \ (i) \\
=& {}_tp_x + {}_tp_y - {}_tp_{\overline{xy}} - {}_tp_{xy} => \text{(ii)}.
\end{aligned}$$

Theorem 8.15. $f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$.

Corollary 8.1.

Example 8.15. *Suppose that* $T_x \perp T_y$ *and*

$q_x = 0.01$	$q_{x+1} = 0.01$	$q_{x+2} = 0.02$
$q_y = 0.02$	$q_{y+1} = 0.03$	$q_{y+2} = 0.03$

Find ${}_kp_{\overline{xy}}$, *for* $k = 1, 2, 3$.

Solution: Formulas: ${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$, ${}_tp_{xy} = {}_tp_x \cdot {}_tp_y$ and ${}_tp_x = 1 - {}_tq_x$.

$$\begin{aligned}
p_{\overline{xy}} =& p_x + p_y - p_{xy} = p_x + p_y - p_xp_y = (0.99) + (0.98) - (0.99)(0.98) = 0.9998 \qquad ({}_2p_x = p_xp_{x+1}), \\
{}_2p_{\overline{xy}} =& {}_2p_x + {}_2p_y - {}_2p_{xy} = {}_2p_x + {}_2p_y - {}_2p_x \cdot {}_2p_y \\
=& (0.99)(0.99) + (0.98)(0.97) - (0.99)(0.99)(0.98)(0.97) = 0.9990169, \\
{}_3p_{\overline{xy}} =& {}_3p_x + {}_3p_y - {}_3p_x \cdot {}_3p_y \\
=& (0.99)(0.99)(0.98) + (0.98)(0.97)(0.97) - (0.99)(0.99)(0.98)(0.98)(0.97)(0.97) \\
=& 0.9969221.
\end{aligned}$$

Let $K_{\overline{xy}} = \lceil T_{\overline{xy}} \rceil$, then

$$\mathbb{P}\{K_{\overline{xy}} = k\} = \mathbb{P}\{k - 1 < T_{\overline{xy}} \le k\} = {}_{k-1|}q_{\overline{xy}} = {}_{k-1}p_{\overline{xy}} - {}_kp_{\overline{xy}} = {}_{k-1}p_{\overline{xy}} \cdot q_{\overline{x+k-1:y+k-1}}.$$

If $T_x \perp T_y$, then $\mathbb{P}\{K_{\overline{xy}} = k\} = {}_kq_x \cdot {}_kq_y - {}_{k-1}q_x \cdot {}_{k-1}q_y$ **Why ?**

Theorem 8.16. $\mathbb{P}\{K_{\overline{xy}} = k\} = \mathbb{P}\{K_x = k\} + \mathbb{P}\{K_y = k\} - \mathbb{P}\{K_{xy} = k\}$.

Theorem 8.17.

The complete expectation of the future lifetime of the entity $(\overline{xy})$ is

$$E(T_{\overline{xy}}) = \mathring{e}_{\overline{xy}} = \int_0^\infty t f_{T_{\overline{xy}}}(t)\, dt = \int_0^\infty {}_tp_{\overline{xy}}\, dt = \int_0^\infty \int_0^\infty (t \vee s) f_{T_x,T_y}(t, s) dtds. \qquad (1)$$

$$E(g(X, Y)) = \int t f_Z(t)dt = \int H'(t){}_tp_Zdt = \int\int g(x, y) f_{X,Y}(x, y)dxdy, \text{ where } Z = g(X, Y).$$

$$E[{T_{\overline{xy}}}^2] = \int_0^\infty t^2 f_{T_{\overline{xy}}}(t)\, dt = \int_0^\infty 2t \cdot {}_tp_{\overline{xy}}\, dt. \textit{ and } \mathrm{Var}(T_{\overline{xy}}) = E[{T_{\overline{xy}}}^2] - (\mathring{e}_{\overline{xy}})^2.$$

Example 8.16. *Suppose that* $f_{T_x,T_y}(s, t) = \frac{6}{(1+s+t)^4}$, $s, t \ge 0$. *Find* $\mathring{e}_{\overline{xy}}$.

Solution: 3 ways by Eq. (1):

$$\text{Method (3).}\quad \overset{\circ}{e}_{\overline{xy}} = \int_0^\infty \int_0^\infty \max(s,t) f_{T_x,T_y}(s,t)\,ds\,dt$$
$$= \int_0^\infty \int_0^t t f_{T_x,T_y}(s,t)\,ds\,dt + \int_0^\infty \int_t^\infty s f_{T_x,T_y}(s,t)\,ds\,dt,$$

$$1st\ integral = \int_0^\infty \int_0^t t f_{T_x,T_y}(s,t)\,ds\,dt = \int_0^\infty \int_0^t t \frac{6}{(1+s+t)^4}\,ds\,dt$$
$$= \int_0^\infty \int_0^t 6t(1+t+s)^{-4}\,ds\,dt = \int_0^\infty -\frac{2t}{(1+t+s)^3}\Big|_0^t\,dt$$
$$= \int_0^\infty \left(-\frac{2t}{(1+2t)^3} + \frac{2t}{(1+t)^3}\right)dt = \int_0^\infty \left(\frac{1-(1+2t)}{(1+2t)^3} + \frac{2(1+t)-2}{(1+t)^3}\right)dt$$
$$= \int_0^\infty \left(\frac{1}{(1+2t)^3} - \frac{1}{(1+2t)^2} + \frac{2}{(1+t)^2} + \frac{-2}{(1+t)^3}\right)dt$$
$$= \left(\frac{-1}{4(1+2t)^2} + \frac{1}{2(1+2t)} - \frac{2}{1+t} + \frac{1}{(1+t)^2}\right)\Big|_0^\infty$$
$$= \frac{1}{4} - \frac{1}{2} + 2 - 1 = \frac{3}{4}.$$

By symmetry, the second integral

$$\int_0^\infty \int_t^\infty s f_{T_x,T_y}(s,t)\,ds\,dt = \int_0^\infty \int_t^\infty s\frac{6}{(1+s+t)^4}\,ds\,dt = \int_0^\infty \int_0^s s\frac{6}{(1+s+t)^4}\,dt\,ds = \frac{3}{4}.$$

Hence, $\overset{\circ}{e}_{xy} = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$.

Method (2). $\overset{\circ}{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}}\,dt$.

$${}_tp_{\overline{xy}} = \mathbb{P}\{T_{\overline{xy}} \ge t\} = \mathbb{P}\{\max(T_x,T_y) \ge t\} = 1 - \mathbb{P}\{T_x < t, T_y < t\}$$
$$=1 - \int_0^t \int_0^t f_{T_x,T_y}(u,v)\,du\,dv = 1 - \int_0^t \int_0^t \frac{6}{(1+u+v)^4}\,du\,dv$$
$$=1 - \int_0^t \frac{-2}{(1+u+v)^3}\Big|_0^t\,dv = 1 - \int_0^t \left(\frac{2}{(1+u)^3} - \frac{2}{(1+u+t)^3}\right)dv$$
$$=1 - \left(\frac{-1}{(1+u)^2} + \frac{1}{(1+u+t)^2}\right)\Big|_0^t = 1 + \frac{1}{(1+t)^2} - \frac{1}{(1+2t)^2} - 1 + \frac{1}{(1+t)^2}$$
$$=\frac{2}{(1+t)^2} - \frac{1}{(1+2t)^2}.$$
$$\overset{\circ}{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}}\,dt = \int_0^\infty \left(\frac{2}{(1+t)^2} - \frac{1}{(1+2t)^2}\right)dt = \left(\frac{-2}{1+t} + \frac{1}{2(1+2t)}\right)\Big|_0^\infty = \frac{3}{2}.$$

Method (1). $\overset{\circ}{e}_{\overline{xy}} = \int_0^\infty t f_{T_{\overline{xy}}}(t)dt = \cdots$, where $f_{T_{\overline{xy}}}(t) = -\frac{d}{dt}{}_tp_{\overline{xy}} = -\frac{d}{dt}(\frac{2}{(1+t)^2} - \frac{1}{(1+2t)^2})$.

Example 8.17. *$T_x \perp T_y$. T_x has mortality rate $\mu_x(t) = \frac{1}{50-t}$, $0 < t < 50$. T_y has mortality rate $\mu_y(t) = \frac{1}{30-t}$, $0 < t < 30$. Compute (1)* $\mathrm{Cov}(T_{xy}, T_x)$ *and (2)* $\mathrm{Cov}(T_{xy}, T_{\overline{xy}})$.

Solution: : T_x has mortality rate $\mu_x(t) = \frac{1}{b-t}$ for $0 < t < b <=> T_x \sim U(0,b)$. $\frac{f(t)}{S(t)} = \frac{\frac{1}{b-0}}{\frac{b-t}{b-0}}$.

$S_{T_x}(t) = \frac{50-t}{50}$, $0 \le t \le 50$. $S_{T_y}(t) = \frac{30-t}{30}$, $0 \le t \le 30$.
(1) $Cov(T_{xy}, T_x) = E(T_{xy}T_x) - E(T_{xy})E(T_x)$.

Need $S_{T_{xy}}(t) = {}_tp_x \cdot {}_tp_y = \frac{(50-t)(30-t)}{1500}$, $0 \le t \le 50$?? or $0 \le t \le 30$??

$$\begin{aligned}
E(T_{xy}) &= \int_0^\infty {}_tp_{xy}\,dt = \int_0^{30} \frac{(50-t)(30-t)}{1500}\,dt = -\int_{30}^0 \frac{(20+x)x}{50 \times 30}\,dx \qquad (x = 30-t)\\
&= \int_0^{30} \frac{20x+x^2}{50 \times 30} dx = 12.\\
E(T_x) &= \frac{50}{2} = 25.\\
\underbrace{E[T_xT_{xy}]}_{\int\int tsf_{T_x,T_{xy}}(t,s)dtds} &= \underbrace{E(T_x(T_x \wedge T_y))}_{\int\int t(t\wedge s)f_{T_x,T_y}(t,s)dtds} = E[T_x(T_x \wedge T_y)(I(T_y < T_x) + I(T_y \ge T_x)]\\
&= E(T_xT_yI(T_x > T_y))) + E(T_x^2 I(T_x \le T_y))\\
&= \int_0^\infty \int_s^\infty tsf_{T_x}(t)f_{T_y}(s)dtds + \int_0^\infty \int_t^\infty t^2 f_{T_x}(t)f_{T_y}(s)dsdt\\
&= \int_0^{30} \int_s^{50} ts\frac{1}{30 \times 50}dtds + \int_0^{50} \int_t^{30} t^2\frac{1}{30 \times 50}dsdt = 352.5.
\end{aligned}$$

So, Cov$(T_x, T_{xy}) = 352.5 - (25)(12) = 52.5$.

(2) $Cov(T_{\overline{xy}}, T_{xy}) = E[T_{\overline{xy}}T_{xy}] - E[T_{\overline{xy}}]E[T_{xy}]$,
$T_{xy}T_{\overline{xy}} = T_xT_y$. $=> E[T_{\overline{xy}}T_{xy}] = E[T_xT_y] = E[T_x]E[T_y]$.
$T_{\overline{xy}} + T_{xy} = T_x + T_y => E(T_{\overline{xy}}) = -E(T_{xy}) + E(T_x) + E(T_y)$. $=>$
Cov$(T_{\overline{xy}}, T_{xy}) = E(T_x)E(T_y) - E(T_{xy})(E(T_x) + E(T_y) - E(T_{xy}))$
$= 15(25) - 12 * (15 + 25 - 12) = 39$.

Note: (T_x, T_{xy}) has a mixed distribution.

Theorem 8.18.

Theorem 8.19.

Theorem 8.20.

Example 8.18.

$\mathring{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}}dt$. $\qquad e_{\overline{xy}} = \sum_{k=1}^\infty {}_kp_{\overline{xy}}$.
$\mathring{e}_{\overline{xy}:\overline{n}|} = \int_0^n {}_tp_{\overline{xy}}dt$. $\qquad e_{\overline{xy}:\overline{n}|} = \sum_{k=1}^n {}_kp_{\overline{xy}}$.
${}_tp_{\overline{xy}} + {}_tp_{xy} = {}_tp_x + {}_tp_y$.
$\mathring{e}_{\overline{xy}} + \mathring{e}_{xy} = \mathring{e}_x + \mathring{e}_y$. $\qquad e_{\overline{xy}} + e_{xy} = e_x + e_y$.
$\mathring{e}_{\overline{xy}:\overline{n}|} + \mathring{e}_{xy:\overline{n}|} = \mathring{e}_{x:\overline{n}|} + \mathring{e}_{y:\overline{n}|}$. $\qquad e_{\overline{xy}:\overline{n}|} + e_{xy:\overline{n}|} = e_{x:\overline{n}|} + e_{y:\overline{n}|}$.

8.4 Joint survival functions

Definition 8.6. ${}_{n}q_{\overset{1}{x}y} = \mathbb{P}\{T_x < T_y, T_x \le n\}$ *(*$= \mathbb{P}\{T_x = T_{xy} \le n\}$*).*

Definition 8.7. ${}_{n}q_{x\overset{1}{y}} = \mathbb{P}\{T_y < T_x, T_y \le n\}$ *(*$= \mathbb{P}\{T_y = T_{xy} \le n\}$*).*

Definition 8.8. ${}_{\infty}q_{\overset{1}{x}y} = \mathbb{P}\{T_x < T_y\}$.

Definition 8.9. ${}_{\infty}q_{x\overset{1}{y}} = \mathbb{P}\{T_y < T_x\}$.

Theorem 8.21. *If* $\mathbb{P}\{T_x = T_y \le n\} = 0$, ${}_{n}q_{\overset{1}{x}y} + {}_{n}q_{x\overset{1}{y}} = {}_{n}q_{xy}$.

Proof.
$$
\begin{aligned}
{}_{n}q_{\overset{1}{x}y} + {}_{n}q_{x\overset{1}{y}} &= \mathbb{P}\{T_x < T_y, T_x \le n\} + \mathbb{P}\{T_y < T_x, T_y \le n\} \\
&= \mathbb{P}\{T_x < T_y, (T_x \wedge T_y) \le n\} + \mathbb{P}\{T_y < T_x, (T_x \wedge T_y) \le n\} \qquad (+\mathbb{P}\{T_x = T_y \le n\} \ \ (=0)) \\
&= \mathbb{P}\{T_x < T_y, (T_x \wedge T_y) \le n\} + \mathbb{P}\{T_y < T_x, (T_x \wedge T_y) \le n\} + \mathbb{P}\{T_y = T_x, (T_x \wedge T_y) \le n\} \\
&= \mathbb{P}\{T_x \wedge T_y \le n\} = \mathbb{P}\{T_{xy} \le n\} = {}_{n}q_{xy}.
\end{aligned}
$$

Example 8.19. *An engineer has estimated the lifetimes of the engines and transmission of a new car. He estimates that the lifetime of the engine has constant mortality force of* 0.05. *The lifetime of the transmission has constant mortality force of* 0.08. *The lifetimes of the engine and transmission are independent random variables.*
(i) probability that the transmission fails before the engine and within 10 years=?
(ii) probability that the transmission fails before the engine=?

Solution: (i) Let T_x be the lifetime of the engine of a new car. $\mu_x = 0.05 = 1/E(T_x)$
Let T_y be the lifetime of the transmission of a new car. $\mu_y = 0.08$ and $E(T_y) < E(T_x) = \frac{1}{\mu_x}$.

$$
\begin{aligned}
{}_{n}q_{x\overset{1}{y}} &= \int_0^n \int_s^\infty f_{T_x,T_y}(t,s)\,dt\,ds = \int_0^n \int_s^\infty \underbrace{f_{T_y}(s) f_{T_x}(t)}_{\textbf{why}??}\,dt\,ds = \int_0^n f_{T_y}(s) \int_s^\infty f_{T_x}(t)\,dt\,ds \\
&= \int_0^n f_{T_y}(s) S_{T_x}(s)\,ds = \int_0^n e^{-0.08t}(0.08)e^{-0.05t}\,dt = \int_0^n (0.08)e^{-0.13t}\,dt = \frac{0.08}{0.13}\left(1 - e^{-0.13n}\right)
\end{aligned}
$$
$$
\begin{aligned}
{}_{10}q_{x\overset{1}{y}} &= \frac{0.08}{0.13}\left(1 - e^{-1.3}\right) = 0.4476727. \\
{}_{\infty}q_{x\overset{1}{y}} &= \frac{0.08}{0.13} = 0.6153846.
\end{aligned}
$$

Theorem 8.22. *If* $\mathbb{P}\{T_x = T_y\} = 0$, ${}_{\infty}q_{\overset{1}{x}y} + {}_{\infty}q_{x\overset{1}{y}} = 1$.

Definition 8.10. ${}_{n}q_{\overset{2}{x}y} = \mathbb{P}\{T_y < T_x \le n\}$.

${}_{n}q_{\overset{2}{x}y} = \mathbb{P}\{T_y < T_x, T_x \le n\}$.
${}_{n}q_{\overset{1}{x}y} = \mathbb{P}\{T_y > T_x, T_x \le n\}$.

Definition 8.11. ${}_nq_{x\overset{2}{y}} = \mathbb{P}\{T_x < T_y \le n\}$.

Definition 8.12. ${}_\infty q_{\overset{2}{x}y} = \mathbb{P}\{T_y < T_x\}$.

Definition 8.13. ${}_\infty q_{x\overset{2}{y}} = \mathbb{P}\{T_x < T_y\}$.

Example 8.20. *Jacob is 40 years old. Emily is 35 years old.*
(a) Jacob and Emily future lifetimes are independent random variables.
(b) Jacob's lifetime follows De Moivre's model with terminal age 80 years.
(c) Emily's lifetime follows De Moivre's model with terminal age 90 years.
(i) the probability that Jacob dies after Emily and within 25 years=?
(ii) the probability that Jacob dies after Emily=?

Solution: Let T_{40} and T_{35} be Jacob's and Emily's future lifetime.

$$
\begin{aligned}
{}_nq_{\overset{2}{x}y} =& \mathbb{P}(T_y < T_x \le n) = \int\int I(0<s<t<n) f_{T_x,T_y}(t,s)\,ds\,dt = \int_0^n \int_0^t \underbrace{f_{T_x}(t) f_{T_y}(s)}_{\textbf{why}?}\,ds\,dt \\
=& \int_0^n f_{T_x}(t) F_{T_y}(t)\,dt = \int_0^n \frac{1}{40}\frac{t}{55}\,dt = \frac{(n)^2}{(2)(40)(55)}, \quad n \le 40. \\
{}_{25}q_{\overset{2}{x}y} =& \frac{(25)^2}{(2)(40)(55)} = 0.1420455 \text{ and } {}_\infty q_{\overset{2}{x}y} = \begin{cases} \frac{(\infty)^2}{(2)(40)(55)} & ? \\ \frac{(40)^2}{(2)(40)(55)} & ? \end{cases}
\end{aligned}
$$

Theorem 8.23. *If* $\mathbb{P}\{T_x = T_y\} = 0$, ${}_nq_{\overset{1}{x}y} + {}_nq_{\overset{2}{x}y} = {}_nq_x$ *and* ${}_nq_{x\overset{1}{y}} + {}_nq_{x\overset{2}{y}} = {}_nq_y$.

Proof. ${}_nq_{\overset{1}{x}y} + {}_nq_{\overset{2}{x}y} = \mathbb{P}\{T_x < T_y, T_x \le n\} + \mathbb{P}\{T_y < T_x \le n\}$
$= \mathbb{P}\{T_x < T_y, T_x \le n\} + \mathbb{P}\{T_x > T_y, T_x \le n\} + \mathbb{P}\{T_x = T_y, T_x \le n\} = \mathbb{P}\{T_x \le n\} = {}_nq_x$. ■

Theorem 8.24. *If* $\mathbb{P}\{T_x = T_y\} = 0$, ${}_nq_{\overset{2}{x}y} + {}_nq_{x\overset{2}{y}} = {}_nq_{\overline{xy}}$.

Proof. ${}_nq_{\overset{2}{x}y} + {}_nq_{x\overset{2}{y}} = \mathbb{P}\{T_y < T_x, T_x \le n\} + \mathbb{P}\{T_x < T_y, T_y \le n\}$

$=\mathbb{P}\{T_y < T_x, \max(T_x,T_y) \le n\} + \mathbb{P}\{T_x < T_y, \max(T_x,T_y) \le n\} + \mathbb{P}\{T_x = T_y, \max(T_x,T_y) \le n\}$
$=\mathbb{P}\{\max(T_x,T_y) \le n\} = \mathbb{P}\{T_{\overline{xy}} \le n\} = {}_nq_{\overline{xy}}$.

Theorem 8.25. *If* $\mathbb{P}\{T_x = T_y\} = 0$,

$$
{}_nq_{x\overset{2}{y}} + \mathbb{P}\{T_x < n < T_y\} = {}_nq_{\overset{1}{x}y} \text{ and } {}_nq_{\overset{2}{x}y} + \mathbb{P}\{T_y < n < T_x\} = {}_nq_{x\overset{1}{y}}.
$$

Proof. ${}_nq_{x\overset{2}{y}} + \mathbb{P}\{T_x < n < T_y\} = \mathbb{P}\{T_x < T_y \le n\} + \mathbb{P}\{T_x < n < T_y\}$
$= \mathbb{P}\{T_x < T_y, T_x < n\} = \mathbb{P}\{T_x < T_y, T_x \le n\} = {}_nq_{\overset{1}{x}y}$. **Why ?**

Theorem 8.26. *If* $\mathbb{P}\{T_x = T_y\} = 0$, ${}_\infty q_{\overset{2}{x}y} + {}_\infty q_{x\overset{2}{y}} = 1$.

Proof. ${}_\infty q_{\overset{2}{x}y} + {}_\infty q_{x\overset{2}{y}} = \mathbb{P}\{T_y < T_x\} + \mathbb{P}\{T_x < T_y\} = 1$. ■

8.5 Common shock model.

A model for dependence is the **common shock model**. This model for two lives assumes:
(1) there are three independent r.v.'s T_x^*, T_y^* and S,
(2) S has an exponential distribution with hazard rate function λ,
(3) $T_x = \min(T_x^*, S)$ and $T_y = \min(T_y^*, S)$.
T_x^* and T_y^* model a force of mortality happening for each live independent of the other one. S models an event which can cause the simultaneous death of both lives such as an accident. The r.v. S is called the **common shock**. Let
${}_tp_x^*$ be the survival function of T_x^*; μ_{x+t}^* be the force of mortality of T_x^*;
${}_tp_y^*$ be the survival function of T_y^*; μ_{y+t}^* be the force of mortality of T_y^*;
${}_tp_x$ be the survival function of T_x; μ_{x+t} be the force of mortality of T_x;
${}_tp_y$ be the survival function of T_y; μ_{y+t} be the force of mortality of T_y.
The common shock model is determined by the "parameter" $(\lambda, {}_tp_x^*, {}_tp_y^*)$.

Example 8.21. *Suppose that $T_x^* \overset{d}{\sim} \mathrm{Exp}(\mu_1)$ and $T_y^* \overset{d}{\sim} \mathrm{Exp}(\mu_2)$, where $\mu_1, \mu_2 > 0$. If λ, μ_1 and μ_2 are given, compute $\mathbb{P}\{T_x = T_y\}$.*

Solution: $T_x = T_x^* \wedge S$ and $T_y = T_y^* \wedge S$.
$\mathbb{P}\{T_x = T_y\} = \mathbb{P}\{T_x^* \wedge S = T_y^* \wedge S\}$
$= \mathbb{P}\{T_x^* \wedge S = T_y^* \wedge S \geq S\} + \mathbb{P}\{T_x^* \wedge S = T_y^* \wedge S < S\}$
$= \mathbb{P}\{T_x^* \geq S, T_y^* \geq S\} + \mathbb{P}\{T_y^* < S, T_x^* < S, T_x^* = T_y^*\}$ *why ?*
$= \mathbb{P}\{T_x^* \geq S, T_y^* \geq S\}$ *why ?* $= \mathbb{P}\{T_x^* \geq S\}\mathbb{P}\{T_y^* \geq S\}$?
$= E(\mathbf{1}\{T_x^* \geq S, T_y^* \geq S\})$ $E(X) = p$ if $X \sim bin(1,p)$
$= E(E(\mathbf{1}\{T_x^* \geq S, T_y^* \geq S\}|S))$ $\int E(\mathbf{1}\{T_x^* \geq w, T_y^* \geq w\}) f_S(w)dw$
$= E(\mathbb{P}\{T_x^* \geq S, T_y^* \geq S|S\}) = E(\mathbb{P}\{T_x^* \geq S\}P\{T_y^* \geq S|S\})$ *why?*
$= E(e^{-\mu_1 S}e^{-\mu_2 S}|S\} = \int_0^\infty \lambda e^{-\lambda t}e^{-\mu_1 t}e^{-\mu_2 t}\,dt = \frac{\lambda}{\mu_1+\mu_2+\lambda}$.

If X is continuous, then $P(X = a) = 0\ \forall\ a$. If T_x and T_y have a jointly continuous distribution, then $\mathbb{P}\{T_x = T_y\} = 0$. The joint distribution of T_x and T_y in $\mathbb{R}^2$ is mixed. It has two parts: a continuous distribution on $\{(s,t) : s,t > 0, s \neq t\}$, and a distribution on the half line $\{(s,t) : s,t > 0, s = t\}$. **(Length of a point =0=area of a curve).**

Theorem 8.27. *The joint survival function of T_x and T_y is given by*

$$S_{T_x,T_y}(s,t) = {}_sp_x^* \cdot {}_tp_y^* e^{-\lambda \max(s,t)}, \quad s,t \geq 0.$$

Proof. $S_{(T_x,T_y)}(s,t) = \mathbb{P}\{T_x > s, T_y > t\} = \mathbb{P}\{(T_x^* \wedge S) > s, (T_y^* \wedge S) > t\}$
$= \mathbb{P}\{T_x^* > s, S > s, T_y^* > t, S > t\}$
$= \mathbb{P}\{T_x^* > s, T_y^* > t, S > \max(s,t)\} = {}_sp_x^* \cdot {}_tp_y^* e^{-\lambda \max(s,t)}$, $s,t \geq 0$. ∎

Theorem 8.28. *The density of the continuous part of the density of (T_x, T_y) is*
$$f_{(T_x,T_y)}(s,t) = \begin{cases} \left(f_{T_x^*}(s) + {}_sp_x^*\lambda\right) f_{T_y^*}(t)e^{-\lambda s} & \text{if } 0 < t < s, \\ f_{T_x^*}(s)\left(f_{T_y^*}(t) + {}_tp_y^*\lambda\right) e^{-\lambda t} & \text{if } 0 < s < t, \end{cases}$$
The "density" of (T_x, T_y) on $\{(t,t) : t > 0\}$ is $f_{T_x,T_y}(t,t) = \lambda e^{-\lambda t}{}_tp_x^* \cdot {}_tp_y^*$.

Skip next two pages.

Recall $f_X(s) = \frac{\partial}{\partial s} F_X(s) = -\frac{\partial}{\partial s} S_X(s)$ if X is continuous.

If (X,Y) is cts, then $f_{X,Y}(s,t) = \frac{\partial^2}{\partial s \partial t} F_{X,Y}(s,t) = \frac{\partial^2}{\partial s \partial t} S_{X,Y}(s,t)$ and $P((X,Y) \in A) = \int\int I((x,y) \in A) f_{X,Y}(x,y) dx dy$.

$$\frac{\partial^2}{\partial s \partial t} S_{X,Y}(s,t) = \frac{\partial^2}{\partial s \partial t} \int_s^\infty \int_t^\infty f_{X,Y}(x,y) dy dx \qquad \textit{skip}$$

$$\left(\text{note } \frac{\partial}{\partial t} \int_{a(t)}^{b(t)} g(x,t) dx = g(b(t),t)b'(t) - g(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} g(x,t) dx\right)$$

$$= \frac{\partial}{\partial s} \frac{\partial}{\partial t} \int_s^\infty \overbrace{\int_t^\infty f_{X,Y}(x,y) dy}^{=g(x,t)} dx$$

$$= \frac{\partial}{\partial s} \int_s^\infty \frac{\partial}{\partial t} \int_t^\infty f_{X,Y}(x,y) dy dx$$

$$= \frac{\partial}{\partial s} \int_s^\infty -f_{X,Y}(x,t) dx$$

$$= f_{X,Y}(s,t) \text{ if } (X,Y) \text{ is cts.}$$

Is (T_x, T_y) **cts ? or** $P(T_x = T_y) = 0$ **?**

$\mathbb{P}\{T_x = T_y \le t\}$
$= \mathbb{P}\{T_x \le t, T_x^* \wedge S = T_y^* \wedge S \ge S\} + \mathbb{P}\{T_x \le t, T_x^* \wedge S = T_y^* \wedge S < S\}$
$= \mathbb{P}\{T_x \le t, T_x^* \ge S, T_y^* \ge S\} + \mathbb{P}\{T_x \le t, T_y^* < S, T_x^* < S, T_x^* = T_y^*\}$
$= \mathbb{P}\{S \le t, T_x^* \ge S, T_y^* \ge S\}$ $\qquad (P(T_x^* = T_y^*) = 0$ as T_x^* and T_y^* are cts)
$= E(\mathbf{1}\{S \le t, T_x^* \ge S, T_y^* \ge S\})$
$= E(E(\mathbf{1}\{S \le t, T_x^* \ge S, T_y^* \ge S\} | S))$
$= E(\mathbb{P}\{S \le t, T_x^* \ge S, T_y^* \ge S | S\})$
$= \int_0^\infty (\mathbb{P}\{s \le t, T_x^* \ge s, T_y^* \ge s | S = s\}) f_S(s) ds$
$= \int_0^\infty [\mathbf{1}(s \le t) {}_s p_x^* \cdot {}_s p_y^*] \lambda e^{-\lambda s} \, ds$
$= \int_0^t {}_s p_x^* \cdot {}_s p_y^* \lambda e^{-\lambda s} \, ds$, or

$$\mathbb{P}\{T_x = T_y \le t\} = \mathbb{P}\{T_x^*, T_y^* \ge S, S \le t\} = \int_0^t \int_u^\infty \int_u^\infty f_{T_x^*}(s) f_{T_y^*}(t) \lambda e^{-\lambda u} \, ds \, dt \, du$$

$$= \int_0^t {}_u p_x^* \cdot {}_u p_y^* \lambda e^{-\lambda u} \, du > 0 \text{ if } t > 0.$$

$$\frac{\partial}{\partial t} \mathbb{P}\{T_x = T_y \le t\} = {}_u p_x^* \cdot {}_u p_y^* \lambda e^{-\lambda u} = f_{T_x, T_y}(t,t) \text{ if } t > 0.$$

$$\frac{\partial^2}{\partial s \partial t} S_{T_x, T_y}(s,t) = \frac{\partial^2}{\partial s \partial t} {}_s p_x^* \cdot {}_t p_y^* e^{-\lambda t} \qquad (\text{if } 0 < s < t)$$

$$= \frac{\partial}{\partial s} {}_s p_x^* \cdot (-f_{T_y^*}(t) e^{-\lambda t} - \lambda {}_t p_y^* e^{-\lambda t})$$

$$= -f_{T_x^*}(t) \cdot (-f_{T_y^*}(t) e^{-\lambda t} - \lambda {}_t p_y^* e^{-\lambda t}) = f_{T_x^*}(t) \cdot (f_{T_y^*}(t) + \lambda {}_t p_y^*) e^{-\lambda t}$$

If $0 < t < s$, then $f_{T_x, T_y}(s.t) = \frac{\partial^2}{\partial s \partial t} S_{T_x, T_y}(s,t) = f_{T_y^*}(t) \cdot (f_{T_x^*}(t) + \lambda {}_t p_x^*) e^{-\lambda t}$.

Another proof: For a set $A \subset \{(s,t) : 0 < s < t\}$,

$$\begin{aligned}
&\mathbb{P}\{(T_x, T_y) \in A\} = \mathbb{P}\{(T_x^* \wedge S, T_y^* \wedge S) \in A\} \quad T_y^* < S \text{ or } T_y^* \geq S \\
=&\mathbb{P}\{(T_x^*, T_y^*) \in A, T_x^* < T_y^* < S\} + \mathbb{P}\{(T_x^*, S) \in A, T_x^* < S \leq T_y^*\} \\
=&\int_A \int_t^\infty f_{T_x^*}(s) f_{T_y^*}(t) \lambda e^{-\lambda u}\, du\, dt\, ds + \int_A \int_u^\infty f_{T_x^*}(s) f_{T_y^*}(t) \lambda e^{-\lambda u}\, dt\, du\, ds \\
=&\int_A f_{T_x^*}(s) f_{T_y^*}(t) e^{-\lambda t}\, dt\, ds + \int_A f_{T_x^*}(s) {}_u p_y^* \lambda e^{-\lambda u}\, du\, ds \\
=&\int_A f_{T_x^*}(s) f_{T_y^*}(t) e^{-\lambda t}\, dt\, ds + \int_A f_{T_x^*}(s) {}_t p_y^* \lambda e^{-\lambda t}\, dt\, ds = \int_A f_{T_x^*}(s) \left(f_{T_y^*}(t) + {}_t p_y^* \lambda \right) e^{-\lambda t}\, dt\, ds.
\end{aligned}$$

Hence, the density of (T_x, T_y) on $\{(s,t) : 0 < s < t\}$ is $f_{T_x^*}(s) \left(f_{T_y^*}(t) + {}_t p_y^* \lambda \right) e^{-\lambda t}$. A similar argument gives that $(T_x, T_y$ on $\{(s,t) : 0 < t < s\}$ is $\left(f_{T_x^*}(s) + {}_s p_x^* \lambda \right) f_{T_y^*}(t) e^{-\lambda t}$.

For a set $A \subset \{(t,t) : t > 0\}$, $\{(T_x, T_y) \in A, T_y^* < S\} = \{T_x^* \wedge S = T_y^*, T_y^* < S\}$
$= \{T_x^* \wedge S = T_y^*, T_y^* < S, T_x^* < S\} + \{T_x^* \wedge S = T_y^*, T_y^* < S, T_x^* \geq S\}$
$= \{T_x^* = T_y^*, T_y^* < S, T_x^* < S\} + \{S = T_y^*, T_y^* < S, T_x^* \geq S\}$
$= \{T_x^* = T_y^*, T_y^* < S, T_x^* < S\} = \{T_x^* = T_y^* < S\}$ w.p.0 as they are independent cts random variables.

$\{(T_x, T_y) \in A, T_y^* \geq S\} = \{T_x^* \wedge S = S, T_y^* \geq S\}$
$= \{T_x^* \wedge S = S, T_y^* \geq S, T_x^* < S\} + \{T_x^* \wedge S = S, T_y^* \geq S, T_x^* \geq S\}$
$= \{T_x^* = S, T_y^* \geq S, T_x^* < S\} + \{S = S, T_y^* \geq S, T_x^* \geq S\}$
$= \{S = S, T_y^* \geq S, T_x^* \geq S\} = \{T_y^* \geq S, T_x^* \geq S\}$

$$\begin{aligned}
\mathbb{P}\{(T_x, T_y) \in A\} =& \mathbb{P}\{T_x^*, T_y^* \geq S, S \in A^*\} \\
=& \int_{A^*} \int_u^\infty \int_u^\infty f_{T_x^*}(s) f_{T_y^*}(t) \lambda e^{-\lambda u}\, ds\, dt\, du \\
=& \int_{A^*} \lambda e^{-\lambda u} {}_u p_x^* \cdot {}_u p_y^*\, du \\
=& \int_{A^*} \lambda e^{-\lambda t} {}_t p_x^* \cdot {}_t p_y^*\, dt.
\end{aligned}$$

Hence, the density of (T_x, T_y) on $\{(t,t) : t > 0\}$ is $e^{-\lambda t} {}_t p_x^* \cdot {}_t p_y^*$.

Theorem 8.29. $\mathbb{P}\{T_x = T_y\} = \int_0^\infty {}_t p_x^* \cdot {}_t p_y^* \lambda e^{-\lambda t}\, dt$.

Theorem 8.30. ${}_t p_x = {}_t p_x^* e^{-\lambda t}, t \geq 0$ *and* ${}_t p_y = {}_t p_y^* e^{-\lambda t}, t \geq 0$.

Theorem 8.31. ${}_t p_{xy} = {}_t p_x^* \cdot {}_t p_y^* e^{-\lambda t} = {}_t p_x \cdot {}_t p_y e^{\lambda t}, t \geq 0$.

Theorem 8.32. ${}_t p_{\overline{xy}} = \left({}_t p_x^* + {}_t p_y^* - {}_t p_x^* \cdot {}_t p_y^* \right) e^{-\lambda t} = {}_t p_x + {}_t p_y - {}_t p_x \cdot {}_t p_y e^{\lambda t}, t \geq 0$.

Example 8.22. *Suppose that* $T_x^* \overset{d}{\sim} \mathrm{Exp}(\mu_1)$ *and* $T_y^* \overset{d}{\sim} \mathrm{Exp}(\mu_2)$*, where* $\mu_1, \mu_2 > 0$*. Find the pdf of each of* T_x*,* T_y*,* T_{xy} *and* $T_{\overline{xy}}$*.*

Solution: $T_x = T_x^* \wedge S$ and $T_x^* \perp S$.
$S_{T_x}(t) = S_{T_x^* \wedge S}(t) = S_{T_x^*}(t) S_S(t) = \exp(-\mu_1 t)\exp(-\lambda t) = \exp(-(\mu_1+\lambda)t)$, $t \geq 0$.
$f_{T_x}(t) = (\mu_1+\lambda)e^{-t(\mu_1+\lambda)}, t \geq 0$.
Similarly, we get $f_{T_y}(t) = (\mu_2+\lambda)e^{-t(\mu_2+\lambda)}, t \geq 0$.
${}_tp_{xy} = \mathbb{P}(T_x > t, T_y > t) = \mathbb{P}(T_x^* > t, S > t, T_y^* > t, S > t) = \mathbb{P}(T_x^* > t, S > t, T_y^* > t)$
$= exp(-(\mu_1+\mu_2+\lambda)t)$, $t > 0$,
$f_{T_{xy}}(t) = (\mu_1+\mu_2+\lambda)e^{-t(\mu_1+\mu_2+\lambda)}, t \geq 0$.

$$\begin{aligned}
f_{T_{\overline{xy}}}(t) + f_{T_{xy}}(t) &= f_{T_x}(t) + f_{T_y}(t) \\
f_{T_{\overline{xy}}}(t) &= f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) \\
&= (\mu_1+\lambda)e^{-t(\mu_1+\lambda)} + (\mu_2+\lambda)e^{-t(\mu_2+\lambda)} - (\mu_1+\mu_2+\lambda)e^{-t(\mu_1+\mu_2+\lambda)}, t \geq 0.
\end{aligned}$$

Example 8.23. *An actuary models the future lifetime of a married couple with ages* 65 *and* 50 *as follows: (1) Each individual life have a future lifetime at birth given by De Moivre's model with terminal age 110. (2) A married couple follows a common shock model with hazard* $\mu = 0.003$. *Find the probability that:*
(i) Both (65) *and* (50) *live more than 20 years.*
(ii) At least of one (65) *and* (50) *lives more than 20 years.*

Solution: (i) ${}_tp_{65}^* = \frac{110-65-t}{110-65} = \frac{45-t}{45}$, ?? and ${}_tp_{50}^* = \frac{110-50-t}{110-50} = \frac{60-t}{60}$, ?? .
The probability that both (65) and (50) live more than 20 years is

$$ {}_{20}p_{xy} = {}_{20}p_x^* \cdot {}_{20}p_y^* e^{-\lambda 20} = \frac{45-20}{45}\frac{60-20}{60}e^{-(0.003)(20)} = 0.3488017$$

(ii) $\mathbb{P}$(at least of one (65) and (50) lives more than 20 years)=

$$ {}_{20}p_{\overline{xy}} = {}_{20}p_x + {}_{20}p_y - {}_{20}p_{xy} = [\frac{45-20}{45} + \frac{60-20}{60} - \frac{25*40}{45*60}]e^{-20\lambda} \approx 0.80.$$

8.6 Insurance for multi–life models

8.6.1 Life insurance for multi–life status Whole life, term insurance, deferred insurance, pure endowment and endowment can be defined for the joint–life status and the last–survivor status in the same way it was done for one life.

Whole life:

$Z_{xy} = v^{K_{xy}}$. Its actuarial present value is $A_{xy} = \sum_{k=1}^\infty v^k \mathbb{P}\{K_{xy} = k\} = \sum_{k=1}^\infty v^k {}_{k-1}|q_{xy}$.
$Z_{\overline{xy}} = v^{K_{\overline{xy}}}$. Its actuarial present value is $A_{\overline{xy}} = \sum_{k=1}^\infty v^k \mathbb{P}\{K_{\overline{xy}} = k\} = \sum_{k=1}^\infty v^k {}_{k-1}|q_{\overline{xy}}$.
$\overline{Z}_{xy} = v^{T_{xy}}$. Its actuarial present value is $\overline{A}_{xy} = \int_0^\infty v^t \cdot f_{T_{xy}}(t)\,dt$.
$\overline{Z}_{\overline{xy}} = v^{T_{\overline{xy}}}$. Its actuarial present value is $\overline{A}_{\overline{xy}} = \int_0^\infty v^t \cdot f_{T_{\overline{xy}}}(t)\,dt$.
$A_{\overline{xy}} + A_{xy} = A_x + A_y$.
$\overline{A}_{\overline{xy}} + \overline{A}_{xy} = \overline{A}_x + \overline{A}_y$.
This relation holds for other actuarial present value variables, for example,

$A_{\overline{xy}:\overline{n}|} + A_{xy:\overline{n}|} = A_{x:\overline{n}|} + A_{y:\overline{n}|}$.

Theorem 8.33. $Z_{\overline{xy}}Z_{xy} = Z_xZ_y$ *and* $E[Z_{\overline{xy}}Z_{xy}] = E[Z_xZ_y]$.

Theorem 8.34. $\mathrm{Cov}(Z_{\overline{xy}}, Z_{xy}) = \mathrm{Cov}(Z_x, Z_y) + (A_x - A_{xy})(A_y - A_{xy})$.

Proof. $\mathrm{Cov}(Z_{\overline{xy}}, Z_{xy}) = E[Z_{\overline{xy}}Z_{xy}] - E[Z_{\overline{xy}}]E[Z_{xy}]$
$= E[Z_xZ_y] - E[Z_{\overline{xy}}]E[Z_{xy}]$
$= \mathrm{Cov}(Z_x, Z_y) + E[Z_x]E[Z_y] - E[Z_{\overline{xy}}]E[Z_{xy}]$
$= \mathrm{Cov}(Z_x, Z_y) + A_xA_y - (A_x + A_y - A_{xy})A_{xy}$ $\quad (A_{\overline{xy}:\overline{n}|} + A_{xy:\overline{n}|} = A_{x:\overline{n}|} + A_{y:\overline{n}|})$
$= \mathrm{Cov}(Z_x, Z_y) + A_xA_y - A_xA_{xy} - A_yA_{xy} + A_{xy}A_{xy}$
$= \mathrm{Cov}(Z_x, Z_y) + (A_x - A_{xy})(A_y - A_{xy})$. ∎

Theorem 8.35. $\mathrm{Cov}(\overline{Z}_{\overline{xy}}, \overline{Z}_{xy}) = \mathrm{Cov}(\overline{Z}_x, \overline{Z}_y) + (\overline{A}_x - \overline{A}_{xy})(\overline{A}_y - \overline{A}_{xy})$.

Example 8.24. *Using the life table in the end of the textbook and* $i = 6\%$, *find* $A^1_{(60:70):\overline{3}|}$ *assuming* $T_{60} \perp T_{70}$.

Solution: Formulas: $A^1_{xy:\overline{n}|} = \sum_{k=1}^n v^k f_{K_{xy}}(k)$. $n = 3$.
$f_{K_x}(k) = P(k-1 < K_x \le k) = {}_{k-1}p_x - {}_kp_x$, $\quad f_{K_{xy}}(k) = \ldots$, $\quad {}_kp_{xy} = {}_kp_x \times {}_kp_y$, $\quad {}_kp_x = \frac{\ell_{x+k}}{\ell_x}$.

$$p_{60:70} = \frac{\ell_{61}}{\ell_{60}}\frac{\ell_{71}}{\ell_{70}} = \frac{87203}{88038}\frac{74507}{76191} = 0.9686227418,$$
$${}_2p_{60:70} = \frac{\ell_{62}}{\ell_{60}}\frac{\ell_{72}}{\ell_{70}} = \frac{86291}{88038}\frac{72717}{76191} = 0.9354651519,$$
$${}_3p_{60:70} = \frac{\ell_{63}}{\ell_{60}}\frac{\ell_{73}}{\ell_{70}} = \frac{85304}{88038}\frac{70811}{76191} = 0.9005260537.$$

$$A^1_{(60:70):\overline{3}|} = v({}_0p_{xy} - p_{60:70}) + v^2(p_{60:70} - {}_2p_{60:70}) + v^3({}_2p_{60:70} - {}_3p_{60:70})$$
$$\approx (1.06)^{-1}(1 - 0.9686) + (1.06)^{-2}(0.9686 - 0.9355) + (1.06)^{-3}(0.9355 - 0.9005) \approx 0.1426.$$

Example 8.25. *Suppose that: (i)* $T_x \overset{d}{\sim} \mathrm{Exp}(\mu_1)$ *and* $T_y \overset{d}{\sim} \mathrm{Exp}(\mu_2)$, *where* $\mu_1, \mu_2 > 0$. *(ii)* $T_x \perp T_y$. *If* μ_1, μ_2 *and* δ *are given, compute* $\overline{A}_{xy}$ *and* $\overline{A}_{\overline{xy}}$.

Solution: $\overline{A}_{xy} = \int_0^\infty v^t \cdot \underbrace{f_{T_{xy}}(t)}_{???}\,dt$, $\quad {}_tp_{xy} = {}_tp_x \cdot {}_tp_y = e^{-\mu_1 t}e^{-\mu_2 t} = e^{-\mu t}$??

$$\overline{A}_{xy} = \int_0^\infty v^t \cdot f_{T_{xy}}(t)\,dt = \int_0^\infty e^{-\delta t}(\mu_1+\mu_2)e^{-(\mu_1+\mu_2)t}\,dt = \int_0^\infty (\mu_1+\mu_2)e^{-\mu t}\,dt = \frac{\mu_1+\mu_2}{\mu_1+\mu_2+\delta}.$$

and $\overline{A}_{\overline{xy}} + \overline{A}_{xy} = \overline{A}_x + \overline{A}_y$ yields

$$\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy} = \frac{\mu_1}{\mu_1+\delta} + \frac{\mu_2}{\mu_2+\delta} - \frac{\mu_1+\mu_2}{\mu_1+\mu_2+\delta}.$$

Example 8.26. *A life insurance pays 200000 at the end of the first death of* (x) *and* (y) *and 100000 at the end of the second death of* (x) *and* (y). *Suppose that*

$$A_x = 0.45, A_y = 0.4, A_{xy} = 0.3.$$

Find the APV of this life insurance.

Solution: $E(B_1 v^{K_{xy}} + B_2 v^{K_{\overline{xy}}}) = B_1 A_{xy} + B_2 A_{\overline{xy}} = \ ?\ B_1,\ B_2,\ A_{xy} = \ ?$

$$A_{\overline{xy}} = A_x + A_y - A_{xy} = 0.45 + 0.4 - 0.3 = 0.55.$$

Hence, the APV of this insurance is

$$200000 A_{xy} + 100000 A_{\overline{xy}} = (200000)(0.3) + (100000)(0.55) = 115000.$$

8.6.2 Life annuities for multi–life status The present value of a whole life annuity immediate paid at the end of the year while both (x) and (y) are alive is $Y_{xy} = a_{\overline{K_{xy}|}} = \sum_{k=1}^{K_{xy}} v^k$. The actuarial present values of immediate case, due case and cts case are

$$\begin{cases} a_{xy} = \sum_{k=1}^{\infty} v^k \cdot {}_k p_{xy} & \textit{immediate case} \\ \ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot {}_k p_{xy} & \textit{due case} \\ \overline{a}_{xy} = \int_0^{\infty} v^t \cdot {}_t p_{xy}\, dt & \textit{cts case.} \end{cases}$$

The present value of a whole life annuity immediate paid at the end of the year while at least one of (x) and (y) are alive is $Y_{\overline{xy}} = a_{\overline{K_{\overline{xy}}|}}$, *etc.*

Their actuarial present values are $\begin{cases} a_{xy} = \sum_{k=1}^{\infty} v^k \cdot {}_k p_{\overline{xy}} & \textit{immediate case} \\ \ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot {}_k p_{\overline{xy}} & \textit{due case} \\ \overline{a}_{xy} = \int_0^{\infty} v^t \cdot {}_t p_{\overline{xy}}\, dt & \textit{cts case.} \end{cases}$

$$a_{\overline{xy}} + a_{xy} = a_x + a_y,$$

$$\ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y.$$

$$\overline{a}_{\overline{xy}} + \overline{a}_{xy} = \overline{a}_x + \overline{a}_y.$$

The present value of a n–year temporary life annuity immediate paid at the end of the year while both (x) and (y) are alive is $Y_{xy:\overline{n}|} = a_{\overline{\min(n,K_{xy})}|}$, *etc.*

Their actuarial present values are $\begin{cases} a_{xy:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_k p_{xy} & \textit{immediate case} \\ \ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_k p_{xy} & \textit{due case} \\ \overline{a}_{xy:\overline{n}|} = \int_0^{n} v^t \cdot {}_t p_{xy}\, dt & \textit{continuous case.} \end{cases}$

The present value of a n–year temporary life annuity immediate paid at the end of the year while at least one of (x) and (y) are alive is $Y_{\overline{xy}:\overline{n}|} = a_{\overline{\min(n,K_{\overline{xy}})}|}$.

The actuarial present values are $\begin{cases} a_{\overline{xy}:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_k p_{\overline{xy}} & \textit{immediate case} \\ \ddot{a}_{\overline{xy}:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_k p_{\overline{xy}} & \textit{due case} \\ \overline{a}_{\overline{xy}:\overline{n}|} = \int_0^{n} v^t \cdot {}_t p_{\overline{xy}}\, dt & \textit{continuous case.} \end{cases}$

Other insurance variables are defined in a similar way.

Example 8.27. *A life annuity pays 40000 at the beginning of the year while both (x) and (y) are alive and 30000 at the beginning of the year while exactly one of (x) and (y) is alive. Suppose that $\ddot{a}_x = 8, \ddot{a}_y = 7$ and $\ddot{a}_{xy} = 5$. Find the APV of this annuity.*

Solution: The payment made at the beginning of the $k+1$–th year ($k = 0, 1, ...$) is

$$\begin{cases} 40000 & \text{if } k < T_{xy} \\ 30000 & \text{if } T_{xy} \le k < T_{\overline{xy}} \end{cases} \begin{aligned} &=(40000)I(T_{xy} > k) + (30000)(I(T_{\overline{xy}} > k \ge T_{xy})) \\ &=(40000)I(T_{xy} > k) + (30000)(I(T_{\overline{xy}} > k) - I(T_{xy} > k)) \\ &=(10000)I(T_{xy} > k) + (30000)I(T_{\overline{xy}} > k). \end{aligned}$$

Its present value is $v^k[(10000)I(T_{xy} > k) + (30000)I(T_{\overline{xy}} > k)]$.
The APV of the annuity:

$$\begin{aligned} &\sum_{k=0}^{\infty} v^k[10000P(T_{xy} > k) + 30000P(T_{\overline{xy}} > k)] \\ =&10^4(\sum_{k=0}^{\infty} v^k {}_kp_{xy} + 3\sum_{k=0}^{\infty} v^k {}_kp_{\overline{xy}}) \\ =&(10000)\ddot{a}_{xy} + (30000)\ddot{a}_{\overline{xy}} \qquad\qquad \ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y \\ =&(10000)\ddot{a}_{xy} + (30000)([\ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}]) \\ =&(10000)5 + (30000)(8 + 7 - 5) = 350000. \end{aligned}$$

Example 8.28. *A continuous annuity pays at an annual rate of* 40000 *while both* (x) *and* (y) *are alive and at annual rate of 10000 while exactly one of* (x) *and* (y) *is alive. Suppose that* $\mu_x = 0.05$ *and* $\mu_y = 0.03$. *Assume that the future lifetimes of* (x) *and* (y) *are independent random variables. Find the APV of this annuity if* $\delta = 0.06$.

Solution: The APV of the annuity is

$$\begin{aligned} &10^4(4\int_0^\infty v^t P(T_{xy} > t)dt + \int_0^\infty v^t P(T_{xy} < t \le T_{\overline{xy}})dt) \\ =\ &10^4(4\int_0^\infty v^t P(T_{xy} > t)dt + \int_0^\infty v^t(P(t \le T_{\overline{xy}}) - P(T_{xy} \ge t))dt) \\ =\ &10^4(4\int_0^\infty v^t {}_tp_{xy}dt + \int_0^\infty v^t {}_tp_{\overline{xy}}dt - \int_0^\infty v^t {}_tp_{xy}dt) = 10^4(3\overline{a}_{xy} + \overline{a}_{\overline{xy}}) \end{aligned}$$

$$\begin{aligned} \overline{a}_x &= \int_0^\infty v^t {}_tp_x\, dt = \int_0^\infty {}_tp_x e^{-\delta t}\, dt = \int_0^\infty e^{-(0.05)t} e^{-(0.06)t}\, dt = \frac{1}{0.11}, \\ \overline{a}_y &= \int_0^\infty {}_tp_y e^{-\delta t}\, dt = \int_0^\infty e^{-(0.03)t} e^{-(0.06)t}\, dt = \frac{1}{0.09}, \\ \overline{a}_{xy} &= \int_0^\infty {}_tp_{xy} e^{-\delta t}\, dt = \int_0^\infty {}_tp_x \cdot {}_tp_y e^{-\delta t}\, dt = \int_0^\infty e^{-(0.08)t} e^{-(0.06)t}\, dt = \frac{1}{0.14}. \end{aligned}$$

$\ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y$. Hence, the APV of this annuity is

$$10^4(4\overline{a}_{xy} + (\overline{a}_{\overline{xy}} - \overline{a}_{xy})) = 10^4(3\overline{a}_{xy} + (\overline{a}_x + \overline{a}_y - \overline{a}_{xy})) = 344877.3.$$

Why there is independent assumption in Ex.8.28, but not in Ex.8.27 ?

8.6.3 Benefit premiums for multi–life status For the multi–life status, benefit premium are defined in exactly the same way as for one life. The annual benefit premium for a whole life insurance paid at the end of the year of the first death is given by $P_{xy} = \frac{A_{xy}}{\ddot{a}_{xy}}$.

The annual benefit premium for a whole life insurance paid at the end of the year of the second death is given by $P_{\overline{xy}} = \frac{A_{\overline{xy}}}{\ddot{a}_{\overline{xy}}}$.

Example 8.29. *A whole life insurance pays 75000 at the end of the year of the second death of (x) and (y). Suppose that $A_{\overline{xy}} = 0.85$ and $i = 7\%$. This life insurance is funded by level premiums made at the beginning of the year while at least one of (x) and (y) is alive. Find the amount of each of these premiums if the equivalence principle is used.*

Solution: Find $P_{\overline{xy}} = \frac{A_{\overline{xy}}}{\ddot{a}_{\overline{xy}}} = ?$ Or $75000P_{\overline{xy}}= ?$ where

$\ddot{a}_{\overline{xy}} = \frac{1-A_{\overline{xy}}}{d}$ and $d = 1 - v$ $(v = 1/(1-i))$.

Each premium is $(75000)P_{\overline{xy}} = (75000)\frac{A_{\overline{xy}}}{\frac{1-A_{\overline{xy}}}{d}} = (75000)\frac{0.85}{\frac{1-0.85}{0.07/1.07}} = 27803.73832$.

8.6.4 Benefit reserves for multi–life status

The benefit reserve at the end of the t–th year before the premium is paid for a whole life insurance paid at the end of the year of the first death is

$$ {}_tV_{xy} = A_{x+t:y+t} - P_{xy}\ddot{a}_{x+t:y+t}. $$

For a whole life insurance paid at the end of the year of the second death, the benefit reserve at the end of the t–th year depends on which life has survived.

$$ {}_tV_{\overline{xy}} = \begin{cases} A_{\overline{x+t:y+t}} - P_{\overline{xy}}\ddot{a}_{\overline{x+t:y+t}} & \text{if both } (x) \text{ and } (y) \text{ have survived at the } t\text{–the year,} \\ A_{x+t} - P_{\overline{xy}}\ddot{a}_{x+t} & \text{if only } (x) \text{ has survived at the } t\text{–th year,} \\ A_{y+t} - P_{\overline{xy}}\ddot{a}_{y+t} & \text{if only } (y) \text{ has survived at the } t\text{–th year.} \end{cases} . $$

8.6.5 Reversionary annuity. A **reversionary annuity** pays only after a determined life has died while the other continuous to survive. Suppose that an unity payment is made to live (y) while it is alive provided that (x) has died. The present value and the APV of this annuity paid at the end of the year is denoted by $Y_{x|y}$ and $a_{x|y}$, respectively.

$$ Y_{x|y} = \sum_{k=K_x}^{K_y-1} v^k = \sum_{k=1}^{\infty} v^k I(K_x \le k < T_y). \qquad (Y_{x|y} = 0 \; if \; 0 < T_x < T_y \le 1) $$

$$ \begin{aligned} \mathbb{P}\{K_x \le k, T_y > k\} = \mathbb{P}\{T_x \le k, T_y > k\} =& \mathbb{P}\{T_y > k\} - \mathbb{P}\{T_x > k, T_y > k\} \\ =& \mathbb{P}\{T_y > k\} - \mathbb{P}\{T_{xy} > k\} = {}_kp_y - {}_kp_{xy}. \end{aligned} $$

$$ a_{x|y} = \sum_{k=1}^{\infty} v^k E(I(K_x \le k < T_y)) = \sum_{k=1}^{\infty} v^k({}_kp_y - {}_kp_{xy}) = a_y - a_{xy}. $$

$$ a_y = a_{x|y} + a_{xy}. $$

A n–year temporary reversionary annuity paid to (y) conditionally to death of (x) limits payments to the first n years. The actuarial present value of this annuity is

$$ a_{x|y:\overline{n}|} = \sum_{k=1}^{n} v^k({}_kp_y - {}_kp_{xy}) = a_{y:\overline{n}|} - a_{xy:\overline{n}|}. $$

A cts reversionary annuity paid to (y) conditionally to death of (x) will make payments at constant rate if (x) dies before (y) from the time of the death of (x) to the time of the death of (y). The actuarial present value of a continuous reversionary annuity with unit rate is

$$\begin{aligned}\overline{a}_{x|y} =& E(\int_{T_x}^{T_y} v^t\,dt) = E(\int I(T_x \le t < T_y) v^t\,dt)\\ =& \int E(I(T_x \le t < T_y))v^t\,dt\\ =& \int_0^\infty \mathbb{P}\{(T_{xy} \le t \le T_y\} v^t\,dt = \int_0^\infty v^t({}_tp_y - {}_tp_{xy})\,dt = \overline{a}_y - \overline{a}_{xy}.\end{aligned}$$

Example 8.30. *The rate of a continuous life annuity to (x) and (y) is:*
(i) 25000 while both (x) and (y) are alive.
(ii) 10000 while (x) is alive and (y) is death.
(iii) 15000 while (x) is death and (y) is alive.
Suppose that $\overline{a}_x = 7$, $\overline{a}_y = 8$ and $\overline{a}_{xy} = 5$. Find the APV of this annuity.

Solution: The APV of this annuity is

$$\begin{aligned}&25000\int_0^\infty v^t P(t < T_{xy})dt + 10000\int_0^\infty v^t P(T_{xy} \le t \le T_x)dt + 15000\int_0^\infty v^t P(T_{xy} \le t \le T_y)dt\\ =&(25000)\overline{a}_{xy} + (10000)(\overline{a}_x - \overline{a}_{xy}) + (15000)(\overline{a}_y - \overline{a}_{xy})\\ =&(25000)(5) + (10000)(7-5) + (15000)(8-5) = 190000.\end{aligned}$$

Theorem 8.36.

8.6.6 Contingent insurance. A contingent insurance paid at the death of (x) only if (x) dies before (y) has present value $v^{T_x} I(T_x < T_y)$. The actuarial present value of this insurance is $\overline{A}^{\,1}_{xy} = E[v^{T_x} I(T_x < T_y)]$. (Compare to $\overline{A}^{\,1}_{xy:\overline{n}|}$).
Similarly, define $\overline{A}_{x\overset{1}{y}} = E[v^{T_y} I(T_x > T_y)]$.

Example 8.31. *Suppose that:*
(i) $T_x \overset{d}{\sim} \mathrm{Exp}(\mu_1)$ and $T_y \overset{d}{\sim} \mathrm{Exp}(\mu_2)$, where $\mu_1, \mu_2 > 0$.
(ii) T_x and T_y are independent random variables.
Assume μ_1, μ_2 and δ are given constants, compute $\overline{A}^{\,1}_{xy}$

Solution: $\overline{A}^{\,1}_{xy} = E[v^{T_x} I(T_x < T_y)] = \int\int [v^t I(t < s) f_{T_x}(t) f_{T_y}(s) dt ds$

$$\begin{aligned}&= \int_0^\infty \int_t^\infty v^t f_{T_x}(t) f_{T_y}(s)\,ds\,dt = \int_0^\infty v^t f_{T_x}(t) S_{T_y}(t)\,dt\\ &= \int_0^\infty e^{-\delta t}\mu_1 e^{-\mu_1 t} e^{-\mu_2 t}\,dt = \frac{\mu_1}{\mu}\int_0^\infty \mu e^{-\mu t} dt = \frac{\mu_1}{\mu_1+\mu_2+\delta}.\end{aligned}$$

Example 8.32. *Daniel is 40 years old. Isabella is 35 years old.*
(i) Daniel and Isabella future lifetime time are independent random variables.

(ii) Daniel's lifetime follows De Moivre model with terminal age 85 years.
(iii) Isabella's lifetime follows De Moivre model with terminal age 90 years.
A life insurance has death benefit of 50000 payable at the Isabella's time of death, if Isabella dies before Daniel. Find the APV of this life insurance if $\delta = 0.065$.

Solution: Let $(x) = (40)$ and $(y) = (35)$. $f_{(x)}(t) = \frac{I(0<t<45)}{85-x}$, $f_{(y)}(t) = \frac{I(0<t<55)}{90-y}$.

$$\begin{aligned}
\overline{A}_{x\overset{1}{y}} =& E(v^{T_y} I(T_x > T_y)) = E(E(v^{T_y} I(T_x > T_y)|T_y)) = E(v^{T_y} S_{T_x}(T_y)) = \int_0^\infty v^t S_{T_x}(t) f_{T_y}(t)dt \\
=& \int_0^{45} e^{-0.065t} \frac{45-t}{45}\frac{1}{55}\, dt = \int_0^{45} e^{-0.065t}\left(\frac{1}{55} - \frac{t}{(45)(55)}\right) dt \\
=& \int_0^{(0.065)(45)} e^{-x}\left(\frac{1}{(0.065)(55)} - \frac{x}{(0.065)^2(45)(55)}\right) dx \quad (x = 0.065t) \\
=& \frac{1}{(0.065)(55)} \int_0^{(0.065)(45)} e^{-x}dx - \frac{1}{(0.065)^2(45)(55)} \int_0^{(0.065)(45)} xe^{-x}\, dx \qquad xe^{-x}dx = -xde^{-x} \\
=& \frac{1}{(0.065)(55)}(-e^{-x})\Big|_0^{(0.065)45} + \frac{1}{(0.065)^2(45)(55)}(e^{-x}x + e^{-x})\Big|_0^{(0.065)45} \\
=& 0.3401969384.
\end{aligned}$$

The APV of this life insurance is $(50000)(0.3401969384) = 17009.84692$.

A contingent insurance paid at the death of (x) only if (x) dies after (y) has present value $v^{T_x} I(T_x > T_y)$. The actuarial present value of this insurance is

$$\overline{A}^{2}_{xy} = E[v^{T_x} I(T_x > T_y)].$$

Similarly, define $\overline{A}_{x\overset{2}{y}} = E[v^{T_y} I(T_x < T_y)]$.

It is easy to see that:

Theorem 8.37. *If* $\mathbb{P}\{T_x = T_y\} = 0$, *then* $\overline{A}^1_{xy} + \overline{A}^2_{xy} = \overline{A}_x$ *and* $\overline{A}_{x\overset{1}{y}} + \overline{A}_{x\overset{2}{y}} = \overline{A}_y$.

Theorem 8.38.

Theorem 8.39. *If* $\mathbb{P}\{T_x = T_y\} = 0$, *then* $\overline{A}_{xy} = \overline{A}^1_{xy} + \overline{A}_{x\overset{1}{y}}$ *and* $\overline{A}_{\overline{xy}} = \overline{A}^2_{xy} + \overline{A}_{x\overset{2}{y}}$.

Proof. We have that

$$\begin{aligned}
&\overline{A}^1_{xy} + \overline{A}_{x\overset{1}{y}} = E[v^{T_x} I(T_x < T_y)] + E[v^{T_y} I(T_x > T_y)] \\
=& E[v^{T_{xy}} I(T_x < T_y)] + E[v^{T_{xy}} I(T_x > T_y)] = E[v^{T_{xy}}] = \overline{A}_{xy}.
\end{aligned}$$

and

$$\begin{aligned}
&\overline{A}^2_{xy} + \overline{A}_{x\overset{2}{y}} = E[v^{T_x} I(T_x > T_y)] + E[v^{T_y} I(T_x < T_y)] \\
=& E[v^{T_{\overline{xy}}} I(T_x > T_y)] + E[v^{T_{\overline{xy}}} I(T_x < T_y)] = E[v^{T_{\overline{xy}}}] = \overline{A}_{\overline{xy}}.
\end{aligned}$$

■

CHAPTER 9
Multiple Decrement Models

In some situations, we are interested in the cause of the removal of an individual from the survivorship group. This cause can be withdrawal, or dispear, or other cause. We also may be interested in the cause of the death. Insurance policies could be made different payments based upon the cause of the death. We need to determine probabilities which depend on the cause that an individual is removed from the survivorship group. Each cause which makes individual to leave a survivorship group is a called a **decrement**. So far, we have studied **single–decrement models**. In this chapter, we consider **multiple–decrement models** (or **competing risks model**), i.e. the individuals in the survivorship group are subject to removal from the group due to several decrements.

The competing risks model or multiple-decrement model assumes:

1. $T_x^{(1)}, ..., T_x^{(m)}$ are independent and cts random variables, with cdf's $F_{T_x^{(j)}}(t)$,
   survival function $S_{T_x^{(j)}}(t)$ (denoted by ${}_tq_x^{'(j)}$ and ${}_tp_x^{'(j)}$, respectively),
   density $f_{T_x^{(j)}}(t)$, force of mortality $\mu_x^{'(j)}(t) = \mu_{x+t}^{'(j)} = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)}$, etc. for $j = 1, ..., m$.
2. $T_x = \min\{T_x^{(1)}, ..., T_x^{(m)}\}$ and $\{J_x = j\} = \{T_x = T_x^{(j)}\}$.
3. Observe (T_x, J_x) (time and main cause of death) but not $T_x^{(1)}, ..., T_x^{(m)}$.

In general, given a cdf F of a cts r.v.,
$S(t) = 1 - F(t)$, $\mu(t) = \frac{f(t)}{S(t)} = -\frac{d}{dt}\ln S(t)$,
$f(t) = F'(t) = \mu(t)S(t)$, $S(t) = exp(-\int_{s\le t}\mu(s)ds)$.
Notations and formulae under the competing risks model:
${}_tq_x^{(\tau)} = F_{T_x}(t)$, ${}_tp_x^{(\tau)} = S_{T_x}(t)$, $\mu_x^{(\tau)}(t)$ $(= \mu_{x+t}^{(\tau)} = \frac{f_{T_x}(t)}{S_{T_x}(t)})$.
${}_tq_x^{(j)} = P(T_x \le t, J_x = j)$ and ${}_tp_x^{(j)} = P(T_x > t, J_x = j)$
$(= P(T_x^{(j)} > t, J_x = j)$ compare to ${}_tp_x^{'(j)} = P(T_x^{(j)} > t))$
$f_{(T_x^{(1)},...,T_x^{(m)})}(t_1, ..., t_m) = \prod_{j=1}^m f_{T_x^{(j)}}(t_j)$,
$f_{(T_x,J_x)}(t, j) = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)} S_{T_x}(t) = \mu_x^{(j)}(t)S_{T_x}(t)$,
$f_{J_x}(j) = \int_0^\infty S_{T_x}(s)\mu_x^{(j)}(s)ds$,
$\mu_x^{(\tau)}(t) = \sum_{j=1}^m \mu_x^{(j)}(t)$,
$\mu_x^{(j)}(t) = \frac{f_{(T_x,J_x)}(t,j)}{S_{T_x}(t)} = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)} = -\frac{d}{dt}\ln S_{T_x^{(j)}}(t)$.

If $m = 1$, it is so-called a **single-decrement model**.

9.1 Deterministic survivorship group

According to the deterministic survivorship group, we are able to observe a cohort of individuals and determine when and why an individual leaves the cohort group. Each reason why an individual leaves the group is called a decrement. Decrements could be different causes of death.

Suppose that there are m decrements. According to the deterministic survivorship group, for each $1 \leq j \leq m$, and each $k = 0, 1, 2, \ldots$, we are able to determine the number of lives who leave the group **during** the k–th year due to decrement j. Equivalently, for each $1 \leq j \leq m$ and each $k = 0, 1, 2, \ldots,$, we are able to determine the number of lives **in** the group at time k who will leave the group due to decrement j.

Definition 9.1. *The total number of lives at time x is denoted by $\ell_x^{(\tau)} = \ell_x$.*

Definition 9.2. *The total number of lives at time x which eventually die due to cause j, $j = 1, 2, \ldots, m$, is denoted by $\ell_x^{(j)}$.*

Definition 9.3. $\ell_0^{(\tau)}$ *$(= \ell_0)$.*

Definition 9.4.

Theorem 9.1. $\ell_x^{(\tau)} = \sum_{j=1}^m \ell_x^{(j)}$ *and* $\ell_0^{(\tau)} = \sum_{j=1}^m \ell_0^{(j)}$.

Theorem 9.2.

Definition 9.5. ${}_td_x^{(\tau)} = {}_td_x$.

Definition 9.6. $d_x^{(\tau)} = {}_1d_x^{(\tau)}$

Theorem 9.3. ${}_td_x^{(\tau)} = \ell_x^{(\tau)} - \ell_{x+t}^{(\tau)}$, $d_x^{(\tau)} = \ell_x^{(\tau)} - \ell_{x+1}^{(\tau)}$ *and* $\ell_x^{(\tau)} = \sum_{k=x}^\infty d_k^{(\tau)}$.

Definition 9.7. *The total number of lives aged (x) which die within t years due to cause j is denoted by ${}_td_x^{(j)}$.*

Theorem 9.4. ${}_td_x^{(j)} = \ell_x^{(j)} - \ell_{x+t}^{(j)}$.

Definition 9.8. $d_x^{(j)} = {}_1d_x^{(j)}$.

Theorem 9.5. $d_x^{(\tau)} = \sum_{j=1}^m d_x^{(j)}$, $d_x^{(j)} = \ell_x^{(j)} - \ell_{x+1}^{(j)}$ *and* $\ell_x^{(j)} = \sum_{k=x}^\infty d_k^{(j)}$.

Definition 9.9.

Example 9.1. *A car company offers a three-year guarantee to new car sales. An actuary models the number of customers in this guarantee as a double decrement models. Decrement 1 is mechanical failure. Decrement 2 is withdrawal. Complete the table*

x	$\ell_x^{(\tau)}$	$\ell_x^{(1)}$	$\ell_x^{(2)}$	$d_x^{(1)}$	$d_x^{(2)}$
0		75	175		
1		70	155		
2		50	75		
3		0	0		

Class exercise.

Solution: Using that $\ell_x^{(\tau)} = \ell_x^{(1)} + \ell_x^{(2)}$, $d_x^{(j)} = \ell_x^{(j)} - \ell_{x+1}^{(j)}$, we get that

x	$\ell_x^{(\tau)}$	$\ell_x^{(1)}$	$\ell_x^{(2)}$	$d_x^{(1)}$	$d_x^{(2)}$
0	250	75	175	5	20
1	225	70	155	20	80
2	125	50	75	50	75
3	0	0	0	0	0

Definition 9.10. ${}_tp_x^{(\tau)} = {}_tp_x = \frac{\ell_{x+t}^{(\tau)}}{\ell_x^{(\tau)}}, \qquad p_x^{(\tau)} = {}_1p_x^{(\tau)} = \frac{\ell_{x+1}^{(\tau)}}{\ell_x^{(\tau)}}$.

Theorem 9.6. ${}_np_x^{(\tau)} = p_x^{(\tau)} p_{x+1}^{(\tau)} \cdots p_{x+n-1}^{(\tau)}$, $n = 0, 1, 2, \dots$

Definition 9.11. ${}_tq_x^{(\tau)} = {}_tq_x, \qquad {}_tq_x^{(\tau)} = 1 - {}_tp_x^{(\tau)} = \frac{\ell_x^{(\tau)} - \ell_{x+t}^{(\tau)}}{\ell_x^{(\tau)}} = \frac{{}_td_x^{(\tau)}}{\ell_x^{(\tau)}}$.

Definition 9.12. $q_x^{(\tau)} = {}_1q_x^{(\tau)}$.

Theorem 9.7.

Definition 9.13. *The proportion of lives aged x who die within t years due to cause j is denoted by* ${}_tq_x^{(j)} = \frac{{}_td_x^{(j)}}{\ell_x^{(\tau)}} = \frac{\ell_x^{(j)} - \ell_{x+t}^{(j)}}{\ell_x^{(\tau)}}$.

Theorem 9.8. ${}_tq_x^{(\tau)} = \sum_{j=1}^m {}_tq_x^{(j)}$.

Definition 9.14. $q_x^{(j)} = {}_1q_x^{(j)}$.

Definition 9.15. ${}_{s|t}q_x^{(\tau)} = {}_{s|t}q_x$

Definition 9.16. ${}_{s|}q_x^{(\tau)} = {}_{s|}q_x$.

Definition 9.17. *The proportion of lives aged x who die within s and $s+t$ years due to cause j is denoted by* ${}_{s|t}q_x^{(j)}$.

$${}_{s|t}q_x^{(\tau)} = \frac{\ell_{x+s}^{(\tau)} - \ell_{x+s+t}^{(\tau)}}{\ell_x^{(\tau)}} = {}_{s+t}q_x^{(\tau)} - {}_sq_x^{(\tau)} = \sum_{j=1}^m {}_{s|t}q_x^{(j)}.$$

Remark: Quiz 452 upto Chapter 9 $\mu_x^{(j)}(t) = \dots$

Theorem 9.9. ${}_{s|t}q_x^{(j)} = {}_sp_x^{(\tau)} \, {}_tq_{x+s}^{(j)}$.

Proof. ${}_sp_x^{(\tau)} \, {}_tq_{x+s}^{(j)} = \frac{\ell_{x+s}^{(\tau)}}{\ell_x^{(\tau)}} \frac{\ell_{x+s}^{(j)} - \ell_{x+s+t}^{(j)}}{\ell_{x+s}^{(\tau)}} = \frac{\ell_{x+s}^{(j)} - \ell_{x+s+t}^{(j)}}{\ell_x^{(\tau)}} = {}_{s|t}q_x^{(j)}$. ■

Definition 9.18. ${}_{s|}q_x^{(j)} = {}_{s|1}q_x^{(j)}$.

Theorem 9.10. *Let n be a positive integer.*
(i) ${}_{n|}q_x^{(j)} = p_x^{(\tau)} p_{x+1}^{(\tau)} \cdots p_{x+n-1}^{(\tau)} q_{x+n}^{(j)}$.
(ii) ${}_{n|}q_x^{(\tau)} = p_x^{(\tau)} p_{x+1}^{(\tau)} \cdots p_{x+n-1}^{(\tau)} q_{x+n}^{(\tau)}$.
(iii) ${}_{n}q_x^{(j)} = \sum_{k=0}^{n-1} {}_{k|}q_x^{(j)}$.
(iv) ${}_{n}q_x^{(\tau)} = \sum_{k=0}^{n-1} {}_{k|}q_x^{(\tau)}$.

Example 9.2. *You are given the following multiple decrement table*

x	$\ell_x^{(1)}$	$\ell_x^{(2)}$
30	10263	8965
31	10128	8896
32	10034	8818
33	9897	8744
34	9764	8687
35	9683	8604

Calculate:
(i) ${}_3p_{31}^{(\tau)}$. **Class exercise.**
(ii) ${}_3q_{30}^{(\tau)}$ **Class exercise..**
(iii) ${}_2q_{31}^{(2)}$.
(iv) ${}_{2|2}q_{31}^{(\tau)}$.
(v) ${}_{2|2}q_{30}^{(1)}$.
(vi) ${}_{2|}q_{31}^{(\tau)}$.
(vii) ${}_{2|}q_{32}^{(1)}$.

Solution: (i) ${}_3p_{31}^{(\tau)} = \frac{\ell_{34}^{(\tau)}}{\ell_{31}^{(\tau)}} = \frac{9764+8687}{10128+8896} = 0.9698801514$.

(ii) ${}_3q_{30}^{(\tau)} = 1 - {}_3p_{30}^{(\tau)}$ *or* $= \frac{\ell_{30}^{(\tau)} - \ell_{33}^{(\tau)}}{\ell_{30}^{(\tau)}} = \frac{10263+8965-(9897+8744)}{10263+8965} = 0.03052839609$.

(iii) ${}_2q_{31}^{(2)} = \frac{\ell_{31}^{(2)} - \ell_{33}^{(2)}}{\ell_{31}^{(\tau)}} = \frac{8896-8744}{10128+8896} = 0.007989907485$.

(iv) ${}_{2|2}q_{31}^{(\tau)} = \frac{\ell_{33}^{(\tau)} - \ell_{35}^{(\tau)}}{\ell_{31}^{(\tau)}} = \frac{(9897+8744)-(9683+8604)}{10128+8896} = 0.01860807401$.

(v) ${}_{2|2}q_{30}^{(1)} = \frac{\ell_{32}^{(1)} - \ell_{34}^{(1)}}{\ell_{30}^{(\tau)}} = \frac{10034-9764}{10263+8965} = 0.01404202205$.

(vi) ${}_{2|}q_{31}^{(\tau)} = \frac{\ell_{33}^{(\tau)} - \ell_{34}^{(\tau)}}{\ell_{31}^{(\tau)}} = \frac{(9897+8744)-(9764+8687)}{10128+8896} = 0.009987384357$.

(vii) ${}_{2|}q_{32}^{(1)} = \frac{\ell_{34}^{(1)} - \ell_{35}^{(1)}}{\ell_{32}^{(\tau)}} = \frac{9764-9683}{10034+8818} = 0.004296626353$.

Example 9.3. *You are given the following multiple decrement table*

x	$q_x^{(1)}$	$q_x^{(2)}$
60	0.005	0.01
61	0.006	0.01
62	0.007	0.02
63	0.008	0.02
64	0.01	0.03

Calculate:
(i) ${}_3p_{61}^{(\tau)}$.
(ii) ${}_3q_{60}^{(\tau)}$.
(iii) ${}_2q_{61}^{(2)}$.
(iv) ${}_{2|}q_{61}^{(\tau)}$. **Class exercise.**
(v) ${}_{2|}q_{62}^{(1)}$.
(vi) ${}_{2|2}q_{61}^{(\tau)}$.
(vii) ${}_{2|2}q_{60}^{(1)}$.

Solution: (i) ${}_3p_{61}^{(\tau)} = p_{61}^{(\tau)} p_{62}^{(\tau)} p_{63}^{(\tau)}$ and $p_x^{(\tau)} = 1 - q_x^{\tau} = 1 - \sum_{j=1}^{m} q_x^{(j)}$.

Need

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$
60	0.005	0.01	0.015	0.985
61	0.006	0.01	0.016	0.984
62	0.007	0.02		
63	0.008	0.02		
64	0.01	0.03		

(i) ${}_3p_{61}^{(\tau)} = p_{61}^{(\tau)} p_{62}^{(\tau)} p_{63}^{(\tau)} = (1-(q_{61}^{(1)}+q_{61}^{(2)}))(1-(q_{62}^{(1)}+q_{62}^{(2)}))(1-(q_{63}^{(1)}+q_{63}^{(2)}))$
$= (1-0.006-0.01)(1-0.007-0.02)(1-0.008-0.02) = 0.930623904.$

(ii) ${}_3q_{60}^{(\tau)} = 1-{}_3p_{60}^{(\tau)} = 1-p_{60}^{(\tau)} p_{61}^{(\tau)} p_{62}^{(\tau)} = 1-(1-q_{60}^{(\tau)})(1-q_{61}^{(\tau)})(1-1_{62}^{(\tau)})$
$= 1-(1-q_{60}^{(1)}-q_{60}^{(2)})(1-q_{61}^{(1)}-q_{61}^{(2)})(1-q_{62}^{(1)}-q_{62}^{(2)})$
$= 1-(1-0.005-0.01)(1-0.006-0.01)(1-0.007-0.02) = 0.05692948.$

(iii) ${}_2q_{61}^{(2)} = {}_0|_1q_{61}^{(2)} + {}_1|_1q_{61}^{(2)} = q_{61}^{(2)} + p_{61}^{(\tau)} q_{62}^{(2)} = 0.01 + (1-0.006-0.01)(0.02) = 0.02968$

(iv) ${}_2|q_{61}^{(\tau)} = {}_2p_{61}^{(\tau)} q_{63}^{(\tau)} = p_{61}^{(\tau)} p_{62}^{(\tau)} q_{63}^{(\tau)} = (1-0.006-0.01)(1-0.007-0.02)(0.008+0.02)$
$= 0.026808096.$

(v) ${}_2|q_{62}^{(1)} = {}_2p_{62}^{(\tau)} q_{64}^{(1)} = p_{62}^{(\tau)} p_{63}^{(\tau)} q_{64}^{(1)} = (1-0.007-0.02)(1-0.008-0.02)(0.01)$
$= 0.00945756$

(vi) ${}_2|_2q_{61}^{(\tau)} = {}_2p_{61}^{(\tau)}\,{}_2q_{63}^{(\tau)} = {}_2|_1q_{61}^{(\tau)} + {}_3|_1q_{61}^{(\tau)} = p_{61}^{(\tau)} p_{62}^{(\tau)} q_{63}^{(\tau)} + p_{61}^{(\tau)} p_{62}^{(\tau)} p_{63}^{(\tau)} q_{64}^{(\tau)}$
$= p_{61}^{(\tau)} p_{62}^{(\tau)} (q_{63}^{(\tau)} + p_{63}^{(\tau)} q_{64}^{(\tau)}) = (1-0.006-0.01)(1-0.007-0.02)[0.008+0.02$
$+ (1-0.008-0.002)(0.01+0.003)] = 0.03913024584.$

(vii) ${}_2|_2q_{60}^{(1)} = {}_2|_1q_{60}^{(1)} + {}_3|_1q_{60}^{(1)} = p_{60}^{(\tau)} p_{61}^{(\tau)} (q_{62}^{(1)} + p_{62}^{(\tau)} q_{63}^{(1)})$
$= (1-0.005-0.01)(1-0.006-0.01)(0.007+(1-0.007-0.002)(0.008))$
$= 0.01446881.$

9.2 Stochastic model for multiple decrements

Definition 9.19.

Definition 9.20.

Definition 9.21.

Definition 9.22.

Definition 9.23.

Theorem 9.11.

Theorem 9.12. *Under the competing risks model,*
$f_{T_x,J_x}(t,j) = {}_tp_x^{(\tau)}\mu_x^{(j)}(t)$, *where* ${}_tq_x^{(\tau)} = \sum_{j=1}^m {}_tq_x^{(j)}$.

Proof. WLOG, let $j=1$. Recall that $f_{T_x,J_x}(t,1) = \frac{d}{dt}\mathbb{P}\{T_x \le t, J_x = 1\}$. Now

$$\begin{aligned}
&\mathbb{P}\{T_x \le t, J_x = 1\} \qquad (= \mathbb{P}\{T_x \le t\}\mathbb{P}\{J_x = 1\}\ ?)\\
=&\mathbb{P}\{T_x = \min(T_x^{(1)},\dots,T_x^{(m)}) \le t, J_x = 1\} = \mathbb{P}\{T_x^{(1)} \le t, T_x^{(1)} \le T_x^{(2)},\dots,T_x^{(1)} \le T_x^{(m)}\}\\
=&\int_0^t f_{T_x^{(1)}}(s_1)\int_{s_1}^\infty f_{T_x^{(2)}}(s_2)\cdots\int_{s_1}^\infty f_{T_x^{(m)}}(s_m)\,ds_m\cdots ds_1\\
=&\int_0^t f_{T_x^{(1)}}(s_1)S_{T_x^{(2)}}(s_1)\cdots S_{T_x^{(m)}}(s_1)\,ds_1 = \int_0^t f_{T_x^{(1)}}(s)S_{T_x^{(2)}}(s)\cdots S_{T_x^{(m)}}(s)\,ds\\
=&\int_0^t \mu_{x+s}^{(1)}{}_sp_x'^{(1)}\cdot{}_sp_x'^{(2)}\cdots{}_sp_x'^{(m)}\,ds = \int_0^t \mu_{x+s}^{(1)}{}_sp_x^{(\tau)}\,ds.
\end{aligned}$$

$f_{T_x,J_x}(t,1) = \frac{d}{dt}\mathbb{P}\{T_x \le t, J_x = 1\} = \frac{d}{dt}\int_0^t \mu_{x+s}^{(1)}{}_sp_x^{(\tau)}\,ds = {}_tp_x^{(\tau)}\mu_{x+t}^{(1)}$ ■

Notice that T_x is a continuous r.v. and J_x is a discrete r.v.

Example 9.4. *A live aged x is subject to two decrements. The mortality rates of these decrements are* $\mu_x^{(1)}(t) = \frac{2}{40-t}$ *and* $\mu_x^{(2)}(t) = \frac{5}{40-t}$, for $0 \le t \le 40$, *Calculate:*
(i) ${}_tp_x^{(\tau)}$; *(ii)* f_{T_x}; *(iii)* f_{T_x,J_x}; *(iv)* ${}_tq_x^{(1)}$; *(v)* ${}_tq_x^{(2)}$; *(vi)* $\mathbb{P}\{J_x = 1\}$; *(vii)* $\mathbb{P}\{J_x = 2\}$.

Solution: Hereafter let $t \in [0,40]$. (Otherwise, we need to specify t each time).
(i) Since ${}_tp_x^{(\tau)} = \exp\left(-\int_0^t \mu_x^{(\tau)}(s)\,ds\right)$ and $\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t) = \frac{7}{40-t}$.
${}_tp_x^{(\tau)} = \exp\left(-\int_0^t \mu_x^{(\tau)}(s)\,ds\right) = \exp\left(-\int_0^t \frac{7}{40-s}\,ds\right) = exp(7(\log(40-s)\big|_0^t))$
$= exp(7(\log(40-t) - \log 40)) = exp(\log((\frac{40-t}{40})^7) = \left(\frac{40-t}{40}\right)^7$.
(ii) $f_{T_x}(t) = -\frac{d}{dt}{}_tp_x^{(\tau)} = \frac{7(40-t)^6}{40^7}$.
(iii) $f_{(T_x,J_x)}(t,j) = {}_tp_x^{(\tau)}\mu_{x+t}^{(j)} = \left(\frac{40-t}{40}\right)^7\mu_{x+t}^{(j)} = \begin{cases}\frac{(2)(40-t)^6}{40^7} & \text{if } j=1,\\ \frac{(5)(40-t)^6}{40^7} & \text{if } j=2,\end{cases}$
(iv) ${}_tq_x^{(1)} = \mathbb{P}\{T_x \le t, J_x = 1\} = \int_0^t f_{T_x,J}(s,1)\,ds = \int_0^t \frac{2(40-s)^6}{40^7}\,ds = \frac{-(2)(40-s)^7}{(7)(40)^7}\Bigg|_0^t = \frac{(2)((40)^7-(40-t)^7)}{(7)(40)^7}$.
(v) ${}_tq_x^{(2)} = \mathbb{P}\{T_x \le t, J_x = 2\} = \int_0^t f_{T_x,J}(s,2)\,ds \qquad = \frac{-(5)(40-s)^7}{(7)(40)^7}\Bigg|_0^t = \frac{(5)((40)^7-(40-t)^7)}{(7)(40)^7}$.
(vi) $\mathbb{P}\{J_x = 1\} = \mathbb{P}\{T_x \le \infty, J_x = 1\} = \int_0^\infty f_{T_x,J}(s,2)\,ds = {}_\infty q_x^{(1)}?? = {}_{40}q_x^{(1)} = \frac{2}{7}$.
(vii) $\mathbb{P}\{J_x = 2\} = {}_{40}q_x^{(2)} = \frac{5}{7}$
or $= 1 - P(J=1) = 5/7$.

Example 9.5. *A live aged x is subject to two decrements. The mortality rates of these decrements are* $\mu_x^{(1)}(t) = 0.01$, for $t \ge 0$, *and* $\mu_x^{(2)}(t) = 0.03$, for $t \ge 0$. *Calculate:*
(i) ${}_tp_x^{(\tau)}$. *(ii)* f_{T_x}. *(iii)* $f_{T_x,J}$. *(iv)* $\mathbb{P}\{J_x = 1\}$. *(v)* $\mathbb{P}\{J_x = 2\}$. *(vi)* ${}_tq_x^{(1)}$. *(vii)* ${}_tq_x^{(2)}$.

Solution: Two independent r.v.s $\sim Exp(\mu_i)$. Let $t > 0$ hereafter.

(i) $\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t) = 0.04$ and ${}_tp_x^{(\tau)} = e^{-0.04t}$ $(= S_{T_x}(t))$.
(ii) $f_{T_x}(t) = 0.04e^{-0.04t}$.
(iii) $f_{(T_x,J_x)}(t,j) = {}_tp_x^{(\tau)}\mu_{x+t}^{(j)} = e^{-0.04t}\mu_{x+t}^{(j)} = \begin{cases} e^{-0.04t}(0.01) & \text{if } j=1, \\ e^{-0.04t}(0.03) & \text{if } j=2, \end{cases}$
(iv) $\mathbb{P}\{J_x = 1\} = \int_0^\infty f_{T_x,J}(t,1)dt = \int_0^\infty e^{-0.04t}0.01dt = \frac{0.01}{0.04}\int_0^\infty e^{-0.04t}0.04dt = \frac{0.01}{0.04} = 1/4$.
(v) $\mathbb{P}\{J_x = 2\} = 1 - P(J=1) = 3/4$.
(vi) ${}_tq_x^{(1)} = \int_0^t {}_sp_x^{(\tau)}\mu_x^{(1)}(s)ds = \int_0^t e^{-0.04s}0.01ds = \frac{0.01}{0.04}\int_0^t e^{-0.04s}0.04ds = (0.25)(1-e^{-0.04t})$.
(vii) ${}_tq_x^{(2)} = \mathbb{P}\{J_x = 2\}{}_tq_x^{(\tau)} = (0.75)(1-e^{-0.04t})$.

Theorem 9.13. *Suppose that a live is subject to m decrements. Decrement j has constant force of mortality $\mu^{(j)}$. Then, for $t > 0$,*
(i) $\mu^{(\tau)} = \sum_{j=1}^m \mu^{(j)}$.
(ii) ${}_tp_x^{(\tau)} = e^{-\mu^{(\tau)}t}$.
(iii) ${}_tq_x^{(\tau)} = 1 - e^{-\mu^{(\tau)}t}$.
(iv) ${}_tq_x^{(j)} = \frac{\mu^{(j)}}{\mu^{(\tau)}}\left(1 - e^{-\mu^{(\tau)}t}\right)$.
(v) $\mathbb{P}\{J_x = j\} = \frac{\mu^{(j)}}{\mu^{(\tau)}}$.
(vi) T_x and J_x are independent r.v.'s.

Proof. (i) Easy. (ii) ${}_tp_x^{(\tau)} = \exp^{-\int_0^t \mu^{(\tau)}\,ds} = e^{-\mu^{(\tau)}t}$.
(iii) ${}_tq_x^{(\tau)} = 1 - {}_tp_x^{(\tau)} = 1 - e^{-\mu^{(\tau)}t}$.
(iv) ${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\mu^{(j)}\,ds = \int_0^t e^{-\mu^{(\tau)}s}\mu^{(j)}\,ds = \frac{\mu^{(j)}}{\mu^{(\tau)}}\left(1 - e^{-\mu^{(\tau)}t}\right)$.
(v) $\mathbb{P}\{J_x = j\} = {}_\infty q_x^{(j)} = \frac{\mu^{(j)}}{\mu^{(\tau)}}$.
(vi) $\mathbb{P}\{T_x \le t, J_x = j\} = {}_tq_x^{(j)} = \frac{\mu^{(j)}}{\mu^{(\tau)}}\left(1 - e^{-\mu^{(\tau)}t}\right) = \mathbb{P}\{T_x \le t\}\mathbb{P}\{J_x = j\}$. ■

9.3 Random survivorship group.

9.4 Associated single decrement tables.

Definition 9.24. *The probability of death within t years due to cause j in the absence of other decrements is denoted by ${}_tq_x'^{(j)}$ $(= P(T_x^{(j)} \le t))$. ${}_tq_x'^{(j)}$ is called the* **absolute rate of death due to decrement** *j. ${}_tq_x'^{(j)}$ is also called the* **net probability of decrement** *and* **independent rate decrement**.

Definition 9.25. $q_x'^{(j)} = {}_1q_x'^{(j)}$.

Definition 9.26. ${}_tp_x'^{(j)} = 1 - {}_tq_x'^{(j)}$.

Definition 9.27. $p_x'^{(j)} = {}_1p_x'^{(j)}$.

Notice that $P(T_x \le t, J_x = j) = {}_tq_x^{(j)} \le {}_tq_x'^{(j)} = P(T_x^{(j)} \le t)$.
${}_tq_x'^{(j)} = \int_0^t {}_sp_x'^{(j)}\mu_{x+s}^{(j)}\,ds$ and

${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)}\, ds.$

Theorem 9.14. ${}_tp_x^{(\tau)} = \prod_{j=1}^m {}_tp_x^{\prime(j)}$.

Theorem 9.15. ${}_tq_x^{(j)} \le {}_tq_x^{\prime(j)} \le {}_tq_x^{(\tau)}$.

Example 9.6. *A live aged x is subject to two decrements. The mortality rates of these decrements are $\mu_x^{(1)}(t) = \frac{2}{40-t}$ and $\mu_x^{(2)}(t) = \frac{5}{40-t}$,* for $0 \le t \le 40$, *(i) Calculate ${}_tp_x^{\prime(1)}$ and ${}_tp_x^{\prime(2)}$. (ii) Calculate the densities of $T_x^{(1)}$ and $T_x^{(2)}$.*

Solution: (i) ${}_tp_x^{\prime(1)} = \exp\left(-\int_0^t \mu_x^{(1)}(s)\, ds\right) = \exp\left(-\int_0^t \frac{2}{40-s}\, ds\right) = \frac{(40-t)^2}{(40)^2}$,
${}_tp_x^{\prime(2)} = \exp\left(-\int_0^t \mu_x^{(2)}(s)\, ds\right) = \exp\left(-\int_0^t \frac{5}{40-s}\, ds\right) = \frac{(40-t)^5}{(40)^5}$, $t \in [0, 40]$.

(ii) $f_{T_x^{(1)}}(t) = -\frac{d}{dt}{}_tp_x^{\prime(1)} = \frac{2(40-t)}{(40)^2}$ and
$f_{T_x^{(2)}}(t) = -\frac{d}{dt}{}_tp_x^{\prime(2)} = \frac{5(40-t)^4}{(40)^5}$, $t \in (0, 40)$.

Example 9.7. *A live aged x is subject to two decrements. The mortality rates of these decrements are $\mu_x^{(1)}(t) = \mu_1 = 0.01$ and $\mu_x^{(2)}(t) = \mu_2 = 0.03$,* for $t \ge 0$.
(i) Calculate ${}_tp_x^{\prime(1)}$ and ${}_tp_x^{\prime(2)}$.
(ii) Calculate the densities of $T_x^{(1)}$ and $T_x^{(2)}$.

Solution: (i) ${}_tp_x^{\prime(j)} = \exp\left(-\int_0^t \mu_x^{(j)}(s)\, ds\right) = e^{-\mu_j t}$, $t \ge 0$.

(ii) $f_{T_x^{(1)}}(t) = 0.01e^{-0.01t}$ and $f_{T_x^{(2)}}(t) = 0.03e^{-0.03t}$ $t \ge 0$.

Remark. $f_{T_x,J}(t, j) = \begin{cases} 0.01e^{-0.04t} & \text{if } j = 1 \\ 0.03e^{-0.04t} & \text{if } j = 2 \end{cases} \quad t > 0.$

Theorem 9.16.

Theorem 9.17.

Theorem 9.18.

Example 9.8.

Example 9.9.

Theorem 9.19.

9.5 Interpolating multiple decrement life tables.

9.5.1 Uniformity in the multiple decrement table. Suppose that decrements are uniformly distributed over each year of age in the multiple decrement table. In this case,

$P(T_x \le t, J_x = j) = {}_tq_x^{(j)} = tq_x^{(j)}, 0 \le t \le 1.$ (**not for** ${}_tq_x^{\prime(j)} = P(T_x^{(j)} \le t)$).

Example 9.10. *Suppose decrements are uniformly distributed over each year of age in the multiple decrement context.*
(a) If $q_x^{(1)} = 0.05$ and $q_x^{(2)} = 0.02$, find $q_x^{\prime(1)}$ and $q_x^{\prime(2)}$.
(b) If $q_x^{\prime(1)} = 0.05$ and $q_x^{\prime(2)} = 0.02$, find $q_x^{(1)}$ and $q_x^{(2)}$.

Solution: Assume ${}_tq_x^{(j)} = tq_x^{(j)}$, $t \in (0,1]$. (a) **Given $q_x^{(j)}$'s how to compute ${}_tq_x^{\prime(j)}$?**
Formulas: (i) ${}_tq_x^{\prime(j)} = 1 - {}_tp_x^{\prime(j)}$ $\quad (= P(T_x^{(1)} \le t))$,
need (ii) ${}_tp_x^{\prime(j)} = \exp\left(-\int_0^t \mu_x^{(j)}(s)\,ds\right)$, **No notation of $\mu_x^{\prime(j)}(s)$!!**
need (iii) $\mu_x^{(j)}(t) = \frac{\frac{d}{dt}{}_tq_x^{\prime(j)}}{{}_tp_x^{\prime(j)}} = \frac{\frac{d}{dt}{}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$ **which to use ?** $= \frac{\frac{d}{dt}{}_tq_x^{(j)}}{1-{}_tq_x^{(\tau)}}$,
need (iv) ${}_tq_x^{(j)} = tq_x^{(j)}$ and ${}_tq_x^{(\tau)} = \sum_j {}_tq_x^{(j)} = t\sum_j q_x^{(j)} = tq_x^{(\tau)}$, $0 \le t \le 1$.
(iii)&(iv) $=> \mu_x^{(j)}(t) = \frac{\frac{d}{dt}{}_tq_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{\frac{d}{dt}(tq_x^{(j)})}{1-{}_tq_x^{(\tau)}} = \frac{q_x^{(j)}}{1-tq_x^{(\tau)}}$,

$$
\begin{aligned}
(ii) \quad => \quad p_x^{\prime(j)} &= \exp\left(-\int_0^1 \mu_x^{(j)}(s)\,ds\right) = \exp\left(-\int_0^1 \frac{q_x^{(j)}}{1-sq_x^{(\tau)}}\,ds\right)\\
&= \exp\left(\frac{q_x^{(j)}}{q_x^{(\tau)}}\int_0^1 \frac{d(1-sq_x^{(\tau)})}{1-sq_x^{(\tau)}}\right) = \exp\left(\frac{q_x^{(j)}}{q_x^{(\tau)}}\log(1-sq_x^{(\tau)})\Big|_0^1\right)\\
p_x^{\prime(j)} &= \exp\Big(\frac{q_x^{(j)}}{q_x^{(\tau)}}\log(1-q_x^{(\tau)})\Big) = \exp\Big(\log\big((1-q_x^{(\tau)})^{\frac{q_x^{(j)}}{q_x^{(\tau)}}}\big)\Big) = (1-q_x^{(\tau)})^{\frac{q_x^{(j)}}{q_x^{(\tau)}}}, \qquad (1)\\
(i) \quad => \quad q_x^{\prime(j)} &= 1 - p_x^{\prime(j)} = 1-(1-q_x^{(\tau)})^{\frac{q_x^{(j)}}{q_x^{(\tau)}}} \qquad (q_x^{(\tau)} = ?)\\
q_x^{(\tau)} &= q_x^{(1)} + q_x^{(2)} = 0.05 + 0.02 = 0.07, \qquad (= q_x^{\prime(1)} + q_x^{\prime(2)}\textbf{???})\\
q_x^{\prime(1)} &= 1-(1-q_x^{(\tau)})^{\frac{q_x^{(1)}}{q_x^{(\tau)}}} = 1-(1-0.07)^{\frac{0.05}{0.07}} = 0.05051562906,\\
q_x^{\prime(2)} &= 1-(1-q_x^{(\tau)})^{\frac{q_x^{(2)}}{q_x^{(\tau)}}} = 1-(1-0.07)^{\frac{0.02}{0.07}} = 0.02052100228,
\end{aligned}
$$

(b) Eq. (1) leads to $\log p_x^{\prime(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}}\log(1-q_x^{(\tau)}) => q_x^{(j)} = q_x^{(\tau)}\frac{\log p_x^{\prime(j)}}{\log(1-q_x^{(\tau)})}\Big|_{q_x^{(\tau)}=??}$.

Also Eq. (1) leads to $p_x^{\prime(1)}p_x^{\prime(2)} = (1-q_x^{(\tau)})^{\frac{q_x^{(1)}}{q_x^{(\tau)}}}(1-q_x^{(\tau)})^{\frac{q_x^{(2)}}{q_x^{(\tau)}}} = (1-q_x^{(\tau)})^{\frac{q_x^{(1)}+q_x^{(2)}}{q_x^{(\tau)}}} = 1-q_x^{(\tau)}$??
$=> q_x^{(\tau)} = 1 - p_x^{\prime(1)}p_x^{\prime(2)} = 1-(1-0.05)(1-0.02) = 0.069.$
$q_x^{(1)} = \frac{(0.069)\log(0.95)}{\log(1-0.069)} = 0.04950259075$ and $q_x^{(2)} = \frac{(0.069)\log(0.98)}{\log(1-0.069)} = 0.01949740925.$

Announcement. Quiz on Friday: 447 and Chapter 7-9 (upto $\mu_x^{(j)} = ...$)

Need help for solutions of 1st midterm ?

9.5.2 Uniformity in the single decrement table. We may assume that decrements are uniformly distributed in the absence of other decrements *i.e.*,

$F_{T_x^{(j)}}(t) = {}_tq_x^{\prime(j)} = tq_x^{\prime(j)}, 0 \le t \le 1$ (v.s. $F_{T_x,J_x}(t,j) = {}_tq_x^{(j)} = tq_x^{(j)}$), where x is an integer.
If this happens, we say that decrements are uniformly distributed in its associated **single** decrement table (v.s. **multiple** decrement table).

Example 9.11. *Suppose decrements are uniformly distributed over each year of age in the associated single decrement table.*
(a) If $q_x^{\prime(1)} = 0.05$ *and* $q_x^{\prime(2)} = 0.02$*, find* $q_x^{(1)}$ *and* $q_x^{(2)}$*.*
(b) If $q_x^{(1)} = 0.05$ *and* $q_x^{(2)} = 0.02$*, find* $q_x^{\prime(1)}$ *and* $q_x^{\prime(2)}$*.*

Solution: (a) Assume ${}_tq_x^{'(j)} = tq_x^{'(j)}$, $t \in (0,1]$, but not ${}_tq_x^{(j)} = tq_x^{(j)}$, $t \in (0,1]$.

$$
\begin{aligned}
q_x^{(1)} =& P(T_x \le 1, J_x = 1) = \int_0^1 f_{T_x,J_x}(s,1)\,ds \\
=& \int_0^1 {}_sp_x^{(\tau)}\mu_{x+s}^{(1)}\,ds \qquad (\mu_{x+s}^{(j)} = \mu_x^{(j)}(s) = \frac{f_{T_x^{(j)}}(s)}{S_{T_x^{(j)}}(s)} = \frac{f_{T_x,J_x}(s,j)}{{}_sp_x^{(\tau)}}) \\
=& \int_0^1 [{}_sp_x^{\prime(1)}\,{}_sp_x^{\prime(2)}]\frac{f_{T_x^{(1)}}(s)}{{}_sp_x^{'(1)}}\,ds \qquad (P(T_x > t) = P(\min_i T_x^{(i)} > t) = P(T_x^{(1)} > t)P(T_x^{(2)} > t)) \\
=& \int_0^1 (1 - sq_x^{\prime(2)})q_x^{'(1)}\,ds \qquad ({}_tq^{\prime(j)} = tq^{\prime(j)} \;=>\; f_{T_x^{(j)}}(t) = q_x^{'(j)},\ 0 \le t \le 1) \\
=& q_x^{'(1)}\int_0^1 1\,ds - q_x^{\prime(2)}q_x^{'(1)}\int_0^1 s\,ds = \left(1 - \frac{s^2}{2}q_x^{\prime(2)}\right)q_x^{'(1)}\Big|_0^1 \quad =>
\end{aligned}
$$

$$
q_x^{(1)} = \left(1 - \frac{1}{2}q_x^{\prime(2)}\right)q_x^{\prime(1)} \text{ and } q_x^{(2)} = \left(1 - \frac{1}{2}q_x^{\prime(1)}\right)q_x^{\prime(2)}. \tag{2}
$$

$$
q_x^{(1)} = \left(1 - \frac{1}{2}(0.02)\right)(0.05) = 0.0495 \text{ and } q_x^{(2)} = \left(1 - \frac{1}{2}(0.05)\right)(0.02) = 0.0195.
$$

(b) Equation (2) in (a) yields

$$
\begin{cases}
0.050 \approx q_x^{(1)} = q_x^{\prime(1)}\left(1 - \frac{1}{2}q_x^{\prime(2)}\right) = q_x^{\prime(1)} - \frac{1}{2}q_x^{\prime(1)}q_x^{\prime(2)} \\
0.020 \approx q_x^{(2)} = q_x^{\prime(2)}\left(1 - \frac{1}{2}q_x^{\prime(1)}\right) = q_x^{\prime(2)} - \frac{1}{2}q_x^{\prime(1)}q_x^{\prime(2)}
\end{cases} \tag{3}
$$

Subtracting the two equations, $0.03 = q_x^{\prime(1)} - q_x^{\prime(2)} \;=>\; q_x^{\prime(1)} = q_x^{\prime(2)} + 0.03$. Then Eq. (3) =>

$$
0 = q_x^{\prime(1)}\left(1 - \frac{1}{2}q_x^{\prime(2)}\right) - 0.05 = (q_x^{\prime(2)} + 0.03)\left(1 - \frac{1}{2}q_x^{\prime(2)}\right) - 0.05 = -(0.5)(q_x^{\prime(2)})^2 + (0.985)q_x^{\prime(2)} - 0.02.
$$

The solutions of this quadratic equation are 0.021 and 1.949. Since $0 \le q_x^{\prime(2)} \le 1$,
$q_x^{\prime(2)} = 0.021$ and $q_x^{\prime(1)} = 0.051$.

9.5.3 Other options. There are other options for the distribution of the decrements.

Example 9.12. *A live aged x is subject to a double–decrement model.*
(i) Decrement 1 is death, which has a uniform distribution over each year of age in the associated single decrement table.
(ii) Decrement 2 is withdrawal, which occurs at the end of the year.
(iii) $q_x^{\prime(1)} = 0.01$ and $q_x^{\prime(2)} = 0.03$.
Calculate $q_x^{(1)}$ and $q_x^{(2)}$.

Solution: Assume: ${}_tq_x^{\prime(1)} = tq_x^{\prime(1)}$, $t \in (0, 1]$, $T_x^{(2)} \in \{1, 2, 3, ...\}$ with $\mathbb{P}\{T_x^{(2)} = 1\} = 0.03$.

$$\begin{aligned} q_x^{(1)} =& \mathbb{P}\{T_x \le 1, J_x = 1\} \overset{??}{=} \mathbb{P}\{T_x^{(1)} \le 1, J_x = 1\} \overset{??}{=} \mathbb{P}\{T_x^{(1)} \le 1, T_x^{(1)} \le T_x^{(2)}\} \\ =& \mathbb{P}\{T_x^{(1)} \le 1, T_x^{(1)} \le T_x^{(2)}, 1 \le T_x^{(2)}\} = \mathbb{P}\{T_x^{(1)} \le 1\} = q_x^{\prime(1)} = 0.01. \\ q_x^{(2)} =& \mathbb{P}\{T_x^{(2)} \le 1, J_x = 2\} = \mathbb{P}\{T_x^{(2)} = 1, J_x = 2\} = \mathbb{P}\{T_x^{(2)} = 1, T_x^{(2)} < T_x^{(1)}\} \\ =& \mathbb{P}\{T_x^{(2)} = 1\}\mathbb{P}\{1 < T_x^{(1)}\} = q_x^{\prime(2)}(1 - q_x^{\prime(1)}) = (0.03)(1 - 0.01) = 0.0297. \end{aligned}$$

Example 9.13. *A live aged x is subject to a double–decrement model.*
(i) Decrement 1 is death, which has a uniform distribution over each year of age in the associated single decrement table.
(ii) Decrement 2 is withdrawal, which occurs at the midpoint of the year.
(iii) $q_x^{\prime(1)} = 0.01$ and $q_x^{\prime(2)} = 0.03$.
Calculate $q_x^{(1)}$ and $q_x^{(2)}$.

Solution: Assume ${}_tq_x^{\prime(1)} = tq_x^{\prime(1)}$, $t \in (0, 1]$.
$T_x^{(2)} \in \{0.5, 1.5, ...\}$. $\mathbb{P}\{T_x^{(2)} = 0.5\} = 0.03$.

$$\begin{aligned} q_x^{(1)} =& \mathbb{P}\{T_x^{(1)} \le 1, J_x = 1\} \\ =& \mathbb{P}\{T_x^{(1)} \le 1, J_x = 1, T_x^{(2)} = 0.5\} + \mathbb{P}\{T_x^{(1)} \le 1, J_x = 1, T_x^{(2)} \ge 1.5\} \\ =& \mathbb{P}\{T_x^{(1)} \le 0.5, T_x^{(2)} = 0.5\} + \mathbb{P}\{T_x^{(1)} \le 1, 1.5 \le T_x^{(2)}\} \\ =& {}_{0.5}q_x^{\prime(1)}(0.03) + {}_1q_x^{\prime(1)}(1 - 0.03) = (0.01t|_{t=0.5})(0.03) + (0.01t|_{t=1})(1 - 0.03) = 0.00985. \\ q_x^{(2)} =& \mathbb{P}\{T_x^{(2)} \le 1, T_x^{(2)} < T_x^{(1)}\} \\ =& \mathbb{P}\{T_x^{(2)} = 0.5 < T_x^{(1)}\} \\ =& \mathbb{P}\{T_x^{(2)} = 0.5\}\mathbb{P}\{0.5 < T_x^{(1)}\} = (0.03)(1 - (0.01t|_{t=0.5})) = 0.02985. \end{aligned}$$

9.6 Insurance for multiple decrement life tables.

Suppose that a life insurance is paid at the end of the year of death if death is due to decrement j. The APV of this insurance is

$$A_x^{(j)} = E(v^{K_x} I(J_x = j)) = \sum_{k=1}^{\infty} v^k f_{K_x,J_x}(k, j) = \sum_{k=1}^{\infty} v^k {}_{k-1}|q_x^{(j)} = \sum_{k=1}^{\infty} v^k {}_{k-1}p_x^{(\tau)} q_{x+k-1}^{(j)}.$$

Example 9.14. *An insurance policy provides a payment at the end of the year of death of \$100000, if the cause of death is natural causes, and \$200000, if the cause of death is accidental death. The force of mortality for a new born due to natural causes is* $\mu_0^{(1)}(t) = 0.01$, *for* $t \geq 0$. *The force of mortality for a new born due to accidents is* $\mu_0^{(2)}(t) = 0.005$, *for* $t \geq 0$. *Find the APV of this insurance if* $\delta = 0.06$.

Solution: $E(B_{J_x} v^{K_x}) = \sum_j B_j E(v^{K_x} I(J_x = j)) = B_1 A_x^{(1)} + B_2 A_x^{(2)} = \ldots\ldots\ B_1,\ A_x^{(1)} = ?$ $A_x^{(j)} = \sum_{k=1}^{\infty} v^k {}_{k-1}|q_x^{(j)} = \sum_{k=1}^{\infty} v^k {}_{k-1}p_x^{(\tau)} q_{x+k-1}^{(j)}$. For $t > 0$,

$$ {}_tp_x^{(\tau)} = {}_tp_x'^{(1)} {}_tp_x'^{(2)} = e^{-0.01t} e^{-0.005t} = e^{-0.015t}, \qquad (P(T_x > t) = P(min(T_x^{(1)}, T_x^{(2)}) > t)) $$

$$ q_x^{(1)} = \int_0^1 f_{T_x,J_x}(t,1) dt = \int_0^1 {}_sp_x^{(\tau)} \mu_x^{(1)}(s)\, ds = \int_0^1 e^{-0.015s}(0.01)\, ds = \frac{2}{3}(1 - e^{-0.015}), $$

$$ q_x^{(2)} = \int_0^1 f_{T_x,J_x}(t,2) dt = \int_0^1 {}_sp_x^{(\tau)} \mu_x^{(2)}(s)\, ds = \int_0^1 e^{-0.015s}(0.005)\, ds = \frac{1}{3}(1 - e^{-0.015}), $$

or $q_x^{(2)} = q_x^{(\tau)} - q_x^{(1)} = 1 - p_x^{(\tau)} - q_x^{(1)} = \ldots$

$$ A_x^{(1)} = \sum_{k=1}^{\infty} v^k {}_{k-1}p_x^{(\tau)} q_{x+k-1}^{(1)} = \sum_{k=1}^{\infty} e^{-0.06k} e^{-0.015(k-1)} \frac{2}{3}(1 - e^{-0.015}) $$

$$ = \frac{2}{3} e^{-0.06}(1 - e^{-0.015}) \sum_{j=0}^{\infty} e^{-0.075j} = \frac{2}{3} e^{-0.06}(1 - e^{-0.015}) \frac{1 - p^{\infty+1}}{1 - p} = 0.1293636291, $$

$$ A_x^{(2)} = \sum_{k=1}^{\infty} v^k {}_kp_x^{(\tau)} q_{x+k-1}^{(2)} = \sum_{k=1}^{\infty} e^{-0.06k} e^{-0.015(k-1)} \frac{1}{3}(1 - e^{-0.015}) $$

$$ = \frac{1}{3} e^{-0.06}(1 - e^{-0.015}) \sum_{j=0}^{\infty} e^{-0.075j} = \frac{1}{3} e^{-0.06}(1 - e^{-0.015}) \frac{1}{1 - e^{-0.075}} = 0.06468181457. $$

Hence, the APV of this insurance is

$E(B_{J_x} v^{K_x}) = B_1 A_x^{(1)} + B_2 A_x^{(2)} = 10^5(0.1293636291 + (2)0.06468181457) \approx 25872.73.$ ▫

Suppose that a life insurance is paid at the moment of death if death is due to decrement j. The APV of this insurance is

$$ E(v^{T_x} I(J_x = j)) = \overline{A}_x^{(j)} = \int_0^{\infty} v^t f_{T_x,J_x}(t,j)\, dt = \int_0^{\infty} v^t {}_tp_x^{(\tau)} \mu_x^{(j)}(t)\, dt. \qquad (2) $$

Example 9.15. *An insurance policy provides a payment at the time of death. The policy is the same as in the previous example.* ${}_tp_x^{(\tau)} = e^{-0.015t}$. *Find the APV of this insurance.*

Solution: $E(B_{J_x} v^{T_x}) = \sum_j B_j E(v^{T_x} I(J_x = j)) = B_1 \overline{A}_x^{(1)} + B_2 \overline{A}_x^{(2)} = \ldots\ldots$

$$ \overline{A}_x^{(1)} = \int_0^{\infty} v^t {}_tp_x^{(\tau)} \mu_x^{(1)}(t)\, dt = \int_0^{\infty} e^{-0.06t} e^{-0.015t}(0.01)\, dt = \frac{0.01}{0.075} = \frac{2}{15}, $$

$$ \overline{A}_x^{(2)} = \int_0^{\infty} v^t {}_tp_x^{(\tau)} \mu_x^{(2)}(t)\, dt = \int_0^{\infty} e^{-0.06t} e^{-0.015t}(0.005)\, dt = \frac{0.005}{0.075} = \frac{1}{15}. $$

Hence, the APV of this insurance is $E(B_{J_x} v^{T_x}) = (100000)\frac{2}{15} + (200000)\frac{1}{15} \approx 26666.67.$

$q_x^{(1)}=\int_0^1(1-sq_x'^{(2)})q_x'^{(1)}\,ds=q_x'^{(1)}(\int_0^1 1\,ds-q_x'^{(2)}\int_0^1 s\,ds)=q_x'^{(1)}\left(1-\frac{s^2}{2}q_x'^{(2)}\right)\Big|_0^1=q_x'^{(1)}\left(1-\frac{1}{2}q_x'^{(2)}\right)$

Example 9.16. *A 50–year old buys a 3–year term fully discrete insurance with face value 20000 payable at the end of the year of death. An actuary prices this insurance using a double decrement model:*
(i) Decrement 1 is death. Decrement 2 is withdrawal.
(ii)

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
50	10000	*54*	–
51	9923	*75*	–
52	9847	*80*	–

(iii) There are no withdrawal benefits.
(iv) $i=0.07$
Calculate the level annual benefit premium.

Solution: Solve BP, where $B=20000$,
$E(B_{J_x}v^{K_x}I(K_x\le 3))-P\ddot{a}_{x:\overline{3}|}=0$ and $B_{J_x}=I(J_x=1)$.
Let $A=E(B_{J_x}v^{K_x}I(K_x\le 3))=E(v^{K_x}I(J_x=1,K_x\le 3))$,
$=> A=P\ddot{a}_{x:\overline{3}|}$,
$=> P=\frac{A}{\ddot{a}_{x:\overline{3}|}}$.

$$
\begin{aligned}
A=&\sum_{k=1}^{3}v^k{}_{k-1}|q_x^{(1)}=vq_{50}^{(1)}+v^2{}_1|q_{50}^{(1)}+v^3{}_2|q_{50}^{(1)}\\
=&(1.07)^{-1}\frac{54}{10000}+(1.07)^{-2}\frac{75}{10000}+(1.07)^{-3}\frac{80}{10000}=0.0181279.\\
\ddot{a}_{x:\overline{3}|}=&\sum_{k=0}^{2}v^k{}_kp_x^{(\tau)}=1+vp_{50}^{(\tau)}+v^2\cdot{}_2p_{50}^{(\tau)}\\
=&1+(1.07)^{-1}\frac{9923}{10000}+(1.07)^{-2}\frac{9847}{10000}=2.787458293.
\end{aligned}
$$

Hence, $BP=\frac{BA}{\ddot{a}_{x:\overline{3}|}}=\frac{20000\times 0.0181279}{2.787458293}\approx 130.07$.

9.7 Asset shares $({}_t$AS=the premiums$-(benefits+expenses))$

We consider a two decrement model for insurance including expenses. Decrement 1 is death. Decrement 2 is withdrawal. Suppose that the benefit depends on the decrement and are paid at the end of the death year and expenses are paid at the beginning of each year alive. For $k\ge 1$ let b_k be the benefit paid for death during year k;

${}_kCV$ be the cash value payable upon for withdrawal during year k;

G be the augmented premium;

A fixed expense of e_{k-1} is paid at year k.

Another expense is an r_{k-1} percentage of the augmented premium.

Under the equivalence principle,

$$\underbrace{G\sum_{k=0}^{\infty} v^k {}_kp_x^{(\tau)}}_{premium} = \underbrace{\sum_{k=1}^{\infty} b_k v^k {}_{k-1|}q_x^{(1)}}_{benifit} + \underbrace{\sum_{k=1}^{\infty} {}_kCV v^k {}_{k-1|}q_x^{(2)}}_{refund} + \underbrace{\sum_{k=0}^{\infty}(r_kG+e_k)v^k {}_kp_x^{(\tau)}}_{expenses}.$$

$$G\ddot{a}_x^{(\tau)} = \sum_{k=1}^{\infty} b_k v^k {}_{k-1}p_x^{(\tau)} q_{x+k-1}^{(1)} + \sum_{k=1}^{\infty} {}_kCV v^k {}_{k-1}p_x^{(\tau)} q_{x+k-1}^{(2)} + \sum_{k=0}^{\infty}(r_kG+e_k)v^k {}_kp_x^{(\tau)}.$$

The following recurrence relation is useful:

$$({}_kAS + G - Gr_k - e_k) = v[b_{k+1}q_{x+k}^{(1)} + {}_{k+1}CV q_{x+k}^{(2)} + p_{x+k}^{(\tau)} \cdot {}_{k+1}AS]. \tag{3}$$

Recall simple case: $({}_tV_x + P_x) = v[bq_{x+t} + {}_{t+1}V_x p_{x+t}]$,

where ${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$. Or ${}_tV_x = \underbrace{\sum_{k=1}^{\infty} b_{k+t}v^k {}_{k-1}p_{x+t}q_{x+t+k-1}}_{benifit} - \underbrace{P_x\sum_{k=0}^{\infty} v^k {}_kp_{x+t}}_{premium}$.

Example 9.17. *A fully discrete whole life insurance with face value of 50000 was issued to* (50)*. This life insurance provides cash value payments at the end of the year if the policyholder cancels the policy. An actuary models this life insurance with two decrements: deaths and withdrawal. You are given:*
(i) The annual gross premium is 550.
(ii) Expenses are 6% of the gross premium and paid at the beginning of the year.
(iii) $q_{60}^{(\text{death})} = 0.04$
(iv) $q_{60}^{(\text{withdrawal})} = 0.05$
(v) $i = 0.065$
(vi) ${}_{10}AS = 6700$ $\quad$ *(*${}_tAS$*= (the premiums−*$(benefits + expenses)$ *at time* t*).*
(vii) ${}_{11}CV = 6000$.
Calculate the asset shares at the end of year 11.

Solution: Eq. (3) yields $({}_{10}AS + G - Gr_{10} - e_{10})(1+i) = b_{11}q_{x+10}^{(1)} + {}_{11}CV q_{x+10}^{(2)} + p_{x+10}^{(\tau)} \cdot {}_{11}AS$.
$(6700 + 550 - (550)(0.06))(1.065) = (50000)(0.04) + (6000)(0.05) + (1 - 0.04 - 0.05)_{11}AS$.

$${}_{11}AS = \frac{(6700 + 550 - (550) * (0.06)) * (1.065) - (50000) * (0.04) - (6000) * (0.05)}{1 - 0.04 - 0.05} \approx 5918.80.$$

Under the competing risks model, $T_x^{(1)}, ..., T_x^{(m)}$ are independent and cts, with cdf's ${}_tq_x^{'(j)}$ and ${}_tp_x^{'(j)} = 1 - {}_tq_x^{'(j)}$, $\mu_x^{(j)}(t) = \mu_{x+t}^{(j)} = \frac{f_{X_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)}$.
$\{T_x = t, J_x = j\} = \{\min\{T_x^{(1)}, ..., T_x^{(m)}\} = t, T_x = T_x^{(j)}\}$, $(F_{T_x}(t), S_{T_x}(t)) = ({}_tq_x^{(\tau)}, {}_tp_x^{(\tau)})$.
$\mu_x(t) = \mu_{x+t} = \frac{f_{T_x}(t)}{S_{T_x}(t)}$,

Q. (1) $P(T_x \le 1, J_x = 1) + P(T_x \le 1, J_x = 2) = q_x^{(1)} + q_x^{(2)} = q_x^{(\tau)} = P(T_x \le 1)$

(2) $q_x^{(1)} + p_x^{(1)} = 1$?

${}_tq_x^{(1)} = P(T_x \le t, J_x = 1)$ and ${}_tp_x^{(1)} = P(T_x > t, J_x = 1)$, $q_x^{(1)} + p_x^{(1)} = P(J_x = 1) < 1$.

CHAPTER 10
Pension.

Pension is a series of periodic payments, usually for life, payable monthly or at other specified intervals. If the annual pension benefit B_x is paid monthly at the beginning of each month, the APV at retired age x of whole life pension is $B_x\ddot{a}_x^{(12)}$. Thus it is somewhat similar to the annuity. The term is frequently used to describe the part of a retirement allowance financed by employer contributions. Whereas the annuity is often purchased after retirement.

10.1 Pension Plans.

Two major categories of employer sponsored pension plans are
the **defined contribution (DC)** and the **defined benefit (DB)**.

Defined Contribution Plan. A pension plan in which the contributions are made to an individual account for each employee, as a percentage of salary. The retirement benefit is dependent upon the account balance at retirement. The balance depends upon amounts contributed during the employee's participation in the plan and the investment experience on those contributions.

Defined Benefit Plan. The DB plan specifies a level of benefit, usually in relation to salary near retirement (final salary plans), or to salary throughout employment (career average salary plans). The contributions are accumulated to meet the benefit level. The pension plan actuary monitors the plan funding on a regular basis to assess whether the contributions need to be changed. A pension plan provides a definite benefit formula for calculating benefit amounts - such as a flat amount per year of service; a percentage of salary; or a percentage of salary in the final years of service, and years of service.

The difference between these two plans: A defined benefit plan, most often known as a pension, is a retirement account for which your employer ponies up (pays out) all the money and promises you a set payout when you retire. A defined contribution plan, like a 401(k) or 403(b), requires you to put in your own money.

In general, **defined benefit plans** come in two varieties: traditional pensions and cash-balance plans. In both cases, you just show up for work and, assuming you meet basic eligibility rules, you are automatically enrolled in the plan. (In some instances, however, you are not enrolled until you have completed your first year on the job.) You also need to stick around on the job for several years, typically five, to be fully "vested" in the plan. The difference is in how the benefits are calculated; in a pension, it's based on a formula that takes into account how long you have been on the job and your average salary during your last few years of employment. The cash-balance plan credits your account with a set percentage of your salary each year. Another key difference: If you leave the company before retirement age, you may take the contents of your cash-balance plan as a lump sum and roll it into an IRA. A traditional pension is not portable.

Some employers offer both defined benefit plans and defined contribution plans. If yours does, you should definitely participate in the defined contribution plan as well. That is because more often than not, the amount of your defined benefit plan won't be enough to allow you to live comfortably in retirement.

Hereafter assume that the plan member is employed during the period of concern. Define the **annual rate of salary at age** y, denoted by A_y, through the expression

$$S_y = \int_y^{y+1} A_t dt, \tag{10.1}$$

where S_y is the salary received in year of age y to $y+1$. Notice that S_y and A_y vary person by person, which is not convenient. Define the **rate of salary function** $\overline{s}_y$, $y \geq x_o$, by

$$\overline{s}_y/\overline{s}_x = A_y/A_x \text{ (or } A_y = A_x\overline{s}_y/\overline{s}_x\text{), where } x_o \leq x < y \text{ and } \overline{s}_{x_o} = 1. \tag{10.2}$$

The **salary scale**, $\{s_n\}_{n \geq x_o}$, is defined by

$$\frac{s_y}{s_x} = \frac{\int_0^1 \overline{s}_{y+t}dt}{\int_0^1 \overline{s}_{x+t}dt}, \text{ for } x_o \leq x < y. \tag{10.3}$$

Thus

$$s_y/s_x = S_y/S_x, \text{ for } x_o \leq x < y, \text{ as} \tag{10.4}$$

$$\begin{aligned}
S_y =& \int_y^{y+1} A_t dt && (by\ (10.1)) \\
=& \int_y^{y+1} A_x \frac{\overline{s}_t}{\overline{s}_x} dt && (by\ (10.2)) \\
=& \frac{A_x}{\overline{s}_x} \int_y^{y+1} \overline{s}_t dt \\
=& A_x \frac{s_y}{\overline{s}_x} \frac{\int_0^1 \overline{s}_{x0+t}dt}{s_{x0}} && (by\ (10.3)) \quad => \frac{S_y}{S_x} = \frac{s_y}{s_x}.
\end{aligned}$$

s_y is not uniquely defined by Eq. (10.3) or Eq. (10.4) neither, unless setting $s_{x_o} = 1$ or $s_{x_o} = \int_0^1 \overline{s}_{x_o+t}dt$. It is desirable to set both $s_{x_o} = 1$ and $s_{x_o} = \int_0^1 \overline{s}_{x_o+t}dt$. Is it always possible ?

Example 10.1. *Let* $\overline{s}_x = e^{x-20}$, $x \geq 20 = x_o$. *Then*
$s_{20} = s_{x_o} = \int_0^1 \overline{s}_{x_o+t}dt = \int_0^1 e^{(x_o+t)-20}dt = \int_0^1 e^{(20+t)-20}dt = \int_0^1 e^t dt = e - 1$ *then* $s_{x_o} > 1$.

Remark. The example indicates that we can only impose one constraint, say, $s_{x_o} = \int_0^1 \overline{s}_{x_o+t}dt$ or $s_{x_o} = 1$, but not both most of the time. If we set $s_{x_o} = \int_0^1 \overline{s}_{x_o+t}dt$ then it implies $s_x = \int_0^1 \overline{s}_{x+t}dt$, as $s_x/s_{x_o} = \int_0^1 \overline{s}_{x+t}dt / \int_0^1 \overline{s}_{x_o+t}dt$.

Formulae (10.1) – (10.4) should be memorized.

Example 10.2. *An employee aged 30 whose current annual salary rate is \$30,000.*
(a) Suppose $\overline{s}_y = 1.04^{y-20}$, $y \geq 20$.

(i) What is A_t, his annual rate of salary at age $t \geq 30$?
(ii) What is S_{40}, his salary for the year of age 40 to 41 ?
(b) Suppose that each year the rate increases by 4%, 3 months after his birthday and then remains constant for a year. Answer the previous two questions.

Sol. (a) (A_t, S_{40}) ?
(1) $\underline{S_y} = \int_y^{y+1} A_t dt$, (2) $\overline{s}_y/\overline{s}_x = \underline{A_y/A_x}$, (3) $\overline{s}_{x_o} = \underline{\ 1\ }$, (4) $\frac{s_y}{s_x} = \frac{\int_0^1 \overline{s}_{y+t}dt}{\int_0^1 \overline{s}_{x+t}dt} = \frac{S_y}{S_x}$.
$A_{30} = 30000$ & $\overline{s}_t = 1.04^{t-20}$ => $A_t/30000 = \overline{s}_t/\overline{s}_{30} = 1.04^{t-20}/1.04^{30-20} = 1.04^{t-20-(30-20)}$
=> $A_t = 30000 \times 1.04^{t-30} = \frac{30000}{1.04^{30}} \times 1.04^t$, $t \geq 30$.
$\underline{S_y} = \int_y^{y+1} A_t dt$, => $S_{40} = \frac{30000}{1.04^{30}} \int_{40}^{41} 1.04^t dt = \frac{30000}{1.04^{30}} \frac{(1.04)^t}{\log(1.04))}\Big|_{40}^{41} = 45290.$ $\quad \int a^x dx = \frac{a^x}{\ln a}$
(b) Let $h(t) = \overline{s}_t/\overline{s}_{30} = 1.04^{\lfloor t-29.25 \rfloor}$, ($\lfloor x \rfloor$ ($\lceil x \rceil$) is called the floor (ceiling) function), *i.e.*,

$$h(t) \overset{def}{=} \begin{cases} 1 & \text{if } t \in [30, 30.25) \\ 1.04 & \text{if } t \in [30.25, 31.25) \\ 1.04^2 & \text{if } t \in [31.25, 32.25) \\ \cdots \end{cases} \text{ where } \lfloor x \rfloor = \text{the largest integer which} \leq x.$$

$$\begin{aligned} S_{40} &= \int_{40}^{41} A_t dt = \int_{40}^{41} A_{30}\frac{\overline{s}_t}{\overline{s}_{30}} dt = 30000 \int_{40}^{41} h(t)dt \\ &= 30000(\int_{40}^{40.25} + \int_{40.25}^{41})(1.04)^{\lfloor t-29.25 \rfloor}dt \\ &= 30000(\int_{40}^{40.25}(1.04)^{\lfloor t-29.25 \rfloor}dt + \int_{40.25}^{41}(1.04)^{\lfloor t-29.25 \rfloor}dt) \\ &= 30000(\int_{40}^{40.25}(1.04)^{10}dt + \int_{40.25}^{41}(1.04)^{11}dt) \\ &= 30000(0.25 \times 1.04^{10} + 0.75 \times 1.04^{11}) = 45739.55. \end{aligned}$$

Notice that $S_{41} = 30000 \int_{41}^{42} h(t)dt = 30000(0.25 \times 1.04^{11} + 0.75 \times 1.04^{12})$. It is interesting to see that $S_{41}/S_{40} = 1.04$ and $\overline{s}_{41}/\overline{s}_{40} = 1.04$. It follows that for $(x, y) = (40, 41)$,

$$s_y/s_x = S_y/S_x = \overline{s}_y/\overline{s}_x = A_y/A_x \text{ (versus } \overline{s}_y/\overline{s}_x = A_y/A_x \text{ and } s_y/s_x = S_y/S_x). \tag{10.5}$$

Is it true in general ?

Counterexample 1 to Eq. (10.5). Let $\overline{s}_t = 1 + at$ for $t > 0$, where $a = 1/1000$. Then

$$\frac{s_{y+1}}{s_y} = \frac{\int_0^1 \overline{s}_{t+1}dt}{\int_0^1 \overline{s}_t dt} = \frac{\int_0^1 1 + a(t+1)dt}{\int_0^1 1 + atdt} = \frac{((1+a)t + at^2/2)\big|_0^1}{(t + at^2/2)\big|_0^1} = \frac{1 + 3a/2}{1 + a/2} \neq 1 + a = \frac{\overline{s}_{y+1}}{\overline{s}_y}.$$

Counterexample 2. Skip it !! Under assumption (b) in Example 10.2. If $x \in [40, 40.25)$, then

$$\begin{aligned} \int_0^1 h(x+t)dt =& (\int_0^{40.25-x} + \int_{40.25-x}^1)h(x+t)dt \\ =& \int_0^{40.25-x} 1.04^{10}dt + \int_{40.25-x}^1 1.04^{11}dt \\ =& 1.04^{10}[(40.25 - x - 0) + 1.04(1 - (40.25 - x)] \\ =& 1.04^{10}[-0.04 * 40.25 + 1.04 + 0.04x] \\ =& 1.04^{10}[0.04 * (x - 40.25) + 1.04]. \end{aligned}$$

If $x \in [40.25, 41)$, then

$$\int_0^1 h(x+t)dt = 1.04^{11}[41.25 - x + 1.04(1 + x - 41.25]$$
$$= 1.04^{11}[0.04 * (x - 41.25) + 1.04].$$

In general, letting $w(x) = x - \lfloor x \rfloor$,

$$\int_0^1 h(x+t)dt = \begin{cases} 1.04^{\lfloor x-29.25 \rfloor}[0.25 - w(x) + 1.04(0.75 + w(x))] & \text{if } w(x) < 0.25 \\ 1.04^{\lfloor x-29.25 \rfloor}[1.25 - w(x) + 1.04(w(x) - 0.25)] & \text{if } w(x) \geq 0.25 \end{cases}$$
$$= \begin{cases} 1.04^{\lfloor x-29.25 \rfloor}[0.25 + 1.04 * 0.75 + 0.04w(x))] & \text{if } w(x) < 0.25 \\ 1.04^{\lfloor x-29.25 \rfloor}[1.25 - 1.04 * 0.25 + 0.04w(x))] & \text{if } w(x) \geq 0.25 \end{cases}$$
$$= \begin{cases} 1.04^{\lfloor x-29.25 \rfloor}[1 + 0.04 * 0.75 + 0.04w(x))] & \text{if } w(x) < 0.25 \\ 1.04^{\lfloor x-29.25 \rfloor}[1 - 0.04 * 0.25 + 0.04w(x))] & \text{if } w(x) \geq 0.25. \end{cases}$$

$\int_0^1 h(40.5+t)dt = 1.04^{11}[1 - 0.01 + 0.02] = 1.04^{11} \times 1.01.$

$\int_0^1 h(30+t)dt = 1 + 0.03 = 1.03.$

$s_{40.5}/s_{30} = 1.04^{11}1.01/1.03.$

$\overline{s}_{40.5}/\overline{s}_{30} = 1.04^{11}.$

Thus it is a counterexample to (10.1) even under the assumption of Example 10.2.

Theorem 10.1. *If the rate of salary function is $\overline{s}_y = ca^y$, where $y \geq x_o$ and $a, c > 0$, then Eq. (10.5) holds. Moreover, if one further defines $\overline{s}_{x_o} = s_{x_o} = 1$, then $\overline{s}_y = s_y$.*

Proof. $\vdash$: (10.5) $s_y/s_x = S_y/S_x = \overline{s}_y/\overline{s}_x = A_y/A_x$ $\qquad s_y/s_x = \frac{\int_0^1 \overline{s}_{y+t}dt}{\int_0^1 \overline{s}_{x+1}dt}$.

$\int_0^1 \overline{s}_{y+t}dt = c\int_0^1 a^{y+t}dt = ca^{y+t}\Big|_0^1/\text{ln}a = ca^y(a-1)/\text{ln}a$. Thus $\frac{s_y}{s_x} = \frac{\int_0^1 \overline{s}_{y+t}dt}{\int_0^1 \overline{s}_{x+1}dt} = a^{y-x} = \frac{\overline{s}_y}{\overline{s}_x}$.
By (10.2) and (10.4), $s_y/s_x = S_y/S_x$ and $A_y/A_x = \overline{s}_y/\overline{s}_x$. Thus Eq. (10.5) holds.

Notice that $\overline{s}_y/\overline{s}_{x_o} = s_y/s_{x_o} = a^{y-x_o}$ if $\overline{s}_y = ca^y$, $\forall\ y \geq x_o$. If one further defines

$\overline{s}_{x_o} = s_{x_o} = 1$, then $\overline{s}_y/\overline{s}_{x_o} = s_y/s_{x_o}$ yields $\overline{s}_y = s_y$. ▫ ■

Corollary. If (1) $\overline{s}_x = s_x$ for some x, and (2) either $s_y = ca^y$ or $\overline{s}_y = ca^y$, where $y \geq x_o$ and $a, c > 0$, then $\overline{s}_y = s_y$.

Remark. Under the assumptions in Theorem 1, if $\overline{s}_y = ca^y$ for all $y \geq x_o$, then $\overline{s}_{x_o} = s_{x_o}$ iff $\frac{s_y - \overline{s}_y}{a^{y-x_o}}$ for all $y \geq x_o$. Thus, $\overline{s}_{x_o} = s_{x_o}$ is a necessary condition, which is not always valid. Hereafter, we shall assume that $\overline{s}_x = s_x = 1$ for some $x = x_o$ in such a case.

Example 10.3. *The final average salary for the pension benefit provided by a pension plan is defined as the average salary in the 3 years before retirement.*
(a) A member aged exactly 35 at the valuation date received \$75,000 in salary in the year to the valuation date. Calculate his predicted final average salary assuming retirement at age 65.
(Is S_{34} or S_{35} given ?)

(b) A member aged exactly 55 at the valuation date was paid salary at a rate of \$100,000 per year at that time. Calculate her predicted final average salary assuming retirement at age 65. Make two different assumptions:

(i) the salary scale is $s_y = 1.04^y$;

(ii) the integer age salary scale is given in Table 1 as follows.

x	s_x	x	s_x	x	s_x	x	s_x
30	1	40	2.005	50	2.97	60	3.484
31	1.082	41	2.115	51	3035	61	3.536
32	1.169	42	2.225	52	3.091	62	3.589
33	1.26	43	2.333	53	3.139	63	3.643
34	1.359	44	2.438	54	3.186	64	3.698
35	1.461	45	2.539	55	3.234		
36	1.566	46	2.637	56	3.282		
37	1.674	47	2.73	57	3.332		
38	1.783	48	2.816	58	3.382		
39	1.894	49	2.897	59	3.432		

Table 1. Salary scale

Sol. Formulas: $\underline{S_y} = \int_y^{y+1} A_t dt$, $\overline{s}_y/\overline{s}_x = \underline{A_y/A_x}$, $\overline{s}_{x_o} = \underline{1}$, and $\frac{s_y}{s_x} = \frac{\int_0^1 \overline{s}_{y+t} dt}{\int_0^1 \underline{\overline{s}_{x+t}} dt} = \frac{S_y}{S_x}$.

(a) $\sum_{y=62}^{64} S_y/3$=? Since $S_{34} = 75000$ (**why** $\neq S_{35}$ **?**) and $S_y/S_{34} = s_y/s_{34}$,

$$\sum_{y=62}^{64} S_y/3 = S_{34}\frac{s_{62}+s_{63}+s_{64}}{3s_{34}} = 75000\frac{s_{62}+s_{63}+s_{64}}{3s_{34}}. \tag{10.6}$$

(i) Use the salary scale $s_y = 1.04^y$ in (10.6) (using R program):

```
> i=62:64
> 75000*sum(1.04**i)/1.04**34/3
```

$75000\frac{1.04^{62}+1.04^{63}+1.04^{64}}{3\times 1.04^{34}}$

```
[1] 234018.8
```

(ii) Use the salary table:

```
> 75000*(3.589+3.643+3.698)/3/1.359
[1] 201067
```

(b) Given $A_{55} = 10^5$, by Eq.(10.2), $A_t/A_{55} = \overline{s}_t/\overline{s}_{55}$.

(i) let $x_o = 0$, then $s_0 = (1.04)^0 = 1$. $\overline{s}_0 = 1$ by convention. Thus $\overline{s}_y = s_y = 1.04^y$ by the previous corollary.

$S_x = \int_x^{x+1} A_t dt = \int_x^{x+1} A_{55}(\overline{s}_t/\overline{s}_{55})dt = A_{55}\int_x^{x+1} 1.04^{t-55}dt = A_{55}1.04^{-55}\int_x^{x+1} 1.04^t dt$
$= A_{55}1.04^{x-55}\frac{1.04-1}{\log 1.04} \approx A_{55}1.04^{x-55}$. $\qquad \int a^x dx = a^x/\ln a$

Thus the final average salary is

$$\sum_{y=62}^{64} S_y/3 \approx A_{55}(s_{62}+s_{63}+s_{64})/(3\overline{s}_{55}) = \frac{10^5(s_{62}+s_{63}+s_{64})}{3\overline{s}_{55}}. \tag{10.7}$$

```
> i=62:65
> 10**5*sum(1.04**i)/1.04**55/3
  [1] 186268.6 # the final average salary.
```

(ii). Let $x_o = 30$, then $s_{30} = 1$. Since $\overline{s}_{30}$ is not specified, one can define $\overline{s}_t = 1$ if $t \in [30, 31)$, then $\int_0^1 \overline{s}_{30+t}dt = 1$ and $s_y = s_y/s_{30} = \frac{\int_0^1 \overline{s}_{y+t}dt}{\int_0^1 \overline{s}_{30+t}dt} = \int_0^1 \overline{s}_{y+t}dt$ and

$S_x = \int_x^{x+1} A_t dt = A_{55} \int_x^{x+1} (\overline{s}_t/\overline{s}_{55})dt = A_{55}s_x/\overline{s}_{55}$.

Thus the final average can also be Eq. (10.7) but $\overline{s}_{55}$ is unknown now. Notice the difference between Equations (10.6) and (10.7).

Since $\overline{s}_{55}$ is unknown, there are **two steps** of approximations:

Step (1) Letting $\overline{s}_{55} \approx s_{54.5}$, *i.e.*,

$$\overline{s}_t \approx s_{t-0.5} \tag{10.8}$$

Step (2) linear interpolation $s_{54.5} = (s_{54} + s_{55})/2$.

The final average salary

$$\sum_{y=62}^{64} S_y/3 \approx 100000\frac{s_{62}+s_{63}+s_{64}}{3s_{54.5}} \approx 100000\frac{s_{62}+s_{63}+s_{64}}{1.5(s_{54}+s_{55})}.$$

```
> 10**5*(3.589+3.643+3.698)/(3*(3.186+3.282)/2)
  [1] 112657.2
```

Justification of (10.8): *If $\overline{s}_{x_o} = 1 = s_{x_o}$ and $\overline{s}_u$ is continuous and non-decreasing in u, $s_t/s_{x_o} = \frac{\int_0^1 \overline{s}_{t+u}du}{\int_0^1 \overline{s}_{x_o+u}du} = \frac{\overline{s}_x}{\overline{s}_\xi}$ for an $x \in [t, t+1]$ and $\xi \in [0,1]$ by the mean value theorem. That is, $s_t \cdot \overline{s}_\xi = \overline{s}_x$. It follows that $s_t \le \overline{s}_x$, where $x \in [t, t+1]$. Actually, it is often that $\overline{s}_\xi \approx 1$. Thus $s_t \approx \overline{s}_x$. Then choose $x = t + 0.5 => s_t = \overline{s}_{t+0.5}$ or $s_{x-0.5} = \overline{s}_x$.*

Remark. Under assumption (i) but using Eq. (10.8).

```
> i=62:65
> 10**5*sum(1.04**i)/1.04**54.5/3
  [1] 189957.4 # compare to [1] 186268.6, the original outcome
```

Remark. From the previous example, one can see that if the expressions of $\overline{s}_y$ and s_y are not both given, then one can define $s_y = \int_y^{y+1} \overline{s}_t dt$ if it is needed.

Definition. The **pension replacement ratio** is
$R = \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}}$ **(used in the problems of this chapter)**,
or $R = \frac{\text{pension income in the year after retirement}}{\text{final salary before retirement}}$, where
we assume that the plan member survives the year following retirement. In a DC pension plan, a certain amount of money is put into an account, the amount is proportional to salary, which rate is called the **contribution rate**.

annual salary S_y, annual salary rate A_y, salary rate function $\overline{s}_y$, salary scale s_y,

$A_y/A_x = \overline{s}_y/\overline{s}_x$ and $S_y/S_x = s_y/s_x$, one sets $s_y = \int_y^{y+1} \overline{s}_t dt$ if no relation about $\overline{s}_y$ and s_y.

Example 10.4. *An employer establishes a DC pension plan. On withdrawal from the plan before retirement age of 65, for any reason, the proceeds of the invested contributions are paid to the employee's survivors. The contribution rate is set using the following assumptions.*
(1) The employee will use the proceeds at retirement to purchase a pension for his lifetime, plus a reversionary annuity for his wife at 60% of the employee's pension (the wife gets pension after she survives the member).
(2) At age 65, the employee is married and the age of his wife is 61.
(3) the target replacement ratio is $R = 65\%$.
(4) The salary rate function is $\overline{s}_y = 1.04^y$ and salaries are assumed to increased continuously.
(5) Contributions are payable monthly in arrears at a fixed percentage of the salary <u>*rate*</u> **at that time**.
(6) Contributions are assumed to earn investment returns of 10% per year.
(7) Annuity purchased at retirement are priced assuming an interest rate of 5.5% per year.
(8) Survival: Makeham's law $\mu_x = \begin{cases} 0.0004 + 4 \times 10^{-6} \times 1.13^x & male \\ 0.0002 + 10^{-6} \times 1.135^x & female \end{cases}$
(9) Members and their spouses are independent w.r.t. mortality.
Consider a male new entrant aged 25.
(a) Calculate his contribution rate required to meet the target replacement ratio.
(b) To be specified later.

Sol. (a) Find the contribution rate c through $T(c) = E \times B$.
$T(c)$= the total contributions to pension at retirement=? E=unit annuity=?
B=annual pension, $B = R \cdot S_{64}$, $R = 0.65$ (see (3)). (T(c), E, S_{64})=?

$$T(c) = \frac{1}{12} \sum_{i=1}^{(65-25)12} \underbrace{cA_{25+i/12}}_{by\ (5)} \underbrace{1.1^{65-25-i/12}}_{by\ (6)} = \frac{1}{12} \sum_{i=1}^{(65-25)12} cA_{25} \frac{\overline{s}_{25+i/12}}{\overline{s}_{25}} 1.1^{40-i/12}, \quad \frac{A_y}{A_x} = \frac{\overline{s}_y}{\overline{s}_x}, \quad (x,y) = ?$$

$$T(c) = \frac{cA_{25}}{12} \sum_{i=1}^{(65-25)12} \underbrace{\frac{1.04^{25+i/12}}{1.04^{25}}}_{by\ (?)} 1.1^{40-i/12} = \frac{cA_{25}}{12} 1.1^{40} \sum_{i=1}^{(65-25)12} (\frac{1.04}{1.1})^{i/12} = 719.6316cA_{25}, \quad (10.4.1)$$

$$S_y = \int_y^{y+1} A_t dt, \quad A_y/A_x = \overline{s}_y/\overline{s}_x = S_y/S_x = s_y/s_x = 1.04^{y-x} \text{ as } \overline{s}_y = 1.04^y \text{ by (4).} \quad (10.4.2)$$

$$S_{64} = 1.04^{39} S_{25}, \text{ as } S_{64}/S_{25} = \overline{s}_{64}/\overline{s}_{25} = 1.04^{39}$$

$$S_{25} = \textstyle\int_{25}^{26} A_t dt = A_{25} \int_{25}^{26} 1.04^{t-25} dt = A_{25} \int_0^1 1.04^t dt = A_{25} \frac{1.04^1 - 1.04^0}{\log 1.04} \approx A_{25}.$$

$$B = RS_{64} = 0.65 \times 1.04^{39} S_{25} = 3.000638 S_{25} \approx 3.000638 A_{25}.$$

$$E = (\ddot{a}_{65}^{(12)} + 0.6(\ddot{a}_{61}^{(12)} - \ddot{a}_{65:61}^{(12)})) = 10.5222 + 0.6 \times (13.92 - 10.01) \ \ (see\ (1)) \quad (10.4.3)$$

(10.4.3) is proved later on.

$$\begin{aligned} T(c) =& E \times B = (10.5222 + 0.6 \times (13.92 - 10.01)) 3.000638 A_{25} \\ =& 38.61785 A_{25} = 719.6316 c A_{25} \qquad by\ (10.4.1). \end{aligned}$$

It yields $c \approx 5.37\%$, the contribution rate. Now the details of (10.4.3):

$\ddot{a}^{(12)}_{65:61} = \frac{1}{12}\sum_{i=0}^{\infty} \underbrace{1.055^{-i/12}}_{by\ (7)}\,{}_{i/12}p_{65:61}$, $\ddot{a}^{(12)}_{65} = \frac{1}{12}\sum_{i=0}^{\infty} v^{i/12}\,{}_{\frac{i}{12}}p_{65}$, and $\ddot{a}^{(12)}_{61} = \frac{1}{12}\sum_{i=0}^{\infty} v^{i/12}\,{}_{\frac{i}{12}}p_{61}$,

with $v = 1/1.055$, and $({}_tp_{65}, {}_tp_{61}, {}_tp_{65:61})$ are derived as follows.

$$
\begin{aligned}
{}_tp_{65} &= \exp(-\int_0^t \mu_{65+x,male}dx) \\
&= \exp(-\int_0^t (0.0004 + 4*10^{-6}*1.13^{65+x})dx) \qquad ?? \\
&= \exp(-0.0004t - 4*10^{-6}*1.13^{65}\frac{1.13^t-1}{\log(1.13)}) \qquad \int a^x dx = a^x/\ln a \\
{}_tp_{61} &= \exp(-\int_0^t \mu_{61+x,female}dx) \qquad ?? \\
&= \exp(-\int_0^t (0.0002 + 10^{-6}*1.135^{61+x})dx) \\
&= \exp(-0.0002t - 10^{-6}*1.135^{61}\frac{1.135^t-1}{\log(1.135)}) \\
{}_tp_{65:61} &= {}_tp_{65}\,{}_tp_{61} = \exp(a)\exp(b) = \exp(a+b) \\
&= \exp(-0.0004t - 4*10^{-6}*1.13^{65}\frac{1.13^t-1}{\log(1.13)} - 0.0002t - 10^{-6}*1.135^{61}\frac{1.135^t-1}{\log(1.135)})
\end{aligned}
$$

R program for computing the above quantities:

```
v=(1/1.055)
t=(0:1000)/12
a=v**t*exp(-0.0004*t-0.000004*1.13**65*(1.13**t-1)/log(1.13))
b=v**t*exp(-0.0002*t-0.000001*1.135**61*(1.135**t-1)/log(1.135))
c=v**t*exp(-0.0004*t-0.000004*1.13**65*(1.13**t-1)/log(1.13) -0.0002*t-0.000001
*1.135**61*(1.135**t-1)/log(1.135))
sum(a)/12                # ä^(12)_65
sum(b)/12                # ä^(12)_61
sum(c)/12                # ä^(12)_65:61
```

Additional Homework 1. Redo part (a) in Example 10.4 by revise the condition (8) in Example 10.4 as (8) Survival: Makeham's law $\mu_x = \begin{cases} 0.0003 + 4\times 10^{-6}\times 1.132^x & male \\ 0.0001 + 10^{-6}\times 1.137^x & female \end{cases}$

Example 10.5. (b) Assume now that
(1) the contribution rate will be 5.5% of salary, (denoted by c),
(2) and that over the member's career, his salary will actually increase by 5% per year,
(3) investment return will be only 8% per year,
(4) the interest rate for calculating annuity values at retirement will be 4.5% per year.
Calculate the actual replacement ratio for the member.

Sol. Solve for R from Eq.s $BE = T$ and $B = RS_{64}$, where B =annual pension, E=unit annuity,

and T is the total contribution to pension. Thus $R = T/(ES_{64})$.
By Theorem 10.1 (page 90), the salary received in the year before retirement S_{64} is

$$\begin{aligned}
S_{64} =& S_{25}\overline{s}_{64}/\overline{s}_{25} \approx A_{25}\overline{s}_{64}/\overline{s}_{25} = 1.05^{64-25}A_{25} = 6.704751A_{25} \\
T =& \frac{1}{12}\sum_{i=1}^{(65-25)12} cA_{25+i/12}1.08^{40-i/12} = \frac{1}{12}\sum_{i=1}^{(65-25)12} cA_{25}\frac{\overline{s}_{25+i/12}}{\overline{s}_{25}}1.08^{40-i/12} \\
=& \frac{cA_{25}}{12}\sum_{i=1}^{(65-25)12} 1.05^{i/12}1.08^{40-i/12} \qquad\qquad as\ \ \overline{s}_y = 1.05^y \\
=& \frac{0.055A_{25}}{12}(1.08^{40})r\frac{1-r^{480}}{1-r}\Big|_{r=(1.05/1.08)^{1/12}} \\
\approx& 28.636A_{25}. \\
E =& \ddot{a}_{65}^{(12)} + 0.6(\ddot{a}_{61}^{(12)} - \ddot{a}_{65:61}^{(12)}) = 11.3576 - 0.6(15.4730 - 10.7579) = 14.1867
\end{aligned}$$

as $\ddot{a}_{65:61}^{(12)} = \frac{1}{12}\sum_{i=0}^{\infty} 1.045^{i/12}{}_{i/12}p_{65:61}$, ...

$R = T/ES_{64} = \frac{28.636A_{25}}{14.1867\times 6.704751A_{25}} \approx 0.301 = 30.1\%$.

The next derivation using $\overline{s}_t \approx s_{t-0.5}$ (Eq. (10.8)) rather than $s_y/s_x = \overline{s}_y/\overline{s}_x$ (Th.10.1). There are minor difference in results.

$$\begin{aligned}
S_{64} =& A_{25}\underbrace{s_{64}}_{unknown}/\overline{s}_{25} \qquad\qquad letting\ s_y = \int_y^{y+1}\overline{s}_t dt \\
\approx& A_{25}\overline{s}_{64.5}/\overline{s}_{25} \quad \text{(or } A_{25}s_{64}/s_{24.5} \text{ by Eq. (10.8))} \\
=& 1.04^{39.5}A_{25} = 4.7078A_{25}.
\end{aligned}$$

The target pension benefit is $B = RS_{64} = 0.65 \times 4.7078A_{25} = 3.0601A_{25}$.

$T = B(\ddot{a}_{65}^{(12)} + 0.6(\ddot{a}_{61}^{(12)} - \ddot{a}_{65:61}^{(12)})) = B\cdot(10.5222 + 0.6\times 3.9128) = 39.3826A_{25}$.
The total contributions at retirement is $T(c) = 719.6316cA_{25}$.
$T(c) = P$ yields $c = 5.4725\%$ per year.
(b) By Eq.(10.4), the total contributions at retirement with $c = 5.5\%$ is

$T = \frac{1}{12}\sum_{i=1}^{(65-25)12} cA_{25}\frac{\overline{s}_{25+i/12}}{\overline{s}_{25}}1.08^{40-i/12} \approx 28.636A_{25}$ as the previous derivation.

By Eq. (10.8), the salary received in the year before retirement S_{64} is

$S_{64} = A_{25}s_{64}/\overline{s}_{25} \approx A_{25}s_{64}/s_{24.5} = 1.05^{64-24.5}A_{25} = 6.8703A_{25}$.
The target pension benefit is $6.8703A_{25}R = B$.
The APV at retirement of total pension is

$T = B(\ddot{a}_{65}^{(12)} + 0.6(\ddot{a}_{61} - \ddot{a}_{65:61}^{(12)}))$.

$\ddot{a}_{65:61}^{(12)} = \frac{1}{12}\sum_{i=0}^{\infty} 1.045^{i/12}{}_{i/12}p_{65:61}$,
where ${}_{i/12}p_{65:61} = P((65)$ and (61) survive $i/12$ years).
$T = B(11.3576 - 0.6(15.4730 - 10.7579)) = 14.1867B = 14.1867 \times 6.8703A_{25}R$.

$14.1867 \times 6.8703A_{25}R = T = 28.636A_{25}$
Thus $R = 29.38\%$.

In a DB pension plan, a member of the plan may either retire at the normal age or exit from the plan for various ways, such as withdraw (to take another job), disability retirement, normal retirement (or age retirement), died in service. We assume that the exit modes are independent of each other.

The demographic elements of the basis for pension plan calculations include assumptions about survival model for members and their spouses, and about the exit patterns from employment.

The members might withdraw to take another job with a different employer at earlier ages, and may be offered a range of ages at which they may retire with the pension that they have accumulated. Some will die while in employment and another group leave early through disability retirement.

In a DC plan, the benefit on exit is the same, regardless the reason for the exit. Thus there is no need to model the member employment patterns. But there is the need for the DB plan.

10.2 Multiple decrement model for a DB pension plan

Hereafter, we use the following notations:

00: no exit;
01: withdrawn from the pension plan;
02: disability retirement;
03: age retirement;
04: died in service.
${}_tp_x^{01} = P(\text{withdrawal exit time} > x+t|\text{the member survives } x)$,
${}_tp_x^{02} = P(\text{disable exit time} > x+t|\text{the member survives } x)$,
${}_tp_x^{03} = P(\text{retirement time} > x+t|\text{the member survives } x)$,
${}_tp_x^{04} = P(\text{death exit time} > x+t|\text{the member survives } x)$.

Remark. Under the multiple decrement model, we have $T_x^{(1)}$, $T_x^{(2)}$, $T_x^{(3)}$, $T_x^{(4)}$ and let $T_x^{(\tau)} = min_i T_x^{(i)}$, distinguishing from $T_x = (X-x)|(X>x)$, where X is the survival time of a person. $T_x^{(\tau)}$ is the exist time of a person from the employment since x.
Death may happen after withdrawal or other exit. So

$${}_tp_x^{04} \neq {}_tp_x = P(T_x > t) \neq P(T_x^{(\tau)} > t).$$

$${}_tp_x^{0i} = {}_tp_x^{'(i)} = P(T_x^{(i)} > t) = P(T_x^{(i)} > t, J_x = i) + P(T_x^{(i)} > t, J_x \neq i) \neq P(T_x^{(\tau)} > t, J_x = i).$$

$${}_tp_x^{(i)} = P(T_x^{(\tau)} > t, J_x = i).$$

$${}_tp_x^{00} = {}_tp_x^{(\tau)} = P(T_x^{(\tau)} > t) = P(\text{exit time} > x+t|(x)).$$

$${}_tp_x = P(T_x > t) = P(\text{age at death} > x+t|(x)) = P(X > x+t|X > x).$$

Using the notation before, if the survival time is continuous, then
${}_tp_x^{01} = {}_tp_x^{'(1)} = \exp(-\int_0^t \mu_{x+s}^{01} ds) = \exp(-\int_x^{x+t} \mu_s^{01} ds)$? or $= \exp(-\int_x^t \mu_s^{01} ds)$?
${}_tp_x^{02} = {}_tp_x^{'(2)} = \exp(-\int_0^t \mu_{x+s}^{02} ds)$,
${}_tp_x^{03} = {}_tp_x^{'(3)} = \exp(-\int_0^t \mu_{x+s}^{03} ds)$,
${}_tp_x^{04} = {}_tp_x^{'(4)} = \exp(-\int_0^t \mu_{x+s}^{04} ds)$.

Example 10.5. *Assume in a pension plan that*
(1) 30% of the members surviving in employment to age 60 retire at 60 and all members who remain in employment to age 65 retire then;

(2) $\mu_x^{01} = \mu_x^w = \begin{cases} 0.1 & \text{for } x \in [0,35) \\ 0.05 & \text{for } x \in [35,45) \\ 0.02 & \text{for } x \in [45,60); \end{cases}$

(3) $\mu_x^{02} = \mu_x^i = 0.001$ *(we actually only need to know for* $x \in (0,65)$*);*

(4) $\mu_x^{03} = \mu_x^r = \begin{cases} 0 & \text{for } x \in [0,60) \\ 0.3 & \text{discrete at } x = 60 \\ 0.1 & \text{for } x \in (60,65]; \end{cases}$

(5) $\mu_x^{04} = \mu_x^d = A + B \times c^x$*,* $A = 0.00022$*,* $B = 2.7 \times 10^{-6}$ *and* $c = 1.124$*.*
(a) P(retiring at age 65 |age 35) = ?
(b) For each mode of exit, calculate the probability that a member currently aged 35 exits employment by that mode.
(c) Find the probability that a member currently aged 35 survival to age 65 (assuming $\mu_x = \mu_x^{04}$*).*

Sol. (a) P(retiring at age 65|age 35) = ?

P(retiring at age 65|age 35) < P(being retired by age 65|age 35) = 1,

P(retiring at age 65|age 35) =P($T_{35}^{(\tau)} > 30$) $= {}_{30}p_{35}^{00}$.

$${}_tp_x^{00} = {}_tp_x^{(\tau)} = P(T_x^{(\tau)} > t) = \underbrace{{}_tp_x^{01} \cdot {}_tp_x^{02} \cdot {}_tp_x^{03} \cdot {}_tp_x^{04}}_{method\ 1} \qquad (= \textstyle\prod_{i=1}^4 P(T_x^{(i)} > t))$$

$$= \underbrace{\exp(-\int_0^{(x+t)\wedge 60} \sum_{i=1}^4 \mu_{x+s}^{0i} ds) \times [(1-0.3)\exp(-\int_{60}^{(x+t)} \sum_{i=1}^4 \mu_{x+s}^{0i} ds)]^{\mathbf{1}((x+t)>60)}}_{method\ 2}.$$

Method 1: ${}_{30}p_{35}^{00} = {}_{30}p_{35}^{01} \cdot {}_{30}p_{35}^{02} \cdot {}_{30}p_{35}^{03} \cdot {}_{30}p_{35}^{04} = ?$ $\qquad ({}_tp_x^{0i} = \exp(-\int_0^t \mu_{x+s}^{0i} ds))$

For $t \in [0,30]$,
$$\begin{aligned}
{}_tp_{35}^{01} &= \exp(-\int_0^t \mu_{35+x}^{01} dx) = \exp(-\int_0^{t\wedge 10} 0.05dx - \int_{t\wedge 10}^{t\wedge 25} 0.02dx) \quad by\ (2) \\
&= \exp\{-0.05(t\wedge 10) - 0.02[(t\wedge 25) - (t\wedge 10)]\}; \\
{}_tp_{35}^{02} &= \exp(-\int_0^t \mu_{35+x}^{02} dx) = \cdots = e^{-0.001t}; \\
{}_tp_{35}^{04} &= \exp(-\int_0^t \mu_{35+x}^{04} dx) = e^{-At - B\frac{c^x}{\ln c}\Big|_0^t} = e^{-At - B\frac{c^t - 1}{\ln c}}; \\
{}_tp_{35}^{03} &= \begin{cases} 1 & t < 25 \\ 0.7\exp(-\int_{25}^t \mu_{35+x}^{03} dx) = 0.7e^{-0.1(t-25)} & 25 \le t < 30, \end{cases} \quad by\ (4) \\
&= I(t < 25) + I(t \in [25,30))0.7e^{-0.1(t-25)}
\end{aligned}$$

Method 2: ${}_{30}p_{35}^{00} = {}_{10}p_{35}^{00} \cdot {}_{15}p_{45}^{00} \cdot {}_{5}p_{60}^{00} = \exp(-(\int_{35}^{45} + \int_{45}^{60} + \int_{60}^{65})\mu_x^{00}dx) = ?$

$$\mu_x^{00} = \mu_x^{01} + \mu_x^{02} + \mu_x^{03} + \mu_x^{04} = \begin{cases} 0.05 + 0.001 + 0 + A + Bc^x & x \in (35, 45) \\ 0.02 + 0.001 + 0 + A + Bc^x & x \in (45, 60) \text{ \textbf{why open ?}} \\ 0 + 0.001 + 0.1 + A + Bc^x & x \in (60, 65) \end{cases}$$

$$\begin{aligned}
{}_{t}p_{35}^{00} =& \exp(-\int_{35}^{35+t}(0.05 + 0.001 + A + Bc^x)dx),\ t \in [0, 10) \\
& {}_{10}p_{35}^{00} = \exp(-\int_{35}^{45}(0.051 + A + Bc^x)dx) = \exp(-[10(0.051 + A) + B\frac{c^{45} - c^{35}}{\ln c}]) \ = \text{(a) in R} \\
{}_{t}p_{45}^{00} =& \exp(-\int_{45}^{45+t}(0.021 + A + Bc^x)dx), \quad t \in [0, 15), \\
& {}_{15-}p_{45}^{00} = \exp(-\int_{45}^{60}(0.021 + A + Bc^x)dx) = {}_{15}p_{45}^{00} \qquad \textbf{???} \qquad (10.5.1) \\
{}_{15}p_{45}^{00} =& 0.7\,{}_{15-}p_{45}^{00} = 0.7\exp(-[15(0.021 + A) + B\frac{c^{60} - c^{45}}{\ln c}]). \qquad = \text{(0.7b), b is in R} \\
{}_{t}p_{60}^{00} =& \exp(-\int_{60}^{60+t}(0.101 + A + Bc^x)dx), \quad t \in (0, 5), \\
& {}_{5}p_{60}^{00} = \exp(-[5(0.101 + A) + B\frac{c^{65} - c^{60}}{\ln c}]) \qquad \textit{see R next}
\end{aligned}$$

Using R to compute:
> A=0.00022
> B=0.0000027
> C=1.124
> (a=exp(-(10*(0.05+0.001+A)+B*(C**45-C**35)/log(C))))
  [1] 0.5973421 # ${}_{10}p_{35}^{00}$
> b=exp(-(15*(0.02+0.001+A)+B*(C**60-C**45)/log(C)))
> a*b
  [1] 0.4253701 # ${}_{25-}p_{35}^{00}$
> 0.7*a*b
  [1] 0.297759 # ${}_{25}p_{35}^{00}$
> c=exp(-(5*(0.1+0.001+A)+B*(C**65-C**60)/log(C)))
> c*0.7*a*b
  [1] 0.1758789 # ${}_{30}p_{35}^{00} = 0.1759$, **the final answer.**
Answer to (a): P(retiring at age 65|age 35) $\approx$ 18%.

(b) P(a member currently aged 35 exits employment by mode j) $= {}_{0}p_x^{(j)} = ?\ j \in \{1, 2, 3, 4\}$.
${}_{0}p_x^{(j)} = P(T_x^{(\tau)} > 0, J_x = j) = \int f^{0j}(t)dt$, where f^{0j} is the corresponding density.

$f^{0j}(t) = \underbrace{{}_{t-}p_{35}^{00}}_{see\ (a)} \times \underbrace{\mu_{35+t}^{0j}}_{given}$ **Why ?**

The force of mortality or hazard function of a random variable X: $\mu_X(t) = \frac{f_X(t)}{S_X(t-)}$

$f_X(t) = S_X(t-)\mu_X(t)$ yieds ${}_{t-}p_x^{00} \cdot \mu_{x+t}^{0j}$ $(={}_tp_{35}^{00}\mu_{35+t}^{0j}$ if ${}_tp_{35}^{00}$ is continuous at t).

There are 4 modes. Start with Mode (3) (normal retired):

$$(3) \quad {}_0p_x^{(3)} = \underbrace{0.3 \cdot {}_{25-}p_{35}^{00}}_{30\%\ retire} + \underbrace{1 \cdot {}_{30-}p_{35}^{00}}_{100\%\ retire} + \int_{25}^{30} {}_tp_{35}^{00} \times 0.1dt \qquad \textbf{why ??}$$

$$= \underbrace{0.3 \cdot {}_{25-}p_{35}^{00} + {}_{30-}p_{35}^{00} + 0.1{}_{25}p_{35}^{00}}_{known} \underbrace{\int_0^5 {}_tp_{60}^{00}dt}_{\text{by R next}} \approx 0.4193, \qquad \text{see R codes below}$$

$$\int_0^5 {}_tp_{60}^{00}dt = \int_0^5 \exp(-\int_{60}^{60+t}(0 + 0.001 + 0.1 + A + Bc^x)dx)dt \qquad \mu\ for\ 4\ modes$$

$$= \int_0^5 \underbrace{\exp(-(0.001 + 0.1 + A)t - \frac{Bc^{60}}{\ln c}(c^t - 1))}_{=g} dt \qquad \text{computed by R :}$$

```
> t=(1:5000)/1000
> g=exp(-(0.1+0.001+A)*t- B*C**60 * (C**t-1) /log(C))
> 0.3*a*b+ c*0.7*a*b +0.1*0.7*a*b*sum(g)/1000
[1] 0.4193452
```

$(\int_a^b g(x)dx = \sum_{i=1}^n g(a + i\frac{b-a}{n})\frac{b-a}{n})$

a*b =0.4253701 (see (a) and 0.7b in R codes in p.98) $(a, b, n) = ?$

(1) The probability that a member currently aged 35 withdraws

$$={}_0p_x^{(1)} = P(T_x^{(\tau)} > 0, J_x = 1) = \int_0^{30} {}_tp_{35}^{00}\mu_t^{01}dt$$

$$= \int_0^{10} {}_tp_{35}^{00}\mu_t^{01}dt + \int_{10}^{25} {}_tp_{35}^{00}\mu_t^{01}dt + \int_{25}^{30} {}_tp_{35}^{00}\mu_t^{01}dt = 0.05\int_0^{10} {}_tp_{35}^{00}dt + 0.02\int_{10}^{25} {}_tp_{35}^{00}dt$$

$$=0.05\int_0^{10} \exp(-[t(0.05 + 0.001 + A) + B\frac{c^{t+35} - c^{35}}{\ln c}])dt + 0.02{}_{10}p_{35}^{00}\int_0^{15} {}_tp_{45}^{00}dt \qquad (1.1)$$

$$=0.05\int_0^{10} \exp(-[t(0.05 + 0.001 + A) + B\frac{c^{t+35} - c^{35}}{\ln c}])dt$$

$$+ 0.02\exp(-[10(0.05 + 0.001 + A) + B\frac{c^{45} - c^{35}}{\ln c}])\int_0^{15} \exp(-[t(0.021 + A) + B\frac{c^{t+45} - c^{45}}{\ln c}])dt$$

≈ 0.5432

(2) The probability that a member currently aged 35 disability exits

$$={}_0p_x^{(2)} = P(T_x^{(\tau)} > 0, J_x = 2) = \int_0^{30} {}_tp_{35}^{00}\mu_t^{02}dt = 0.001\int_0^{30} {}_tp_{35}^{00}dt = 0.001[\int_0^{10} + \int_{10}^{25} + \int_{25}^{30}]{}_tp_{35}^{00}dt$$

$$=0.001[\underbrace{\int_0^{10} {}_tp_{35}^{00}dt}_{see\ (1.1)\ above} + \underbrace{{}_{10}p_{35}^{00}\int_0^{15} {}_tp_{45}^{00}dt}_{see\ (1.1)\ above} + \underbrace{{}_{25}p_{35}^{00}\int_0^5 {}_tp_{60}^{00}dt}_{?}]$$

≈ 0.0166 **how ?**

(4) The probability that a member currently aged 35 death exits

$$={}_0p_x^{(4)} = P(T_x^{(\tau)} > 0, J_x = 4) = \int_0^{30} {}_tp_{35}^{00}\mu_t^{04}dt$$

$$=A \underbrace{\int_0^{30} {}_tp_{35}^{00}dt}_{known\ from\ (2)} + B \underbrace{\int_0^{30} {}_tp_{35}^{00}c^t dt}_{homework} \approx 0.0208.$$

Notice $0.4193 + 0.5432 + 0.0166 + 0.0208 = 0.9999$.

$\sum_{i=1}^4$ P(a member currently aged 35 exits employment by mode j)=1

Why $\neq 0.9999$ **?**

$\sum_{i=1}^4$ P(a member currently aged 35 exits employment by mode j)
= P(a member currently aged 35 exits employment).

$\sum_{i=1}^4$ P(a member in mode j currently aged 35 exits employment)= 4 or 1 ?

(c) The probability that a member currently aged 35 survival to age 65 (assuming $\mu_x = \mu_x^{04}$).

${}_{30}p_{35} = \exp(-\int_{35}^{65} \mu_x^{04}dx) = \exp(-30A - B\frac{c^{65}-c^{35}}{\ln c})$ ($\mu_x = A + B \times c^x$).

${}_{30}p_{35} = P(T_{35} > 30)$?? $\quad T_x^{(\tau)} = \min_{i\in\{1,2,3,4\}} T_x^{(i)}$=exit time of (x)$\neq$survival time of (x).

The service table

Notations:

l_x = total number of persons at age x;

w_x = number of persons withdraw at age x;

i_x = number of persons retired due to illness or disability at age x;

r_x = number of persons retired normally at age x;

d_x = number of persons died at age x.

$l_{x_o+k} = l_{x_o} \cdot {}_kp_{x_o}^{00}$,

$w_{x_o+k} = l_{x_o}{}_kp_{x_o}^{00} \cdot p_{x_o+k}^{01}$,

$i_{x_o+k} = l_{x_o}{}_kp_{x_o}^{00} \cdot p_{x_o+k}^{02}$,

$r_{x_o+k} = l_{x_o}{}_kp_{x_o}^{00} \cdot p_{x_o+k}^{03}$,

$d_{x_o+k} = l_{x_o}{}_kp_{x_o}^{00} \cdot p_{x_o+k}^{04}$.

x	l_x	w_x	i_x	r_x	d_x	x	l_x	w_x	i_x	r_x	d_x
20	1000000	95104	951	0	237	44	137656	6708	134	0	95
21	903707	85846	859	0	218	45	130719	2586	129	0	100
22	816684	77670	777	0	200	46	127904	2530	127	0	106
23	738038	70190	702	0	184	47	125140	2476	124	0	113
24	666962	63430	634	0	170	48	122428	2422	121	0	121
25	602728	57321	573	0	157	49	119763	2369	118	0	130
26	544677	51800	518	0	145	50	117145	2317	116	0	140
27	492213	46811	468	0	134	51	114572	2266	113	0	151
28	444800	42301	423	0	125	52	112042	2216	111	0	163
29	401951	38226	382	0	117	53	109553	2166	108	0	176
30	363226	34543	345	0	109	54	107102	2118	106	0	190
31	328228	31215	312	0	102	55	104688	2070	103	0	206
32	296599	28201	282	0	96	56	102308	2023	101	0	224
33	268014	25488	255	0	91	57	99960	1976	99	0	243
34	242181	23031	230	0	86	58	97642	1930	96	0	264
35	218834	10665	213	0	83	59	95351	1884	94	0	288
36	207872	10131	203	0	84	60−	93085?	0	0	27926	0
37	197455	9623	192	0	84	60+	65160?	0	62	6188	210
38	187555	9141	183	0	85	61	58700	0	56	5573	212
39	178147	8682	174	0	86	62	52860	0	50	5018	213
40	169206	8246	165	0	87	63	47579	0	45	4515	214
41	160708	7832	157	0	89	64	42805	0	41	4061	215
42	152631	7438	149	0	90	65−	38488	0	0	38488	0
43	144954	7064	141	0	93						

Table 2. Pension plan service table

In Table 2, 60− and 65− indicate the cases that the retirement occurs exactly at ages 60 and 65. 60+ indicates that the case that the event happens during age $(60, 61)$. l_x **is not the # of people alive, but staying on**. Check $l_x = w_x + i_x + r_x + d_x + l_{x+1} + 1$ for $x = 20$. Why ?

Example 10.5 (continued). Use the service table information to answer the questions in (a) and (b). (c) Moreover, estimate ${}_{30}p_{35}$, ${}_{2}p_{35}$ and ${}_{2.5}p_{35}$.

Sol. (a) P(a member retiring at age 65|age 35)= $P(T_{35} > 30) = {}_{30}p_{65}^{00} = \frac{l_{65-}}{l_{35}} = ...$

(b) ${}_0p_x^{(j)} = P(T_x^{(\tau)} > 0, J_x = j)$, P(a member currently aged 35 exits employment by mode j)

P(one withdraws|(35))= ${}_0p_{35}^{(1)} = \sum_{i=35}^{65-} w_i/l_{35} = \sum_{i=35}^{59} w_i/l_{35} = \frac{10665+\cdots+1884}{218834} \approx 0.54.$

P(one retires in ill health|(35)) $= {}_0p_{35}^{(2)} = \sum_{i=35}^{65-} i_i/l_{35} = \sum_{i=35}^{64} i_i/l_{35} = \frac{213+\cdots+41}{218834} \approx 0.02.$

P(one normally retires|(35)) $= {}_0p_{35}^{(3)} = \sum_{i=35}^{65-} r_i/l_{35} = \sum_{i>59}^{65-} r_i/l_{35} = \frac{27926+\cdots+38488}{218834} \approx 0.42.$

P(one dies in service|(35)) $= {}_0p_{35}^{(4)} = \sum_{i=35}^{64} d_i/l_{35} = \sum_{i=35}^{65-} d_i/l_{35} = \frac{83+\cdots+215}{218834} \approx 0.02.$

(c) estimate ${}_{30}p_{35}$, ${}_{2}p_{35}$ and ${}_{2.5}p_{35}$.

${}_{30}p_{35} = \frac{l_{65}}{l_{35}} = \frac{l_{65-}}{l_{35}} = 38488/218834 \approx 0.18$?

No ! l_x **is not the # of people alive, but # of people stay in employment** at time x.

Q: How to find ${}_tp_x$?

Ans: Recall the **Kaplan-Meier estimator** (KME) (in Math 450) or the **product-limit-**

estimator (PLE) with observations (Z_i, δ_i), $i = 1, ..., n$, $Z_i = X_i \wedge C_i$ and $\delta_i = \mathbf{1}(X_i \le C_i)$:

$$\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \frac{d_k}{R_k}),$$

where $t_1 < \cdots < t_m$ are distinct values of Z_i's with $\delta_i = 1$, d_k is the number of person died at time t_k, and R_k is the number of person at risk at time t_k $(= \sum_{i=1}^n I(Z_i \ge t_k))$. An estimator of $\sigma^2_{\hat{S}_{pl}(t)}$ is $\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \frac{1}{n}(\hat{S}_{pl}(t))^2 \sum_{k:\ t_k \le t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}$. A 95% CI of $S_X(t)$ is $\hat{S}_{pl}(t) \pm 1.96\hat{\sigma}_{\hat{S}_{pl}(t)}$. Based on definition of the data in Table 2, one estimates $s(20-) = S_X(20-) = 1$. Moreover, d_k is not the people died at time k, but in the time interval $(k, k+1)$. Thus it is better to denoted by d_{k+}, rather than d_k, whereas l_k remains the same. Then the PLE should be modified as

$$\hat{S}_{pl}(t) = \prod_{k<j}(1 - \frac{d_k}{l_k}), \text{for } t = j \in \{1, 2, 3, ...\}. \qquad \text{(rather than } \hat{S}_{pl}(t) = \textstyle\prod_{k \le j}(1 - \frac{d_k}{l_k}))$$

$$\begin{aligned}
{}_{30}p_{35} =& \frac{s(65)}{s(35)} = \frac{\prod_{t_k<65}(1 - d_k/l_k)}{\prod_{t_k<35}(1 - d_k/l_k)} = \prod_{k:35 \le t_k < 65}(1 - d_k/l_k) \\
=& (1 - \frac{83}{218834})(1 - \frac{84}{207872}) \cdots (1 - \frac{215}{42805}) \approx 0.999. \\
{}_2p_{35} =& \prod_{35 \le t_k < 37}(1 - d_k/l_k) = (1 - \frac{83}{218834})(1 - \frac{84}{207872}). \\
{}_{2.5}p_{35} =& \prod_{k:\ 35 \le k < 37.5}(1 - d_k/l_k) = {}_2p_{35}. \\
{}_jp_x =& \prod_{x \le k < x+j}(1 - \frac{d_k}{l_k}), \quad \text{rather than} \prod_{x<k \le j}(1 - \frac{d_k}{l_k}).
\end{aligned} \tag{10.5.2}$$

If assume UDD, ${}_{2.5}p_{35} = {}_2p_{35}[(1 - r) + r(1 - \frac{d_{38}}{l_{38}})] = {}_2p_{35}[1 - r\frac{d_{38}}{l_{38}}]$ with $r = 0.5$, as

$${}_{2.5}p_{35} = (1-r){}_2p_{35}+r{}_3p_{35} = (1-r)\prod_{k:\ 35 \le k < 37}(1-\frac{d_k}{l_k})+r\prod_{k:\ 35 \le k < 38}(1-\frac{d_k}{l_k}) = {}_2p_{35}[1-r+r(1-\frac{d_{37}}{l_{37}})].$$

Use R program:

```
> (1-83/218834)*(1-84/207872) # 0.9992168
> q()
    L=c(1000000,903707, ..., 38488)
    d=c(237,218, ..., 0)
    a=prod(1-d/L)
    b=prod(1-d[1:15]/L[1:15])
    a/b
    prod(1-d[16:46]/L[16:46])
```

Example 10.6. *Employees in a pension plan pay contribution of 6% of their previous month's salary at each month end. Calculate the APV at entry of contributions for a new*

entrant aged 35, with a starting salary rate of \$100,000 using
(a) exact calculation using the multiple decrement model specified in Example 10.5,
(b) the values in Table 2, adjusting the APV of an annuity payable annually under UDD.
Other assumptions:
Salary rate function: Salaries increase at 4% per year continuously; ($\bar{s}_y$)
Interest: 6% per year effective.

Sol. Formulas: $S_y = \int_y^{y+1} A_t dt$, $S_y/S_x = s_y/s_x$, $= A_y/A_x = \bar{s}_y/\bar{s}_x$ $(= 1.04^{y-x})$, by Th10.1.
Q: 10^6 is A_{35} or S_{35} ?

(a) Let Y be the exit time in years. APV of total contribution for (35)= ?

$$Ans = E(\sum_{k=1}^{(Y-35)12} \frac{1}{12} 0.06 S_{35+k/12} v^{k/12}) = \frac{0.06}{12} \sum_{k=1}^{30\times 12} S_{35+k/12} v^{k/12} {}_{k/12}p_{35}^{00} = ?$$

where ${}_tp_x^{00}$ is given in Ex.10.5, $v = 1/1.06$, $\bar{s}_y = s_y = 1.04^y$, $\frac{S_y}{S_x} = \frac{s_y}{s_x} = 1.04^{y-x}$,

$S_y = 1.04^{y-x} S_x = 1.04^{y-35} S_{35}$ and $S_y = \int_y^{y+1} A_t dt$,

$S_{35} = \int_{35}^{35+1} A_t dt = \int_{35}^{36} A_{35} \frac{\bar{s}_t}{\bar{s}_{35}} dt = \int_0^1 A_{35} 1.04^t dt = A_{35} \frac{1.04-1}{\log 1.04} = 1.02 A_{35} \approx A_{35} = 10^5$.

Three periods: (35,60), 60, $(60, 65]$. $l_{60-} < l_{60}$. **Why 3 ?**

${}_{25-}p_{35}^{00} > 0.7 {}_{25}p_{35}^{00} = {}_{25}p_{35}^{00}$.

l_{60-} = # of people alive at 60.

l_{60} = # of people remain in work in (60,61).

So k in $\sum_k$ of Ans: 1:(25 × 12 − 1), 300, 301:360.

$Ans = \frac{0.06}{12} \sum_{k=1}^{30\times 12} {}_{k/12}p_{35}^{00} S_{35+k/12} v^{k/12} = \frac{0.06}{12} \sum_{k=1}^{30\times 12} {}_{k/12}p_{35}^{00} S_{35} (1.04)^{k/12} v^{k/12}$ $\quad (S_{35} \approx A_{35})$

$$\approx \frac{10^5 \times 0.06}{12} [\sum_{k=1}^{299} {}_{\frac{k}{12}}p_{35}^{00} 1.04^{k/12} v^{k/12} + {}_{25-}p_{35}^{00} 1.04^{25} v^{25} + \sum_{k=301}^{360} {}_{\frac{k}{12}}p_{35}^{00} 1.04^{k/12} v^{k/12}] = ? \quad ({}_tp_x = {}_tp_x^{00})$$

By Eq.(10.5.1) in Ex.10.5
$$\begin{cases} {}_tp_{35}^{00} = \exp(-t(0.05 + 0.001 + A) - Bc^{35}(c^t - 1)) & t \in [0, 10) \\ {}_tp_{45}^{00} = \exp(-t(0.02 + 0.001 + A) - Bc^{45}(c^t - 1)) & t \in [0, 15) \\ {}_tp_{60}^{00} = \exp(-t(0.1 + 0.001 + A) - Bc^{60}(c^t - 1)) & t \in [0, 5) \\ {}_tp_{35}^{00} = {}_{10}p_{35}^{00} \cdot {}_{t-10}p_{45}^{00} \text{ (see 2 lines above)} & t \in [10, 25) \\ {}_tp_{35}^{00} = {}_{25}p_{35}^{00} \cdot {}_{t-25}p_{60}^{00} \quad ({}_{25}p_{35}^{00} = 0.7 {}_{25-}p_{35}^{00}) & t \in [25, 30) \end{cases}$$

$$Ans \approx \frac{0.06 \times 10^5}{12} [\sum_{k=1}^{299} {}_{\frac{k}{12}}p_{35}^{00} (\frac{1.04}{1.06})^{k/12} + {}_{25-}p_{35}^{00} (\frac{1.04}{1.06})^{25} + \sum_{k=301}^{360} {}_{\frac{k}{12}}p_{35}^{00} (\frac{1.04}{1.06})^{k/12}]$$

$$= \frac{0.06 \times 10^5}{12} \Bigg[\sum_{k=1}^{120} {}_{\frac{k}{12}}p_{35}^{00} (\frac{1.04}{1.06})^{k/12} + {}_{10}p_{35}^{00} \sum_{k=121}^{299} {}_{\frac{k}{12}-10}p_{45}^{00} (\frac{1.04}{1.06})^{k/12} \qquad ({}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m})$$

$$+ {}_{25-}p_{35}^{00} (\frac{1.04}{1.06})^{25} + {}_{25}p_{35}^{00} \sum_{k=301}^{360} {}_{\frac{k}{12}-25}p_{60}^{00} (\frac{1.04}{1.06})^{k/12} \Bigg] \quad ({}_tp_x^{00} \text{ is given above}) \qquad (10.6.1)$$

One has to use program to compute Ans, as $\sum_k e^{c^{k/12}}$ has no simple expression.

(b) With ${}_tp_x^{00}$ based on Table 2, compute $E(\sum_{k=1}^{(Y-35)12} \frac{1}{12} c S_{35+k/12} v^{k/12})$.

The expression of Ans is as in Eq. (10.6.1), but ${}_tp_{35}^{00}$ is different.
Since ${}_tp_{35}^{00}$ is discrete if using the service table, one has to use program to compute it. ${}_tp_{35}^{00} = \frac{l_{x+t}}{l_x}$, for $t \in \{0, 1, ..., 25-, 25, ..., 29, 30-\}$.

$$
{}_{\frac{i}{12}+j}p_{35}^{00} = \begin{cases} {}_jp_{35}^{00}(1 - \frac{i}{12}) + {}_{j+1}p_{35}^{00}\frac{i}{12} & \text{for } i \in \{1, ..., 12\} \text{ and } j \in \{0, 1, ..., 28\}, \text{ except 25 and 29} \\ {}_jp_{35}^{00}(1 - \frac{i}{12}) + {}_{(j+1)-}p_{35}^{00}\frac{i}{12} & \text{for } i \in \{1, ..., 12\} \text{ and } j \in \{24, 29\} \end{cases}
$$

due to UDD.

$$
Ans = \frac{0.06 \times 10^5}{12} \sum_{j=0}^{29} \sum_{i=1}^{12} {}_{\frac{i}{12}+j}p_{35}^{00}(\frac{1.04}{1.06})^{j+i/12} \tag{10.6.2}
$$

Additional Homework 2: Use R to compute Ans. in Eq. (10.6.1) and (10.6.2).

10.3 Valuation of benefits

The term **accrued benefit** means a benefit calculated using both past service and the average salary as of the determination date. There are two methods for evaluation. The approach which uses salaries projected to the exit date is called the **projected unit credit** (PUC) method. Valuating the accrued benefit with no allowance for the future salary increases is called the **traditional unit credit** (TUC) method, or **unit credit cost method**, or **accrued benefit cost method**, or **current unit** method.
In a DB final salary pension plan, let B_r be the basic annual age retirement pension benefit

$$
B_r = nS_{Fin}\alpha \text{ or } nB \ (B \neq S_{Fin}\alpha), \text{ where} \tag{10.3.1}
$$

(1) n is the total number of year of service,
(2) S_{Fin} is the average salary in a specific period before retirement (1 or 3 years);
(3) α is the accrual rate, typically between 0.01 and 0.02.
(4) B is the fixed pension benefit per year.

$$
\text{The APV of the age (x) retirement pension is } B_r\ddot{a}_x^{(12)}.
$$

$B_r = n\alpha S_{Fin}$ is used in the next two sections and $B_r = nB$ is used in the last section.

For an employee who has been a member of the plan for all his/her working life, say $n = 40$ years, this typically gives a replacement ratio R in the range of 40-80% ($n\alpha = 40(1\% \sim 2\%)$). It can be interpreted that this benefit formula as that the employee earns a pension of $100\alpha\%$ of final average salary for each year of employment.

For a member who is currently aged y, and who joined the pension plan at age e ($\leq y$) with retirement age 60, an estimate of her annual pension at retirement is $(60 - e)\hat{S}_{Fin}\alpha$. It can be split as

$$
(60 - e)\hat{S}_{Fin}\alpha = \underbrace{(y - e)\hat{S}_{Fin}\alpha}_{\textbf{accrued benefit}} + \underbrace{(60 - y)\hat{S}_{Fin}\alpha}_{\textit{to be earned}}.
$$

Example 10.7. *A pension plan in Example 10.5 offers an age retirement pension of* 1.5% *of final average salary for each year of service, where final average salary is defined as the*

earning in the 3 years before retirement. Estimate the APV of the accrued age retirement pension for a member aged 55 with 20 years of service, whose salary prior to the valuation date was \$50000. Basic assumptions:
(1) The pension is paid monthly in advance for life, with no spouse's benefit.
(2) Interest rate is 5% per year.
(3) Salary scale s_y is given in Table 1 with the linear interpolation.
(4) Post-retirement survival: $\mu_x = A + Bc^x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$.
(5) Exit age is Y, which is age retiring ($Y \in \mathcal{W}$ based on Table 2).

Sol. The accrued age retirement pension is $B_r \ddot{a}_Y^{(12)}$, where $B_r = n\alpha S_{Fin}$ (10.3.1), $n = 20$, $\alpha = 0.015$, $\hat{S}_{Fin} = 50000\frac{z_Y}{s_{54}}$ and $z_Y = \frac{s_{Y-3}+s_{Y-2}+s_{Y-1}}{3}$. Thus write $B_Y \ddot{a}_Y^{(12)} = n\alpha S_{Fin}\ddot{a}_Y^{(12)}$. Its present value is $B_Y \ddot{a}_Y^{(12)} v^{Y-55}$, $v = ...$

Solve: $Ans = E(B_Y \ddot{a}_Y^{(12)} v^{Y-55} I((55) \text{ age retiring}))=...$

$Ans = \sum_{k\in\mathcal{W}} \frac{B_k}{12}\ddot{a}_k^{(12)} v^{k-55} \mathrm{P}((55) \text{ age retiring at } k) = \sum_{k\in\mathcal{W}} \frac{B_k}{12}\ddot{a}_k^{(12)} v^{k-55}\,{}_{k-55|}q_{55}^{03} =...,$

Need to specify or derive $\mathcal{W}$, ${}_{k-55|}q_{55}^{03}$, $\ddot{a}_k^{(12)}$ and B_k !

1. $\mathcal{W} = \{60-, 65-\} \cup (60, 65)$,
2. ${}_{k-55|}q_{55}^{03} = \frac{r_k}{l_{55}}$ is based on Table 2.
3. $\ddot{a}_k^{(12)} = \sum_{i=0}^{\infty} v^{i/12}\,{}_{\frac{i}{12}}p_k$. 3 ways for ${}_{\frac{i}{12}}p_k$ or $\ddot{a}_x^{(12)}$:
   (1) direct derivation by ${}_tp_x = e^{-\int_0^t \mu_x dx}$, $\mu_x = A + Bc^x$, $x > 0$ (see assumption (4)),
   (2) direct derivation by Table D.1 (p.996),
   (3) Table D.3 (see p.998) (not quite convenient here for $\ddot{a}_{35.5}$ etc. (see Eq.(10.7.2) below)).
4. B_Y: If the member retires at exactly age 60 (*i.e.* 60−, the accrued benefit, based on 20 years' past service and an accrual rate of 1.5%, is an annual pension payable monthly in advance from age 60: $(B_Y = n\alpha S_{Fin})$

$$B_{60-} = 20 \times 0.015 \times 50000\frac{z_{60}}{s_{54}} = 15000\frac{z_{60}}{s_{54}} = 15000\frac{3.332 + 3.382 + 3.432}{3 \times 3.186} = 15,922.79$$

   based on Table 1, with probability $f^{03}(60)$ $(= \frac{r_{60-}}{l_{55}}$ (from Table 2)).

If the member retires at age 60+, that is, in (60,61), the accrued benefit, based on 20 years' past service and an accrual rate of 1.5%, is an annual pension payable monthly in advance from age 60 approximated by

$$B_{60+} = 15000\frac{z_{60.5}}{s_{54}} = 15000\frac{0.5(3.332 + 3.382) + 0.5(3.382 + 3.432) + 0.5(3.432 + 3.484)}{3 \times 3.186}$$

based on Table 1 and UDD, with probability ${}_{60+|}q_{55}^{03}$, estimated by $\frac{r_{60+}}{l_{55}}$.

If the member retires at age 61,62, 63, 64...

The APV of the accrued age retirement pension is

$$Ans \approx 15000\Big(\sum_{k\in\{60-,65-\}} \frac{z_k}{s_{54}} v^{k-55}\ddot{a}_k^{(12)} \cdot \frac{r_{k-}}{l_{55}} + \sum_{k=60}^{64} \frac{z_{0.5+k}}{s_{54}} v^{.5+k-60}\ddot{a}_{0.5+k}^{(12)} \cdot \frac{r_{k+}}{l_{55}}\Big) \approx 137,508. \quad (10.7.1)$$

based on the assumptions in Example 10.5. Here

r_{k+} and r_{k-} are from Table 2, *e.g.*, $r_{60-} = 27926$ and $r_{60+} = 6188$, and

$s_{60.5} = (s_{60} + s_{61})/2$ from Table 1.

Remark 1. Table D.3 gives $\ddot{a}_x$. For a period of length $\frac{1}{m}$:

(i) the interest factor is $(1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$.

(ii) the effective rate of interest is $(1+i)^{1/m} - 1 = \frac{i^{(m)}}{m}$.

(iii) the discount factor is $(1+i)^{-1/m} = v^{1/m} = (1-d)^{1/m} = 1 - \frac{d^{(m)}}{m}$.

(iv) the effective rate of discount is $1 - v^{1/m} = \frac{d^{(m)}}{m}$.

Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)}d^{(m)}}. \tag{10.7.2}$$

Additional Homework 3. Write an R code to derive Ans in Eq.(10.7.1) by directly deriving $\ddot{a}_x$ and by Table D.1 (p.996).

The term **accrued benefit** means a benefit calculated using both past service and the average salary as of the determination date. In a DB final salary pension plan, let B_r be the basic annual age retirement pension benefit

$$B_r = nS_{Fin}\alpha \text{ or } nB\ (B \neq S_{Fin}\alpha), \text{ where} \tag{10.3.1}$$

(1) n is the total number of year of service,
(2) S_{Fin} is the average salary in a specific period before retirement (1 or 3 years);
(3) α is the accrual rate, typically between 0.01 and 0.02.
(4) B is the fixed pension benefit per year.

$$\text{The APV of the age (x) retirement pension is } B_r\ddot{a}_x^{(12)}.$$

$B_r = n\alpha S_{Fin}$ is used in the next two sections and $B_r = nB$ is used in the last section.

Ex. 10.7. A pension plan in Example 10.5 offers an age retirement pension of 1.5% of final average salary for each year of service, where final average salary is defined as the earning in the 3 years before retirement. Estimate the APV of the accrued age retirement pension for a member aged 55 with 20 years of service, whose salary prior to the valuation date was \$50000.
Basic assumptions:
(1) The pension is paid monthly in advance for life, with no spouse's benefit.
(2) Interest rate is 5% per year.
(3) Salary scale s_y is given in Table 1 with the linear interpolation.
(4) Post-retirement survival: $\mu_x = A + Bc^x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$.
(5) Exit age is Y, which is age retiring ($Y \in \mathcal{W}$ based on Table 2).

Sol. The accrued age retirement pension is $B_r\ddot{a}_Y^{(12)}$, where $B_r = n\alpha S_{Fin}$ (10.3.1), $n = 20$, $\alpha = 0.015$, $\hat{S}_{Fin} = 50000\frac{z_Y}{s_{54}}$ and $z_Y = \frac{s_{Y-3}+s_{Y-2}+s_{Y-1}}{3}$. Thus write $B_Y\ddot{a}_Y^{(12)} = n\alpha S_{Fin}\ddot{a}_Y^{(12)}$.

Its present value is $B_Y\ddot{a}_Y^{(12)}v^{Y-55}$,

$Ans = E(B_Y\ddot{a}_Y^{(12)}v^{Y-55}I(M = 01)))$, where M is the mode ($M \in \{01, 02, 03, 04\}$),

$Ans = \sum_{k\in\mathcal{W}} \frac{B_k}{12}\ddot{a}_k^{(12)} v^{k-55} \mathrm{P}((55)$ age retiring at $k) = \sum_{k\in\mathcal{W}} \frac{B_k}{12}\ddot{a}_k^{(12)} v^{k-55}{}_{k-55|}q_{55}^{03} = ...,$
Need to specify or derive $\mathcal{W}$, ${}_{k-55|}q_{55}^{03}$, $\ddot{a}_k^{(12)}$ and B_k !

Ex10.7 is for the accrued age retirement pension. There are other modes: withdraw, ill leave, and death in service.

Withdrawal pension

When an employee leaves employment before being eligible to take an immediate pension, the usual benefit in a DB plan is a deferred pension, *i.e.*,

accrual rate×service×final average salary,

but would not be paid until the member attains the normal retirement age, where the final average salary is based earnings in the years immediately proceeding withdrawal. Some adjustments are called **cost of living adjustments** (COLAs).

Example 10.8. *A final salary pension plan offers an accrual rate of* 2%*, and the normal retirement age is 65. Final average salary is the average salary in the three years before retirement or withdrawal. Pensions are paid monthly in advance for the life from age 65, with no spouse's benefit, and are guaranteed for 5 years (if the member died before age 70).*
(a) Estimate the APV of the accrued withdrawal pension for a life now aged 35 with 10 years of service whose salary in the past year was $\$10^5$*, in the following two different cases*
(i) with no COLA,
(ii) with a COLA in deferment of 3% *per year.*
(b) On death during deferment, a lump sum benefit of five times the accrued annual pension, with a COLA of 3% *per year, is paid immediately. Estimate the APV of this benefit. Basis:*
(1) Service table: Table 2.
(2) Salary scale: From Table 1;
(3) Interest rate is 5% *per year.*
(4) Post-retirement survival: the **standard ultimate survival model**
http://people.math.binghamton.edu/qyu/ftp/tableD.pdf;

Remark. In this example, the evaluation method is the projected unit credit (PUC) method, as it allows future salary increases. It also related to the accrued benifit (annual) pension $B_Y = \underline{n\alpha S_{Fin}}$. In Ex.10.7, solve $E(B_Y \ddot{a}_Y^{(12)} v^{Y-55} I(\mathrm{M{=}03}))$.

Sol. (a) (i) With no COLA and $v = 1/1.05$, solve $E(B_Y \ddot{a}_{\overline{65:\overline{5}}}^{(12)} v^{65-35} I(M = 01)A)$, where $A = I((35)$ is alive at age 65), M is the mode ($M \in \{01, 02, 03, 04\}$), Y is the exit age,

$$B_Y = 10\alpha S_{Fin},\ S_{Fin} = 10^5 \frac{\frac{s_{y-3}+s_{y-2}}{2} + \frac{s_{y-2}+s_{y-1}}{2} + \frac{s_{y-1}+s_y}{2}}{3s_{34}},\ s_y \text{ see (2) (similar to } B_{60+} \text{ in Ex.10.7)}$$

Reason: According to the service table assumption, the member can withdraw at any age up to 60, that is, ages in $\{35, 36, ..., 59\}$. Withdrawal between ages t and $t + 1$ (*i.e.* in $[t, t + 1)$) is **treated as** $t + 0.5$, and $t \in \{35, ..., 59\}$. That is, the withdrawal age is treated as a discrete r.v. $Y \in \mathcal{W} = \{35.5, 36.5, ..., 59.5\}$.

$$E(B_Y \ddot{a}_{\overline{65:\overline{5}}}^{(12)} v^{30} A I(M = 01)) = v^{30} \ddot{a}_{\overline{65:\overline{5}}}^{(12)} \sum_{t=0}^{24} B_{35.5+t} \cdot {}_{29.5-t}p_{35.5+t}^{00} \cdot {}_{t+0.5|}q_{35}^{01}. \quad (1)$$

Two issues: (I) Why ? (II) $\ddot{a}^{(12)}_{\overline{65:\overline{5}}}$=? $B_{35.5+t}$=? ${}_{29.5-t}p^{00}_{35.5+t}$=? ${}_{t+0.5|}q^{01}_{35}$= ?

$$
\begin{aligned}
(I) \qquad \text{Ans} =& E(B_Y \ddot{a}_{35} v^{30} I(M=01)A) = v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}} E(B_Y I(M=01)A) \\
=& v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}} E(E(B_Y I(M=01)A|D)) \qquad D = \lfloor Y-35 \rfloor \\
=& v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}} E(B_Y I(M=01) \underbrace{{}_{29.5-D}p^{00}_{35.5+D}}_{P(A=1|D)}) \\
=& v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}} \sum_{t=0}^{24} B_{35.5+t} \underbrace{{}_{29.5-t}p^{00}_{35.5+t}}_{P(A=1,M=01|D=t)} \cdot \underbrace{{}_{t+0.5|}q^{01}_{35}}_{P(D=t)}
\end{aligned}
$$

$$
\begin{aligned}
(II)\ B_{35.5+t} =& n\alpha \cdot S_{Fin} = 10 \times 0.02 \times 10^5 \times \frac{z_{35+t+0.5}}{s_{34}}, \qquad \text{(n=10 years service)} \\
z_{y+0.5} =& \frac{s_{y-3}+2s_{y-2}+2s_{y-1}+s_y}{2\cdot 3} \ (s_y \text{ from Table 1}), \\
{}_{29.5-t}p^{00}_{35.5+t} =& \frac{l_{65}}{(l_{35+t}+l_{36+t})/2} \qquad \text{from Table 2} \\
{}_{t+0.5|}q^{01}_{35} =& \frac{w_{35+t}}{l_{35}} \text{ from Table 2, as withdraw is really between } 35+t \text{ and } 36+t. \\
\ddot{a}^{(12)}_{\overline{65:\overline{5}}} =& \ddot{a}^{(12)}_{\overline{5}} + {}_5p_{65}v^5\ddot{a}^{(12)}_{70} \text{ (3 ways for } \ddot{a}^{(12)}_{70}\text{: } \mu_x\text{, Tables D.1 \& D.3) in my website).} \\
\text{Table D.1: } \ddot{a}^{(12)}_{\overline{65:\overline{5}}} =& \frac{1}{12}\frac{1-v^5}{1-v^{1/12}} + (\frac{l_{70}}{l_{65}}v^5)\frac{1}{12}\sum_{i=0}^{\infty} v^{i/12}{}_{i/12}p_{70} \ (l_y \text{ from Table D.1}) \\
=& \frac{1}{12}\frac{1-v^5}{1-v^{1/12}} + \frac{1}{12}(\frac{l_{70}}{l_{65}}v^5)\sum_{i=0}^{\infty}\frac{l_{70+i/12}}{l_{70}}v^{i/12} \qquad (\textit{see Remark below}) \\
=& \frac{1}{12}\frac{1-v^5}{1-v^{1/12}} + \frac{1}{12}(\frac{l_{70}}{l_{65}}v^5)\sum_{i=0}^{100-70\overbrace{-1}^{??}}\sum_{k=0}^{11}\frac{l_{70+i}-d_{70+i}k/12}{l_{70}}v^{i+k/12} \ \text{(UDD) (3)} \\
\approx& 4.45 + 0.75 \times 11.55 \approx 13.16.
\end{aligned}
$$

$$
\begin{aligned}
Ans =& E(B_Y \ddot{a}_{35} v^{30} I(M=01)A) = v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}}\sum_{t=0}^{24} B_{35.5+t}\,{}_{29.5-t}p^{00}_{35.5+t} \cdot {}_{t+0.5|}q^{01}_{35} \\
=& v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}}\sum_{t=0}^{24} \underbrace{10\cdot 0.02\cdot 10^5 \frac{z_{35.5+t}}{s_{34}}}_{=B_{35.5+t}=n\alpha S_{Fin}} \underbrace{\frac{l_{65}}{(l_{35+t}+l_{36+t})/2}}_{Table\ 2\ or\ D.1\ ?} \underbrace{\frac{w_{35+t}}{l_{35}}}_{Table\ 2} = 48246.
\end{aligned}
$$

Remark. If use Table D.1, ${}_tp^{00}_x = \frac{l_{x+t}}{l_x}$.

$\sum_{j\ge x} {}_{j-x|}q_x = \sum_{j\ge x}\frac{d_j}{l_x} = 1 => {}_1p_x = \frac{l_{x+1}}{l_x} = \frac{l_x-d_x}{l_x} = 1-\frac{d_x}{l_x} = 1-{}_1q_x$.

$$
and \quad \frac{l_{x+t}}{l_x} = \frac{l_x-d_x-d_{x+1}-\cdots-d_{x+t-1}}{l_x} = 1-\frac{\sum_{i=0}^{t-1}d_{x+i}}{l_x} = 1-{}_tq_x = {}_tp_x.
$$

R codes for $\sum_{i=0}^{100-70}\sum_{k=0}^{11}\frac{l_{70+i}-d_{70+i}k/12}{l_{70}}v^{i+k/12}$ in (3):

L=c(91082.43, l_{71}, ..., l_{100})

d=L[1:30]-L[2:31]

d=c(d,L[31])

a=1:31

for(i in 0:30)

a[i+1]=sum((L[i+1]-((0:11)/12)*d[i+1])*v**(i+(0:11)/12))

sum(a)/L[1]

(ii) With a COLA in deferment of 3% per year, solve

$$E(B_Y\ddot{a}_{35}v^{30}1.03^{29.5-D}AI(M=01)) \qquad (> E(B_Y\ddot{a}_{35}v^{30}AI(M=01)) \text{ as } D\geq 0)$$

$$Ans = v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}}}\sum_{t=0}^{24} 2\times 10^4\frac{z_{35.5+t}}{s_{34}}\frac{l_{65}}{l_{35.5+t}}\frac{w_{35+t}}{l_{35}}\times 1.03^{29.5-t} = 88853.$$

Remark. Recall that under the multiple decrement model before section 2, $T_x^{(i)}$ is the death time since x due to i-th cause, $i\in\{1,2,...,m\}$ and $T_x^{(\tau)} = \min_{i=1}^n T_i$ is the survival time since x, same as T_x.

${}_tp_x^{(i)} = P(T_x^{(\tau)} > t, J_x = i) = P(T_x^{(i)} > t, J_x = i)$?

${}_tp_x^{'(i)} = P(T_x^{(i)} > t)$.

${}_tp_x = P(T_x > t) = \prod_{i=1}^m P(T_x^{(i)} > t)$.

Since Section 2, we use the following notations:

00: no exit;

01: withdrawn from the pension plan;

02: disability retirement;

03: age retirement;

04: died in service.

${}_tp_x^{01} = P(\text{withdrawal exit time} > x+t|\text{the member survives } x) = P(T_x^{(1)} > t)$,

${}_tp_x^{02} = P(\text{disable exit time} > x+t|\text{the member survives } x) = P(T_x^{(2)} > t)$,

${}_tp_x^{03} = P(\text{retirement time} > x+t|\text{the member survives } x) = P(T_x^{(3)} > t)$,

${}_tp_x^{04} = P(\text{death exit time} > x+t|\text{the member survives } x) = P(T_x^{(4)} > t)$.

$T_x^{(\tau)}$ is the exist time of a person from the employment since x, distinguishing from T_x, the survival time of a person since x.

${}_tp_x^{04} \neq {}_tp_x = P(T_x > t) \neq P(T_x^{(\tau)} > t)$ (different from the statement in Remark).

${}_tp_x^{0i} = {}_tp_x^{'(i)} = P(T_x^{(i)} > t) = P(T_x^{(i)} > t, J_x = i) + P(T_x^{(i)} > t, J_x \neq i) \neq P(T_x^{(\tau)} > t, J_x = i)$.

${}_tp_x^{(i)} = P(T_x^{(\tau)} > t, J_x = i)$.

${}_tp_x^{00} = {}_tp_x^{(\tau)} = P(T_x^{(\tau)} > t) = P(\text{exit time} > x+t|(x))$.

${}_tp_x = P(T_x > t) = P(\text{survival time} > x+t|(x)) = P(X > x+t|X > x)$. X =survival time.

Using the notation before, if the survival time is continuous, then
$_tp_x^{0i} = {}_tp_x^{'(i)} = \exp(-\int_0^t \mu_{x+s}^{0i} ds)$,
$_tp_x^{00} = {}_tp_x^{(\tau)} = \exp(-\int_0^t \sum_{i=1}^4 \mu_{x+s}^{0i} ds) = \prod_{i=1}^4 \exp(-\int_0^t \mu_{x+s}^{0i} ds)$,

Remark about midterm.

Problem 1. A continuous two-life annuity pays: 100 while both (30) and (40) are alive; 70 while (30) is alive but (40) is dead; and 50 while (40) is alive but (30) is dead. The actuarial present value of this annuity is 1180. Continuous single life annuities paying 100 per year are available for (30) and (40) with actuarial present values of 1200 and 1000, respectively. Calculate the actuarial present value of a two-life continuous annuity that pays 100 while at least one of them is alive.

Sol. $100\bar{a}_{\overline{30:40}} = \underline{100}(\bar{a}_{30} + \bar{a}_{40} - \bar{a}_{30:40}) = ?$ $\quad (\bar{a}_{30},\ \bar{a}_{40}$ and $\bar{a}_{30:40}) = ?$
Formulas: $P_x = \frac{\bar{A}_x}{\bar{a}_x}$ and $\underline{\bar{a}_x = \int_0^\infty v^t{}_tp_x}dt$. $=> \bar{a}_{30:40} = \int_0^\infty v^t{}_tp_{30:40}dt$ and $\bar{a}_{\overline{30:40}} = \int_0^\infty v^t{}_tp_{\overline{30:40}}dt$.

Thus $\bar{a}_{30} = \frac{\bar{A}_x}{\bar{a}_x} = 1200/100 = 12$, $\bar{a}_{40} = 1000/100 = 10$ and $\bar{a}_{30:40}$ can be found from

$$\begin{aligned}
1180 =& 100\bar{a}_{30:40} + 70\int_0^\infty v^t({}_tp_{30} - {}_tp_{30:40})dt + 50\int_0^\infty v^t({}_tp_{40} - {}_tp_{30:40})dt \\
=& 100\bar{a}_{30:40} + 70(\int_0^\infty v^t{}_tp_{30}dt - \int_0^\infty v^t{}_tp_{30:40}dt) + 50(\int_0^\infty v^t{}_tp_{40}dt - \int_0^\infty v^t{}_tp_{30:40}dt) \\
=& 100\bar{a}_{30:40} + 70(\bar{a}_{30} - \bar{a}_{30:40}) + 50(\bar{a}_{40} - \bar{a}_{30:40}) \\
=& 100\bar{a}_{30:40} + (70)(12) - 70\bar{a}_{30:40} + 50(10) - 50\bar{a}_{30:40}.
\end{aligned}$$

Hence, $\bar{a}_{30:40} = \frac{(70)(12)+500-1180}{70+50-100} = 8$. The actuarial present value of a two-life continuous annuity that pays 100 while at least one of them is alive is

$$100\bar{a}_{\overline{30:40}} = 100(\bar{a}_{30} + \bar{a}_{40} - \bar{a}_{30:40}) = 100(12 + 10 - 8) = 1400.$$

Problem 2. For a special fully continuous whole life insurance on (x), you are given:
(i) Mortality follows a double decrement model. (ii) The death benefit for death due to cause 1 is 3. (iii) The death benefit for death due to cause 2 is 1. (iv) $\mu_x^{(1)}(t) = 0.02$, $t \geq 0$. (v) $\mu_x^{(2)}(t) = 0.04$, $t \geq 0$. (vi) The force of interest, δ, is a positive constant.
Calculate (1) (2pt) $f_{T_x^{(i)}}(t)$, (2) (1pt) $S_{T_x^{(\tau)}}(t)$, (3) (2pt) $f_{T_x^{(\tau)},J_x}(t,i)$
(4) (30pt) the benefit premium for this insurance.

Sol. Assume $t \geq 0$. (1) $f_{T_x^{(i)}}(t) = \mu_x^{(i)} e^{-\mu_x^{(i)} t}$, by formulas:
$\mu_X(x) = \mu(x) = \mu_x = \underline{\frac{f_X(x)}{S_X(x-)}}$, or $f_{T_x^{(i)}}(t) = \mu_x S_X(x) = \mu_x^{(i)}(t) e^{-\int_0^t \mu_x^{(i)}(y)dy}$.
23. $X \sim \mathcal{G}(\alpha, \beta)$. $f(x) = \underline{\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}}$, if $x > \underline{0}$, $\mu = \underline{\alpha\beta}$, $\sigma^2 = \underline{\alpha\beta^2}$, $\Gamma(\alpha+1) = \underline{\alpha\Gamma(\alpha)}$
24. $Exp(\lambda) = \underline{\mathcal{G}(1, \lambda)}$,
(2) $S_{T_x}^{(\tau)}(t) = e^{-\mu_1 t} e^{-\mu_2 t} = e^{-(\mu_1+\mu_2)t}$ by the formulas:
If $\underline{T_x \perp T_y}$, then ${}_t\underline{q_{\overline{xy}}} = {}_tq_{xt}q_y$, $\underline{{}_tp_{xy}} = {}_tp_{xt}p_y$ and $\underline{\mu_{xy}(t)} = \mu_x(t) + \mu_y(t)$.
(3) $f_{T_x^{(\tau)},J_x}(t,i) = \mu_x^{(i)} e^{-\mu t}$, where $\mu = \mu_x^{(1)} + \mu_x^{(2)}$, by the formula:
$\underline{f_{(T_x^{(\tau)},J_x)}(t,j)} = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)} S_{T_x^{(\tau)}}(t) = \mu_x^{(j)}(t) S_{T_x^{(\tau)}}(t)$,

(4) $\overline{P}_x = \frac{\overline{A}_x}{\overline{a}_x} = ?$

$$\begin{aligned}
\overline{A}_x =& 3\overline{A}_x^{(1)} + \overline{A}_x^{(2)} = \int_0^\infty v^t \cdot [3f_{T_x^{(\tau)},J_x}(t,1) + f_{T_x^{(\tau)},J_x}(t,2)]\,dt \\
=& \int_0^\infty e^{-\delta t} \cdot (3\mu_x^{(1)}(t) + \mu_x^{(2)}(t))e^{-(\mu_x^{(1)}+\mu_x^{(2)})}\,dt \qquad \text{see (3) above} \\
=& \int_0^\infty e^{-\delta t} e^{-0.06t}(3(0.02) + (0.04))\,dt = \frac{0.1}{\delta + 0.06}, \\
\overline{a}_x =& \int_0^\infty e^{-\delta t}\,{}_tp_x^{(\tau)}\,dt = \int_0^\infty e^{-\delta t}e^{-0.06t}\,dt = \frac{1}{\delta+0.06}, \\
\overline{P}_x =& \frac{\overline{A}_x}{\overline{a}_x} = 0.1. \square
\end{aligned}$$

5. The force of mortality is $\mu_X(x) = \mu(x) = \mu_x = \underline{\frac{f_X(x)}{S_X(x-)}}$. $\mu_{T(x)}(t) = \underline{\mu_x(t)}$. If X is cts, $\underline{\mu(x)} = -\frac{d}{dx}\ln S_X(x)$, $\underline{S_X(x)} = \exp\big(-\int_0^x \mu(t)\,dt\big)$, $\underline{f_{T(x)}(t)} = {}_tp_x\mu(x+t)$. $\mu_x(t) = \underline{\mu(x+t)}$.

7. The multiple-decrement model:

1. $T_x^{(1)}, ..., T_x^{(m)}$ are *independent* and cts r.v.
2. $\underline{T_x^{(\tau)}} = \min\{T_x^{(1)}, ..., T_x^{(m)}\}$ and $\underline{\{J_x = j\}} = \{T_x^{(\tau)} = T_x^{(j)}\}$.

$\underline{{}_tp_x^{(\tau)}} = S_{T_x^{(\tau)}}(t)$, $\mu_x^{(\tau)}(t) = \mu_{x+t}^{(\tau)} = \frac{f_{T_x^{(\tau)}}(t)}{S_{T_x^{(\tau)}}(t)} = \underline{\sum_{j=1}^m \mu_x^{(j)}(t)}$,

$\underline{{}_tp_x^{(j)}} = P(T_x^{(\tau)} > t, J_x = j)$. $\underline{{}_tp_x^{'(j)}} = S_{T_x^{(j)}}(t)$.

Example 10.8. (b) On death during deferment, a lump sum benefit of five times the accrued annual pension, with a COLA of 3% per year, is paid immediately. Estimate the APV of this benefit. Basis:

(1) Service table: Table 2.
(2) Salary scale: From Table 1;
(3) Interest rate is 5% per year.
(4) Post-retirement survival: the **standard ultimate survival model**

Sol. Solve $Q = E((5B_X)v^{X-35+T_X}I(T_X \le 65-X)I(M=01)\cdot 1.03^{T_X})$, where
X is exit time, treat $X \in \{35.5, ..., 59.5\}$ due to Table 2 and UDD.
$B_X = n\alpha S_{Fin}$. n=10 years of service
M is exit mode,
T_x is the survival length after (x),
let $D = X - 35.5 \in \{0, ..., 24\}$,

$$\begin{aligned}
Q =& E((5B_X)v^{D+0.5}(1.03v)^{T_X}I(T_X \le 65-X)I(M=01)) \\
=& E((5B_X)v^{D+0.5}I(M=01)E((\tfrac{1.03}{1.05})^{T_X}I(T_X \le 65-X)|X)) \\
=& E((5B_X)v^{D+0.5}I(M=01)\overline{A}^{\,1}_{35.5+D:\overline{29.5-D}|j}) \text{ where } j = 0.02/1.05 = 1 - 1.03/1.05. \\
=& \textstyle\sum_{t=0}^{24}(5\times 10\times 0.02\times 10^5\frac{z_{35.5+t}}{s_{34}})v^{t+0.5}\times \overline{A}^{\,1}_{35.5+t:\overline{29.5-t}|j}\frac{w_{35+t}}{l_{35}} \\
=& 10^5\textstyle\sum_{t=0}^{24}\frac{z_{35.5+t}}{s_{34}}v^{t+0.5}\times \overline{A}^{\,1}_{35.5+t:\overline{29.5-t}|}\frac{w_{35+t}}{l_{35}}
\end{aligned}$$

Here $\overline{A}^{1}_{x:\overline{n}|} = E(v^{T_x}\mathbf{1}(T_x \le n))$. Using basis (1) and (3), we have
$\overline{A}^{1}_{35.5+t:\overline{29.5-t}|} = \int_0^{29.5-t} v^x \mu^{(01)}_{35.5+x} {}_{35+x}p^{00}_{35} dx = \int_0^{29.5-t} v^x (A + Bc^{35.5+x}) \frac{l_{35+x}}{l_{35}} dx$

$$Q = \frac{10^5 v^{0.5}}{s_{34} l_{35}} \sum_{t=0}^{24} z_{35.5+t} w_{35+t} v^t \overline{A}^{1}_{35.5+t:\overline{29.5-t}|} \approx 1813.$$

Ignore the rest of this page !
(b.2). Using basis (3) $\overline{A}^{1}_{35.5+t:\overline{29.5-t}|} = \int_0^{29.5-t} v^x (\exp(-\int_0^t \mu_y dy))\mu_x dx$. This approach ignores that μ_x is related to $I(M=01)$.
(b.3). Using basis (1) and (3) (Table D.1) and using UDD,

$$\begin{aligned}\overline{A}^{1}_{35.5+t:\overline{29.5-t}|j} &= \int_0^{29.5-t} (\frac{1.03}{1.05})^x f_{T_{35.5+t}}(x)dx \text{ (for } t \in \{0,...,24\}) \\ &= \sum_{i=1}^{\lfloor 29.5-t \rfloor} \int_{i-1}^{i} (\frac{1.03}{1.05})^x \frac{d_{35+t+i-1}}{l_{35.5+t}} dx + \int_{\lfloor 29.5-t \rfloor}^{29.5-t} (\frac{1.03}{1.05})^x \frac{d_{59}}{l_{35.5+t}} dx\end{aligned}$$

This approach mistakes ${}_{t|}q^{04}_x$ for post retirement df.
(b.4) $Q = \int((5B_x)v^{x-35}E(v^{T_x} I(T_x \le 65-x)\cdot 1.03^{T_x})|X=x) f^{01}_X(x)dx$,
with $B_X = 2\times 10^4 \frac{z_{35.5+t}}{s_{35}}$ and $v^{x-35} = v^{t+0.5}$ for $D = X-35 \in (t, t+1]$, $t \in \{0,1,...,24\}$.

$$Q = 10^5 \sum_{t=0}^{24} \frac{z_{35.5+t}}{s_{34}} v^{t+0.5} \times \overline{A}^{1}_{35+t:\overline{30-t}|j} \frac{w_{35+t}}{l_{35}}, \text{ where}$$

$$\begin{aligned}\overline{A}^{1}_{35+t:\overline{30-t}|j} &= \sum_{i=1}^{30-t} \int_{i-1}^{i} (\frac{1.03}{1.05})^x \prod_{k:\ 35+t \le k < 35+t+i-1} (1-\frac{d_k}{l_k}) \frac{d_{35+t+i-1}}{l_{35+t+i-1}} dx \\ &= \sum_{i=1}^{30-t} \frac{(\frac{1.03}{1.05})^i (1-(\frac{1.03}{1.05})^{-1})}{\log\frac{1.03}{1.05}} \prod_{k:\ 35+t \le k < 35+t+i-1} (1-\frac{d_k}{l_k}) \frac{d_{35+t+i-1}}{l_{35+t+i-1}} \\ &\approx \sum_{i=1}^{30-t} \frac{(\frac{1.03}{1.05})^i (\frac{0.02}{1.03})}{0.02} \prod_{k:\ 35+t \le k < 35+t+i-1} (1-\frac{d_k}{l_k}) \frac{d_{35+t+i-1}}{l_{35+t+i-1}}\end{aligned}$$

Similar mistake as in (b.3).

Career Average earnings (CAE) plans

This plan differs from the standard one in replacing S_{Fin} by $(TPE)_x/n$, where
TPE= total pensionable earnings during the service and x is the age at evaluation.

$$B_x = \alpha n \frac{(TPE)_x}{n} \qquad (= \alpha n CAE).$$

A popular variation of the CAE plan is the career average revalued earnings plan, in which an inflation adjustment of the salary is made before average. The accrual principle is the same.

Example 10.9. *A pension plan offers a retirement benefit of 4% of career average earnings for each year of service. The pension benefit is paid monthly in advance for life, guaranteed for 5 years, with no spousal benefit. On withdrawal, a deferred pension is payable from age 65. The multiple decrement model in Example 10.5 is appropriate for this pension plan, including the assumption that members can retire at exact ages 60 and 65. Consider a member now aged 35 who has 10 years of service, with total past earnings of $525000.*
(a) Write down an integral formula for an accurate calculation of the APV of his accrued age and withdrawal benefits.
(b) Use Table 2 to estimate the APV of his accrued age and withdrawal benefits assuming that
(1) Post-retirement survival is the Standard Ultimate Survival Model,
(2) Interest rate is 5% per year.

Sol. (a) APV of his accrued age retirement and withdrawal benefits are

$$E(H_D v^D I(M=03)) \text{ and } E(H_D v^{30} I(M=01)A), \text{ respectively, where}$$

X is the exit age, $D = X - 35$, M is the exit mode,
$((X-35, M) = (M_{35}^{(\tau)}, J_{35})$ under the multiple decrement model). $f_D^{03}(t) \stackrel{def}{=} f_{M_{35}^{(\tau)}, J_{35}}(t, 3)$.
$H_t = \ddot{a}^{(12)}_{\overline{35+t:\overline{5}|}} B_t$ (total pension), ([17] in 450)
$B_t = \alpha$TPE$= \alpha n$CAE $= 0.04 \times 525000 = 21000$ (annual pension), $(\alpha = 0.04)$
$A = I((35)$ survives to age 65).
APV of his accrued age retirement benefits) is

$$E(H_D v^D I(M=03)) = \int_{t\in(25,30)} H_t v^t f_D^{03}(t)dt + \sum_{t\in\{25,30\}} H_t v^t \cdot f_D^{03}(t)$$

$$=21000\Big(\int_{25}^{30} v^t \ddot{a}^{(12)}_{\overline{35+t:\overline{5}|}} f_D^{03}(t)dt + v^{25}\ddot{a}^{(12)}_{\overline{60:\overline{5}|}} f_D^{03}(25) + v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}|}} f_D^{03}(30)\Big) \qquad (10.9.1)$$

$$=21000\Big(\int_{25}^{30} v^t \ddot{a}^{(12)}_{\overline{35+t:\overline{5}|}} {}_tp_{35}^{00}\mu_{35+t}^{03}dt + v^{25}\ddot{a}^{(12)}_{\overline{60:\overline{5}|}} 0.3\,{}_{25-}p_{35}^{00} + v^{30}\ddot{a}^{(12)}_{\overline{65:\overline{5}|}} {}_{30-}p_{35}^{00}\Big)$$

$$E(H_D v^{30} I(M=01)A) = E(E(H_D v^{30} I(M=01)A|D))$$

$$=E(H_D v^{30} I(M=01)E(A|D)) = E(H_D v^{30} I(M=01)\,{}_{30-t}p_{35+t}^{00})$$

$$=21000\ddot{a}^{(12)}_{\overline{65:\overline{5}|}} v^{30}\int_0^{25} {}_{30-t}p_{35+t}^{00} f_D^{01}(t)dt \qquad (10.9.2)$$

$$=21000\ddot{a}^{(12)}_{\overline{65:\overline{5}|}} v^{30}\int_0^{25} {}_{30-t}p_{35+t}^{00}\cdot {}_tp_{35}^{00}\mu_{35+t}^{01}dt$$

(b) By (10.9.1), APV of his accrued age retirement benefits is $E(H_D v^D I(M=03))$

$$
\begin{aligned}
=&21000\Big(\int_{25}^{30} v^t \ddot{a}^{(12)}_{\overline{35+t:5}|} f_D^{03}(t)dt + v^{25}\ddot{a}^{(12)}_{\overline{60:5}|} f_D^{03}(25) + v^{30}\ddot{a}^{(12)}_{\overline{65:5}|} f_D^{03}(30)\Big)\\
\approx&21000\Big(\sum_{k=0}^{4}\int_k^{k+1} v^{25.5+k}\ddot{a}^{(12)}_{\overline{60.5+k:5}|}\frac{r_{60+k}}{l_{35}}dt + v^{25}\ddot{a}^{(12)}_{\overline{60:5}|}\frac{r_{60-}}{l_{35}} + v^{30}\ddot{a}^{(12)}_{\overline{65:5}|}\frac{r_{65}}{l_{35}}\Big) \qquad \textbf{Why ?}\\
=&21000\Big(\sum_{k=0}^{4} v^{25.5+k}\ddot{a}^{(12)}_{\overline{60.5+k:5}|}\frac{r_{60+k}}{l_{35}} + \frac{r_{60-}}{l_{35}}v^{25}\ddot{a}^{(12)}_{\overline{60:5}|} + \frac{r_{65}}{l_{35}}v^{30}\ddot{a}^{(12)}_{\overline{65:5}|}\Big) \approx 31666.
\end{aligned}
$$

APV of his accrued withdrawal benefits is

$$
\begin{aligned}
E(H_D v^{30} I(M=01)A) =& 21000\ddot{a}^{(12)}_{\overline{65:5}|}v^{30}\int_0^{25} {}_{30-t}p^{00}_{35+t} f_D^{01}(t)dt \qquad \textit{by (10.9.2)}\\
=& 21000\Big(\frac{w_{35}}{l_{35}}({}_{29.5}p^{00}_{35.5}) + \frac{w_{36}}{l_{35}}({}_{28.5}p_{36.5}) + \cdots + \frac{w_{59}}{l_{35}}({}_{5.5}p_{59.5})\Big) = 33173.
\end{aligned}
$$

10.4 Benefit reserves

In a typical DB pension plan the employee pays a fixed contribution, and the balance of the cost of the employee benefits is funded by the employer. The employer's contribution is set at the regular actuarial valuations and is expressed as a percentage of salary.

The t-th **actuarial (accrued) liability**, denoted by ${}_tV$, is the value at the valuation date t (often at the beginning of a year), of the **pension benefit accrued** from the date of entry into the plan to the date of valuation, taking into consideration all the appropriate benefits. ${}_tV$ is also called the **reserve** or the $t-$**th benefit reserve**. The **normal cost** is the present value of a single year's accrual.

$${}_tV = E(v^{R-t}(t-e)\alpha S_{Fin}\ddot{a}^{(12)}_R), \text{ where } R \text{ is the retiring age and } e \text{ is the entering age.}$$

$$
(10.9) \qquad {}_tV = \underbrace{(t-e)\alpha\hat{S}_{Fin}}_{annual\ PB} \times \underbrace{\ddot{a}^{(12)}_r}_{1\ \$/year} \times \underbrace{{}_{r-t}p_t v^{r-t}}_{present\ value} \qquad \text{if } P(R=r)=1,
$$

as $P(R=r|T_x=t) = P(T_t > r-t)$.
For the moment, we assume for simplicity that
(1) all employer contributions are paid at the start of the year,
(ii) no employee contributions,
(iii) any benefits payable during the year are paid exactly half-way of the year.

With these assumptions, the **normal contribution** (or **normal cost**), denoted by C_t, due at the start of the year t to $t+1$ for a member aged x at time t, is found from

$$
(10.10) \qquad C_t = \text{APV of mid-year exists benifits (MYEB)} + v\cdot {}_1p^{00}_x \cdot {}_{t+1}V - {}_tV.
$$

Example 10.10. *A member aged 50 has 20 years past service. His salary in the year to valuation was \$50000. Calculate the value of his accrued pension benefit and the normal contribution due at the start of the year assuming valuation uses "final pensionable earnings" (before age retirement) at the valuation date, and assuming*

(a) projected unit credit (PUC) funding, and (b) TUC funding, (see Section 10.3),
The pension plan:
accrual rate 1.5%*,*
normal retirement age 65,
life annuity payable monthly in advance,
no benefit due on death in service.
Assumptions are as follows:
no exist other than death before normal retirement age;
interest rate is 5% *per year effectively;*
salaries increase at 4% *per year (in case of PUC);*
mortality always follows $\mu_x = A + Bc^x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$.

Sol. Compute ${}_{50}V$ and C_{50}, where ${}_tV = (t-e)\alpha \hat{S}_{Fin} \times {}_{r-t}p_t v^{r-t} \ddot{a}_r^{(12)}$ (see (10.9)), C_t is as in (10.10), e = entering age =30, $t = 50$ or 51, $r = 65$, $v = 1/1.05$, ${}_sp_x = \exp(-\int_0^s A + Bc^{x+y}dy)$, $\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{i=0}^{\infty} v^{i/12}{}_{65+i/12}p_{65} = 13.087$, $\alpha = 0.015$ and with $\underline{different}$ $\hat{S}_{Fin}$ in (a)&(b).
(a) **The PUC funding.** $\hat{S}_{Fin} = S_{49}\frac{s_{64}}{s_{49}} = 50000 \times 1.04^{15}$?? $= 90047$

The accrued pension benefit $${}_{50}V = 20 \times 1.5\% \times \hat{S}_{Fin} \times \overbrace{{}_{15}p_{50}}^{??} \times v^{15} \times \ddot{a}_{65}^{(12)} = 163161,$$
$${}_{51}V = 21 \times 1.5\% \times \hat{S}_{Fin} \times {}_{14}p_{51} \times v^{14} \times \ddot{a}_{65}^{(12)}. \tag{1}$$
$$v_1p_{50} \cdot {}_{51}V = 21 \times 1.5\% \times \hat{S}_{Fin} \times {}_{15}p_{50} \times v^{15} \times \ddot{a}_{65}^{(12)} = \frac{21}{20}{}_{50}V.$$

The normal contribution (by Eq. (10.10)) is
$$C_{50} = \underbrace{0}_{no\ midyear\ exit} + v_1p_{50} \cdot {}_{51}V - {}_{50}V = {}_{50}V/20 = 8158. \tag{2}$$

(b) **The TUC funding.** $\hat{S}_{Fin} = S_{t-1}$, not $S_{49}\frac{s_{64}}{s_{49}}$.
The accrued liability and the accrued pension benefit
$${}_{50}V = 1.5\% \times 20 \times S_{49} \times {}_{15}p_{50} \times v^{15} \times \ddot{a}_{65}^{(12)}.$$
The accrued liability next year
$${}_{51}V = 1.5\% \times 21 \times S_{50} \times {}_{14}p_{51} \times v^{14} \times \ddot{a}_{65}^{(12)} \quad (1.04S_{49} = S_{50} \text{ compare to Eq. (1)}).$$
$v \cdot {}_1p_{50} \cdot {}_{51}V = 1.5\% \times 21 \times S_{50} \times {}_{15}p_{50} \times v^{15} \times \ddot{a}_{65}^{(12)}$
The normal cost $C_{50} = {}_{50}V(\frac{21}{20}\frac{S_{50}}{S_{49}} - 1) = 8335$ (compare to Eq. (2)).

Example 10.11. *A pension plan offers a pension benefit of $1000 per year of service, with fractional years counting proportionally. A member aged 61 has 35 years past service. Determine the normal cost rate payable in respect of age retirement benefits using the following plan information and valuation assumptions.*
(1) Age retirement are permitted at any age between 60 and 65.
(2) The pension is paid monthly in advance for life.
(3) Contributions are paid annually at the start of each year.
(4) Assumptions:
Exits follow the service table given in Table 2.

Interest rate: 6% per year effective.
All lifes taking age retirement exit exactly half-way through the year of age (except at 65).
Survival after retirement: follows the Standard Ultimate Survival Model.

Sol. Solve $C_{61} = MYEB + v_1p_{61}^{00} \cdot {}_{62}V - {}_{61}V = ?$
where ${}_tV = E((t-e)\alpha S_{Fin} \times \ddot{a}_R^{(12)} v^{R-t})$, and other terms to be explained.
We do not need to specify whether we use PUC or TUC method as $\alpha\hat{S}_{Fin} = 10^3$.
$t - e = 35$ years of past service. (The entring age $e = 61 - 35 = 26$). $R = r \in (61, 65]$. Then ${}_{61}V = 35 \times 10^3 E(\ddot{a}_R^{(12)} v^{R-t}) = 35 \times 10^3 \sum_{r \in \mathcal{W}} v^{r-t} \ddot{a}_r^{(12)} f_{T_{61}^{(\tau)}, J_{61}}(r, 3)$ and $t = 61$.

$$
\begin{aligned}
{}_{61}V &= 35 \times 10^3 \Big(\sum_{k=1}^{4} \frac{r_{60+k}}{l_{61}} v^{k-0.5} \ddot{a}_{60.5+k}^{(12)} + \frac{r_{65}}{l_{61}} v^4 \ddot{a}_{65}^{(12)}\Big) = 345307, \\
&\text{as } R \in D = \{61.5, 62.5, 63.5, 64.5, 65\}. \\
v \times {}_1p_{61}^{00} \times {}_{62}V &= 10^3 \times 36 \Big(\sum_{k=2}^{4} \frac{r_{60+k}}{l_{61}} v^{k-0.5} \ddot{a}_{60.5+k}^{(12)} + \frac{r_{65}}{l_{61}} v^4 \ddot{a}_{65}^{(12)}\Big) = 312863. \\
MYEB &= \underbrace{10^3 \times 35.5}_{\alpha\hat{S}_{Fin}(t-e)} \times \frac{r_{61}}{l_{61}} v^{0.5} \ddot{a}_{61.5}^{(12)} = 41723.
\end{aligned}
$$

Hence, Eq. (10.10) yields

$$C_{61} = MYEB + v_1p_{6162}V - {}_{61}V = 41723 + 312863 - 345307 = 9278.$$

10.5 The traditional unit credit (TUC) method.

The TUC method is most often used with pension plans that provide a flat pension benefit, such as \$30/month for each year of service. If the entry age is 35 and the retirement age is 65, the annual pension benefit commencing at retirement will be $B_{65} = 30 \times 30 \times 12 = 10800$ dollars. In this section, we shall consider the case that $B_r = (r - e)B$, where B is the annual pension benefit, which is constant in the whole service time. If a participant is to retire at age r with an expected annual pension of B_r, with one-twelfth of B_r payable at the beginning of each month, then $B_r\ddot{a}_r^{(12)}$ is sufficient to fund this pension at age r.

Let us consider a pension fund valuation for (x) at time 0. The annual pension benefit which has accrued from (entering) age e to age x is usually a certain number of dollars per month (*e.g.* \$30) for each year of service.
The **annual benefit that accrued to age** x is denoted by B_x.

B_x =pension benefit per month×12× years from age e to age x.
e.g. $B_x = B \times (t - e) = 30 \times 12 \times 30$. (This is the credit earned by the employee, which will be paid after retirement (annually)). The **actuarial liability** at age x is the value of the pension benefit accrued from age e to age x:

${}_xV = B_x \frac{D_r^{(r)}}{D_x^{(r)}} \ddot{a}_r^{(12)}$, where $D_r^{(r)}$ ($D_x^{(r)}$) is given in a service table (*i.e.* D_x in the next example).
If <u>there no other decrement</u>, then $D_x^{(r)} = D_x$.

Compare to ${}_xV = \underbrace{\alpha n\hat{S}_{Fin}}_{B_x} \cdot {}_{r-x}p_x \cdot v^{r-x}\ddot{a}_r^{(12)}$ (see Eq. (10.10)).

The **total actuarial liability (TAL)** at time 0 for all active participants in a pension plan is

$$TAL_0 = \sum_{x:\ age\ of\ all\ participants} {}_xV.$$

Let b_x be the piece of the total pension benefit that is earned (accrued) in the year following age x. The simplest case is $b_x = B_r/(r-e)$ (r is retirement age).

The **normal cost (or normal contribution)** at the beginning of each year is the cost of the pension benefit that is earned in that year. That is,

$C_x = b_x \frac{D_r^{(\tau)}}{D_x^{(\tau)}} \ddot{a}_r^{(12)}$ (compare to $C_x = MYEB + v \cdot {}_1p_{x x+1}V - {}_xV$).

In single-decrement situation, we will often use

$C_x = b_x v^t {}_tp_x \ddot{a}_r^{(12)}$, where $t = r - x$, and ${}_xV = C_x(x-e)$.

Reason: ${}_xV = (x-e)b_x v^{r-x} \cdot {}_{r-x}p_x \cdot \ddot{a}_r^{(12)}$.

${}_{x+1}V = (x+1-e)b_{x+1}v^{r-x-1} \cdot {}_{r-x-1}p_{x+1} \cdot \ddot{a}_r^{(12)}$.

Since there is no midyear exit (no other decrement), the normal contribution

$C_x = v \cdot {}_1p_x \cdot {}_{x+1}V - {}_xV = b_x v^{r-x} \cdot {}_{r-x}p_x \cdot \ddot{a}_r^{(12)}$.

Example 10.12. *The servise table is as follows.*

Age x	participants	D_x
25	8	16
35	0	8
45	2	4
55	0	2
65	0	1

Table 3. Service Table

Plan effective date: 1/1/84 and census date on 1/1/94,

Normal retirement benefit: \$30 per month for each year of service

All employees were hired at age 25.

Retired or terminated vested participants: None

Preretirement terminations other than by death: None

Selected annuity value: $\ddot{a}_{65}^{(12)} = 10$

Commutation functions are given in Table 3.

What is the TUC actuarial liability (TAL_0) and normal cost (TNC_0) as of 1/1/94 ?

Note that there are no decrements other than death, we use appropriate single-decrement table (mortality only). That is, we use D_x rather than $D_x^{(\tau)}$.

Sol. Let 1/1/94 be time 0. $TAL_0 = \sum_x {}_xV = ?$ $\quad TNC_0 = \sum_x C_x = ?$ $\quad$ where ${}_xV = B_x \frac{D_r^{(\tau)}}{D_x^{(\tau)}} \ddot{a}_r^{(12)}$, $C_x = b_x \frac{D_r^{(\tau)}}{D_x^{(\tau)}} \ddot{a}_r^{(12)}$, $\ddot{a}_r^{(12)} = 10$, $B_x = (x-e)b_x$, $e = 25$, and $b_x = 30 \times 12\ \forall\ x$.

From the table, there are 8 $x = 25$ and 2 $x = 45$, but none of other (x).

$B_{25} = 30 \times 12(t-25) = 0$ and $B_{45} = 30 \times 12(t-25) = 30 \times 12 \times 20 = 7200$,
${}_{25}V = B_{25}\frac{D_{65}}{D_{25}}\ddot{a}_{65}^{(12)} = 0$. ${}_{45}V = B_{45}\frac{D_{65}}{D_{45}}\ddot{a}_{65}^{(12)} = 18000$.

$$TAL_0 = \sum_x {}_xV = 8_{25}V + 2_{45}V = 0 + 2(18000) = 36000.$$

$$TNC_0 = \sum_x C_x = \sum b_x \frac{D_r}{D_x}\ddot{a}_r^{(12)} = b_x\ddot{a}_r^{(12)} \sum_x \frac{D_r}{D_x} \qquad (b_x, \ddot{a}_r, r) = (30 \times 12, 10, 65))$$
$$= (30 \times 12) \times 10 \times (8(1/16) + 2(1/4)) = 3600.$$

TUC actuarial liability (TAL_0) and normal cost (TNC_0) are 36000 and 3600, respectively.

Example 10.13. *Normal retirement benefit: \$10/month for each year of service Actuarial cost method is TUC. Actuarial assumptions:*

(1) Interest: 6%

(2) Preretirement terminations other than deaths: None

(3) Retirementage: 65

(4) Participants as of 1/1/93: 100 active employees, all age 60

(5) Normal cost for 1993 as of 1/1/93: \$100,000 *(for 100 participant)*

(6) Selected mortality value: $q_{60} = 0.04$

Calculate the normal cost for 1994 as of 1/1/94 in each of the following cases:

(a) per survivor,

(b) for the total group if 92 participants are alive at 1/1/94,

(c) if 96 participants are alive, and

(d) if all participants are alive.

Sol. Formula before Ex.10.12: $C_t = b_x(v^{r-t})_{r-t}p_t\ddot{a}_r^{(12)}$, and $TNC_x = nC_x$, where

$b_x = 10 \times 12$, $r = 65$, $v = 1/1.06$, $\ddot{a}_x^{(12)} = ?$ ${}_tp_x = ?$

The normal cost per participant at age 60 is

$C_{60} = (10 \times 12)v^5{}_5p_{60}\ddot{a}_{65}^{(12)} = 100,000/100$ (given in (5)).

(a) $C_{61} = (10 \times 12)v^4{}_4p_{61}\ddot{a}_{65}^{(12)} = \frac{C_{60}}{0.96/1.06} = 1104.17$ (as ${}_5p_{60} = p_{60} \cdot {}_4p_{61}$),

(b) $TNC_{61} = 92C_{61} = 101583$

(c) $TNC_{61} = 96C_{61} = 106000$

(d) $TNC_{61} = 100C_{61} = 110417$. ▫

Note: $\frac{D_r^{(\tau)}}{D_x^{(\tau)}}$ is replaced by $v^{r-x} \cdot {}_{r-x}p_x$, due to assuming single-decrement and constant rates.

If F_0 is the amount of the pension fund at time 0 and TAL_0 represents the plan's total actuarial liability for all active, retired and terminated vested participants at time 0, then the surplus at that time is $F_0 - TAL_0$.

Traditionally, most plans had, and many plans still have, a negative sur-plus, called the **unfunded actuarial liability** (UAL$_0$), where $UAL_0 = TAL_0 - F_0$.

Example 10.14. *Refer to the data given in Example 10.12. Under the Traditional Unit Credit cost method, what is the unfunded actuarial liability as of 1/1/94 if the plan assets amount to \$5000 at that time?*

Sol. $UAL_0 = TAL_0 - F_0 = 36000 - 5000 = 31000.$ □

The fund balance at the beginning of the year (BOY), which we have denoted by F_0, will increase during the year by actual investment income and contributions to the fund. It will be diminished by amounts withdrawn from the fund as benefits. At time 0, we can calculate what we expect the unfunded actuarial liability to be at time 1 as

$$^{exp}UAL_1 = (UAL_0 + C_0)(1+i) - {}^{i}C,$$

where ${}^{i}C$ is the contribution plus the interest earned during the year on the contribution using the actuarial interest assumption. If the contribution is made at the end of the year (EOY), then ${}^{i}C = C$, and if it is made at BOY, then ${}^{i}C = C(1+i)$. A total **experience gain**, ${}^{tot}G$, will result if the actual unfunded actuarial liability (${}^{act}UAL$) is less than ${}^{exp}UAL$. That is,

$$^{tot}G_1 = {}^{exp}UAL_1 - {}^{act}UAL_1.$$

A negative gain is called a loss.

Example 10.15. *Actuarial cost method: TUC. Assumed interest rate: 6%*
Valuation results as of 1/1/93:

Actuarial liability: \$100000

Actuarial value of assets: \$50000

Normal cost as of 12/31/93: \$10000
Valuation results as of 1/1/94:

Actuarial liability: \$115000

Actuarial value of assets: \$70000
Contributions: (*≠normal cost=normal contribution*)

\$13910 at 12/31/93

\$15587 at 12/31/94
What is the total experience gain for 1993 ?

Sol. Let 1/1/93 be time 0 and 1/1/94 be time 1. ${}^{tot}G_1 = ?$
There is a gain when the actual unfunded liability turns out to be less than the expected. Since the normal cost is at 12/31/93, ${}^{i}C = C$. Then the expected unfunded liability at time 1 is

$$\begin{aligned} {}^{exp}UAL_1 =& UAL_0(1+i) + NC - {}^{i}C \qquad (1)\\ =&(100000 - 50000)(1.06) + 10000 - 13910 = 49090\\ {}^{act}UAL_1 =& AL_1 - F_1 = 115000 - 70000 = 45000\\ {}^{tot}G =& {}^{exp}UAL - {}^{act}UAL = 49090 - 45000 = 4090 \quad \square \end{aligned}$$

Note that the normal cost is usually at BOY but occasionally it is at EOY; the contribution is usually at EOY but occasionally it is at BOY or mid-year.

Example 10.16. *Normal retirement benefit: \$10 per month for each year of service*
Vesting eligibility: 100% after 5 years of service
Preretirement death benefit: None
Actuarial cost method: Tranditional Unit Credit
Actuarial assumptions:

Interest rate: 7% per year

Preretirement terminations other than deaths: EOY

Retirement age: 65

Selected annuity value: $\ddot{a}_{65}^{(12)} = 8.736$.
Data for sole participant:

Date of birth: 1/1/31

Date of hire: 1/1/89

Status as of 1/1/94: Active

Selected probability

x	$q_x^{(\tau)}$	$q_x^{(d)}$
63	0.069	0.019
64	0.081	0.021
65	0.023	0.023

$$q_x^{(d)} = P(T_x^{(\tau)} \le 1, J_x = 4).$$

What is the normal cost for 1994 as of 1/1/94 ?

Sol. The participant is 63 years old on 1/1/94 and will retire at $r = 65$. Then

$C_t = b_r(v^{r-t})_{r-t}p_t\ddot{a}_r^{(12)}$.

$C_{63} = 10 \times 12 \times v^{r-63}{}_{r-63}p_{63}^{(d)}\ddot{a}_{65} = 120 \times (1/1.07)^2 \times (0.981)(0.979)(8.736) = 879.3$

In this defined benefit plan with no participant contributions, 100% vesting means that the participant is entitled to 100% of the retirement benefit accrued to the date of withdrawal. If withdrawal occurs before five years of service, no retirement benefit is payable because there is no vesting. 60% vesting, for example, would mean that the participant is entitled to 60% of the accrued benefit.

Since deaths occur at EOY, $q_x^{(d)} = q_x^{\prime(d)}$, and we discount for mortality because the death benefit is zero. If, alternatively, the terminal reserve were paid out on death, we would not discount for mortality. Similarly, we do not discount for withdrawal because vesting is 100%, and the unit credit liability is not released.

The benefit formulas mentioned so far did not make use of projected future salary. If a salary scale is not being used in the benefit projection, it would appear that the actuary is assuming that salaries are not expected to increase. The effect of such an assumption is normally to shift a portion of the costs from the present to the future. This is often not appropriate. If the actuary does use a salary scale, it may be a scale independent of age, such as 5% per year, or it may be a more sophisticated scale that depends on age. In any event, it will make due allowance for inflation, actual past salaries, and expected future salaries. We will illustrate the use of a 5% scale by calculating the pension benefits for (a) a 2% final salary plan, (b) a 2% final three-year average plan, and (c) a 2% career average plan.

(a) For a person currently age x with a salary of S_x, the expected (or projected) final salary, at age $r-1$, is $S_{r-1} = (1.05)^{r-1-x} S_x$, and the pension benefit accrued to age x is

$B_x = 0.02(1.05)^{r-1-x} S_x (x-e)$

and the expected final 3-year average salary is

$$
\begin{aligned}
FAS =& \frac{1}{3}[S_{r-3} + S_{r-2} + S_{r-1}] \\
=& \frac{1}{3}[(1.05)^{r-3-x} + (1.05)^{r-2-x} + (1.05)^{r-1-x}] S_x \\
=& \frac{1}{3}(1.05)^{r-1-x} S_x \ddot{a}_{\overline{3}|0.05}.
\end{aligned}
$$

The accrued pension benefit is $B_x = \frac{0.02\ddot{a}_{\overline{3}|0.05}}{3}(1.05)^{r-1-x} S_x (x-e)$

Example 10.17. *Which of the following statements concerning the Traditional Unit Credit cost method are true ?*
1. Under this method, the assumption must be made that each participant will remain in the plan until retirement or prior death. That is, no other withdraw except retiring and death.
2. If the benefit accrual in each year ($b_x \ddot{a}_r (v^{r-x})_{r-x}p_x$) is constant for any given participant, the normal cost for that participant will also remain constant, provided actual experience is in accordance with actuarial assumptions.
3. The actuarial liability of a newly established plan is equal to the present value of the benefits attributable to credited service prior to the effective date of the plan. That is, they all apply the formula: ($b_x \ddot{a}_r (v^{r-x})_{r-x}p_x$) for each additiona year of service.

Sol. 1. False. We could use other decrements such as withdrawal or disability.
2. False. The effect of mortality and interest discount is reducing with age.
3. True.

$$
\begin{aligned}
{}_tV =& E(v^{R-t}(t-e)\alpha S_{Fin} \ddot{a}_R^{(12)}), \text{ where } R \text{ is the retiring age.} \\
{}_tV =& \underbrace{(t-e)\alpha \hat{S}_{Fin}}_{B_t} \times \underbrace{\ddot{a}_r^{(12)}}_{1\ \$/year} \times \underbrace{{}_{r-t}p_t v^{r-t}}_{present\ value} \qquad \text{if } P(R=r)=1. \\
{}_tV =& B_t \frac{D_r^{(r)}}{D_t^{(r)}} \ddot{a}_r^{(12)}. \\
C_t =& b_t \frac{D_r^{(\tau)}}{D_t^{(\tau)}} \ddot{a}_r^{(12)} \text{ (compare to } C_t = MYEB + v \cdot {}_1p_t \cdot {}_{t+1}V - {}_tV).
\end{aligned}
$$

10.6 Problems from actuarial exams

1. (#19. Exam MLC Spring 2018). Mark is covered under a defined benefit pension plan. You are given:

(i) The annual benefit payable as a life annuity-due is 2% of the 3-year final average salary per year of service.

(ii) Mark retires at age 65 with 30 years of service.

(iii) Mark's salary in his final year of employment was 100,000.

(iv) At the start of each of the last 3 years of employment, Mark's salary increased by 3%.

(v) Mortality follows the Illustrative Life Table.

(vi) i = 0.06

(vii) Mark chooses to take his benefit as a 10-year guaranteed whole life annuity-due. Calculate the annual payment he will receive using the equivalence principle.

(A) 52,400 (B) 52,800 (C) 53,200 (D) 53,600 (E) 54,000

2. (#20. Exam MLC Spring 2018). XYZ offers a pension plan with the following lump sum death-in-service benefits, payable immediately on death:

(1) 10,000 for each full year of service on death in service between ages 64 and 65.

(2) 15,000 for each full year of service on death in service between ages 65 and 66.

You are given:

(i) Death is assumed to occur half-way through the year of age.

(ii) Decrements for this pension plan follow the Illustrative Service Table.

(iii) i = 0.05

(iv) XYZ uses the Traditional Unit Credit funding method.

Calculate the normal cost for this benefit for a new employee who is age 50.

(A) 60 (B) 70 (C) 80 (D) 90 (E) 100

3. (#6. Exam MLC Spring 2018). A defined benefit pension plan with two members, Finn and Oscar, provides for a pension benefit paid as a monthly whole life annuity-due. The annual pension benefit is 1.7% of the final one-year's salary for each year of service.

You are given:

(i) Mortality follows the Illustrative Life Table, assuming deaths are uniformly distributed between integer ages.

(ii) Participants reaching age 64.5 retire at that time with probability 50%. All participants reaching age 65 in service retire immediately. There are no other retirements.

(iii) There are no withdrawals from the plan other than by death or retirement.

(iv) i = 0.06

(v) $\ddot{a}^{(12)}_{64.5} = 9.5613$

(vi) Salaries increase every year on January 1. Future salary increases are 2% per year.

(vii) On January 1, 2018, Finn is 25 years old. He is a new employee with no past service. His salary in 2018 is 60,000.

(viii) On January 1, 2018, Oscar is 64 years old and has 29 years of service. His salary in 2017 was 95,000 and in 2018 is 100,000.

(a) Calculate the projected replacement ratios for both Finn and Oscar assuming that they each retire at exact age 65.

(b) Calculate the total accrued liability for the plan on January 1, 2018, under the Traditional Unit Credit (TUC) method.

(c) (i) Calculate the Normal Cost under the TUC method for Finn.

(ii) Calculate the Normal Cost under the TUC method for Oscar.

(d) (i) Without further calculation, state with reasons whether the Normal Cost under the Projected Unit Credit (PUC) method will be greater or less than the TUC for Finn.

(ii) Without further calculation, state with reasons whether the Normal Cost under the PUC will be greater or less than the TUC for Oscar.

10.7 Solution to the Problems from actuarial exams

1. Sol. Let B denote the regular annual pension and B^* the one allowing for a 10-year guarantee.

$B\ddot{a}_{65} = B^*\ddot{a}_{\overline{65:\overline{10|}}} => B\ddot{a}_{65}/\ddot{a}_{\overline{65:\overline{10|}}} = B^* = ?$

$\ddot{a}_{\overline{65:\overline{10|}}} = \ddot{a}_{\overline{10|}} + {}_{10}E_{65}\ddot{a}_{75} = 7.8017 + 2.8864 = 10.6881$ (from the Illustrative Life Table).

$B = n\alpha S_{Fin} = 30(0.02 * 10^5)(\frac{(1.03)^{-2}+1.03^{-1}+1}{3}) = 58269$, where $v = 1/1.06$.

$B^* = \frac{58269\ddot{a}_{65}}{10.6881} = \frac{58269*9.8969}{10.6881} = 53955$. Answer Key E

2. Sol. The normal cost is the APV of a single year's accrual:

$E(10^4 v^{K_x} I(K_x = 64.5 - x) + 1.5(10^4 v^{K_x} I(K_x = 65.5 - x)$

$= 10^4[v^{64.5-x}({}_{64-x}|q_x^{04}) + 1.5v^{65.5-x}({}_{65-x}|q_x^{04})]$, where $x = 50$.

$NC = 10^4 \frac{v^{64.5-50}d_{64}^{(d)} + 1.5v^{65.5-50}d_{65}^{(d)}}{l_{50}} = 90.3.$

Answer Key D

3. Sol. a) $\underline{\text{The pension replacement ratio } R} = \frac{\text{pension income in the year after retirement}}{\text{final salary before retirement}}$.

Finn: $RR = \frac{n\alpha S_{Fin}}{S_{Fin}} = \frac{(65-25)(0.017)S_{25}(1.02)^{65-25-1}}{S_{25}(1.02)^{65-25-1}} = (65-25)(0.017) = 0.68.$

Oscar: $RR = \frac{(29+1)(0.017)S_{64}}{S_{64}} = (29+1)(0.017) = 0.51.$

b) ${}_tV = E((t-e)\alpha S_{Fin} \underline{v^{R-t}\ddot{a}_R^{(12)}}).$

The table only provide $\ddot{a}_x$, not $\ddot{a}_x^{(12)}$. Make use of the following method:

For a period of length $\frac{1}{m}$: Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)}d^{(m)}},$$

where $(1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$ and $(1+i)^{-1/m} = v^{1/m} = (1-d)^{1/m} = 1 - \frac{d^{(m)}}{m}$.

${}_tV = E((t-e)\alpha S_{Fin} \underline{v^{R-t}\ddot{a}_R^{(12)}}).$

Fin: ${}_tV = 0$, as $t - e = 25 - 25 = 0$.

Oscar: ${}_tV = 29(0.017)S_{63}[0.5{}_{0.5}p_{64}v^{0.5}\ddot{a}_{64.5}^{(12)} + 0.5p_{64}v^1\ddot{a}_{65}^{(12)}]$

$p_x = 1 - q_x$ and ${}_{0.5}p_x = 1 - 0.5q_x = 1 - 0.5(0.01952) \approx 0.99$ from the Illustrative Table.

${}_tV = 29(0.017)(95000)[0.5(0.99)1.06^{-0.5}(9.5613) + 0.5(0.98048)1.06^{-1}(9.431551)]$

$= 419644.49.$

c) The normal cost is the APV of a single year's accrual.

(A) $C_t =$ APV of benefits for mid-year exists $+\underline{v \cdot {}_1p_x^{00}} \cdot {}_{t+1}V - {}_tV.$

(B) $C_t = E(\alpha S_{Fin} v^{R-t} \ddot{a}_R)$ using TUC method (without MYEB).

Fin: $NC = 0.017S_{25}[0.5{}_{39.5}p_{25}v^{39.5}\ddot{a}_{64.5}^{(12)} + 0.5{}_{40}p_{25}v^{40}\ddot{a}_{65}^{(12)}]$ from (B), or (A) as ${}_tV = 0 = MYEB$,

${}_{40}p_{25} = \frac{l_{25+40}}{l_{25}} = \frac{7533964}{9565017} = 0.787658.$ and ${}_{39.5}p_{25} = \frac{(7683979+7533964)/2}{9565017} = 0.7955$

$NC = (0.017)(60000)[0.5(0.7955)(1.06)^{-39.5}(9.5613) + 0.5(0.787658)(1.06)^{-40}(9.431551)]$

$= 756.63$ (v) $\ddot{a}_{64.5}^{(12)} = 9.5613$

Oscar: ${}_tV = 419644.49.$

$MYEB = 0.017(29.5)S_{63.5}0.5{}_{0.5}p_{64}v^{.5}\ddot{a}_{64.5}^{(12)} = 224827.81$

${}_{t+1}Vvp_{64} = 0.017(30)S_{64}0.5p_{64}v\ddot{a}_{65}^{(12)} = 222462.17$

$NC = 224827.81 + 222462.17 - 419644,49 = 27645.49.$

(d) (i) Without further calculation, state with reasons whether the Normal Cost under the Projected Unit Credit (PUC) method will be greater or less than the TUC for Finn.

(ii) Without further calculation, state with reasons whether the Normal Cost under the PUC will be greater or less than the TUC for Oscar.

d) (1) The PUC would be more expensive for Finn as the NC includes the impact of future salary increases in the pension cost. Because Finn has many years of service ahead, prefunding salary increases will have a significant impact.

(2) The TUC would be more expensive for Oscar. The TUC requires all past accrued liability to be adjusted for current salary increases. As Oscar is near retirement, this is a significant cost. The PUC prefunds future salary increases, but since Oscar has little time left in employment, this cost is small compared with the salary upgrade in TUC.

Additional Homework 3.

A1. Use R program to verify the answers in part (b) of **Example 10.5** (see this page).

A2. Use R program to derive the answers in both (a) and (b) of Example 10.5 (continued) (see last two pages)

Example 10.5. A pension plan member is entitled to a lump sum benefit on death in service of 4 times the salary paid in the year up to death. Assume that

(1) 30% of the members surviving in employment to age 60 retire at 60 and

all members who remain in employment to age 65 retire then;

(2) $\mu_x^{01} = \mu_x^w = \begin{cases} 0.1 & \text{for } x \in [0,35) \\ 0.05 & \text{for } x \in [35,45) \\ 0.02 & \text{for } x \in [45,60) \\ arbitrary\ (\geq 0) & \text{for } x \geq 60 \text{ (as we only need to know up to age 60);} \end{cases}$

(3) $\mu_x^{02} = \mu_x^i = 0.001$ (we actually only need to know for $x \in (0,65)$);

(4) $\mu_x^{03} = \mu_x^r = \begin{cases} 0 & \text{for } x \in [0, 60) \\ 0.3 & \text{discrete at } x = 60 \\ 0.1 & \text{for } x \in (60, 65]; \end{cases}$

(5) $\mu_x^{04} = \mu_x^d = A + B \times c^x$, $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$.

(a) P(retiring at age 65 |age 35) = ?

(b) For each mode of exit, calculate the probability that a member currently aged 35 exits employment by that mode.

Sol. (a) ......

$$\begin{aligned}(b)(3)\quad & P(\text{a member retires at 60 or 65 or in (60,65)}) \\ = & \underbrace{0.3 \cdot {}_{25-}p_{35}^{00}}_{30\%\ retire} + \underbrace{1 \cdot {}_{30-}p_{35}^{00}}_{100\%\ retire} + \int_{25}^{30} {}_tp_{35}^{00} \times 0.1dt \qquad \textbf{why ??} \\ = & \underbrace{0.3 \cdot {}_{25-}p_{35}^{00} + {}_{30-}p_{35}^{00} + 0.1{}_{25}p_{35}^{00}}_{known} \underbrace{\int_0^5 {}_tp_{60}^{00}dt}_{\text{by R next}} \approx 0.4193,\end{aligned}$$

$$\begin{aligned}\int_0^5 {}_tp_{60}^{00}dt = & \int_0^5 \exp(-\int_{60}^{60+t}(0.1 + 0.001 + A + Bc^x)dx)dt \\ = & \int_0^5 \underbrace{\exp(-(0.1 + 0.001 + A)t - \frac{Bc^{60}}{\ln c}(c^t - 1))}_{=g} dt \qquad \text{computed by R program:}\end{aligned}$$

$$g = \exp(-\int_{60}^{60+t}(0.1 + 0.001 + A + Bc^x)dx) = \exp(-(0.1 + 0.001 + A)t - \frac{Bc^{60}}{\ln c}(c^t - 1))$$

```
> t=(1:5000)/1000
> g=exp(-(0.1+0.001+A)*t- B*C**60 * (C**t-1) /log(C))
> 0.3*a*b+ c*0.7*a*b +0.1*0.7*a*b*sum(g)/1000
[1] 0.4193452
```

$(\int_a^b g(x)dx \approx \sum_{i=1}^n g(a + \frac{b-a}{n})\frac{b-a}{n})$

$(a, b, n) = ?$

(1) The probability that a member currently aged 35 withdraws

$$\begin{aligned} = & \int_0^{25} {}_tp_{35}^{00}\mu_t^{01}dt \qquad \text{why not } \int_0^{30} ? \\ = & \int_0^{10} {}_tp_{35}^{00}\mu_t^{01}dt + \int_{10}^{25} {}_tp_{35}^{00}\mu_t^{01}dt \\ = & 0.05\int_0^{10} {}_tp_{35}^{00}dt + 0.02\int_{10}^{25} {}_tp_{35}^{00}dt \\ = & 0.05\int_0^{10} \exp(-[t(0.05 + 0.001 + A) + B\frac{c^{t+35} - c^{35}}{\ln c}])dt + 0.02{}_{10}p_{35}^{00}\int_0^{15} {}_tp_{45}^{00}dt\end{aligned}$$

$$=0.02\exp(-[10(0.05+0.001+A)+B\frac{c^{45}-c^{35}}{\ln c}])\int_0^{15}\exp(-[t(0.021+A)+B\frac{c^{t+45}-c^{45}}{\ln c}])dt$$
$$+0.05\int_0^{10}\exp(-[t(0.05+0.001+A)+B\frac{c^{t+35}-c^{35}}{\ln c}])dt\approx 0.5432$$

(2) The probability that a member currently aged 35 disability exits

$$=\int_0^{30}{}_tp_{35}^{00}\mu_t^{02}dt=0.001\int_0^{30}{}_tp_{35}^{00}dt$$
$$=0.001[\underbrace{\int_0^{10}{}_tp_{35}^{00}dt}_{see\ (1)}+\underbrace{{}_{10}p_{35}^{00}\int_0^{15}{}_tp_{45}^{00}dt}_{see\ (1)}+\underbrace{{}_{25}p_{35}^{00}\int_0^{5}{}_tp_{60}^{00}dt}_{?}]$$

≈ 0.0166 **how ?**

The probability that a member currently aged 35 death exits

$$=\int_0^{30}{}_tp_{35}^{00}\mu_t^{04}dt \qquad \text{(see assumption (5))}$$
$$=A\underbrace{\int_0^{30}{}_tp_{35}^{00}dt}_{known\ from\ (2)}+B\underbrace{\int_0^{30}{}_tp_{35}^{00}c^tdt}_{homework}\approx 0.0208.$$

Additional problem 2. Employees in a pension plan pay contribution of 6% of their previous month's salary at each month end. Calculate the APV at entry of contributions for a new entrant aged 55, with a starting salary rate of $100,000 using
(a) exact calculation using the multiple decrement model specified in additonal problem 1,
(b) the values in Table 2, adjusting the APV of an annuity payable annually under UDD.
Other assumptions:

Salary rate function: Salaries increase at 4% per year continuously;

Interest: 5% per year effective.

Sol. (a) Let Y be the exit time in years. APV of total contribution for (55).

$Ans = E(\sum_{k=1}^{(Y-55)12} \frac{1}{12} 0.06 S_{55+k/12} v^{k/12}) = \frac{0.06}{12} \sum_{i=1}^{10\times 12} S_{55+i/12} v^{i/12} {}_{i/12}p_{55}^{00} = ?$

Given: $A_{55} = 10^5$,

$1.04^y = \bar{s}_y = s_y$.

$v = 1/1.05$.

$S_{55} = \int_0^1 A_{55} 1.04^t dt = A_{55} \frac{1.04-1}{\log 1.04} = 1.02 A_{55} \approx A_{55}$.

Three periods: (55,60), 60, $(60, 65]$.

$$Ans = \frac{1.02(10^5)(0.06)}{12} [\sum_{k=1}^{59} {}_{\frac{k}{12}}p_{55}^{00} 1.04^{k/12} v^{k/12} + {}_{5-}p_{55}^{00} 1.04^5 v^5 + \sum_{k=61}^{120} {}_{\frac{k}{12}}p_{55}^{00} 1.04^{k/12} v^{k/12}] \quad (1)$$

$Ans = \frac{10^5 \times 0.06}{12} [\sum_{k=1}^{59} {}_{\frac{k}{12}}p_{55}^{00} 1.04^{k/12} v^{k/12} + {}_{5-}p_{55}^{00} 1.04^5 v^5 + \sum_{k=61}^{120} {}_{\frac{k}{12}}p_{55}^{00} 1.04^{k/12} v^{k/12}] =?$

$$\begin{cases} {}_tp_{55}^{00} = \exp(-0.031t) & t \in [0,5), \\ {}_tp_{55}^{00} = {}_5p_{55}^{00} \cdot {}_{t-5}p_{60}^{00} & t \in [5,10), \\ {}_tp_{60}^{00} = \exp(-0.131t) & t \in [0,5), \\ {}_5p_{55}^{00} = 0.7 {}_{5-}p_{55}^{00} = 0.7\exp(-0.155) & \end{cases}$$

$$\begin{aligned} Ans =& \frac{0.06 \times 10^5}{12} [\sum_{k=1}^{59} {}_{\frac{k}{12}}p_{55}^{00} (\frac{1.04}{1.05})^{k/12} + {}_{5-}p_{55}^{00} (\frac{1.04}{1.05})^5 + \sum_{k=61}^{120} {}_{\frac{k}{12}}p_{55}^{00} (\frac{1.04}{1.05})^{k/12}] \\ =& \frac{0.06 \times 10^5}{12} [\sum_{k=1}^{60} \exp(-\frac{0.031k}{12})(\frac{1.04}{1.05})^{k/12} \\ & + 0.7\exp(-0.155) \sum_{k=61}^{120} \exp(-0.131(\frac{k}{12} - 5))(\frac{1.04}{1.05})^{k/12}] \\ =& \frac{0.06 \times 10^5}{12} [\sum_{k=1}^{60} (\exp(-0.031) \cdot \frac{1.04}{1.05})^{k/12} \\ & + 0.7\exp(-0.155 + 0.131 * 5) \sum_{k=61}^{120} (\exp(-0.131) \cdot \frac{1.04}{1.05})^{k/12}] \\ =& 500[x \frac{1-x^{60}}{1-x}\Big|_{x=(\exp(-0.031)\cdot\frac{1.04}{1.05})^{1/12}} + 0.7\exp(0.5) t^{61} \frac{1-t^{60}}{1-t}\Big|_{t=(\exp(-0.131)\cdot\frac{1.04}{1.05})^{1/12}}] \\ =& 39349.31 \end{aligned}$$

```
x=(exp(-0.031)*1.04/1.05)**(1/12)
u=x*(1-x**60)/(1-x)
t=(exp(-0.131)*1.04/1.05)**(1/12)
v=t**61*(1-t**60)/(1-t)
500*(u+0.7*exp(0.5)*v)
[1] 39349.31
```

(b) APV at entry of total contribution to Pension of (x)= $E(\sum_{k=1}^{(Y-x)12} \frac{1}{12} c S_{x+k/12} v^{k/12})$. Ans is as in Eq. (1), but ${}_tp_{55}^{00}$ is different. ${}_tp_x^{00}$ is estimated by $\frac{l_{x+t}}{l_x}$. Under the assumption of UDD,

$$
{}_{\frac{i}{12}+j}p_{55}^{00} = {}_jp_{55}^{00}(1-r) + {}_{j+1}p_{55}^{00} r \qquad r = \frac{i}{12} \in [0,1]
$$

$$
= \begin{cases} \frac{l_{55+j}}{l_{55}}(1-\frac{i}{12}) + \frac{l_{55+j+1}}{l_{55}}\frac{i}{12} & \text{if } j \in \{55,...,64\} \setminus \{59,60\} \\ \frac{l_{59}}{l_{55}}(1-\frac{i}{12}) + \frac{l_{60-}}{l_{55}}\frac{i}{12} & \text{if } j = 59 \\ \frac{l_{60+}}{l_{55}}(1-\frac{i}{12}) + \frac{l_{61}}{l_{55}}\frac{i}{12} & \text{if } j = 60 \end{cases} \overset{??}{=} \begin{cases} \frac{l_{55+j}-d_{55+j}(i/12)}{l_{55}}\frac{i}{12} & \text{if } j \in \{55,... \\ \frac{l_{59}-d_{59}(i/12)}{l_{55}}\frac{i}{12} & \text{if } j = 59 \\ \frac{l_{60+}-d_{60+}(i/12)}{l_{55}}\frac{i}{12} & \text{if } j = 60 \end{cases}
$$

$$
\begin{aligned}
Ans =& E(\sum_{k=1}^{(Y-55)12} \frac{1}{12} 0.06 S_{55+k/12} v^{k/12}) \\
=& \frac{0.06}{12} \sum_{i=1}^{10\times 12} S_{55+i/12} v^{i/12} {}_{i/12}p_{55}^{00} \\
=& \frac{0.06\times 10^5}{12} \sum_{k=1}^{10\times 12} {}_{k/12}p_{55} (\frac{1.04}{1.05})^{k/12} \\
=& 500 \sum_{j=0}^{9} \sum_{i=1}^{12} {}_{j+i/12}p_{55} (\frac{1.04}{1.05})^{j+i/12} = 41130.89
\end{aligned}
$$

```
l=c(104688,102308 ,99960, 97642, 95351, 93085, 65160 ,58700, 52860, 47579 ,42805 ,38488)
p=1:10
p[1:6]=l[1:6]/l[1]
p[7:12]=l[7:12]/l[1]
p
s=0
r=1.04/1.05
for (j in 1:5)
s=s+sum((((1:12)/12)*(p[j+1]-p[j])+p[j]) *r**(j-1+(1:12)/12))
for (j in 7:11)
s=s+sum((((1:12)/12)*(p[j+1]-p[j])+p[j]) *r**(j-2+(1:12)/12))
(s=500*s)
```

The solution below mistakes ${}_tp_x^{00}$ as ${}_tp_x^{'(3)}$.

$Ans = E(\sum_{k=1}^{(Y-55)12} \frac{1}{12} 0.06 S_{55+k/12} v^{k/12}) = \frac{0.06}{12} \sum_{i=1}^{10\times 12} S_{55+i/12} v^{i/12} {}_{i/12}p_{55}^{00}$
Ans is as in Eq. (1), but ${}_tp_{55}^{00}$ is different.

Since ${}_tp_{55}^{00}$ is discrete if using the service table,

one has to use program to compute it. In Table 2, there are censoring to death d_x due to (w_x, r_x, i_x), thus ${}_tp_x^{00} \neq \frac{l_{x+t}}{l_x}$, **use the PLE** instead. The difference is very large !

$l_{65}/l_{55} = 0.37$,

${}_{10}p_{55} = \prod_{55<k\le 65} \approx 0.68$

$$\begin{cases} {}_jp_{55}^{00} = \prod_{k:\ 55<k\le 55+j}(1-d_k/l_k) & \text{for } j \in \{0,1,2,3,4\}, \\ {}_5p_{55}^{00} = 0.7\prod_{k:\ 55<k\le 55+4}(1-d_k/l_k) & \\ {}_jp_{55}^{00} = 0.7\prod_{k:\ 55<k\le 55+j}(1-d_k/l_k) & \text{for } j \in \{5,6,...,10\}, \\ {}_jp_{55}^{00} = {}_{j-1}p_{55}^{00}(1-d_{55+j}/l_{55+j}) & \text{for } j \in \{1,2,...,4,5-,6,...,10\}, \\ {}_{\frac{i}{12}+j}p_{55}^{00} = {}_jp_{55}^{00}(1-\frac{i}{12}) + {}_{j+1}p_{55}^{00}\frac{i}{12} & \text{for } i \in \{1,...,12\} \text{ and } j \in \{0,1,2,3,5,...,9\}, \\ {}_{\frac{i}{12}+4}p_{55}^{00} = {}_4p_{55}^{00}(1-\frac{i}{12}) + {}_{5-}p_{55}^{00}\frac{i}{12} & \text{for } i \in \{1,...,11\}, \\ {}_{\frac{i}{12}+j}p_{55}^{00} = {}_jp_{55}^{00}[1-\frac{i}{12}\frac{d_{j+56}}{l_{j+56}}] & \text{for } i \in \{1,...,12\} \text{ and } j \in \{0,1,2,...,9\}. \end{cases}$$

Note that under the assumption of UDD,

$$\begin{aligned} {}_{\frac{i}{12}+j}p_{55}^{00} =& {}_jp_{55}^{00}(1-r) + {}_{j+1}p_{55}^{00}r \qquad\qquad r = \frac{i}{12} \in [0,1] \\ =& {}_jp_{55}^{00}(1-\frac{i}{12}) + {}_{j+1}p_{55}^{00}\frac{i}{12} \\ =& {}_jp_{55}^{00}(1-\frac{i}{12}) + \frac{i}{12}{}_jp_{55}^{00}(1-\frac{d_{j+56}}{l_{j+56}}) \\ =& {}_jp_{55}^{00}[(1-\frac{i}{12}) + \frac{i}{12}(1-\frac{d_{j+56}}{l_{j+56}})] \\ =& {}_jp_{55}^{00}[1-\frac{i}{12}\frac{d_{j+56}}{l_{j+56}}]. \end{aligned}$$

However, since d_j actually occured between $[j, j+1)$, it is better to linearize d_j in the interval $[j, j+1)$. This way yields ${}_{\frac{i}{12}+j}p_{55}^{00} = {}_{j-1}p_{55}^{00}[1-\frac{i}{12}\frac{d_{j+55}}{l_{j+55}}]$ for $i \in \{1,2,...,12\}$, then

$$\begin{aligned} Ans =& \frac{0.06\times 10^5}{12}[\sum_{j=0}^{4} + \sum_{j=5}^{9}]\sum_{i=1}^{12} {}_{\frac{i}{12}+j}p_{55}^{00}(\frac{1.04}{1.05})^{j+i/12} \\ =& 500[\sum_{j=0}^{4} + 0.7\sum_{j=5}^{9}]\sum_{i=1}^{12}(1-\frac{i}{12}\frac{d_{j+55}}{l_{j+55}})(\frac{1.04}{1.05})^{j+i/12} \prod_{k:\ 55<k\le 55+j}(1-\frac{d_k}{l_k}) \\ =& 500[\sum_{j=1}^{5} + 0.7\sum_{j=6}^{10}]\sum_{i=1}^{12}(1-\frac{i}{12}\frac{d_{j+54}}{l_{j+54}})(\frac{1.04}{1.05})^{j-1+i/12} \prod_{k:\ 55<k\le 54+j}(1-\frac{d_k}{l_k}) \qquad (1) \\ =& 48248.82 \end{aligned}$$

```
x1=c(55,104688,2070,103,0,206)
x2=c(56,102308,2023,101,0,224)
x3=c(57,99960,1976,99,0,243)
x4=c(58,97642,1930,96,0,264)
x5=c(59 , 95351,1884, 94, 0,288)
x6=c(60,93085,0,0,27926 , 0)
x7=c(60,65160,0,62,6188,210)
x8=c(61,58700,0,56,5573,212)
x9=c(62,52860,0, 50,5018,213)
x10=c(63,47579,0,45,4515,214)
x11=c( 64,42805,0,41,4061,215)
x12=c(65,38488,0,0,38488,0)
x=c(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12)
dim(x)=c(6,12)
x=t(x)
d=x[,6]
l=x[,2]
p=1:12
for (i in 2:6)
p[i]=p[i-1]*(1-d[i]/l[i])
p[7]=0.7*p[6]
for (i in 8:12)
p[i]=p[i-1]*(1-d[i]/l[i])
p=c(p[1:5],p[7:11]) # 0:4, 5:9
d=c(d[1:5],d[7:11])
l[12]/l[1]
p
s=0
r=1.04/1.05
for (j in 1:10)
s=s+sum((1-((1:12)/12)*d[j]/l[j])*r**(j-1+(1:12)/12)*p[j]) # compare Eq. (1)
(s=500*s)
```

Additional Homework 1. Assume that

(1) 30% of the members surviving in employment to age 60 retire at 60 and all members who remain in employment to age 65 retire then;

(2) $\mu_x^{01} = \mu_x^w = \begin{cases} 0.1 & \text{for } x \in [0,35) \\ 0.05 & \text{for } x \in [35,45) \\ 0.02 & \text{for } x \geq 45; \end{cases}$

(3) $\mu_x^{02} = \mu_x^i = 0.001$;

(4) $\mu_x^{03} = \mu_x^r = \begin{cases} 0 & \text{for } x \in [0,60) \\ 0.3 & \text{discrete at } x = 60 \\ 0.1 & \text{for } x \in (60,\infty); \end{cases}$

(5) $\mu_x^{04} = \mu_x^d = 0.01$.

(a) P(retiring at age 65 |age 55) = ?

(b) For each mode of exit, calculate the probability that a member currently aged 55 exits employment by that mode.

Sol. (a) P(retiring at age 65|age 55) =P(each exit time> 65|age 55) = ${}_tp_{55}^{00}\big|_{t=10}$.

${}_tp_x^{00}$ =P(a member's exit time> $x+t$|age x)

=P(each exit time> $x+t$|age x)) = $\underbrace{{}_tp_x^{01} \cdot {}_tp_x^{02} \cdot {}_tp_x^{03} \cdot {}_tp_x^{04}}_{method\ 1} = \underbrace{\exp(-\int_0^t \sum_{i=1}^4 \mu_{x+s}^{0i} ds)}_{method\ 2}$.

Method 2: ${}_{10}p_{55}^{00} = {}_5p_{55}^{00} \cdot {}_5p_{60}^{00} = \exp(-(\int_{55}^{60} + \int_{60}^{65})\mu_x^{00} dx) = ?$

$$\mu_x^{00} = \mu_x^{01} + \mu_x^{02} + \mu_x^{03} + \mu_x^{04} = \begin{cases} 0.02 + 0.001 + 0.01 = 0.031 & x \in (55,60) \\ 0.02 + 0.001 + 0.1 + 0.01 = 0.131 & x \in (60,65) \end{cases}$$

$${}_tp_{55}^{00} = \exp(-\int_{55}^{55+t} (0.031) dx) = \exp(-0.031t), \quad t \in [0,5),$$

$${}_{5-}p_{55}^{00} = \exp(-0.155)$$

$${}_5p_{55}^{00} = 0.7\,{}_{5-}p_{55}^{00}$$

$${}_tp_{60}^{00} = \exp(-\int_{60}^{60+t} (0.131) dx) = \exp(-0.131t), \quad t \in (0,5),$$

$${}_5p_{60}^{00} = \exp(-5(0.131)) = \exp(-0.655).$$

Using R to compute:

b=exp(-0.155)

c=exp(-0.655)

c*0.7*b # ${}_{10}p_{55}^{00} = 0.3114006$ **the final answer.**

The probability of (55) being retired by 65 is about 31%.

(b) The probability that a member currently aged 55 exits employment by mode j

$= \int f^{0j}(t)dt$, where f^{0j} is the corresponding density, $j \in \{1,2,3,4\}$.

$f^{0j}(t) = \underbrace{{}_{t-}p_{55}^{00}}_{see\ (a)} \times \underbrace{\mu_{55+t}^{0j}}_{given}$ **Why ?**

$f_X(t) = S_X(t-)\mu_X(t)$ yieds ${}_{t-}p_x^{00} \cdot \mu_{x+t}^{0j}$ $(={}_tp_{55}^{00}\mu_{55+t}^{0j}$ if ${}_tp_{55}^{00}$ is continuous at t).

There are 4 modes. Start with the 3rd mode (normal retired):

$$\begin{aligned}
(3)\quad & P(\text{a member retires at 60 or 65 or in (60,65)}) \\
&= \underbrace{0.3 \cdot {}_{5-}p_{55}^{00}}_{30\%\ retire} + \underbrace{1 \cdot {}_{10-}p_{55}^{00}}_{100\%\ retire} + \int_5^{10} {}_tp_{55}^{00} \times 0.1dt \\
&= 0.3 \cdot {}_{5-}p_{55}^{00} + {}_{10-}p_{55}^{00} + 0.1{}_5p_{55}^{00} \int_0^5 \exp(-0.131t)dt \\
&= 0.3 \cdot {}_{5-}p_{55}^{00} + {}_{10-}p_{55}^{00} + 0.1{}_5p_{55}^{00} \frac{1-\exp(-0.655)}{0.131} = 0.7882412
\end{aligned}$$

0.3*b+ c*0.7*b +0.1*0.7*b*(1-c)/0.131

(2) The probability that a member currently aged 55 disability exits

$$\begin{aligned}
&= \int_0^{10} {}_tp_{55}^{00}\mu_t^{02}dt \\
&= 0.001 \int_0^{10} {}_tp_{55}^{00}dt \\
&= 0.001[\underbrace{\int_0^5 {}_tp_{55}^{00}dt}_{see\ (1)} + \underbrace{{}_5p_{55}^{00} \int_0^5 {}_tp_{60}^{00}dt}_{?}] \\
&= 0.001[\frac{1-\exp(-0.031*5)}{0.031} + {}_5p_{55}^{00}\frac{1-\exp(-0.655)}{0.131}] \approx 0.006830929
\end{aligned}$$

0.001*((1-exp(-0.031*5))/0.031 +b*0.7*(1-exp(-0.655))/0.131)

(4) The probability that a member currently aged 55 death exits

$$\begin{aligned}
&= \int_0^{10} {}_tp_{55}^{00}\mu_t^{04}dt \\
&= 0.01 \int_0^{10} {}_tp_{55}^{00}dt \\
&\approx 0.06830929
\end{aligned}$$

(1) The probability that a member currently aged 55 withdraws

$$= \int_0^{10} {}_tp_{55}^{00}\mu_t^{01}dt = 0.02 \int_0^5 {}_tp_{55}^{00}dt \approx 2 * 0.06830929 = 0.1366186$$

Additional problem 3. A pension plan offers an age retirement pension of 1.5% of final average salary for each year of service, where final average salary is defined as the earning in the three years before retirement. Estimate the APV of the accrued age retirement pension for a member aged 55 with 20 years of service, whose salary prior to the valuation date was $50000. Basic assumptions:
(1) The pension is paid monthly in advance for life, with no spouse's benefit.
(2) Interest rate is 5% per year.
(3) Salary scale s_y is given in Table 1 with the linear interpolation.
(4) Post-retirement survival: $\mu_x = \mu_x^{04}$ given by additional problem 1, as well as the multiple decrement model assumption.
(5) Use Table 2

Sol. Exit age is Y. $E(B_Y \ddot{a}_Y^{(12)} v^{Y-55} I(\text{age retiring}))= \sum_{k\in\mathcal{D}} \frac{B_k}{12}\ddot{a}_k^{(12)} v^{k-55}{}_{k-55|}q_{55}^{03}$?
where B_Y is the accrued annual pension at age 55 and the exit age is Y,

$B_Y = \underbrace{(y-e)\hat{S}_{Fin}\alpha}_{\textbf{accrued benefit}} = 20\hat{S}_{Fin}\alpha$, $(y,e)=(55,35)$.

$\hat{S}_{Fin} = \hat{S}_{Fin}(Y) = 50000\frac{z_Y}{s_{54}}$, the **projected** final average salary,

$z_Y = \frac{s_{Y-3}+s_{Y-2}+s_{Y-1}}{3}$, $\alpha = 0.015$.

If the member retires at exactly age 60 (*i.e.* 60−, the accrued benefit, based on 20 years' past service and an accrual rate of 1.5%, is an annual pension payable monthly in advance from age 60:

$$B_{60-} = 20\times 0.015\times 50000\frac{z_{60}}{s_{54}} = 15000\frac{z_{60}}{s_{54}} = 15000\frac{3.332+3.382+3.432}{3\times 3.186} = 15,922.79$$

based on Table 1. If the member retires at age 60+, that is, in (60,61), the accrued benefit, based on 20 years' past service and an accrual rate of 1.5%, is an annual pension payable monthly in advance from age 60:

$$B_{60+}\cdot {}_{60+|}q_{55}^{03} = 15000\frac{z_{60.5}}{s_{54}}\cdot {}_{5+|}q_{55}^{03} = 15000\frac{3.332+2*3.382+2*3.432+3.484}{2*3\times 3.186}\frac{r_{60+}}{l_{55}}$$

based on Tables 1 and 2. The APV of the accrued age retirement pension is

$$15000\Big(\frac{z_{60}}{s_{54}}v^{60-55}\ddot{a}_{60}^{(12)}\cdot 0.3\,{}_5p_{55}^{00} + \frac{z_{65}}{s_{54}}v^{65-55}\ddot{a}_{65}^{(12)}\,{}_{10}p_{55}^{00} + \sum_{k=60}^{64}\frac{z_{0.5+k}}{s_{54}}v^{0.5+k-55}\ddot{a}_{0.5+k}^{(12)}\cdot {}_{k|}q_{55}^{03}\Big) = 170488.1$$

Here

r_{k+} and r_{k-} are from Table 2, *e.g.*, $r_{60-} = 27926$ and $r_{60+} = 6188$, and
$s_{60.5} = (s_{60}+s_{61})/2$ from Table 1.

$$\begin{aligned}
\ddot{a}_k^{(12)} &= \frac{1}{12}\sum_{i=0}^{\infty} v^{i/12}\,{}_{\frac{i}{12}}p_k \\
&= \frac{1}{12}\sum_{i=0}^{\infty}((\exp(-0.01)/1.05)^{1/12})^k \\
&= \frac{1}{12}\frac{1}{1-x}\Big|_{(\exp(-0.01)/1.05)^{1/12}} = 17.05135
\end{aligned}$$

R codes for problem 3:

```
x1=c(55,104688,2070,103,0,206)
x2=c(56,102308,2023,101,0,224)
x3=c(57,99960,1976,99,0,243)
x4=c(58,97642,1930,96,0,264)
x5=c(59 , 95351,1884, 94, 0,288)
x6=c(60,93085,0,0,27926 , 0)
x7=c(60,65160,0,62,6188,210)
x8=c(61,58700,0,56,5573,212)
x9=c(62,52860,0, 50,5018,213)
x10=c(63,47579,0,45,4515,214)
x11=c( 64,42805,0,41,4061,215)
x12=c(65,38488,0,0,38488,0)
x=c(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12)
dim(x)=c(6,12)
x=t(x)
l=x[,2] # 55-65
d=x[,6]
r=x[,5]
z=c(3.186, 3.332 ,3.382 ,3.432 ,3.484 , 3.536 ,3.589 ,3.643 ,3.698) # 54,57-64
a=17.05135
v=1/1.05
s=mean(z[2:4])/z[1]*v**5*r[6]/l[1]
s=s+mean(z[7:9])/z[1]*v**10*r[12]/l[1]
for (i in 0:4)
s=s+mean(z[c((2+i):(5+i),3+i,4+i)])/z[1]*v**(5.5+i)*r[7+i]/l[1]
s*15000*a
```

R codes for solution (a) of Problem 4:

```
w=x[,3]
a=(1-v**5)/(1-v**(1/12))/12
b= exp(-0.001)*v
a=a+b**5/(1-b**(1/12))/12 # ä^(12)_{65:5̄}
s=0
for (i in 0:4)
s=s+mean(z[c((2+i):(5+i),3+i,4+i)])/z[1]*prod(1-(d[(i+2):12]/l[(i+2):12]))*w[i+1]/l[1]
v**10*2*10000*a*s
```

Additional Problem 4. A final salary pension plan offers an accrual rate of 2%, and the normal retirement age is 65. Final average salary is the average salary in the three years before exit (retirement or withdrawal or death). Pensions are paid monthly in advance for the life from age 65, with no spouse's benefit, and are guaranteed for 5 years (if the member died before age 70).
(a) Estimate the APV of the accrued withdrawal pension for a life now aged 55 with 10 years of service whose salary in the past year was $\$10^5$,
 (i) with no COLA,
 (ii) with a COLA in deferment of 3% per year.
(b) On death exit, a lump sum benefit of five times the accrued annual pension, with a COLA of 3% per year, is paid immediately. Estimate the APV of this benefit.
Basic assumptions are as previous problems. (1) Service table: Table 2.
 (2) Salary scale: From Table 1;
 (3) Post-retirement survival: as in Problem 3.
 (4) Interest rate is 5% per year.

Sol. (a) At withdrawal age X, with no COLA, $Q = E(B_D \ddot{a}^{(12)}_{\overline{65:\overline{5}|}} v^{10} AI(M=01)) = ?$
 For simplicity, assume $X \in \{55.5, 56.5, 57.5, 58.5, 59.5\}$.
Notations: $D = X - 55.5 \in \{0,1,2,3,4\}$,
 M is the mode ($M \in \{01, 02, 03, 04\}$),
 $A = I((55)$ is alive at age $65)$

$$\begin{cases} S_{Fin} & \text{the average salary before withdrawal age } X; \\ B_D = n\alpha S_{54}\frac{z_X}{s_{54}} & \text{the accrued withdrawal (annual) pension with withdrawal age } X, \\ B_D\ddot{a}^{(12)}_{\overline{65:\overline{5}|}} & \text{the accrued total pension with withdrawl age } X; \end{cases}$$

$$B_t = n\alpha \cdot {}_tS_{Fin} = 10 \times 0.02 \times 10^5 \times \frac{z_{55+t+0.5}}{s_{54}},$$

$$z_{y+0.5} = \frac{s_{y-3} + 2s_{y-2} + 2s_{y-1} + s_y}{2 \cdot 3} \ (s_y \text{ from Table 1}),$$

$$\begin{aligned}
Q =& E(B_D \ddot{a}^{(12)}_{\overline{65:\overline{5}|}} v^{10} I(M=01)_{5-D}p'^{(4)}_{55+D}) \\
=& v^{10} \cdot 10 \cdot 0.02 \cdot 10^5 \cdot \ddot{a}^{(12)}_{\overline{65:\overline{5}|}} \sum_{i=0}^{4} E(I(X-55 \in [i,i+1))\frac{z_X}{s_{54}} I(M=01)_{65-X}p'^{(4)}_X) \\
\approx& v^{10} \cdot 10 \cdot 0.02 \cdot 10^5 \cdot \ddot{a}^{(12)}_{\overline{65:\overline{5}|}} \sum_{i=0}^{4} E(I(D \in [i,i+1))\frac{z_{i+55.5}}{s_{54}} I(M=01)_{9.5-i}p'^{(4)}_{55.5+i}) \\
=& v^{10} \cdot 2 \cdot 10^4 \cdot \ddot{a}^{(12)}_{\overline{65:\overline{5}|}} \sum_{i=0}^{4} \frac{s_{55.5+i-3} + 2s_{55.5+i-2} + 2s_{55.5+i-1} + s_{55.5+i}}{6s_{54}} {}_{9.5-i}p'^{(4)}_{55.5+i} \cdot {}_{i+0.5|}q^{01}_{55} \\
=& v^{10} \cdot 2 \cdot 10^4 \cdot \ddot{a}^{(12)}_{\overline{65:\overline{5}|}} \sum_{i=0}^{4} (s_{55.5+i-3} + 2s_{55.5+i-2} + 2s_{55.5+i-1} + s_{55.5+i}) \frac{1}{6s_{54}} {}_{9.5-i}p'^{(4)}_{55.5+i} \frac{w_{55+i}}{l_{55}} = 25045.8
\end{aligned}$$

$_{i+0.5|}q_{55}^{01} = \dfrac{w_{55+i}}{l_{55}}$ from Table 2, as withdraw is really between $55+i$ and $56+i$.

$_{9.5-i}p_{55.5+i}^{'(4)} = \prod_{55.5+i<k\le 65}(1-\dfrac{d_k}{l_k})$ use Table 2.

$\ddot{a}_{\overline{65:\overline{5}|}}^{(12)} = \ddot{a}_{\overline{5}|}^{(12)} + {}_5p_{65}v^5\ddot{a}_{70}^{(12)} = \ddot{a}_{\overline{5}|}^{(12)} + (ve^{-0.001})^5\ddot{a}_{70}^{(12)}$

$\ddot{a}_{\overline{5}|}^{(12)} = \dfrac{1}{12}\sum_{i=0}^{59} v^{i/12} = \dfrac{1}{12}\dfrac{1-v^5}{1-v^{1/12}}\Big|_{v=1/1.05}$

$\ddot{a}_{70}^{(12)} = \dfrac{1}{12}\sum_{i=0}^{\infty} v^{i/12}e^{-0.001i/12} = \dfrac{1}{12}\sum_{i=0}^{\infty}((ve^{-0.001})^{(1/12)})^i = \dfrac{1}{12}\dfrac{1}{1-(ve^{-0.001})^{1/12}}$

(ii). With COLA, $Q = E(B_D\ddot{a}_{\overline{65:\overline{5}|}}^{(12)}v^{10}I(M=01)_{5-D}p_{55+D}1.03^{10-D})$
$= v^{10}\cdot 2\cdot 10^4\cdot \ddot{a}_{\overline{65:\overline{5}|}}^{(12)}\sum_{i=0}^4 \frac{s_{55.5+i-3}+2s_{55.5+i-2}+2s_{55.5+i-1}+s_{55.5+i}}{6s_{54}}1.03^{9.5-i}{}_{9.5-i}p_{55.5+i}\frac{w_{55+i}}{l_{55}}$

Under the competing risks model $T_x^{(1)},...,T_x^{(m)}$ are independent and cts, with cdf' ${}_tq_x^{'(j)}$.

${}_tp_x^{'(j)} = 1 - {}_tq_x^{'(j)}$ and $\mu_x^{(j)}(t) = \mu_{x+t}^{(j)} = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)}$.

$\{T_x = t, J_x = j\} = \{\min\{T_x^{(1)},...,T_x^{(m)}\} = t, T_x = T_x^{(j)}\}$,

$(F_{T_x}(t),\, S_{T_x}(t)) = ({}_tq_x^{(\tau)},\, {}_tp_x^{(\tau)})$.

$\mu_x(t) = \mu_{x+t} = \frac{f_{T_x}(t)}{S_{T_x}(t)}$, $f_{(T_x,J_x)}(t,j)$.

$P(T_x\le t, J_x = j) = {}_tq_x^{(j)}$ and ${}_tp_x^{(j)} = 1 - {}_tq_x^{(j)}$

$f_{(T_x^{(1)},...,T_x^{(m)})}(t_1,...,t_m) = \prod_{j=1}^m f_{T_x^{(j)}}(t_j)$,

$f_{(T_x,J_x)}(t,j) = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}}S_{T_x}(t) = \mu_x^{(j)}(t)S_{T_x}(t)$,

$f_{J_x}(j) = \int_0^\infty S_{T_x}(s)\mu_x^{(j)}(s)ds$,

$\mu_x(t) = \sum_{j=1}^m \mu_x^{(j)}(t)$,

$$\mu_x^{(j)}(t) = \frac{f_{(T_x,J_x)}(t,j)}{S_{T_x}(t)} = \frac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)} = -\frac{d}{dt}\ln S_{T_x}^{(j)}(t).$$

The multiple-decrement model:

1. $T_x^{(1)}, ..., T_x^{(m)}$ are $\underline{\textit{independent}}$ and cts r.v.
2. $\underline{T_x} = \min\{T_x^{(1)}, ..., T_x^{(m)}\}$ and $\underline{\{J_x = j\}} = \{T_x = T_x^{(j)}\}$.

$\underline{{}_tp_x^{(\tau)}} = S_{T_x}(t)$,

$\mu_x^{(\tau)}(t) = \mu_{x+t}^{(\tau)} = \frac{f_{T_x}(t)}{S_{T_x}(t)} = \underline{\sum_{j=1}^m \mu_x^{(j)}(t)}$,

$\underline{{}_tp_x^{(j)}} = P(T_x > t, J_x = j)$.

$\underline{{}_tp_x^{\prime(j)}} = S_{T_x^{(j)}}(t)$.

Multiple decrement model for a DB pension plan

00: no exit;

01: withdrawn from the pension plan;

02: disability retirement;

03: age retirement;

04: died in service.

${}_tp_x^{01} = P(\text{withdrawal time} > x+t | \text{the member survives } x)$,

${}_tp_x^{02} = P(\text{disable time} > x+t | \text{the member survives } x)$,

${}_tp_x^{03} = P(\text{retirement time} > x+t | \text{the member survives } x)$,

${}_tp_x^{04} = P(\text{survival time} > x+t | \text{the member survives } x)$.

Remark. T_x = exit time for (x),

$T_x^{(j)}$ = exit time for (x) in mode $J_x = j \in \{1, 2, 3, 4\}$ or $(w, i, r, d\}$.

${}_tp_x^{\prime(j)} = P(T_x^{(j)} > t)$,

${}_tp_x^{(j)} = P(T_x > t, J_x = j) = {}_tp_x^{0j}$.

${}_tp_x^{00} = {}_tp_x^{(\tau)} = P(T_x > t)$.

Using Table 2,

${}_tp_x^{00} = \frac{l_{x+t}}{l_x}$,

$({}_t|q_x^{01}, {}_t|q_x^{02}, {}_t|q_x^{03}, {}_t|q_x^{04}) = (\frac{w_{x+t}}{l_x}, \frac{i_{x+t}}{l_x}, \frac{r_{x+t}}{l_x}, \frac{d_{x+t}}{l_x})$,

${}_tp_x^{\prime(4)} = \prod_{x<k\le x+t}(1 - \frac{d_k}{l_k})$,

${}_t|q_x^{\prime(4)} = \prod_{x<k<x+t}(1 - \frac{d_k}{l_k})\frac{d_{x+t}}{l_{x+t}}$,

If the exit time is continuous, then

${}_tp_x^{01} = \exp(-\int_0^t \mu_{x+s}^{01} ds)$ $(= \frac{\exp(-\int_0^{x+t} \mu_s^{01} ds)}{\exp(-\int_0^x \mu_s^{01} ds)} = \exp(-\int_x^{x+t} \mu_s^{01} ds))$,

${}_tp_x^{02} = \exp(-\int_0^t \mu_{x+s}^{02} ds)$,

${}_tp_x^{03} = \exp(-\int_0^t \mu_{x+s}^{03} ds)$,

${}_tp_x^{04} = \exp(-\int_0^t \mu_{x+s}^{04} ds)$.

4/5/2019.

Additional Homework 1. Assume that

(1) 30% of the members surviving in employment to age 60 retire at 60 and all members who remain in employment to age 65 retire then;

(2) $\mu_x^{01} = \mu_x^w = \begin{cases} 0.1 & \text{for } x \in [0, 35) \\ 0.05 & \text{for } x \in [35, 45) \\ 0.02 & \text{for } x \geq 45; \end{cases}$

(3) $\mu_x^{02} = \mu_x^i = 0.001;$

(4) $\mu_x^{03} = \mu_x^r = \begin{cases} 0 & \text{for } x \in [0, 60) \\ 0.3 & \text{discrete at } x = 60 \\ 0.1 & \text{for } x \in (60, \infty); \end{cases}$

(5) $\mu_x^{04} = \mu_x^d = 0.01.$

(a) P(retiring at age 65 |age 55) = ?

(b) For each mode of exit, calculate the probability that a member currently aged 55 exits employment by that mode.

Additional problem 2. Employees in a pension plan pay contribution of 6% of their previous month's salary at each month end. Calculate the APV at entry of contributions for a new entrant aged 55, with a starting salary rate of $100,000 using

(a) exact calculation using the multiple decrement model specified in additonal problem 1,

(b) the values in Table 2, adjusting the APV of an annuity payable annually under UDD.

Other assumptions:

Salary rate function: Salaries increase at 4% per year continuously;

Interest: 5% per year effective.

Additional problem 3. A pension plan offers an age retirement pension of 1.5% of final average salary for each year of service, where final average salary is defined as the earning in the three years before retirement. Estimate the APV of the accrued age retirement pension for a member aged 55 with 20 years of service, whose salary prior to the valuation date was $50000. Basic assumptions:

(1) The pension is paid monthly in advance for life, with no spouse's benefit.

(2) Interest rate is 5% per year.

(3) Salary scale s_y is given in Table 1 with the linear interpolation.

(4) Post-retirement survival: $\mu_x = \mu_x^{04}$ given by additional problem 1.

(5) Use Table 2.

Additional Problem 4. A final salary pension plan offers an accrual rate of 2%, and the normal retirement age is 65. Final average salary is the average salary in the three years before exit (retirement or withdrawal or death). Pensions are paid monthly in advance for the life from age 65, with no spouse's benefit, and are guaranteed for 5 years (if the member died before age 70).

(a) Estimate the APV of the accrued withdrawal pension for a life now aged 55 with 10 years of service whose salary in the past year was $\$10^5$,

(i) with no COLA,

(ii) with a COLA in deferment of 3% per year.

(b) On death exit, a lump sum benefit of five times the accrued annual pension, with a COLA of 3% per year, is paid immediately. Estimate the APV of this benefit.

Basic assumptions are as previous problems.: (1) Service table: Table 2.

(2) Salary scale: From Table 1;

(3) Post-retirement survival: as in Problem 3.

(4) Interest rate is 5% per year.

CHAPTER 11

Markov Chains

11.1 Stochastic processes.

Definition 11.1. *A* **stochastic process** $\{X_t : t \in T\}$ *is a collection of r.v.'s defined in the same sample space* Ω*, where* T *is a set.*

T is called the **parameter set**, often a **time set**.
If T is discrete, $\{X_t : \ t \in T\}$ is called a **discrete–time process**. Usually, $T = \{0, 1, \dots\}$.
If T is an interval, $\{X_t : \ t \in T\}$ is called a **continuous–time process**. Usually, $T = [0, \infty)$.
Let $\mathbb{R}^T$ be the collection of functions from T into $\mathbb{R}$. A stochastic process $\{X_t : t \in T\}$ defines a function from Ω into $\mathbb{R}^T$. That is,

$X_{\bullet}$ is a map $\omega \mapsto X_{\bullet}(\omega) \in \mathbb{R}^T \ \forall \ \omega \in \Omega$, where $X_{\bullet}(\omega) = (X_t(\omega) : \ t \in T)$.
X_t denotes a random variable, and $X_{\bullet}$ denotes a stochastic processes. $X_{\bullet}(\omega) = X_t(\omega)$?

Recall for a function f, f or $f(\cdot) = f(x)$?

Recall the definitions of a random variable X and a random vector $\mathbf{X} = (X_1, ..., X_n)$.

X is a map $\omega \mapsto X(\omega) \in \mathbb{R} \ \forall \ \omega \in \Omega$,

$\mathbf{X}$ is a map $\omega \mapsto \mathbf{X}(\omega) \in \mathbb{R}^n \ \forall \ \omega \in \Omega$, where $\mathbf{X}(\omega) = (X_1(\omega), ..., X_n(\omega))$.

Remark. Pay attention to the difference between X, $X_{\bullet}$, $\mathbf{X}$ and X_t.

A stochastic processes associates a function to each outcome ω. A stochastic process is a random function (a generalized random variable with values in a space of functions).

Some actuarial concepts involve the study of a stochastic process. A stochastic process can be used to model:

(a) X_t is the total amount of policies in effect held by an insurance company until time t.

(b) X_t is the number of death of policy holders of an life insurance policy at the t-th year.

(c) X_t is the evolution of an insurance company's investments over time t.

(d) X_t is the surplus of an insurance company at time t.

(e) X_t is the price of a stock at time t.

11.2 Markov chains.

The rest of the chapter is dedicated to the study of a type of stochastic process appearing in many applications.

Definition 11.2. *A* **discrete time Markov chain** $\{X_n : n = 0, 1, 2, \dots\}$ *is a stochastic process with values in a countable space* E *such that*

$$\mathrm{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} = \mathrm{P}\{X_{n+1} = j | X_n = i_n\} \ \forall \ i_0, i_1, \dots, i_n, j \in E,$$

where E is called the **state space**. Often $E = \{0, 1, 2, \dots\}$, or $E = \{0, 1, 2, \dots, m\}$. Each element of E is called a **state**. If $X_n = k \in E$, we say that the Markov chain $\{X_n\}_{n=0}^m$ or $\{X_n\}_{n=0}^\infty$ is at state k at (time) stage n.

Example 11.1. *A fair coin is tossed repeatedly. Let X_n be the total number of heads obtained in the first n throws, $n = 0, 1, 2 \dots$. Notice that $X_0 = 0$, $X_n \sim bin(n, 0.5)$, and state space is $E = \{0, 1, 2, \dots\}$. $\{X_n\}_{n=0}^\infty$ is a Markov chain because for each $i_0, i_1, \dots, i_n, j \in E$*

$$\mathrm{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n\} \qquad i_0 \mathbf{= ?},$$
$$= \begin{cases} \frac{1}{2} & \text{if } j = i_n \textit{ or } i_n + 1, \textit{ with } i_1 - i_0, \dots, i_n - i_{n-1} \in \{0, 1\} \textbf{ why ??} \\ 0 & \text{otherwise.} \end{cases}$$

Markov chains can be used to study the mortality models presented in the previous chapters.

First, we consider the basic survival model.

Theorem 11.1. *Consider an individual. Let* $X_n = \begin{cases} 0 & \textit{if the individual is alive at time } n, \\ 1 & \textit{if the individual is dead at time } n. \end{cases}$
The sequence of r.v.'s $\{X_n\}_{n=0}^\infty$ is a Markov chain with state space $E = \{0, 1\}$.

Q: $\mathrm{P}\{X_{n+1} = 0 | X_0 = i_0?\ X_1 = i_1?\ \cdots, X_{n-1} = i_{n-1}?\ X_n = i_n?\ \} = \mathbf{??}$

$\mathrm{P}\{X_{n+1} = ? | X_0 = i_0?\ X_1 = i_1?\ \cdots, X_{n-1} = i_{n-1}?\ X_n = 1\ \} = \mathbf{??}$

We also can model multiple decrement models.

Theorem 11.2. *Consider an individual, which may die due to causes $1, \dots, m$. Let*

$$X_n = \begin{cases} 0 & \textit{if the individual is alive at time } n, \\ j & \textit{if the individual is dead at time } n \textit{ and died due to cause } j. \end{cases}$$

The sequence of r.v.s $\{X_n\}_{n=0}^\infty$ is a Markov chain with state space $E = \{0, 1, \dots, m\}$.

Q: $\mathrm{P}\{X_{n+1} = 0 | X_0 = i_0?\ X_1 = i_1?\ \cdots, X_{n-1} = i_{n-1}?\ X_n = i_n?\ \} = \mathbf{??}$

$\mathrm{P}\{X_{n+1} = ? | X_0 = i_0?\ X_1 = i_1?\ \cdots, X_{n-1} = i_{n-1}?\ X_n = 3\ \} = \mathbf{??}$

We also can model multiple life models.

Theorem 11.3. *Consider a pair of individuals. Let*

$$X_n = \begin{cases} 1 & \textit{if both individuals are alive at time } n, \\ 2 & \textit{if the first individual is alive and the second one dead at time } n, \\ 3 & \textit{if the first individual is dead and the second one alive at time } n, \\ 4 & \textit{if both individuals are dead at time } n, \end{cases}$$

The sequence of r.v.'s $\{X_n\}_{n=0}^\infty$ is a Markov chain with state space
$E = \{1, 2, 3, 4\}$ or $E = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

For a Markov chain the conditional distribution of a future state X_{n+1} given the past states $X_0, X_1, \dots, X_n$ depends only on X_n.

$$\mathrm{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} = \mathrm{P}\{X_{n+1} = j | X_n = i_n\}.$$

This property is a special case of conditional independence, defined as follows:

Definition 11.3. *Given events A, B and C, we say that A and B are (conditional) independent given C if* $\mathrm{P}\{A \cap B | C\} = \mathrm{P}\{A|C\}\mathrm{P}\{B|C\}$.

Theorem 11.4. *Given events A, B and C such that* $\mathrm{P}\{B \cap C\} > 0$*, A and B are independent given C iff* $\mathrm{P}\{A | B \cap C\} = \mathrm{P}\{A|C\}$.

By the previous theorem, for a Markov chain $\{X_n : n = 0, 1, 2, \dots\}$, $\forall\, i_0, \dots, i_{n+1} \in E$, given that $X_n = i_n$, $\{(X_0, X_1, \dots, X_{n-1}) = (i_0, \dots, i_{n-1})\}$ and $\{X_{n+1} = i_{n+1}\}$ are independent. In general, for a Markov chain, past and future are independent given the present (see Theorem 11.11 in page 150).

Example 11.2. *Suppose that you are playing blackjack in a casino. At a certain moment, you have \$200 in chips. How good would you be after that does not depend on the number of money you started with.*

It is possible to define Markov chain for a continuous time, *i.e.* there are Markov chains of the form $\{X_t\}_{t \in T}$, where T is an interval.

Definition 11.4. *The stochastic process $\{X_t : t \in [0, \infty)\}$ is a Markov chain, if it takes values in a countable space E and $\forall\, i_0, i_1, \dots, i_n, j \in E$ and whenever $0 \le t_0 < t_1 < \cdots < t_{n+1}$,*

$$\mathrm{P}\{X_{t_{n+1}} = j | X_{t_0} = i_0, \dots, X_{t_{n-1}} = i_{n-1}, X_{t_n} = i_n\} = \mathrm{P}\{X_{t_{n+1}} = j | X_{t_n} = i_n\}.$$

Since X_n takes values in the countable set E, X_n has a discrete distribution.
Let $\alpha_n(i) = \mathbb{P}\{X_n = i\}$, $i \in E$. $\alpha_n(i)$, $i \in E$, is the density function of the r.v. X_n, Notice that $\alpha_n(i) \ge 0$ and $\sum_{i \in E} \alpha_n(i) = 1$.
Denote the row vector $(\alpha_n(i))_{i \in E}$ by α_n, *e.g.*, $\alpha_n = (\alpha_n(0), \alpha_n(1), \dots, \alpha_n(k))$ if $E = \{0, 1, \dots, k\}$.
α_0 or $(\alpha_0(i))_{i \in E}$ is called the **initial distribution** of the Markov chain.
Denote $Q_n(i, j) = \mathbb{P}\{X_{n+1} = j | X_n = i\}$, where $i, j \in E$.
$Q_n(i, j)$ is called the **one–step transition probability** from state i into state j at stage n.
Denote $Q_n = (Q_n(i, j))_{i, j \in E}$.

For example, $Q_n = \begin{pmatrix} Q_n(0,0) & Q_n(0,1) & \cdots & Q_n(0,k) \\ Q_n(1,0) & Q_n(1,1) & \cdots & Q_n(1,k) \\ \cdots & \cdots & \cdots & \cdots \\ Q_n(k,0) & Q_n(k,1) & \cdots & Q_n(k,k) \end{pmatrix}$ if $E = \{0, 1, \dots, k\}$.

Or write $Q_n =$

$$\text{departing states}\quad \begin{array}{c} \\ 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ k \end{array} \begin{array}{c} \text{arriving states} \\ \begin{array}{cccc} 0 & 1 & \cdots & k \end{array} \\ \begin{pmatrix} Q_n(0,0) & Q_n(0,1) & \cdots & Q_n(0,k) \\ Q_n(1,0) & Q_n(1,1) & \cdots & Q_n(1,k) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Q_n(k,0) & Q_n(k,1) & \cdot & Q_n(k,k) \end{pmatrix} \end{array}$$

Example 11.3. *Consider a Markov chain with* $E = \{0,1,2\}$ *and* $Q_6 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$.

Class exercise: *Find* $\mathrm{P}\{X_7 = j | X_6 = 0\}$ *and* $\mathrm{P}\{X_7 = 1 | X_6 = j\}$ *for all* j.

Solution: $(\mathrm{P}\{X_7 = 0|X_6 = 0\}, \mathrm{P}\{X_7 = 1|X_6 = 0\}, \mathrm{P}\{X_7 = 2|X_6 = 0\}) = (0.2, 0.3, 0.5)$.

$$\begin{pmatrix} \mathrm{P}\{X_7 = 1|X_6 = 0\} \\ \mathrm{P}\{X_7 = 1|X_6 = 1\} \\ \mathrm{P}\{X_7 = 1|X_6 = 2\} \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.5 \\ 0.1 \end{pmatrix}.$$

Theorem 11.5. *The one–step transition probabilities* $Q_n(i,j)$*,* $i, j \in E$*, satisfy that* $Q_n(i,j) \geq 0$ and $\sum_{j\in E} Q_n(i,j) = 1$.

Example 11.4. *Let (i)* $Q_0 = \begin{pmatrix} -1 & 2 \\ 0.5 & 0.5 \end{pmatrix}$ *and (ii)* $Q_0 = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.7 \end{pmatrix}$.
Which of the following are legitimate one-step transition probability matrices ?

Definition 11.5. *A state* j *is called an absorbing state if* $Q_n(j,j) = 1$ **for each** $n \geq 0$.

If j is an absorbing state and $X_n = j$, then $X_{n+m} = j$ for each $m \geq 1$.

For the Markov chain for a survival model (see Theorem 11.1 in page 142), state 1 (death) is an absorbing state. For (x) after n years,

$$Q_n = \begin{pmatrix} Q_n(0,0) & Q_n(0,1) \\ Q_n(1,0) & Q_n(1,1) \end{pmatrix} = \begin{pmatrix} p_{x+n} & q_{x+n} \\ 0 & 1 \end{pmatrix} \textbf{ why ? } Q_{n+1} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

For the Markov chain for a multiple decrement model (see Theorem 11.2 in page 142), states $1, \dots, m$ are absorbing states. For (x) after time n,

$$Q_n = \begin{pmatrix} p_{x+n}^{(\tau)} & q_{x+n}^{(1)} & \cdot & \cdots & \cdot & q_{x+n}^{(m)} \\ 0 & 1 & \cdot & \cdots & \cdot & 0 \\ \cdot & \cdot & \ddots & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \ddots & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdots & \cdot & 0 & 1 \end{pmatrix} \overset{e.g.\ in\ Ch10}{=} \begin{pmatrix} p_{x+n}^{00} & q_{x+n}^{01} & q_{x+n}^{02} & q_{x+n}^{03} & q_{x+n}^{04} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the Markov chain for a multiple life model (see Theorem 11.3 in page 142), that is, $((l,l),(l,d),(d,l),(d,d))$, state 4 is an absorbing state. If (x) and (y) are independent lives,

$$Q_n = \begin{pmatrix} p_{x+n}p_{y+n} & p_{x+n}q_{y+n} & q_{x+n}p_{y+n} & q_{x+n}q_{y+n} \\ 0 & p_{x+n} & 0 & q_{x+n} \\ 0 & 0 & p_{y+n} & q_{y+n} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote $Q_n^{(k)}(i,j) = P\{X_{n+k} = j | X_n = i\}$.

$Q_n^{(k)}(i,j)$ is called the k–step transition probability from state i into state j at time n.

$$Q_n^{(k)}(i,j) \geq 0 \text{ and } \sum_{j \in E} Q_n^{(k)}(i,j) = 1.$$

Denote the matrix $(Q_n^{(k)}(i,j))_{i,j\in E}$ by $Q_n^{(k)}$.

Lemma 11.1. *(***Sequential conditioning***)* $\forall\ B, A_1, A_2, \dots, A_n \subseteq \Omega$ *with* $\mathrm{P}\{B\} > 0$, $\mathrm{P}\{\cap_{i=1}^n A_i | B\} = \mathrm{P}\{A_1|B\}\mathrm{P}\{A_2 \mid B \cap A_1\} \cdots \mathrm{P}\{A_n \mid B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}$.

Corollary 11.1. *For each* $A_1, A_2, \dots, A_n \subseteq \Omega$, $\mathrm{P}\{A_1 \cap A_2 \cap \cdots \cap A_n\} = \mathrm{P}\{A_1\}\mathrm{P}\{A_2 \mid A_1\} \cdots \mathrm{P}\{A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}$.

Why is $P(A_1) > 0$ missing (see Lemma 11.1) ?

Example 11.5. *From a deck of 52 cards, you withdraw three cards one after another.* $\mathbb{P}$*(the first two cards are spades and the third one is a club)= ?*

Solution: Let $A_i = \{\text{i} - \text{thcard is a spade}\}$, $i = 1,\ 2$, and let $A_3 = \{\text{third card is a club}\}$. Then $\mathrm{P}\{A_1 \cap A_2 \cap A_3\} = \mathrm{P}\{A_1\}\mathrm{P}\{A_2|A_1\}\mathrm{P}\{A_3|A_1 \cap A_2\} = \frac{13}{52}\frac{12}{51}\frac{13}{50} \approx 0.0153$.

Next theorem shows that using the initial distribution and the one–step transition probabilities, we can find the distribution of a Markov chain.

Theorem 11.6. *(Basic theorem for Markov chains) Let* $\{X_n : n = 0,1,2,\dots\}$ *be a Markov chain with state space* E*. Then,* $\forall\ i_0, i_1, \dots, i_n \in E$,

(i) $\mathrm{P}\{X_j = i_j,\ j = 0,1,...,n\} = \alpha_0(i_0)Q_0(i_0,i_1)Q_1(i_1,i_2)\cdots Q_{n-1}(i_{n-1},i_n)$.

(ii) $\alpha_n(i_n) = \sum_{i_0,i_1,\dots,i_{n-1}\in E} \alpha_0(i_0)Q_0(i_0,i_1)Q_1(i_1,i_2)\cdots Q_{n-1}(i_{n-1},i_n) \quad (= P(X_n = i_n))$.

We will say that

the Markov chain does $i_0 \mapsto i_1 \mapsto \cdots \mapsto i_{n-1} \mapsto i_n$, if $(X_0, X_1, \dots, X_n) = (i_0, i_1, \dots, i_n)$.
Using matrix notation (setting $E = \{0,1,\dots,k\}$), Theorem 11.6 (ii) implies that

$\alpha_n = \alpha_0 Q_0 Q_1 \cdots Q_{n-1} = \alpha_1 Q_1 Q_2 \cdots Q_{n-1} = \alpha_{n-1}Q_{n-1}$.

Example 11.6. *Consider a Markov chain with*
$E = \{0,1,2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$, $Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$. *Find:*
(i) $\mathrm{P}\{X_0 = 1\}$. *(ii)* $\mathrm{P}\{X_1 = 1\}$. *(iii)* $\mathrm{P}\{X_2 = 1\}$. *(iv)* $\mathrm{P}\{X_0 = 1, X_1 = 2\}$.
(v) $\mathrm{P}\{X_0 = 1, X_1 = 0\}$. *(vi)* $\mathrm{P}\{X_0 = 1, X_1 = 2, X_2 = 2\}$.
(vii) $\mathrm{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$. *(viii)* $\mathrm{P}\{X_0 = 1, X_2 = 2\}$.

Sol: Formulas: (1) $Q_n(i,j) = P(X_{n+1} = j | X_n = i)$,
(2) $Q_n^{(k)}(i,j) = P(X_{n+k} = j | X_n = i)$,
(3) $\mathrm{P}\{X_j = i_j,\ j = 0, 1, ..., n\} = \alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n)$,
(4) $(P(X_n = 0), P(X_n = 1), P(X_n = 2)) = \alpha_n = (\alpha_n(0), \alpha_n(1), \alpha_n(2))$,
(5) $\alpha_n = \alpha_0 Q_0 Q_1 \cdots Q_{n-1} = \alpha_1 \prod_{j=1}^{n-1} Q_j = \alpha_{n-1} Q_{n-1}$.
(i) $\mathrm{P}\{X_0 = 1\} = \alpha_0(1) = 0.4$ by (4) and $\alpha_0 = (0.3, 0.4, 0.3)$,
(ii) $\mathrm{P}\{X_1 = 1\} = \alpha_1(1)$, where $\alpha_1 = (\alpha_1(0), \alpha_1(1), \alpha_1(2))$.

$$\alpha_1 = \alpha_0 Q_0 = (0.3, 0.4, 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} = (0.21, 0.32, 0.47) \ \ => \ \ \alpha_1(1) = 0.32$$

Or simpler way: $\mathrm{P}\{X_1 = 1\} = \alpha_0 Q_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0.3, 0.4, 0.3) \begin{pmatrix} 0.3 \\ 0.5 \\ 0.1 \end{pmatrix} = 0.32,$

(iii) $\mathrm{P}\{X_2 = 1\} = \alpha_2(1)$, where $\alpha_2 = (\alpha_2(0), \alpha_2(1), \alpha_2(2)) = \alpha_0 Q_0 Q_1 =$
$(0.3, 0.4, 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix} = (0.21, 0.32, 0.47) \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$
$= (0.258, 0.416, 0.326)$. $\alpha_2(1) = ?$
Or simpler way: $\mathrm{P}\{X_2 = 1\} = \alpha_0 Q_0 Q_1 (0,1,0)^t$
$= (0.3, 0.4, 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0.21, 0.32, 0.47) \begin{pmatrix} 0.1 \\ 0.5 \\ 0.5 \end{pmatrix} = 0.416.$
(iv) $\mathrm{P}\{X_0 = 1, X_1 = 2\} = P(X_0 = 1) P(X_1 = 2 | X_0 = 1) = \alpha_0(1) Q_0(1,2) = (0.4)(0.2) = 0.08.$
(v) $\mathrm{P}\{X_0 = 1, X_1 = 0\} = P(1 \mapsto 0) = \alpha_0(1) Q_0(1,0) = (0.4)(0.3) = 0.12.$
(vi) $\mathrm{P}\{X_0 = 1, X_1 = 2, X_2 = 2\}$
$= \alpha_0(1) Q_0(1,2) Q_1(2,2) = (0.4)(0.2)(0.2) = 0.016.$
(vii) $\mathrm{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$
$= \alpha_0(2) Q_0(2,1) Q_1(1,0) = (0.3)(0.1)(0.3) = 0.009.$
(viii) $\mathrm{P}\{X_0 = 1, X_2 = 2\} = \sum_{j=0}^{2} \mathrm{P}\{X_0 = 1, X_1 = j, X_2 = 2\}$
$= (0.4)(0.3)(0.8) + (0.4)(0.5)(0.2) + (0.4)(0.2)(0.2) = 0.152$, or

$$\mathrm{P}\{X_0 = 1, X_2 = 2\} = \alpha_0(1) \underbrace{Q_0^{(2)}(1,2)}_{2\ step} = \alpha_0(1)(0,1,0) Q_0 Q_1 (0,0,1)^t = 0.4(0.3, 0.5, 0.2) \begin{pmatrix} 0.8 \\ 0.2 \\ 0.2 \end{pmatrix}.$$

Example 11.7. *An actuary models the life status of an individual with lung cancer using a Markov chain model with states: State1 : alive; and State 2: dead. The transition probability matrices are* $Q_0 = \begin{pmatrix} 0.6 & 0.4 \\ 0 & 1 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.4 & 0.6 \\ 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 0.2 & 0.8 \\ 0 & 1 \end{pmatrix}, Q_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. *A death benefit of* 10^5 *is paid at the end of the year of the death. The annual effective rate of interest is* 5%. *The insured is alive at the beginning of year zero. Calculate the actuarial present value of this life insurance.*

Solution: The APV is $E(10^5 v^{K_x}) = 10^5 \sum_{n=1}^{\infty} v^n \overbrace{f_{K_x}(n)}^{?} = ?$ Notice:

1. f_{K_x} was given before for $U(0,m)$, or $Exp(1/\mu)$, or lifetable, but is MC here. Formulas:

   $\alpha_n(i_n) = \sum_{i_0,i_1,...,i_{n-1}\in E} \alpha_0(i_0)Q_0(i_0,i_1)Q_1(i_1,i_2)\cdots Q_{n-1}(i_{n-1},i_n)$. $(i_0, i_1, i_2, ...) =$ **???**)

   $\alpha_n = \alpha_0 Q_0 Q_1 \cdots Q_{n-1} = \alpha_1 Q_1 Q_2 \cdots Q_{n-1} = \alpha_{n-1}Q_{n-1}$ $(\alpha_0 = (\alpha_0(0), \alpha_0(1))$ **?**)

   $\mathrm{P}\{X_j = i_j,\ j = 0,1,...,n\} = \mathrm{P}\{X_0 = i_0, ..., X_n = i_n\} = \alpha_0(i_0)Q_0(i_0,i_1)\cdots Q_{n-1}(i_{n-1},i_n)$.

2. After four years, an individual is dead with probability one (**why ?**)

1st approach:

$f_{K_x}(1)$	$f_{K_x}(2)$	$f_{K_x}(3)$	$f_{K_x}(4)$
$P(1 \mapsto 2)$	$P(1 \mapsto 1 \mapsto 2)$	$P(1 \mapsto 1 \mapsto 1 \mapsto 2)$	$P(1 \mapsto 1 \mapsto 1 \mapsto 1 \mapsto 2)$
$1 * 0.4$	$1 * 0.6 * 0.6$	$1 * 0.6 * 0.4 * 0.8$	$1 * 0.6 * 0.4 * 0.2 * 1$
0.4	0.36	0.192	0.048

The APV of this life insurance is $10^5 \sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$
$= 10^5\{(0.4)(1.05)^{-1} + (0.36)(1.05)^{-2} + (0.192)(1.05)^{-3} + (0.048)(1.05)^{-4}\} = 91282.95.$

2nd approach (standard): $f_{K_x}(n) = \mathrm{P}\{X_n = 2\} - \mathrm{P}\{X_{n-1} = 2\} = F_X(n) - F_X(n-1)$ (**Why ?**).

$\{X_n = 2\} = \{$ a person is dead (not necessarily died) in the n-th year$\}$.

$P(X_n = 2)$ $(= \alpha_n(2))$ and $\alpha_n = \alpha_0 Q_1 Q_2 \cdots Q_{n-1}$.

Since at the beginning the individual is alive, $\alpha_0 = (1,0) = (\alpha_0(1), \alpha_0(2))$.

$$\alpha_0(2) = 0$$

$$\alpha_1(2) = \alpha_0 Q_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1,0)\begin{pmatrix} 0.4 \\ 1 \end{pmatrix} = 0.4,$$

$$\alpha_2(2) = \alpha_0 Q_0 Q_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0.6, 0.4)\begin{pmatrix} 0.6 \\ 1 \end{pmatrix} = 0.76,$$

$$\alpha_3(2) = \alpha_0 Q_0 Q_1 Q_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0.24, 0.76)\begin{pmatrix} 0.8 \\ 1 \end{pmatrix} = 0.952,$$

$$\alpha_4(2) = \alpha_0 Q_0 Q_1 Q_2 Q_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0.048, 0.952)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1.$$

dead in year	1	2	3	4
Probability	0.4	$0.76 - 0.4 = 0.36$	$0.952 - 0.76 = 0.192$	$1 - 0.952 = 0.048$

Next theorem shows how to find probabilities for a Markov chain given the initial state.

Theorem 11.7. *Let $\{X_n : n = 0, 1, 2, \dots\}$ be a Markov chain and $i_0, \dots, i_n \in E$. Then*
(i) $\mathrm{P}\{X_1 = i_1, \dots, X_n = i_n | X_0 = i_0\} = Q_0(i_0, i_1)Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n)$,
(ii) $Q_0^{(n)}(i_0, i_n) = \sum_{i_1,\dots,i_{n-1}\in E} Q_0(i_0, i_1)Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n)$.

Using matrix notation, Theorem 11.7 (ii) states that

$$Q_0^{(n)} = Q_0Q_1 \cdots Q_{n-1}. \tag{11.1}$$

Equation (11.1) is one of the **Kolmogorov–Chapman equations** of a Markov chain.

Next theorem generalizes Theorem 11.6 by considering a general first time.

Theorem 11.8. *(i)* $\mathrm{P}\{X_n = i_n, X_{n+1} = i_{n+1}, \dots, X_{n+m} = i_{n+m}\}$
$= \alpha_n(i_n)Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.
(ii) $\alpha_{n+m}(i_{n+m}) = \sum_{i_n,i_{n+1},\dots,i_{n+m-1}\in E} \alpha_n(i_n)Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.

Q: $\alpha_{0+4}(i_{0+4}) = \sum_{i_0,i_1,i_2,i_3\in E} \alpha_0(i_0)Q_0(i_0, i_1)Q_1(i_1, i_2)Q_2(i_2, i_3)Q_3(i_3, i_4)$?
$\alpha_{3+1}(i_{3+1}) = \sum_{i\in E} \alpha_3(i)Q_3(i, i_4)$?
$\alpha_{0+4}(i_{0+4}) = \alpha_{3+1}(i_{3+1}) = \alpha_{2+2}(i_{2+2})$?
In matrix notation, Theorem 11.8 (ii) says that

$$\alpha_{n+m} = \alpha_nQ_nQ_{n+1} \cdots Q_{n+m-1}.$$

When we condition in a general time, we have that:

Theorem 11.9. *Let $\{X_n : n = 0, 1, 2, \dots\}$ be Markov chain. For each $i_n, \dots, i_{n+m} \in E$,*
(i) $\mathrm{P}\{X_{n+1} = i_{n+1}, \dots, X_{n+m-1} = i_{n+m-1}, X_{n+m} = i_{n+m}|X_n = i_n\}$
$= Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.
(ii) $Q_n^{(m)}(i_n, i_{n+m}) = \sum_{i_{n+1},\dots,i_{n+m-1}\in E} Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.

Previous theorem follows from Theorem 11.8. Theorem 11.9 (ii) implies that

$$Q_n^{(m)} = Q_nQ_{n+1} \cdots Q_{n+m-1}. \tag{11.2}$$

Equation (11.2) is called the Kolmogorov–Chapman equation.

Main formulas:
(1) $\alpha_n(i) = P(X_n = i)$,
$\alpha_n = (\dots, \alpha_n(i), \alpha_n(i + 1), \dots)$,
(2) $Q_n^{(m)}(i, j) = P(X_{n+m} = j|X_n = i)$,
$Q_n^{(m)} = Q_nQ_{n+1} \cdots Q_{n+m-1}$.
(3) $\mathrm{P}\{X_j = i_j,\ j = n, n + 1, \dots, n + m\}$
$= \alpha_n(i_n)Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.
(4) $\mathrm{P}\{X_j = i_j,\ j = n + 1, \dots, n + m|X_n = i_n\}$
$= Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.
(5) $\alpha_n = \alpha_0Q_0Q_1 \cdots Q_{n-1} = \alpha_1Q_1Q_2 \cdots Q_{n-1} = \alpha_{n-1}Q_{n-1} = \alpha_k \prod_{j=k}^{n-1} Q_j,\ k \in \{0, 1, \dots, n - 1\}$

Example 11.8. *Consider a Markov chain with* $E = \{1, 2\}$,
$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}$ *and* $Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$. *Suppose that* $X_0 = 1$.
(i) Find the probability that the first time the chain is in state 2 is stage 2.
(ii) Find the probability that at stage 2 the chain is in state 2.

Solution: (i) If the first time the chain is in state 2 is stage 2, then the Markov chain does $1 \mapsto 1 \mapsto 2$. $\mathbb{P}\{X_1 = 1, X_2 = 2 | X_0 = 1\} = ?$

$$\mathbb{P}\{X_1 = 1, X_2 = 2 | X_0 = 1\} = Q_0(1,1)Q_1(1,2) = (0.5)(0.8) = 0.4$$

(ii) The Markov chain can be at stage 2 in state 2, if any of the following transitions occur

$$1 \mapsto 1 \mapsto 2, \ or \ 1 \mapsto 2 \mapsto 2.$$

$Q_0^{(2)}(1,2) = \mathbb{P}\{X_2 = 2 | X_0 = 1\} = ?$
$Q_0^{(2)}(1,2) = P(1 \mapsto 1 \mapsto 2, \ or \ 1 \mapsto 2 \mapsto 2 | X_0 = 1) = (0.5)(0.8) + (0.5)(0.4) = 0.6.$
Standard approach:

$$Q_0^{(2)}(1,2) = (1,0)Q_0Q_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1,0) \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0.5, 0.5) \begin{pmatrix} 0.8 \\ 0.4 \end{pmatrix} = 0.6.$$

Example 11.9. *Consider a Markov chain with*
$E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$, $Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$, $Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}$.
Find: (i) $\mathrm{P}\{X_3 = 2, X_4 = 1\}$. *(ii)* $\mathrm{P}\{X_3 = 1, X_4 = 1, X_5 = 2\}$. *(iii)* $\mathrm{P}\{X_5 = 2\}$.

Solution: (i) $\mathrm{P}\{X_3 = 2, X_4 = 1\} = \alpha_3(2)Q_3(2,1) = (0.8)(0.3) = 0.24$.
(ii) $\mathrm{P}\{X_3 = 1, X_4 = 1, X_5 = 2\} = \alpha_3(1)Q_3(1,1)Q_4(1,2) = (0.2)(0.6)(0.8) = 0.096$.
(iii) $\alpha_5(2) = \alpha_3 Q_3 Q_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0.2, 0.8) \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= (0.36, 0.64) \begin{pmatrix} 0.8 \\ 0.3 \end{pmatrix} = 0.48.$

Example 11.10. *Consider a Markov chain with*
$E = \{1, 2\}$, $Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$, $Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}$.
Find: (i) $\mathrm{P}\{X_4 = 2 | X_3 = 1\}$. *(ii)* $\mathrm{P}\{X_4 = 2, X_5 = 1 | X_3 = 1\}$. *(iii)* $\mathrm{P}\{X_5 = 1 | X_3 = 1\}$.

Solution: (i) $\mathrm{P}\{X_4 = 2 | X_3 = 1\} = Q_3(1,2) = 0.4$.
(ii) $\mathrm{P}\{X_4 = 2, X_5 = 1 | X_3 = 1\} = Q_3(1,2)Q_4(2,1) = (0.4)(0.7) = 0.28$.
(iii) $\mathrm{P}\{X_5 = 1 | X_3 = 1\} = Q_3^{(2)}(1,1) = 0.4$, as

$$Q_3^{(2)}(1,1) = (1,0)Q_3Q_4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0.6, 0.4) \begin{pmatrix} 0.2 \\ 0.7 \end{pmatrix}$$

Theorem 11.10.

Theorem 11.11. *Let $\{X_n : n = 0, 1, 2, \dots\}$ be Markov chain. Then,*

$$\begin{aligned}&\mathrm{P}\{(X_{n+1}, \dots, X_{n+m}) \in B | (X_0, \dots, X_{n-1}) \in A, X_n = i_n\}\\ =&\mathrm{P}\{(X_{n+1}, \dots, X_{n+m}) \in B | X_n = i_n\} \text{ for each } A \in \mathbb{R}^n \text{ and } B \in \mathbb{R}^m.\end{aligned}$$

Example 11.11. *Consider a Markov chain with*
$E = \{1, 2, 3\}$, $Q_2 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.2 & 0.7 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$, $Q_3 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \end{pmatrix}$.
Find $P\{X_3 = 3, X_4 = 2 | X_0 = 2, X_1 = 3, X_2 = 1\}$.

Solution: $P\{X_3 = 3, X_4 = 2 | X_0 = 2, X_1 = 3, X_2 = 1\}$
$= P\{X_3 = 3, X_4 = 2 | X_2 = 1\} = Q_2(1, 3)Q_3(3, 2) = (0.3)(0.1) = 0.03.$

Example 11.12. *Consider a Markov chain (MC) with $E = \{1, 2\}$,*
$Q_7 = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$, $Q_8 = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}$. *Find $P\{X_9 = 1 | X_5 = 2, X_7 = 1\}$.*

Solution: $P\{X_9 = 1 | X_5 = 2, X_7 = 1\} = P\{X_9 = 1 | X_7 = 1\} = Q_7^{(2)}(1, 1)$,

$$Q_7^{(2)}(1, 1) = (1, 0) \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0.4, 0.6) \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix} = 0.48.$$

Main formulas:
(1) $\alpha_n(i) = P(X_n = i), \qquad \alpha_n = (\dots, \alpha_n(i), \alpha_n(i + 1), \dots)$,
(2) $Q_n^{(m)}(i, j) = P(X_{n+m} = j | X_n = i), \qquad Q_n^{(m)} = Q_n Q_{n+1} \cdots Q_{n+m-1}$.
(3) $\mathrm{P}\{X_j = i_j,\ j = n, n + 1, \dots, n + m\}$
$= \alpha_n(i_n)Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.
(4) $\mathrm{P}\{X_j = i_j,\ j = n + 1, \dots, n + m | X_n = i_n\}$
$= Q_n(i_n, i_{n+1})Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m})$.
(5) $\alpha_n = \alpha_0 Q_0 Q_1 \cdots Q_{n-1} = \alpha_1 Q_1 Q_2 \cdots Q_{n-1} = \alpha_{n-1} Q_{n-1}$.
A Markov chain satisfying $Q_n(i, j) = Q_0(i, j)$, $\forall\ n \geq 1$, is called a **homogeneous MC** .
Define
$P(i, j) = \mathrm{P}\{X_{n+1} = j | X_n = i\} = Q_n(i, j)$ for all n.
$P = (P(i, j))_{i,j \in E}$ or $= (p_{ij})_{i,j \in E}$
$P^{(m)}(i, j) = \mathrm{P}\{X_{n+m} = j | X_n = i\} = Q_n^{(m)}(i, j)$ for each n.
$P^{(m)} = (P^{(m)}(i, j))_{i,j \in E}$ or $= (p_{ij}^{(m)})_{i,j \in E}$
We have that $P^{(1)} = P$. The matrix $P^{(n)}$ must satisfy that:
(i) for each $i, j \in E$, $P^{(n)}(i, j) \geq 0$.
(ii) for each $i \in E$, $\sum_{j \in E} P^{(n)}(i, j) = 1$.

Main formulas for a homogeneous Markov chain,
(2) $P^{(m)}(i, j) = P(X_{n+m} = j | X_n = i), \qquad P^{(m)} = \underbrace{PP \cdots P}_{m \text{ factors}}$.
(3) $\mathrm{P}\{X_j = i_j,\ j = n, n + 1, \dots, n + m\} = \alpha_n(i_n)P(i_n, i_{n+1})P(i_{n+1}, i_{n+2}) \cdots P(i_{n+m-1}, i_{n+m})$.
(4) $\mathrm{P}\{X_j = i_j,\ j = n + 1, \dots, n + m | X_n = i_n\} = P(i_n, i_{n+1})P(i_{n+1}, i_{n+2}) \cdots P(i_{n+m-1}, i_{n+m})$.
(5) $\alpha_n = \alpha_0 P^{(n)} = \alpha_1 P^{(n-1)} = \cdots = \alpha_{n-1} P$.

Example 11.13. *Suppose that a homogeneous Markov chain has state space* $E = \{1, 2, 3\}$*, transition matrix* $P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$*, and initial distribution* $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(i) $P^{(2)}$,
(ii) $P^{(3)}$,
(iii) $\mathrm{P}\{X_2 = 2\}$,
(iv) $\mathrm{P}\{X_0 = 1, X_3 = 3\}$,
(v) $\mathrm{P}\{X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 1\}$,
(vi) $\mathrm{P}\{X_2 = 3 | X_1 = 3\}$,
(vii) $\mathrm{P}\{X_{12} = 1 | X_5 = 3, X_{10} = 1\}$,
(viii) $\mathrm{P}\{X_3 = 3, X_5 = 1 | X_0 = 1\}$,
(ix) $\mathrm{P}\{X_3 = 3 | X_0 = 1\}$.

Solution: (i) $P^{(2)} = PP$

$$= \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix}$$

(ii) $P^{(3)} = PPP = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix}$

$$= \begin{pmatrix} 0.3425926 & 0.3796296 & 0.2777778 \\ 0.3888889 & 0.2569444 & 0.3541667 \\ 0.3506944 & 0.3159722 & 0.3333333 \end{pmatrix}$$

(iii) $\mathrm{P}\{X_2 = 2\} = \alpha_2(2) = \alpha_0 Q_0 Q_1 (0, 1, 0)^t$, thus
$\alpha_2(2) = \alpha_0 P^{(2)} (0, 1, 0)^t = (1/2, 1/3, 1/6)(0.2222, 0.4583, 0.2917)^t = 0.3125.$

(iv) $\mathrm{P}\{X_0 = 1, X_3 = 3\} = \mathrm{P}\{X_0 = 1\}\mathrm{P}\{X_3 = 3 | X_0 = 1\}$
$= \alpha_0(1)(Q_0 Q_1 Q_2)(1, 3)$ (or $= \alpha_0(1)(1, 0, 0)(Q_0 Q_1 Q_2)(0, 0, 1)^t$)
$= \alpha_0(1) P^{(3)}(1, 3) = (0.5)(0.2777778) = 0.1388889.$

(v) $\mathrm{P}\{X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 1\} = Q_0(1, 2) Q_1(2, 3) Q_2(3, 1) = \frac{2}{3}\frac{1}{2}\frac{1}{4} = \frac{1}{12}.$

(vi) $\mathrm{P}\{X_2 = 3 | X_1 = 3\} = Q_1(3, 3) = P(3, 3) = 1/2.$

(vii) $\mathrm{P}\{X_{12} = 1 | X_5 = 3, X_{10} = 1\} = \mathrm{P}\{X_{12} = 1 | X_{10} = 1\} = P^{(2)}(1, 1)$

$$= (1, 0, 0) P^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0.444444.$$

(viii) $\mathrm{P}\{X_3 = 3, X_5 = 1 | X_0 = 1\} = \mathrm{P}\{X_3 = 3 | X_0 = 1\}\mathrm{P}\{X_5 = 1 | X_3 = 3\}$
$= P^{(3)}(1, 3) P^{(2)}(3, 1) = (0.277778)(0.3333) = 0.09258341.$

(ix) $\mathrm{P}\{X_3 = 3 | X_0 = 1\} = P^{(3)}(1, 3) = (1, 0, 0) P^3 (0, 0, 1)^t = 0.277778.$

Theorem 11.12.

Theorem 11.13.

Theorem 11.14.

Theorem 11.15.

11.6. Some actuarial applications. In this section, we consider the APV of different types of cashflows. Let v_t be the t–year discount factor, i.e. the present value of \$1 received t years in the future is v_t. Consider a Markov chain with state space $E = \{1, 2, \dots, m\}$. Let $Q_n^{(k)}(i,j) = \mathrm{P}\{X_{n+k} = j | X_n = i\}$. Suppose that a payment of ${}_{n+k}C(j)$ is made at time $n+k$ if the Markov chain is in state j at time $n+k$. If $X_n = i$, then the APV of this cashflow at time n is

$$A = \sum_{k=0}^{\infty} Q_n^{(k)}(i,j) \cdot {}_{n+k}C(j) v_{n+k} v_n^{-1}. \tag{1}$$

Example 11.14. *An employer offers a pension plan for its employees. Assume that the status of an employee is allowed to change at the end of the year. The status of employees is modeled by a non-homogeneous Markov Chain with three states: Healthy (1), Disabled (2), and Gone (3). The transition-probability matrices for a new entrant (time 0) are*

$$Q_0 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.7 & 0.1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.5 & 0.2 & 0.3 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.1 & 0.8 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (= Q_k)\ k \geq 4,\ \textit{implicitly}).$$

Suppose that $i = 6\%$. Every employee makes a contribution of \$ 10^4 at the beginning of the year while he is in state 1. Calculate the APV of these contributions.

Solution: By Eq. (1), the APV of these contributions is

$$\begin{aligned} A =& \sum_{k=0}^{\infty} Q_0^{(k)}(1,1) \cdot {}_kC(1) v_k \\ =& (10^4)[Q_0^{(0)}(1,1) + Q_0^{(1)}(1,1)v + Q_0^{(2)}(1,1)v^2 + Q_0^{(3)}(1,1)v^3] \\ =& (10^4)[1 + (0.8)(1.06)^{-1} + (0.37)(1.06)^{-2} + (0.092)(1.06)^{-3}] \approx 2.161261 \times 10^4. \end{aligned}$$

Reason: $Q_0^{(0)}(1,1) =$ **??** $\qquad Q_0^{(1)} = Q_0, \qquad Q_0^{(2)} = Q_0 Q_1 = \begin{bmatrix} 0.37 & 0.18 & 0.45 \\ 0.33 & 0.16 & 0.51 \\ 0 & 0 & 1 \end{bmatrix}$

$$Q_0^{(3)} = Q_0 Q_1 Q_2 = \begin{bmatrix} 0.092 & 0.129 & 0.779 \\ 0.082 & 0.115 & 0.803 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Q_0^{(4)} = Q_0 Q_1 Q_2 Q_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Simpler way: $Q_0^{(2)}(1,1) = (1,0,0) Q_0 Q_1 (1,0,0)^t = (0.8, 0.1, 0.1)(0.4, 0.5, 0)^t$.

$Q_0^{(3)}(1,1) = (1,0,0) Q_0^{(3)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (0.8, 0.1, 0.1) \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.5 & 0.2 & 0.3 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0.2 \\ 0.1 \\ 0 \end{pmatrix} = (0.8, 0.1, 0.1) \begin{pmatrix} 0.1 \\ 0.12 \\ 0 \end{pmatrix} =$ 0.092

Suppose that a payment of ${}_{n+k}C(i,j)$ is made at time $n+k$ if the Markov chain satisfies $X_{n+k-1}=i$ and $X_{n+k}=j$. If $X_n=h$, then the APV of this cashflow at time n is

$$\sum_{k=0}^{\infty} Q_n^{(k)}(h,i)Q_{n+k}(i,j)\cdot {}_{n+k}C(i,j)v_{n+k+1}v_n^{-1}.$$

Example 11.15. *Under the information on Example 11.14, the employer makes a benefit payment of \$5000 for transition from state 1 to state 2 (get sick).*
(i) Calculate the APV of these benefit payments.
(ii) Suppose that these benefit payments are funded by a level premium made while an employee is at state 1. Calculate the level benefit premium.

Solution: (i) The APV of these contributions is

$$\begin{aligned}
&\sum_{k=0}^{\infty} Q_0^{(k)}(1,1)Q_k(1,2)\cdot {}_kC(1,2)v_{k+1} \qquad\qquad v_0^{-1}=(v^{-1})^0\\
=&(5000)[Q_0(1,2)v+Q_0(1,1)Q_1(1,2)v^2+Q_0^{(2)}(1,1)Q_2(1,2)v^3]\\
=&(5000)[(0.1)(1.06)^{-1}+(0.8)(0.2)(1.06)^{-2}+(0.37)(0.3)(1.06)^{-3}]\\
=&1649.684.
\end{aligned}$$

(ii) Let P be the level benefit premium. By Example 11.14, the APV of premiums is $2.161261P$. Hence, $P=\frac{1649.684}{2.162}=763.30$.

However, since the last year, everyone will go, the level benefit premium is

$$Q_0^{(0)}(1,1)+Q_0^{(1)}(1,1)v+Q_0^{(2)}(1,1)v^2=1+(0.8)(1.06)^{-1}+(0.37)(1.06)^{-2}=2.084.$$

Hence, $P=\frac{1649.684}{2.084}=791.59$.

$$Q_0=\begin{bmatrix}0.8&0.1&0.1\\0.7&0.1&0.2\\0&0&1\end{bmatrix},\quad Q_1=\begin{bmatrix}0.4&0.2&0.4\\0.5&0.2&0.3\\0&0&1\end{bmatrix},$$
$$Q_2=\begin{bmatrix}0.2&0.3&0.5\\0.1&0.1&0.8\\0&0&1\end{bmatrix},\quad Q_3=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}.$$

Theorem 11.16.

11.3 Random walk.

Let $\{\epsilon_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with
$\mathrm{P}\{\epsilon_j=1\}=p$ and $\mathrm{P}\{\epsilon_j=-1\}=1-p$, where $0<p<1$.
Let $X_n=X_0+\sum_{j=1}^{n}\epsilon_j$ for $n\geq 1$, where X_0 is a r.v., often $=0$.
The stochastic process $\{X_n : n\geq 0\}$ is called a **random walk**.

Imagine a drunkard coming back home at night. We assume that the drunkard goes in a straight line, but, he does not know which direction to take. After giving one step in one direction, the drunkard ponders which way to take home and decides randomly which direction to take. The drunkard's path is a random walk.

A random walk is a homogeneous Markov chain with states $E = \{0, \pm 1, \pm 2, \dots\}$ and $P(i, i+1) = P(X_{n+1} = i+1 | X_n = i) = p$ and $P(i, i-1) = 1 - p \ \forall \ n \geq 0$.

Formulas:

$S_n = \frac{X_n - X_0 + n}{2} \sim b(n, p)$,

$X_n = X_0 + 2S_n - n$.

For each $1 \leq m \leq n$,

(i) $X_m \perp X_n - X_m$.

(ii) $X_n - X_m \sim X_{n-m}$ **why does it mean ?**

(iii) $\mathrm{Cov}(X_m, X_n) = 4mp(1-p)$.

Example 11.16. *Suppose that* $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* 0.55 *of a step to the right, find:*
(i) $\mathrm{P}\{X_4 = -2\}$. *(ii)* $E[X_4]$. *(iii)* $\mathrm{Var}(X_4)$.

Solution: (i) $\mathrm{P}\{X_4 = -2\} = \mathrm{P}(2S_4 - 4 = -2) = \mathrm{P}\{S_4 = 1\} = \binom{4}{1}(0.55)^1(0.45)^3 \approx 0.20$.

(ii) $E[X_4] = E(0 + 2S_4 - 4) = 2np - 4 = 2 * 4 * 0.55 - 4 = 0.4$.

(iii) $\mathrm{Var}(X_4) = \mathrm{Var}(0 + 2S_4 - 4)) = 4npq = (4)(4)(0.55)(0.45) = 3.96$.

Example 11.17. *Suppose that* $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* p *of a step to the right, find:*
(i) $\mathrm{P}\{X_3 = -1, X_6 = 2, \}$
(ii) $\mathrm{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\}$

Solution: Only do (ii).

$$\text{(ii)} \quad \begin{aligned} & \mathrm{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\} \\ = \ & \mathrm{P}\{X_5 = 1, X_{10} - X_5 = 3, X_{16} - X_{10} = -2\} \\ = \ & \mathrm{P}\{X_5 = 1\}\mathrm{P}\{X_{10} - X_5 = 3\}\mathrm{P}\{X_{16} - X_{10} = -2\} \\ = \ & \mathrm{P}\{X_5 = 1\}\mathrm{P}\{X_5 = 3\}\mathrm{P}\{X_6 = -2\} \qquad X_n = 0 + 2S_n - n \ \text{ and } \ S_n = \mathrm{Binom}(n, p) \\ = \ & \mathrm{P}\{\mathrm{Binom}(5, p) = (5+1)/2\}\mathrm{P}\{\mathrm{Binom}(5, p) = (5+3)/2\}\mathrm{P}\{\mathrm{Binom}(6, p) = (6-2)/2\} \\ = \ & \binom{5}{3}p^3q^2\binom{5}{4}p^4q^1\binom{6}{2}p^2q^4 = 750p^9q^7. \end{aligned}$$

Example 11.18. *Suppose that* $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* p *of a step to the right and* $0 \leq m \leq n$ *Show that:*
(i) $\mathrm{Cov}(X_m, X_n) = m4p(1-p)$.
(ii) $\mathrm{Var}(X_n - X_m) = (n-m)4p(1-p)$.
(iii) $\mathrm{Var}(aX_n + bX_m) = (a+b)^2 m4p(1-p) + a^2(n-m)4p(1-p)$.

Solution: (i) Since $X_m \perp X_n - X_m$,

$$\begin{aligned} & \mathrm{Cov}(X_m, X_n) = \mathrm{Cov}(X_m, X_m + X_n - X_m) \\ = \ & \mathrm{Cov}(X_m, X_m) + \mathrm{Cov}(X_m, X_n - X_m) = \mathrm{Var}(X_m) = m4p(1-p). \end{aligned}$$

(ii) Since $X_n - X_m \sim X_{n-m}$,

$$\mathrm{Var}(X_n - X_m) = \mathrm{Var}(X_{n-m}) = \mathrm{Var}(0 + 2S_{n-m} + (n-m)) = 4(n-m)p(1-p).$$

(iii) Since X_m and $X_n - X_m$ are independent r.v.'s,

$$\begin{aligned} & \mathrm{Var}(aX_n + bX_m) = \mathrm{Var}(bX_m + a(X_m + X_n - X_m)) \\ = \ & \mathrm{Var}((a+b)X_m + a(X_n - X_m)) \\ = \ & \mathrm{Var}((a+b)X_m) + \mathrm{Var}(a(X_n - X_m) \\ = \ & (a+b)^2 4mp(1-p) + a^2 4(n-m)p(1-p) \end{aligned}$$

$V(aX_n + bX_m) = a^2 V(X_n) + 2abCov(X_n, X_m) + b^2 V(X_m) = a^2 \cdot 4npq + 2ab \cdot 4mpq + b^2 \cdot 4mpq$ $= a^2 \cdot 4(n-m)pq + a^2 \cdot 4mpq + 2ab \cdot 4mpq + b^2 \cdot 4mpq.$

Skip the next two examples.

Example 11.19. *Suppose that $\{X_n : n \geq 1\}$ is a random walk with $X_0 = 0$ and probability p of a step to the right, find:*
(i) $\mathrm{Var}(-3 + 2X_4)$
(ii) $\mathrm{Var}(-2 + 3X_2 - 2X_5)$.
(iii) $\mathrm{Var}(3X_4 - 2X_5 + 4X_{10})$.

Example 11.20. *Let $\{X_n : n \geq 0\}$ be a random walk with $X_0 = 0$ and $P[X_{n+1} = i+1 \mid X_n = i] = p$ and $\mathrm{P}\{X_{n+1} = i-1 \mid X_n = i\} = 1-p$. Find:*
(i) $\mathrm{P}\{X_{10} = 4]$.
(ii) $\mathrm{P}\{X_4 = -2, X_{10} = 2\}$.
(iii) $\mathrm{P}\{X_3 = -1, X_9 = 3, X_{14} = 6\}$.
(iv) $\mathrm{Var}(X_7)$.
(v) $\mathrm{Var}(5 - X_3 - 2X_{10} + 3X_{20})$.
(vi) $\mathrm{Cov}(-2X_3 + 5X_{10}, 5 - X_4 + X_7)$.

Remark. Let $\{X_n\}_{n\geq 0}$ be a random walk with $X_0 = 0$. Given $k > 0$, let T_k be the first time that $S_n = k$, i.e. $T_k = \inf\{n \geq 0 : S_n = k\}$. Then, T_k has negative binomial distribution,

$\mathrm{P}\{T_k = n\} = \binom{n-1}{k-1}p^k(1-p)^{n-k}, n \geq k$. $E[T_k] = \frac{k}{p}$ and $\mathrm{Var}(T_k) = \frac{kq}{p^2}$.

Example 11.21.

11.4 Hitting probabilities.

In this section, we consider a homogeneous Markov chain $\{X_n\}_{n\geq 0}$ with one-step transition probability matrix $(P(i,j))_{i,j\in E}$ $(= (p_{ij})_{i,j\in E})$.

Definition 11.6. *Given a set $A \subset E$, the hitting probability h_i^A is the probability that the process hits A given that it starts at i.*

$h_i^A = \mathrm{P}\{\{X_n\}_{n\geq 0}$ is ever in $A|X_0 = i\} = \sum_{n=0}^{\infty} \mathrm{P}\{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A|X_0 = i\}$.

Theorem 11.17. $(h_0^A, h_1^A, \ldots)$ *satisfies the recurrence equations*

$$\begin{aligned}
h_i^A &= 1, \text{ for } i \in A, \\
h_i^A &= \sum_{j\in E} P(i,j)h_j^A, \text{ for } i \notin A.
\end{aligned}$$

Proof. $h_i^A = \mathrm{P}\{X_0 \in A|X_0 = i\} + \sum_{n=1}^{\infty} \mathrm{P}\{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A|X_0 = i\}$??
If $i \in A$, $h_i^A = \mathrm{P}\{X_0 \in A|X_0 = i\} = 1$. On the other hand, if $i \notin A$, then

$$\begin{aligned}
h_i^A =& \overbrace{\mathrm{P}\{X_0 \in A|X_0 = i\}}^{=0} + \sum_{n=1}^{\infty} \mathrm{P}\{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A|X_0 = i\} \\
=& \sum_{n=1}^{\infty} \mathrm{P}\{X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in A|X_0 = i\} \\
=& (\sum_{j\in E} + \sum_{j\in E})\mathrm{P}\{X_1 = j, \{X_n\}_{n\geq 1} \text{ is ever in } A|X_0 = i\} \\
=& \sum_{j\in A} \mathrm{P}\{X_1 = j|X_0 = i\} + \sum_{j\notin A} \mathrm{P}\{X_1 = j, \{X_n\}_{n\geq 1} \text{ is ever in } A|X_0 = i\} \\
=& \sum_{j\in A} Q_0(i,j) + \sum_{j\notin A} Q_0(i,j)\mathrm{P}\{\{X_n\}_{n\geq 1} \text{ is ever in } A|X_1 = j\} \\
=& \sum_{j\in A} P(i,j) + \sum_{j\notin A} P(i,j)\mathrm{P}\{\{X_n\}_{n\geq 1} \text{ is ever in } A|X_1 = j\} \\
=& \sum_{j\in A} P(i,j) \underbrace{h_j^A}_{=1} + \sum_{j\notin A} P(i,j)h_j^A \quad \text{as } \{X_n\}_{n\geq 0} \text{ is a homogeneous MC} \\
=& \sum_{j\in E} P(i,j)h_j^A.
\end{aligned}$$

Theorem 11.18. *Suppose that* $(x_0, x_1, \dots)$ *is a nonnegative solution to the system of linear equations,*

$$x_i = 1, \text{ for } i \in A, \qquad (1)$$
$$x_i = \sum_{j\in E} P(i,j)x_j, \text{ for } i \notin A. \qquad (2)$$

Then, for each $i \in E$, $x_i \geq h_i^A$.

It follows from the theorem that $(h_0^A, h_1^A, \dots)$ is the minimal solution of the equations

$$\begin{array}{lll} \text{if } i \in A, & x_i = 1, \text{ for } i \in A, & h_i^A = 1, \\ \text{if } i \notin A. & x_i = \sum_{j\in E} P(i,j)x_j, & h_i^A = \sum_{j\in E} P(i,j)h_j^A, \end{array} \quad \text{and } x_i \geq h_i^A.$$

When $A = \{j\}$, where j is absorbing state, h_i^A $(= 1)$ is the probability that the Markov chain is absorbed by state j. In this case, h_i^A is called the **absorption probability** of state j.

Example 11.22. *An American roulette has 38 pockets numbered:* $0, 00, 1, 2, \dots, 36$. *Peter has \$100 and wants to have \$400 or go broke. Peter bets all his money in each play to the first 18, i.e. to* $1, 2, \dots, 18$. *If after turning the roulette, the ball falls into of the pockets in the first 18, Peter gets a payoff of twice his ante, i.e. Peter gets his ante plus a profit equal to its ante. If the ball does not fall in the first 18, Peter loses all his ante. Calculate the probability that Peter ends up with \$400.*

Solution: Let X_n be Peter's amount of money at time n, with $A = \{400\}$, $h_{100}^A = ?$
$\{X_n\}$ is a Markov chain with state space $E = \{0, 100, 200, 400\}$. The probability that Peter wins in a bet is $\frac{18}{38} = \frac{9}{19}$. $P = \big(P(i,j)\big)_{i,j\in E} = \big(\mathrm{P}(X_{n+1} = j | X_n = i)\big)_{i,j\in E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{10}{19} & 0 & \frac{9}{19} & 0 \\ \frac{10}{19} & 0 & 0 & \frac{9}{19} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Note $\begin{pmatrix} x_0 \\ x_{100} \\ x_{200} \\ x_{400} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{10}{19} & 0 & \frac{9}{19} & 0 \\ \frac{10}{19} & 0 & 0 & \frac{9}{19} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{100} \\ x_{200} \\ x_{400} \end{pmatrix} => \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{10}{19} & -1 & \frac{9}{19} & 0 \\ \frac{10}{19} & 0 & -1 & \frac{9}{19} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{100} \\ x_{200} \\ x_{400} \end{pmatrix}$.

The solution to $\mathbf{x}$ with $x_{400} = 1$ is not unique.

$h_{400}^A = 1$ by Theorem 10.17, as $A = \{400\}$.

$h_0^A = 0$, as $P(0,0) = 1$ *i.e.*, 0 is an absorbing state.

Then by Theorem 10.17, $\mathbf{x} = P\mathbf{x} => x_{100} = (9/19)x_{200}$ and $x_{200} = (9/19)x_{400} = (9/19)$.

So, $h_{100}^A = x_{100} = (9/19)^2 = 0.2243767313$.

Proof of Theorem 10.18. For $i \notin A$,

$$\begin{aligned} x_i = \sum_{j_1\in E} P(i,j_1)x_{j_1} &= \sum_{j_1\in A} P(i,j_1) + \sum_{j_1\notin A} P(i,j_1)x_{j_1} \qquad \textit{by Eq (1)} \\ =&\mathrm{P}\{X_1 \in A | X_0 = i\} + \sum_{j_1\notin A} P(i,j_1) \sum_{j_2\in E} P(j_1,j_2)x_{j_2} \qquad \textit{by Eq (2)} \\ =&\mathrm{P}\{X_1 \in A | X_0 = i\} + \sum_{j_1\notin A}\sum_{j_2\in A} P(i,j_1)P(j_1,j_2) + \sum_{j_1,j_2\notin A} P(i,j_1)P(j_1,j_2)x_{j_2} \\ =&\mathrm{P}\{X_1 \in A | X_0 = i\} + \mathrm{P}\{X_1 \notin A, X_2 \in A | X_0 = i\} + \sum_{j_1,j_2\notin A} P(i,j_1)P(j_1,j_2)x_{j_2}. \end{aligned}$$

By induction, we get that

$$x_i = \sum_{j=1}^{m} \mathrm{P}\{X_1 \notin A, \ldots, X_{j-1} \notin A, X_j \in A | X_0 = i\} + \sum_{j_1,\ldots,j_m \notin A} P(i, j_1)P(j_1, j_2)\cdots P(j_{m-1}, j_m)x_{j_m}$$

Since x_i is nonnegative,

$$\begin{aligned} x_i &\geq \sum_{j=1}^{m} \mathrm{P}\{X_1 \notin A, \ldots, X_{j-1} \notin A, X_j \in A | X_0 = i\} \\ &= \sum_{j=0}^{m} \mathrm{P}\{X_0 \notin A, \ldots, X_{j-1} \notin A, X_j \in A | X_0 = i\} \qquad \text{as } i \notin A \end{aligned}$$

Letting $m \to \infty$, $x_i \geq \sum_{j=o}^{\infty} \mathrm{P}\{X_0 \notin A, \ldots, X_{j-1} \notin A, X_j \in A | X_0 = i\} = h_i^A$. ▫

Example 11.23. *A blind squirrel is in one of the chambers of a 2×3 maze. The squirrel can go any available chamber which is either up, or down, or to the right or to the left at random. In chamber 3, there is a weasel (huangshulang). In chamber 6, there is a raccoon (li2). The raccoon and weasel do not move to any chamber. Once that the squirrel enters a chamber with a predator, it is eaten. Calculate the probability that the squirrel is eaten by a raccoon.*

1	2	3 WEASEL
4	5	6 RACCOON

Solution: The states are $E = \{1, 2, 3, 4, 5, 6\}$. The transition matrix is

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For $A = \{6\}$, we need to find h_i^A or h_i, $i \in \{1, 2, 4, 5\}$. We setup the equations $\mathbf{h} = P\mathbf{h}$ =>

$$\begin{array}{l} h_1 = \frac{1}{2}h_2 + \frac{1}{2}h_4 \\ h_2 = \frac{1}{3}h_1 + \frac{1}{3}h_5 \\ h_3 = 0 \\ h_4 = \frac{1}{2}h_1 + \frac{1}{2}h_5 \\ h_5 = \frac{1}{3}h_4 + \frac{1}{3}h_2 + \frac{1}{3} \\ h_6 = 1 \end{array} \Rightarrow \begin{pmatrix} 1 & -0.5 & -0.5 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & -1/3 & 0 & 0 \\ -0.5 & 0 & 1 & -0.5 & 0 & 0 \\ 0 & -1/3 & -1/3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_4 \\ h_5 \\ h_3 \\ h_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}$$

The solutions are $h_1 = \frac{5}{11}$, $h_2 = \frac{4}{11}$, $h_4 = \frac{6}{11}$, $h_5 = \frac{7}{11}$.

11.5 Gambler's ruin problem.

Skip this section! Imagine a game played by two players. Player A starts with $\$k$ and player B starts with $\$(N-k)$. They play successive games until one of them ruins. In every game, they bet \$1. The probability that A wins is p and the probability that A losses is $1-p$. Assume that the outcomes in different games are independent. Let X_n be player A's money after n games. After one of the player ruins, no more wagers are done.

If $X_n = N$, then $X_m = N$, for each $m \geq n$. If $X_n = 0$, then $X_m = 0$, for each $m \geq n$. For $1 \leq k \leq N-1$,

$$\mathrm{P}\{X_{n+1} = k+1 | X_n = k\} = p, \ \mathrm{P}\{X_{n+1} = k-1 | X_n = k\} = 1-p.$$

$\{X_n\}_{n=1}^{\infty}$ is a homogeneous Markov chain with states $\{0, 1, \ldots, N\}$ and one–step transition probability

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & \cdots & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let P_k be the probability that player A wins (player B gets broke). We will prove that

$$P_k = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{k}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

P_k is the probability that a random walk with $X_0 = k$, reaches N before reaching 0. P_k is the probability that a random walk goes up $N-k$ before it goes down k.

Theorem 11.19. *(Gambler's ruin probability) Consider the Markov chain with states $E = \{0, 1, 2, \ldots, N\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = q = 1-p$ for $1 \leq i \leq N-1$, and $P_{0,0} = P_{N,N} = 1$. Let h_i be the probability that the Markov chain hits N. Then,* $h_i = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2}, \end{cases}$

Proof. To find h_i, we need to find the minimal solution of $h_i \geq 0$ ($\mathbf{h} = P\mathbf{h}$).

$$\begin{aligned}
&h_N = 1 \\
&h_{N-1} = qh_{N-2} + ph_N \quad (=> \quad ph_{N-1} + qh_{N-1} = qh_{N-2} + ph_N) \qquad (*) \\
&h_{N-2} = qh_{N-3} + ph_{N-1} \\
&\cdots \\
&h_2 = qh_1 + ph_3 \\
&h_1 = qh_0 + ph_2 \\
&h_0 = 0
\end{aligned}$$

These equations and (*) imply that

$$\begin{aligned}
h_N - h_{N-1} &= \frac{q}{p}(h_{N-1} - h_{N-2}) \\
h_{N-1} - h_{N-2} &= \frac{q}{p}(h_{N-2} - h_{N-1}) \\
&\cdots \\
h_3 - h_2 &= \frac{q}{p}(h_2 - h_1) \qquad (**) \\
h_2 - h_1 &= \frac{q}{p}(h_1 - h_0).
\end{aligned}$$

$$\begin{aligned}
h_1 - h_0 &= (h_1 - h_0) && (1) \\
h_2 - h_1 &= \left(\frac{q}{p}\right)(h_1 - h_0) && (2) \\
(**) => \quad h_3 - h_2 &= \left(\frac{q}{p}\right)^2 (h_1 - h_0) && (3) \\
&\cdots \\
h_N - h_{N-1} &= \left(\frac{q}{p}\right)^{N-1} (h_1 - h_0) && (N)
\end{aligned}$$

Adding Equations (1) through (N),

$$1 = h_N - h_0 = \left(1 + \left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{N-1}\right)(h_1 - h_0) = h_1 \sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j$$

$$=> \quad h_1 = \frac{1}{\sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j} \quad (= 1/\frac{1 - r^N}{1 - r}\Big|_{r=q/p}).$$

For $i = 1, ..., N-1$, adding Equations (1) through (i) yields

$$h_i = (h_i - h_0) = \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j (h_1 - h_0) = \frac{\sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j}{\sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j} = \begin{cases} \frac{i}{N} & \text{if } p = 1/2 \\ \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq 1/2 \end{cases}$$

∎

Example 11.24. *An American roulette has 38 pockets numbered: $0, 00, 1, 2, \ldots, 36$. Peter has \$100 and wants to have \$400 or go broke. Peter bets \$50 in each play to $1, 2, \ldots, 18$, until he is either broke or has \$400. If after turning the roulette, the ball falls into of the pockets in the first 18, Peter gets a payoff of twice his ante, i.e. Peter gets his ante plus a profit equal to its ante. If the ball does not fall in the first 18, Peter loses all his ante. Calculate the probability that Peter ends up with \$400.*

Solution: The probability that Peter wins in a bet is $\frac{18}{38} = \frac{9}{19}$. Let X_n be Peter's amount of money at time n. $\{\frac{1}{50}X_n\}$ is a Markov chain with state space $E = \{0, 1, \ldots, 8\}$ and one-step transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

With $A = \{8\}$, we need to find $h_{100}^A = P_k$. This is the gambler's ruin problem with $p = \frac{9}{19}$, $k = 2$ and $N = 8$. The probability that Peter ends up with \$400 is $P_k = \frac{1-\left(\frac{10}{9}\right)^2}{1-\left(\frac{10}{9}\right)^8} = 0.1772923206$.

Example 11.25. *Two gamblers, A and B make a series of \$1 wagers where B has 0.55 chance of winning and A has a 0.45 chance of winning on each wager. What is the probability that B wins \$10 before A wins \$5?*

Solution: Here, $p = 0.55$, $k = 5$, $N - k = 10$. So, the probability that B wins \$10 before A wins \$5 is $P_k = \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} = \frac{1-\left(\frac{0.45}{0.55}\right)^5}{1-\left(\frac{0.45}{0.55}\right)^{15}}$.

Example 11.26. *Suppose that on each play of a game a gambler either wins \$1 with probability p or losses \$1 with probability $1-p$, where $0 < p < 1$. The gambler bets continuously until she or he is either winning n or losing m. What is the probability that the gambler quits a winner assuming that she/he starts with \$i?*

Solution: We have to find the probability that a random walk goes up n before it goes down m. So, $N - k = n$ and $k = m$, and the probability is $P_k = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^m}{1-\left(\frac{q}{p}\right)^{m+n}} & \text{if } p \neq \frac{1}{2} \\ \frac{m}{m+n} & \text{if } p = \frac{1}{2} \end{cases}$

Theorem 11.20. *(Gambler's ruin probability against an adversary with an infinite amount of money) Let $\{X_n\}$ be the Markov chain with state space $E = \{0, 1, 2, \ldots\}$ and transition probabilities $P(i, i+1) = p$, and $P(i, i-1) = q = 1-p$ for $i \geq 1$, and $P(0, 0) = 1$. Let P_k be the probability that the Markov chain does not hit 0 given that $X_n = k$, $k \geq 1$. Then,*
$P_k = \begin{cases} 1 - \left(\frac{q}{p}\right)^k & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2} \end{cases}$

Proof. Let h_k be the probability that the Markov chain hits 0 given that $X_n = k$. We need to find the minimal solution of $h_i \geq 0$ ($\mathbf{h} = P\mathbf{h}$)). $h_0 = 1$,

$$\begin{array}{lll} h_1 = qh_0 + ph_2 \quad (=> ph_1 + qh_1 = qh_0 + ph_2) & => & h_2 - h_1 = \frac{q}{p}(h_1 - h_0) \\ h_2 = qh_1 + ph_3, & & h_3 - h_2 = \frac{q}{p}(h_2 - h_1) \\ \cdots & & \cdots \\ h_{N-1} = qh_{N-2} + ph_N, & & h_N - h_{N-1} = \frac{q}{p}(h_{N-1} - h_{N-2}) \end{array}$$

$$=> h_1 - h_0 = (h_1 - h_0),$$

$$h_2 - h_1 = \left(\frac{q}{p}\right)(h_1 - h_0)$$

$$h_3 - h_2 = \left(\frac{q}{p}\right)^2 (h_1 - h_0)$$

$$\cdots$$

$$+)\ h_N - h_{N-1} = \left(\frac{q}{p}\right)^{N-1} (h_1 - h_0)$$

$- -$

$$h_N - h_0 = (h_1 - h_0) \sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j = (h_1 - 1) \sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j. \tag{1}$$

$$h_N = 1 + (h_1 - 1) \sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j \geq 0.$$

$$h_1 \geq 1 - (\sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j)^{-1}.$$

$$h_1 \geq 1 - (\sum_{j=0}^{\infty} \left(\frac{q}{p}\right)^j)^{-1} \text{ (letting } N \to \infty)$$

The minimal solution is the one with $h_1 = 1 - \{\sum_{j=0}^{\infty} \left(\frac{q}{p}\right)^j\}^{-1}$.

If $0 < p \leq 0.5$, then $\sum_{j=0}^{\infty} \left(\frac{q}{p}\right)^j = \infty$, $h_1 = 1$ and $h_N = 1$ by Eq. (1), $\forall\ N \geq 1$.

If $p > 0.5$, $\sum_{j=0}^{\infty} \left(\frac{q}{p}\right)^j = \frac{1}{1-\frac{q}{p}}$, and $h_1 = \frac{q}{p}$ and

$h_N = 1 + (h_1 - 1) \sum_{j=0}^{N} \left(\frac{q}{p}\right)^j = 1 + \left(\frac{q}{p} - 1\right) \sum_{j=0}^{N} \left(\frac{q}{p}\right)^j = \left(\frac{q}{p}\right)^N$. ■

CHAPTER 12

Poisson Processes

12.1 Exponential and gamma distributions.

12.1.1 Exponential distribution

Definition 12.1. *The* **gamma function** *is defined by* $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt, \alpha > 0$.

Theorem 12.1. *The gamma function satisfies the following properties: (i) For each* $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. *(ii) For each integer* $n \geq 1$, $\Gamma(n) = (n-1)!$. *(iii)* $\Gamma(1/2) = \sqrt{\pi}$.

Theorem 12.2. *For each integer* $n \geq 1$, $\int \frac{x^n}{n!}e^{-x}\,dx = -e^{-x}\sum_{j=0}^n \frac{x^j}{j!} + c$.

Definition 12.2. $X \sim \mathrm{Exp}(\lambda)$, $f(x) = \frac{e^{-\frac{x}{\lambda}}}{\lambda}$ if $x \geq 0$.

Theorem 12.3. *Let* $X \sim \mathrm{Exp}(\lambda)$, *then* $S_X(x) = e^{-\frac{x}{\lambda}}$, $x \geq 0$.

$$E[X] = \lambda, \mathrm{Var}(X) = \lambda^2, E[X^k] = \lambda^k k!,\ M(t) = E(e^{Xt}) = \begin{cases} \frac{1}{1-\lambda t} & \text{if } t < \lambda^{-1}, \\ \infty & \text{else.} \end{cases}$$

The exponential distribution satisfies that for each $s, t \geq 0$,

$$\mathrm{P}\{X > s+t | X > t\} = {}_s p_t = S_{T_t}(s) = \frac{\mathrm{P}\{X > s+t\}}{\mathrm{P}\{X > t\}} = \frac{e^{-\frac{s+t}{\lambda}}}{e^{-\frac{t}{\lambda}}} = e^{-\frac{s}{\lambda}} = \mathrm{P}\{X > s\}.$$

This is property is called the **memoryless property of the exponential distribution**.

Example 12.1.

12.1.2 Gamma distribution

Definition 12.3. $X \sim \mathcal{G}(\alpha, \beta)$ *if* $f_X(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}$, $x \geq 0$, *where* $\alpha, \beta > 0$.

$Exp(\lambda) = \mathcal{G}(1, \lambda)$ and $\chi^2(\nu) = \mathcal{G}(\nu/2, 2)$.

Theorem 12.4. *If* $X \sim \mathcal{G}(\alpha, \beta)$, *then*

$$E[X] = \alpha\beta, \mathrm{Var}(X) = \alpha\beta^2, E[X^k] = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}, M(t) = \frac{1}{(1-\beta t)^\alpha},\ \text{if } t < \frac{1}{\beta}.$$

Example 12.2.

Example 12.3.

Theorem 12.5.

Example 12.4.

Example 12.5.

Example 12.6.

Example 12.7.

Additional Formula in 447:

44. If X_1___X_2. **key:** $\perp$,

X_i's $\sim$:	$X_1 + X_2 \sim$:	key
$G(\alpha_i, \beta)$	________	$G(\alpha_1+\alpha_2, \beta)$
$\chi^2(v_i)$	________	$\chi^2(v_1+v_2)$
$Pois(\lambda_i)$	________	$Pois(\lambda_1+\lambda_2)$
$N(\mu_i, \sigma_i^2)$	________	$N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$
$bin(n_i, p)$	________	$bin(n_1+n_2, p)$

By induction, the previous formula implies the results as follows.

Theorem 12.6. *(1) If $X_1, \ldots, X_n$ are independent r.v.'s and $X_i \sim \text{Gamma}(\alpha_i, \beta)$, $1 \le i \le n$, then, $\sum_{i=1}^n X_i \sim \text{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$. (2) If $X_1, \ldots, X_n$ are i.i.d $\sim Exp(\lambda)$, then $\sum_{i=1}^n X_i \sim \mathcal{G}(n, \lambda)$.*

Example 12.8. *Suppose that you arrive at a single–teller office of the Department of Motor Vehicles to find three customers waiting in line and one being served. If the services times are all exponential with rate 2 minutes, calculate the probability that you will have to wait in line more than 10 minutes before being served.*

Solution: By the memory–less property of the exponential distribution, the remaining serving time for the customer which is served is also exponential. Hence, your waiting time is $Y = \sum_{j=1}^4 X_j$, where $\{X_j\}_{j=1}^4$ are i.i.d. $S(t) = e^{-t/2}$, $t > 0$. By Theorem 12.6, $Y \sim G(4, 2)$, Hence, the density of Y is $f_Y(y) = \frac{y^{4-1}e^{-\frac{y}{2}}}{2^4\Gamma(4)} = \frac{y^3 e^{-\frac{y}{2}}}{2^4(3!)}$, $y > 0$. By the change of variable $\frac{y}{2} = z$,

$$\mathbb{P}\{Y > 10\} = \int_{10}^\infty \frac{y^3 e^{-\frac{y}{2}}}{96}\, dy = \int_5^\infty \frac{z^3 e^{-z}}{6}\, dz = \int_5^\infty \frac{z^3}{6}\, d(-e^{-z}) = -e^{-z}\frac{z^3}{6}\Big|_5^\infty - \int_5^\infty (-e^{-z}) d\frac{z^3}{6}$$
$$= \cdots = -\left(\frac{z^3}{6} + \frac{z^2}{2} + z + 1\right) e^{-z} \Big|_5^\infty = \left(\frac{5^3}{6} + \frac{5^2}{2} + 5 + 1\right) e^{-5} = 0.2650259153.$$

Suppose that a system has n parts. The system functions works only if all n parts work. Let X_i be the lifetime of the i–th part of the system. Suppose that $X_1, \ldots, X_n$ be independent r.v.'s and that X_i has an exponential distribution with mean θ_i. Let Y be the lifetime of the system. Then, $Y = \min(X_1, \ldots, X_n)$. Then, for $t > 0$ $\mathrm{P}\{Y > t\}$

$$= \mathrm{P}\{\min(X_1, \ldots, X_n) > t\} = \mathrm{P}\{\cap_{i=1}^n \{X_i > t\}\} = \prod_{i=1}^n \mathrm{P}\{X_i > t\} = \prod_{i=1}^n e^{-\frac{t}{\theta_i}} = e^{-t\sum_{i=1}^n \frac{1}{\theta_i}}.$$

So, Y has an exponential distribution with mean $\frac{1}{\sum_{i=1}^n \frac{1}{\theta_i}}$.

Let $Z = \max(X_1, \dots, X_n)$. Then, for $t > 0$,

$$\mathrm{P}\{Z \le t\} = \mathrm{P}\{\max(X_1, \dots, X_n) \le t\} = \mathrm{P}\{\cap_{i=1}^n \{X_i \le t\}\} = \prod_{i=1}^n P(X_i \le t\} = \prod_{i=1}^n \left(1 - e^{-\frac{t}{\theta_i}}\right).$$

Since $\max(x, y) + \min(x, y) = x + y$,

$$E[\max(X_1, X_2)] = \theta_1 + \theta_2 - \frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}}.$$

Given $x_1, \dots, x_n$, by induction, we can show that $\max(x_1, \dots, x_n)$ $= \sum_{i=1}^n x_i - \sum_{i_1 < i_2} \min(x_{i_1}, x_{i_2}) + \sum_{i_1 < i_2 < i_3} \min(x_{i_1}, x_{i_2}, x_{i_3}) - \cdots + (-1)^{n+1} \min(x_1, \dots, x_n)$.

Example 12.9. *A system consists of 4 components. The lifetime of the 4 components are independent random variables with an exponential distribution and respective means 2,3,4, 10. The system will work only if all four components work. Find the expected lifetime of the system.*

Solution: Let X_i, $1 \le i \le 4$, be the lifetime of the components. Let $T = \min_{1 \le i \le 4} X_i$ be the lifetime of the system. Then $E[T] = \frac{1}{\sum_{i=1}^n \frac{1}{\theta_i}} = \frac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10}} = 0.8450704225$.

Example 12.10. *A remote control has three batteries. The lifetime of these batteries are 20, 30 and 50 days respectively. The lifetimes of the batteries are independent r.v.'s with an exponential distribution. The remote control will work if at least one of the batteries work. Find the average time until the remote control does not work.*

Solution: Let X, Y and Z be the lifetimes of the batteries. We have that

$$\begin{aligned}
&E[\max(X, Y, Z)] \\
=&E[X + Y + Z - \min(X, Y) - \min(X, Z) - \min(Y, Z) + \min(X, Y, Z)] \\
=&20 + 30 + 50 - \frac{1}{\frac{1}{20} + \frac{1}{30}} - \frac{1}{\frac{1}{20} + \frac{1}{50}} - \frac{1}{\frac{1}{30} + \frac{1}{50}} + \frac{1}{\frac{1}{20} + \frac{1}{30} + \frac{1}{50}} = 64.64170507.
\end{aligned}$$

Example 12.11. *A system consists of 3 components. The lifetime of the 3 component are independent, identically distributed random variables with density function $f(x) = \frac{6}{x^7}$, if $1 < x$. The system will work only if all 3 components work. Find the expected lifetime of the system.*

Solution: Let X_i, $1 \le i \le 3$, be the lifetime of the components.
Let $T = \min_{1 \le i \le 3} X_i$ be the lifetime of the system.
Two ways: (1) $E(T) = \int x f_T(x) dx$, (2) $E(T) = \int_0^\infty S_T(t) dt$.

$$\mathbb{P}\{X_i > x\} = \int_x^\infty \frac{6}{t^7}\, dt = \frac{1}{x^6},\ x \ge 1,$$

$$\mathbb{P}\{T > t\} = \mathrm{P}\{\cap_{i=1}^3 \{X_i > t\}\} = \prod_{i=1}^3 \mathrm{P}\{X_i > t\} = (\frac{1}{t^6})^3 = \frac{1}{t^{18}},\ t \ge 1$$

Which way is better ?
2nd way: (a) $E(Y) = \int_0^\infty S_T(t)dt = \int_1^\infty t^{-18}dt = 1/17$. **Is it correct ?**

(b) $E(Y) = \int_0^\infty S_T(t)dt = \int_0^1 1dt + \int_1^\infty t^{-18}dt = 18/17$.
1st way: $f_T(t) = -S_T(t)' = \frac{18}{t^{19}}, t \geq 1$. Thus $E[T] = \int_1^\infty t f_T(t)\,dt = \int_1^\infty \frac{18}{t^{18}}\,dt = \frac{18}{17}$.

Theorem 12.7. *(i) Let $X_1, \ldots, X_n$ be independent r.v.'s such that X_i has an exponential distribution with mean θ_i. Then,* $\mathrm{P}\{X_i = \min_{1\leq j\leq n} X_j\} = \frac{\frac{1}{\theta_i}}{\sum_{j=1}^n \frac{1}{\theta_j}}$.
(ii) Suppose $X \perp Y$, $X \sim Exp(\theta_1)$ and $Y \sim Exp(\theta_2)$. Then $\mathrm{P}\{X < Y\} = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1}+\frac{1}{\theta_2}}$.

Proof. (ii) $f_{X,Y}(x,y) = \frac{e^{-\frac{x}{\theta_1}-\frac{y}{\theta_2}}}{\theta_1\theta_2}, x, y > 0$. So, $\mathrm{P}\{X < Y\} = \int_0^\infty \int_x^\infty \frac{e^{-\frac{x}{\theta_1}-\frac{y}{\theta_2}}}{\theta_1\theta_2}\,dy\,dx$

$$= \int_0^\infty \frac{e^{-\frac{x}{\theta_1}}}{\theta_1} \int_x^\infty \frac{e^{-\frac{y}{\theta_2}}}{\theta_2}\,dy\,dx = \int_0^\infty \frac{e^{-\frac{x}{\theta_1}}}{\theta_1} e^{-\frac{x}{\theta_2}}\,dx = \int_0^\infty \frac{e^{-x[\frac{1}{\theta_1}+\frac{1}{\theta_2}]}}{\theta_1}\,dx = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1}+\frac{1}{\theta_2}}$$

$$(i) \quad \mathrm{P}\{X_i = \min_{1\leq j\leq n} X_j\} = P(X_i < \min_{1\leq j\leq n, j\neq i} X_j) = \frac{\frac{1}{\theta_i}}{\sum_{j=1}^n \frac{1}{\theta_j}},$$

because $Y = \min_{1\leq j\leq n, j\neq i} X_j \sim Exp(\left(\sum_{1\leq j\leq n, j\neq i}^n \frac{1}{\theta_j}\right)^{-1})$, and $X_i \perp Y$. ■

Example 12.12. *A factory has two electricity generators. The smaller of the two generators has expected duration before failure of 20 days. The other generator has an expected duration of* 15 *days. The amount of time which each generator lasts before failing has an exponential distribution. The duration before failure of the two generators are independent r.v.'s.*
(i) Calculate the mean of the time until one of the two generators fails.
(ii) Calculate the mean of the time until both generators breaks down.
(iii) Calculate the probability that the smaller generators fails before the other.

Solution: (i) X and Y be the lifetime of the two generators.
$X \sim Exp(20)$ and $Y \sim Exp(15)$.
(i) $E[\min(X,Y)] = \frac{1}{\frac{1}{\theta_1}+\frac{1}{\theta_2}} = \frac{1}{\frac{1}{20}+\frac{1}{15}} = \frac{60}{7}$.
(ii) $E[\max(X,Y)] = E[X+Y-\min(X,Y)] = 20+15-\frac{60}{7} = \frac{185}{7}$.
(iii) $\mathrm{P}\{X < Y\} = \frac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{15}} = \frac{3}{7}$.

Example 12.13. *Let X_1 and X_2 be i.i.d. from exponential distribution with mean $\lambda > 0$. Let $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$. Find the density function, the mean and the variance of $X_{(1)}$ and $X_{(2)}$.*

Solution: Density: Formula: $f = F' = -S'$. $P(X_{(1)} > t) = P(X_1 > t)P(X_2 > t) = e^{-t/\lambda}e^{-t/\lambda} = e^{-t2/\lambda}$. So $f_{X_{(1)}}(t) = \frac{1}{2\lambda}e^{-\frac{t}{2\lambda}}$, $t > 0$.
$\mathrm{P}\{X_{(2)} \leq x\} = \mathrm{P}\{X_1 \leq x, X_2 \leq x\} = (1-e^{-\frac{x}{\lambda}})(1-e^{-\frac{x}{\lambda}}) = 1-2e^{-\frac{x}{\lambda}}+e^{-\frac{2x}{\lambda}}$.

So, the density function of $X_{(2)}$ is $f_{X_{(2)}}(x) = F'_{X(2)}(t) = \frac{2e^{-\frac{x}{\lambda}}}{\lambda} - \frac{2e^{-\frac{2x}{\lambda}}}{\lambda}$, $x > 0$

Mean: $X_{(1)} \sim Exp(\frac{1}{\frac{1}{\lambda}+\frac{1}{\lambda}})$, $\frac{1}{\frac{1}{\lambda}+\frac{1}{\lambda}} = \frac{\lambda}{2}$. $E[X_{(1)}] = \frac{\lambda}{2}$

$E[X_{(2)}] = \int_0^\infty x(\frac{2e^{-x/\lambda}}{\lambda} - \frac{2e^{-2x/\lambda}}{\lambda}) = \cdots$, or since $X_{(1)} + X_{(2)} = X_1 + X_2$,

$E(X_{(2)}) = \lambda + \lambda - \lambda/2 = 3\lambda/2. \qquad (2)$

Variance: $\mathrm{Var}(X_{(1)}) = \frac{\lambda^2}{4}$.

Since $X_{(1)} + X_{(2)} = X_1 + X_2$ and $X_{(1)}X_{(2)} = X_1X_2$,

$V(X_{(1)} + X_{(2)}) = V(X_1 + X_2) = 2\lambda^2$

$=> V(X_{(1)}) + V(X_{(2)}) + 2cov(X_{(1)}, X_{(2)}) = 2\lambda^2$.

$cov(X_{(1)}, X_{(2)}) = E(X_{(1)}X_{(2)}) - E(X_{(1)})E(X_{(2)}) = E(X_1X_2) - \frac{\lambda}{2}\frac{3\lambda}{2} = \lambda \cdot \lambda - \frac{3\lambda^2}{4} = \frac{\lambda^2}{4}$ by (2).

$V(X_{(2)}) = 2\lambda^2 - \frac{\lambda^2}{4} - 2\frac{\lambda^2}{4} = \frac{5\lambda^2}{4}$

Example 12.14.

Theorem 12.8. *If $X \sim Pois(\lambda)$ then $E[X] = \lambda$, $\mathrm{Var}(X) = \lambda$ and $M(t) = e^{\lambda(e^t-1)}$.*

Theorem 12.9.

12.2 Poisson process.

Definition 12.4. *A stochastic process $\{N(t) : t \geq 0\}$ is said to be a* **counting process** *if $N(t)$ represents the total number of "events" that have occurred up to time t.*

A counting process $N(t)$ must satisfy:
(i) $N(t) \in \{0, 1, 2, ...\}$. (ii) If $s < t$, then $N(s) \leq N(t)$.

For a counting process $\{N(t) : t \geq 0\}$ and $s < t$, $N(t) - N(s)$ is the number of events occurring in the time interval $(s, t]$.

Definition 12.5. *A stochastic process is said to possess* **independent increments** *if $N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots, N(t_m) - N(t_{m-1})$ are independent r.v.'s whenever $0 \leq t_1 < t_2 < \cdots < t_m$.*

Notice that if $s < t$, $N(t) - N(s)$ is the increment of the process in the interval $(s, t]$. Notice that independence is only required for non–overlapping increments. Variables such us $N(7) - N(4)$ and $N(10) - N(5)$ are not necessarily independent.

Definition 12.6. *A counting process is said to have* **stationary increments** *if for each $0 \leq t_1 \leq t_2$, $N(t_2) - N(t_1) \sim N(t_2 - t_1) - N(0)$.*

That is, a counting process has stationary increments if the distribution of an increment depends on its length, independently on its starting time.

The main r.v. to count occurrences is the Poisson distribution (Pois(λ) or $\mathcal{P}(\lambda)$).

Definition 12.7. *A r.v. $X \sim Pois(\lambda)$ if $\mathrm{P}\{X = k\} = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, ...$, $\lambda > 0$.*

Definition 12.8. *A stochastic process $\{N(t) : t \geq 0\}$ is said to be a* **Poisson process** *with rate $\lambda > 0$, if: (i) $N(0) = 0$.*
(ii) The process has independent increments.
(iii) For each $0 \leq s, t$, $N(s + t) - N(s) \sim Pois(\lambda t)$.

Condition (iii) implies that a Poisson process has stationary increments.

The rate of occurrences per unit of time is a constant. The average number of occurrences in the time interval $(s, s + t]$ is λt.

Theorem 12.10. *For each $t \geq 0$, $E[N(t)] = \lambda t$ and $\mathrm{Var}(N(t)) = \lambda t$.*

Theorem 12.11. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Then, for each $0 \leq t_1 < t_2 < \cdots < t_m$, and each $0 \leq k_1 \leq k_2 \leq \cdots \leq k_m$,*

$$\begin{aligned}\mathrm{P}\{\cap_{j=1}^m \{N(t_j) = k_j\}\} =& e^{-\lambda t_1} \frac{(\lambda t_1)^{k_1}}{k_1!} \prod_{j=2}^m e^{-\lambda(t_j - t_{j-1})} \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} \\ & (= \mathrm{P}\{N(t_1) = k_1\} \prod_{j=2}^m P(N(t_j) - N(t_{j-1}) = k_j - k_{j-1})\end{aligned}$$

Announcement: Quiz on Wednesday: 8, 12, 13.

Theorem 12.12. *For each* $0 \le s \le t$, $\mathrm{Cov}(N(s), N(t)) = \lambda s$.

Proof. Since $N(s)$ and $N(t) - N(s)$ are independent, $\mathrm{Cov}(N(s), N(t) - N(s)) = 0$. So,

$$\begin{aligned}\mathrm{Cov}(N(s), N(t)) =& \mathrm{Cov}(N(s), N(s) + N(t) - N(s)) \\ =& \mathrm{Cov}(N(s), N(s)) + \mathrm{Cov}(N(s), N(t) - N(s)) = \mathrm{Var}(N(s)) = \lambda s,\end{aligned}$$

Example 12.15. *Let* $\{N(t) : t \ge 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $\mathrm{P}\{N(5) = 4\}$. *(iii)* $\mathrm{P}\{N(5) = 4, N(6) = 9, N(10) = 15\}$.
(v) $\mathrm{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$. *(vi)* $\mathrm{P}\{N(2) + N(5) = 4\}$.

Solution: (i) $\mathrm{P}\{N(5) = 4\} = \mathrm{P}\{\mathrm{Pois}(5\lambda) = 4\} = \frac{e^{-5\lambda}(5\lambda)^4}{4!} = \frac{e^{-10}(10)^4}{4!}$.
(iii) $\mathrm{P}\{N(5) = 4, N(6) = 9, N(10) = 15\}$
$= \mathrm{P}\{N(5) = 4, N(6) - N(5) = 9 - 4 = 5, N(10) - N(6) = 15 - 9 = 6\}$
$= \mathrm{P}\{N(5) = 4\}P\{N(1) = 5\}\mathrm{P}\{N(4) = 6\} = e^{-10}\frac{10^4}{4!}e^{-2}\frac{2^5}{5!}e^{-8}\frac{8^6}{6!} = e^{-20}\cdots$
(v) $\mathrm{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$
$= \mathrm{P}\{N(5) - N(2) = 3\}\mathrm{P}\{N(7) - N(6) = 4\} = e^{-6}\frac{6^3}{3!}e^{-2}\frac{2^4}{4!} = e^{-8}(24)$.
(vi) $\mathrm{P}\{N(2) + N(5) = 4\}$
$= \mathrm{P}\{N(2) + N(5) - N(2) + N(2) = 4\} = \mathrm{P}\{2N(2) + (N(5) - N(2)) = 4\}$
$= \mathrm{P}\{2N(2)+(N(5)-N(2)) = 4, \textit{ and } N(2) \in \{0, 1, 2\}\}$ (as $2N(2) \le N(2)+N(5) = 4$)
$= \mathrm{P}\{(N(2), N(5) - N(2)) \in \{(0, 4), (1, 3), (2, 2)\}$
$= \mathrm{P}\{N(2) = 0\}P\{N(3) = 4\} + \mathrm{P}\{N(2) = 1\}P\{N(3) = 2\} + \mathrm{P}\{N(2) = 2\}P\{N(3) = 0\}$
$= e^{-4}e^{-6}\frac{6^4}{4!} + e^{-4}\frac{4^1}{1!}e^{-6}\frac{6^2}{2!} + e^{-4}\frac{4^2}{2!}e^{-6}$

Example 12.16. *Example 12.15 (continued)* $(\lambda = 2)$. *Compute:*
(i) $E[2N(3) - 4N(5)]$. *(ii)* $\mathrm{Var}(2N(3) - 4N(5))$. *(iii)* $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$. *(v)* $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Solution: $E(N(t)) = V(N(t)) = t\lambda$, $N(t + s) - N(s) = \mathcal{P}(t\lambda)$, $Cov(N(t + s), N(t)) = t\lambda$.
(i) $E[2N(3) - 4N(5)] = 2E[N(3)] - 4E[N(5)] = (2)(3)(2) - (4)(5)(2) = -28$.
(ii) $\mathrm{Var}(2N(3) - 4N(5)) = \mathrm{Var}((2 - 4)N(3) - 4(N(5) - N(3)))$
$= (-2)^2\mathrm{Var}(N(3)) + (-4)^2\mathrm{Var}(N(5 - 3)) = (-2)^2(2)(3) + (-4)^2(2)(2) = 88$.
(iii) $E[N(5) - 2N(6) + 3N(10)] = (5)(2) - (2)(6)(2) + (3)(10)(2) = 10 - 24 + 60 = 46$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$
$= \mathrm{Var}((1 - 2 + 3)N(5) + (-2 + 3)(N(6) - N(5)) + 3(N(10) - N(6)))$
$= \mathrm{Var}(2N(5) + (N(6) - N(5)) + 3(N(10) - N(6)))$
$= 4\mathrm{Var}(N(5)) + \mathrm{Var}(N(1)) + 9\mathrm{Var}(N(4)) = (4)(5)(2) + (1)(2) + (9)(4)(2) = 114$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10)) = \mathrm{Cov}(N(5), 3N(10)) - \mathrm{Cov}(2N(6), 3N(10))$
$= (3)(5)(2) - (6)(6)(2) = 30 - 72 = -42$.

Example 12.17.

Theorem 12.13.

Theorem 12.14. *(Markov property of the Poisson process) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $0 \leq t_1 < t_2 < \cdots < t_m < s$ and let $k_1 \leq k_2 \leq \cdots \leq k_m \leq j$. Then,* $\mathrm{P}\{N(s) = j | N(t_1) = k_1, \dots, N(t_m) = k_m\} = \mathrm{P}\{N(s) = j | N(t_m) = k_m\}$.

Previous theorem says that a Poisson process is a Markov chain with continuous time and state space $E = \{0, 1, \dots\}$ (see Definition 11.4, page 143). Previous theorem implies that for $0 \leq t_1 < t_2 < \cdots < t_m < s_1 < s_2 < \cdots s_n$ and for $k_1 \leq k_2 \leq \cdots \leq k_m \leq j_1 \leq \cdots \leq j_n$,

$$\begin{aligned}
&\mathrm{P}\{N(s_1) = j_1, \dots, N(s_n) = j_n | N(t_1) = k_1, \dots, N(t_m) = k_m\} \\
=&\mathrm{P}\{N(s_1) = j_1, \dots, N(s_n) = j_n | N(t_m) = k_m\}
\end{aligned}$$

Theorem 12.15. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $t_0 > 0$ and let $j \geq 0$. Then, the distribution of $\{N(t) - N(t_0) : t \geq t_0\}$ conditional on $N(t_0) = j$ is that of a Poisson process with rate λ. In particular, for each $t_0 < s_1 < \cdots < s_m$ and each $j \leq k_1 \leq \cdots \leq k_m$, $P\{N(s_1) = k_1, \dots, N(s_m) = k_m | N(t_0) = j\} = \prod_{i=1}^m P\{N(s_i - t_0) = k_i - j\}$.*

$$N(s+t)|(N(s) = j) = (N(s+t) - N(s) + N(s))|(N(s) = j) \sim N(s+t) - N(s) + j \sim N(t) + j$$

$$E[N(s+t)|N(s) = j] = \lambda t + j \text{ and } \mathrm{Var}(N(s+t)|N(s) = j) = \lambda t.$$

Previous theorem says that the number of occurrences from one moment on is a Poisson process. In some sense, the process starts anew at every time. Given a particular time, future occurrences from that time on follow a Poisson process with the same rate as the original process.

Example 12.18. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Compute:*
(i) $\mathrm{P}\{N(5) = 7 | N(3) = 2\}$. *(ii)* $E[2N(5) - 3N(7) | N(3) = 2)$. *(iii)* $\mathrm{Var}(N(5) | N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3)$. *(v)* $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2)$.

Solution: (i) $\mathrm{P}\{N(5) = 7 | N(3) = 2\} = \mathrm{P}(N(5) - N(3) = 7 - 2 | N(3) = 2)$
$= \mathrm{P}(N(2) = 5) = e^{-6} \frac{6^5}{5!}$.
(ii) $E[2N(5) - 3N(7) | N(3) = 2] = E[2N(5) - 3(N(7) - N(5)) - 3N(5) | N(3) = 2]$
$= E[-(N(5) - N(3)) - N(3) - 3(N(7) - N(5)) | N(3) = 2] = -2(3) - 2 - 3(2)(3) = -26$.
(iii) $\mathrm{Var}(N(5) | N(2) = 3) = \mathrm{Var}(N(5) - N(2) + 3 | N(2) = 3)$ **Why ?**
$= \mathrm{Var}(N(5) - N(2) | N(2) = 3) = \mathrm{Var}(N(3)) = (3)(3) = 9$.
(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3) = \mathrm{Var}(N(5) - N(2)) = Var(N(3)) = (3)(5 - 2) = 9$.
(v) $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2) = \mathrm{Var}(2N(5) - 3(N(7) - N(5) + N(5)) | N(3) = 2)$
$= \mathrm{Var}((2 - 3)(N(5) - N(3) + N(3)) - 3(N(7) - N(5)) | N(3) = 2)$
$= \mathrm{Var}(-(N(5) - N(3)) - 3(N(7) - N(5))) = 2(3) + 3^2(2)(3) = 60$

Next we consider conditioning on the future.

Theorem 12.16. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $s, t \geq 0$. Then,* $\mathrm{P}\{N(t) = k | N(s+t) = n\} = \binom{n}{k} \left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{n-k}$,
i.e. $N(t) | N(s+t) = n \sim bin(n, p)$, where $p = \frac{t}{t+s}$.

Proof $\mathrm{P}\{N(t)=k|N(s+t)=n\}$
$=\mathrm{P}\{N(t)=k, N(s+t)=n\}/\mathrm{P}\{N(s+t)=n\}$
$=\mathrm{P}\{N(t)=k, N(s+t)-N(t)=n-k\}/\mathrm{P}\{N(s+t)=n\}$

$$=\frac{e^{-\lambda t}\frac{(\lambda t)^k}{k!}e^{-\lambda s}\frac{(\lambda s)^{n-k}}{(n-k)!}}{e^{-\lambda(s+t)}\frac{(\lambda(s+t))^n}{n!}}=\binom{n}{k}\left(\frac{t}{t+s}\right)^k\left(\frac{s}{t+s}\right)^{n-k}$$

Previous theorem can be extended as follows, given $0\le t_1<t_2<\cdots<t_m$, the conditional distribution of $(N(t_1), N(t_2)-N(t_1),\ldots,N(t_m)-N(t_{m-1}))$ given $N(t_m)=n$ is $Multi(n,(\frac{t_1}{t_m},\frac{t_2-t_1}{t_m},\ldots,\frac{t_m-t_{m-1}}{t_m}))$. Given $N(t_m)=n$, we know that events happens in the interval $[0,t_m]$, each of these events happens independently and the probability that one of these events happens in particular interval is the fraction of the total length of this interval.

Example 12.19. *Customers arrive at a store according to a Poisson process with a rate* 40 *customers per hour. Assume that three customers arrived during the first 15 minutes. Calculate the probability that no customer arrived during the first five minutes.*

Solution: Let $N(t)$ be the number of customers arriving in the first t minutes. $N(t)$ is a Poisson process with rate 2/3. We have that

$$\mathrm{P}\{N(5)=0|N(15)=3\}=\binom{n}{x}p^x(1-p)^{n-x}=\binom{3}{0}\left(\frac{5}{15}\right)^0\left(\frac{10}{15}\right)^3=8/27.$$

Example 12.20. *Let* $\{N(t):t\ge 0\}$ *be a Poisson process with rate* $\lambda=3$. *Compute:*
(i) $E[N(1)|N(3)=2]$.
(ii) $\mathrm{Var}(N(1)|N(3)=2)$.
(iii) $E[2N(1)-3N(7)|N(3)=2]$.

Solution: (i) $bin(n,\frac{t}{t+s})$ $(bin(2,1/3))$. $E[N(1)|N(3)=2]=np=2\frac{1}{3}=2/3$.
(ii) $\mathrm{Var}(N(1)|N(3)=2)=npq=(2)\frac{1}{3}\left(1-\frac{1}{3}\right)=\frac{4}{9}$.
(iii) $E[2N(1)-3N(7)|N(3)=2]$
$=E(2N(1)|N(3)=2)-3E(N(7)-N(3)+N(3)|N(3)=2)$
$=(2)(2)(1/3)-(3)((4)(3)+2)=\frac{-122}{3}$.
For a Poisson process with rate $\lambda>0$,
(i) $N(0)=0$.
(ii) The process has independent increments.
(iii) For each $0\le s,t$, $N(s+t)-N(s)\sim Pois(\lambda t)$.
(iv) $\mathrm{Cov}(N(s),N(t))=\lambda(s\wedge t)$.
Question: $Var(N(1)-2N(2)+N(3))$
$=Var(\underbrace{(N(3)-N(2))}_{\sim N(1)}-\underbrace{(N(2)-N(1))}_{\sim N(1)})=Var(N(1)-N(1))=0$ **??**

12.3 Interarrival times.

For $n \geq 1$, let S_n be the arrival time of the n–th event, i.e. $S_n = \inf\{t \geq 0 : N(t) = n\}$.

$\{S_n \leq t\} = \{$the $n-$th occurrence happens before time $t\} =$
$\{N(t) \geq n\} = \{$there are n or more occurrences in the interval $[0,t]\}$.
$\{S_n > t\} = \{N(t) < n\}$. (1)
$\{N(t) = n\} = \{N(t) \geq n\} \cap \{N(t) < n+1\} = \{S_n \leq t\} \cap \{S_{n+1} > t\} = \{S_n \leq t < S_{n+1}\}$.

Let $X \sim Gamma(n,1)$ and $Y \sim Pois(t)$.
Abusing notation, write $X = Gamma(n,1)$ and $Y = Pois(t)$.

Theorem 12.17. $\mathrm{P}\{\mathrm{Gamma}(n,1) > t\} = \mathrm{P}\{\mathrm{Pois}(t) < n\}$, $\forall\ t \geq 0$ *and* $n \in \{1,2,...\}$.

$\mathrm{P}\{\mathrm{Gamma}(n,1) \geq t\} = \mathrm{P}\{\mathrm{Pois}(t) \leq n\}$??
$\mathrm{P}\{\mathrm{Gamma}(n,1) > t\} = \mathrm{P}\{\mathrm{Pois}(t) \leq n-1\}$??
$\mathrm{P}\{\mathrm{Gamma}(n,1) \geq t\} = \mathrm{P}\{\mathrm{Pois}(t) < n\}$??

Theorem 12.18. $S_n \sim \mathcal{G}(n, 1/\lambda)$

Proof. Several ways: Check f_{S_n}, F_{S_n}, S_{S_n} or M_{S_n}.
$\mathbb{P}\{S_n > t\} = \mathbb{P}\{N(t) < n\}$ (by Eq. (1))

$$= \underbrace{\mathrm{P}\{\mathrm{Pois}(\lambda t) < n\}}_{\text{(by Poisson Process)}} = \underbrace{\mathrm{P}\{\mathrm{Gamma}(n,1) > \lambda t\}}_{\text{(by Theorem 12.17)}} = \int_{\lambda t}^{\infty} \frac{x^{n-1}e^{-x}}{\Gamma(n)}dx$$

$= \int_t^\infty \frac{(y\lambda)^{n-1}e^{-y\lambda}}{\Gamma(n)}dy\lambda$ $\quad (x = y\lambda,\ x/\lambda \mapsto y$ and $\lambda t/\lambda \mapsto t)$
$= \int_t^\infty \frac{y^{n-1}e^{-y/\beta}}{\beta^n\Gamma(n)}dy$ $\quad (\beta = 1/\lambda)$
$= \mathrm{P}\left\{\mathrm{Gamma}\left(n, \frac{1}{\lambda}\right) > t\right\}$ ■

Theorem 12.19. $f_{S_1,\ldots,S_n}(s_1,\ldots,s_n) = \lambda^n e^{-\lambda s_n}$, if $0 < s_1 < s_2 < \cdots < s_n$.

Remark. The distribution of $(S_1,\ldots,S_{n-1})$ given $S_n = s_n$ is uniform in the region $0 < s_1 < s_2 < \cdots < s_n$, i.e.

$$\begin{aligned} f_{S_1,\ldots,S_{n-1}|S_n}(s_1,\ldots,s_{n-1}|s_n) &= \frac{f_{S_1,\ldots,S_n}(s_1,\ldots,s_n)}{f_{S_n}(s_n)} = \frac{\lambda^n e^{-\lambda s_n}}{\frac{\lambda^n s_n^{n-1}e^{-\lambda s_n}}{(n-1)!}} \\ &= \frac{(n-1)!}{s_n^{n-1}}, \text{ for } 0 < s_1 < s_2 < \cdots < s_n. \end{aligned}$$

Example 12.21. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Calculate:*
(i) $\mathrm{P}\{S_3 > 5\}$.
(iii) Find the expected value and the variance of S_3.
(iv) $\mathrm{P}\{S_2 > 3, S_5 > 7\}$.

Solution: Formulas: 13.6. $S_n \sim$ ____________key: $\mathcal{G}(n, 1/\lambda)$ $(\mathcal{G}(3, 1/3))$

13.11. $\{N(t) < n\} = \{Pois(t\lambda) < n)\} =$ ____________key: $\{S_n > t\}$.

(i) $\mathrm{P}\{S_3 > 5\} = \int_5^\infty \frac{t^{3-1}e^{-\lambda t}}{(1/\lambda)^3\Gamma(3)}dt = \mathrm{P}\{N(5) < 3\} = e^{-5\lambda}\left(1 + 5\lambda + \frac{(5\lambda)^2}{2}\right)$ **which is better ?**

(iii) $E[S_3] = \alpha\beta = 3(1/3) = 1$ and $\mathrm{Var}(S_3) = \alpha\beta^2 = 3(1/3)^2 = 1/3$.

$$
\begin{aligned}
(iv)\quad & \mathrm{P}\{S_2 > 3, S_5 > 7\} \\
=&\mathrm{P}\{N(3) < 2, N(7) < 5\} \\
=&\mathrm{P}\{N(3) = 0, N(7) \le 4\} + \mathrm{P}\{N(3) = 1, N(7) \le 4\} \\
=&\mathrm{P}\{N(3) = 0\}\mathbb{P}\{N(7) - N(3) \le 4 - 0\} + \mathrm{P}\{N(3) = 1\}\mathbb{P}\{N(7) - N(3) \le 4 - 1\} \\
=&\mathrm{P}\{N(3) = 0\}\mathbb{P}\{N(4) \le 4\} + \mathrm{P}\{N(3) = 1\}\mathbb{P}\{N(4) \le 3\} \\
=&e^{-3\lambda} \cdot \sum_{j=0}^{4} e^{-4\lambda}\frac{(4\lambda)^j}{j!} + e^{-3\lambda}(3\lambda) \cdot \sum_{j=0}^{3} e^{-4\lambda}\frac{(4\lambda)^j}{j!} \\
=&e^{-9}e^{-12}\left(1 + 12 + \frac{12^2}{2} + \frac{12^3}{6} + \frac{12^4}{24}\right) + e^{-9}(9)e^{-12}\left(1 + 12 + \frac{12^2}{2} + \frac{12^3}{6}\right) \\
\approx&3.4 \times 10^{-6}.
\end{aligned}
$$

Let $T_n = S_n - S_{n-1}$ be the time elapsed between the $(n-1)$–th and the n–th event (the **waiting time** for the next event). T_n is called the **interarrival** between the $(n-1)$–th and the n–th event.

Theorem 12.20. *T_1, ..., T_n are i.i.d. $\sim Exp(1/\lambda)$, and $E(T_n) = \frac{1}{\lambda}$.*

Theorem 12.21.

Example 12.22. *Let $\{N(t) : t \ge 0\}$ be a Poisson process with rate $\lambda = 3$. Calculate:*
(i) The density of T_6.
(ii) Find the expected value and the variance of T_6.
(iii) Find $\mathrm{Cov}(T_3, T_8)$.
(iv) Find $\mathrm{Cov}(S_2, S_9)$.

Solution: (i) $T_6 \sim Exp(1/3) => f_{T_6}(t) = 3e^{-3t}, t \ge 0$.
(ii) $E[T_6] = \alpha\beta = (1/3), \mathrm{Var}(T_6) = \alpha\beta^2 = (1/3)^2 = 1/9$.
(iii) $\mathrm{Cov}(T_3, T_8) = 0$.
(iv) $S_2 \not\perp S_9$, but $S_2 \perp S_9 - S_2$,
as $S_2 = T_1 + T_2$, $S_9 - S_2 = T_3 + T_4 + \cdots + T_9$, and T_i's are i.i.d.

$$\mathrm{Cov}(S_2, S_9) = \mathrm{Cov}(S_2, S_2 + S_9 - S_2) = \mathrm{Cov}(S_2, S_2) + \mathrm{Cov}(S_2, S_9 - S_2) = 2(1/3)^2 + 0 = 2/9.$$

Theorem 12.22. *Given that $N(t) = n$, the n arrival times $S_1, \dots, S_n$ have conditional density* $f_{S_1,\dots,S_n|N(t)=n}(s_1, \dots, s_n) = \frac{n!}{t^n}, 0 < s_1 < s_2 < \dots < s_n < t$.

It follows from the previous theorem that
$f_{T_1,\dots,T_n|N(t)=n}(t_1, \dots, t_n) = \frac{n!}{t^n}, 0 < t_1, t_2, \dots, t_n, t_1 + t_2 + \dots + t_n < t$.

Example 12.23. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Calculate: (i)* $\mathrm{P}\{S_3 > 7|N(4) = 1\}$. *(ii)* $E[S_2|N(4) = 3]$. *(iii)* $E[S_3|N(4) = 1]$. *(iv)* $\mathrm{P}\{T_3 > 5|N(4) = 1\}$.

Sol: (i) $\mathrm{P}\{S_3 > 7|N(4) = 1\}$ $\qquad S_n = \inf\{t \geq 0 : N(t) = n\}$.
$= \mathrm{P}\{N(7) < 3|N(4) = 1\}$
$= \mathrm{P}\{N(7) - N(4) < 3 - 1|N(4) = 1\}$
$= \mathrm{P}\{N(3) \leq 1\} = \mathrm{P}\{Pois(3\lambda) \leq 1\}$
$= e^{-9}(9^0/0! + 9^1/1!) = 0.001234$.
(ii) $E[S_2|N(4) = 3] = \int y f_Y(y)dy = \int g(x)f_X(x)dx$?
Formula: $f_{S_1,\dots,S_n|N(t)=n}(s_1, \dots, s_n) = \frac{n!}{t^n}, 0 < s_1 < s_2 < \dots < s_n < t$.
$E[S_2|N(4) = 3] = \int_0^4 \int_0^{s_3} \int_0^{s_2} s_2 \frac{3!}{4^3} ds_1 ds_2 ds_3$
$= \int_0^4 \int_0^{s_3} s_2^2 \frac{3!}{4^3} ds_2 ds_3$
$= \int_0^4 s_3^3 \frac{3!}{3 \cdot 4^3} ds_3$
$= 4^4 \frac{3!}{4 \cdot 3 \cdot 4^3} = 2$
(iii) $E[S_3|N(4) = 1] = \int_0^\infty \mathrm{P}(S_3 > s|N(4) = 1)ds$ $\qquad E(X) = \int_0^\infty S_X(t)dt$ if $X \geq 0$.
$\{N(4) = 1\} = \{S_1 \leq 4 < S_2 \leq S_3\}$ => $S_3 - 4 \geq 0$ if $N(4) = 1$.
$E[S_3|N(4) = 1]$
$= \int_0^4 1ds + \int_4^\infty \mathrm{P}(S_3 > s|N(4) = 1)ds$
$= 4 + \int_4^\infty \mathrm{P}(N(s) < 3|N(4) = 1)ds$
$= 4 + \int_4^\infty \mathrm{P}(N(s) - N(4) < 3 - 1|N(4) = 1)ds$
$= 4 + \int_4^\infty \mathrm{P}(N(s - 4) < 2)ds$
$= 4 + \int_4^\infty \mathrm{P}(S_2 > s - 4)ds$
$= 4 + \int_0^\infty \mathrm{P}(S_2 > t)dt$ $\qquad S_2 \sim \mathcal{G}(n, 1/\lambda)$
$= 4 + E(S_2) = 4 + \alpha\beta = 4 + 2/\lambda = 4\frac{2}{3}$.
(iv) $\mathrm{P}\{T_3 > 5|N(4) = 1\} = \mathrm{P}\{T_3 > 5|T_1 \leq 4 < T_1 + T_2\} = \mathrm{P}\{T_3 > 5\} = e^{-15}$. $\quad (T_3 = S_3 - S_2)$

Exam 3 of Math 452, Spring, 2023. Name: ___________

1. The notations are the same as in Chapter 10, with $T_x^{(\tau)} = \min_{i \in \{1,\dots,4\}} T_x^{(i)}$. That is, there are 4 exit modes: withdraw, illness, retiring normally and death in service. Assume that

   (1) 30% of the members surviving in employment to age 60 retire at 60 and

   all members who remain in employment to age 65 retire then;

   (2) $\mu_x^{01} = \mu_x^w = \begin{cases} 0.1 & \text{for } x \in [0, 35) \\ 0.05 & \text{for } x \in [35, 45) \\ 0.02 & \text{for } x \in (45, 60); \end{cases}$

   (3) $\mu_x^{02} = \mu_x^i = 0.001$, $x \in (0, 65)$;

(4) $\mu_x^{03} = \mu_x^r = \begin{cases} 0 & \text{for } x \in [0, 60) \\ 0.1 & \text{for } x \in (60, 65); \end{cases}$

(5) $\mu_x^{04} = \mu_x^d = 0.01$, $x > 0$.
(a) Calculate P(existing at age 61 |age 58) and P(existing at age 65 |age 58), *i.e.*, $P(T_{58}^{(\tau)} = 3)$ and $P(T_{58}^{(\tau)} = 7)$.
(b) For each of the last 2 modes of exit, find the probability that a member currently aged 58 exits by that mode, *i.e.*, $P(T_{58} \leq 7, J_{58} = i)$, $i = 3, 4$.
(c) Calculate the probability that **(58)** is still alive by **age 61**.

Hint: 1. The problem is almost identical to Ex. 10.5 in my notes (around p.97).

2. $P(T_x > t) = P(X > x + t | X > x) = \frac{S_X(x+t)}{S_X(x)}$.

3. If $S_X(t) = \begin{cases} 1 & if\ t < 0 \\ e^{-t} & if\ t \in [0, 5) \\ e^{-6} & if\ t \in [5, 8) \end{cases}$, then

$P(T_2 > 1) = ?$ $P(T_2 > 2) = ?$ $P(T_2 > 3) = ?$ $P(T_6 > 1) = ?$ $P(T_6 > 3) = ?$

$P(X = 0) = ?$ $P(X = 3) = ?$ $P(X = 5) = ?$ $P(X = 6) = ?$ $P(X = 8) = ?$

1. Sol. (a) P(existing at age 61|age 58)$= {}_tp_{58}^{00}\big|_{t=3-} - {}_tp_{58}^{00}\big|_{t=3} = 0$.

$$\text{P(existing at age 65|age 58)} = {}_tp_{58}^{00}\big|_{t=7-} - {}_tp_{58}^{00}\big|_{t=7} \approx 0.38.$$

Reason: $\mu_x^{00} = \mu_x^{01} + \mu_x^{02} + \mu_x^{03} + \mu_x^{04} \begin{cases} 0.1 + 0.001 + 0.01 & if\ x \in [0, 35) \\ 0.05 + 0.001 + 0.01 & if\ x \in [35, 45) \\ 0.02 + 0.001 + 0.01 & if\ x \in [45, 60) \\ 0.001 + 0.1 + 0.01 & if\ x \in (60, 65) \end{cases}$

$$= \begin{cases} 0.111 & if\ x \in [0, 35) \\ 0.061 & if\ x \in [35, 45) \\ 0.031 & if\ x \in [45, 60) \\ 0.111 & if\ x \in (60, 65) \end{cases}$$

$$S_X(x) = \begin{cases} 1 & if\ x < 0 \\ exp(-\int_0^x 0.111dt) & if\ x \in [0, 35) \\ \exp(-0.111(35) - 0.061(x - 35)) & if\ x \in [35, 45) \\ \exp(-0.111(35) - 0.061(45 - 35) - 0.031(x - 45)) & if\ x \in [45, 60) \\ \exp(-0.111(35) - 0.061(45 - 35) - 0.031(15) - 0.111(x - 60))\ ?? & if\ x \in (60, 65) \\ 0.7[\exp(-0.111(35) - 0.061(45 - 35) - 0.031(15)] \exp(-0.111(x - 60))\ ? & if\ x \in (60, 65) \\ ?? & otherwise \end{cases}$$

$$
{}_tp_{58}^{00} = \frac{S_X(58+t)}{S_X(58)} = \begin{cases} 1 & if\ t<0 \\ {}_tp_{58}^{00}\ (=?) & if\ t\in[0,2) \\ 0.7\times {}_{2-}p_{58}^{00}\underbrace{{}_tp_{60}^{00}}_{=?} & if\ t\in[2,7) \\ 0 & otherwise \end{cases}
$$

$$
\begin{aligned}
{}_tp_{58}^{00} =& \frac{S_X(58+t)}{S_X(58)} = \exp(-\int_{58}^{58+t}(0.031)dx) = \exp(-0.031t), \quad t\in[0,2), \\
{}_{2-}p_{58}^{00} =& \exp(-0.062) \\
{}_2p_{58}^{00} =& 0.7\times {}_{2-}p_{58}^{00} \\
{}_tp_{60}^{00} =& \frac{S_X(60+t)}{S_X(60)} = \exp(-\int_{60}^{60+t}(0.111)dx) = \exp(-0.111t), \quad t\in(0,5), \\
{}_{5-}p_{60}^{00} =& \exp(-5(0.111)) = \exp(-0.555), \\
{}_5p_{60}^{00} =& 0 \\
{}_7p_{58}^{00} =& {}_2p_{58}^{00}\cdot {}_5p_{60}^{00} = 0/
\end{aligned}
$$

Thus P(existing at age 61|age 58)= ${}_tp_{58}^{00}\big|_{t=3-} - {}_tp_{58}^{00}\big|_{t=3} = e^{-0.111(3-3)} = 0.$

P(existing at age 65|age 58) = ${}_tp_{58}^{00}\big|_{t=7-} - {}_tp_{58}^{00}\big|_{t=7} \approx 0.38 - 0 = 0.38.$

Using R to compute:

b=exp(-0.062)

c=exp(-0.555)

c*0.7*b # ${}_7p_{58}^{00} = 0.3776925$ **the final answer.**

The probability of (58) being existed by 65 is about 38%.

$$
\begin{aligned}
(b)\quad & P(\text{a member retires at 60 or 65 or in (60,65)})\\
=& \underbrace{0.3 \cdot {}_{2-}p_{58}^{00}}_{30\%\ retire} + \underbrace{1 \cdot {}_{7-}p_{58}^{00}}_{100\%\ retire} + \int_2^7 f_{58}^{03}(t)dt\\
=& 0.3 \cdot {}_{2-}p_{58}^{00} + 1 \cdot {}_{7-}p_{58}^{00} + \int_2^7 {}_tp_{58}^{00} \times 0.1dt\\
=& 0.3 \cdot {}_{2-}p_{58}^{00} + {}_{7-}p_{58}^{00} + 0.1{}_2p_{58}^{00} \int_0^5 \exp(-0.111t)dt\\
=& 0.3 \cdot {}_{2-}p_{58}^{00} + {}_{7-}p_{58}^{00} + 0.1{}_2p_{58}^{00} \frac{1-\exp(-0.555)}{0.111}\\
=& 0.9121128\\
& 0.3 * b + c * 0.7 * b + 0.1 * 0.7 * b * (1-c)/0.111
\end{aligned}
$$

The probability that a member currently aged 58 death exits

$$
\begin{aligned}
=& \int_0^7 {}_tp_{58}^{00} \mu_t^{04} dt\\
=& 0.01 \int_0^7 {}_tp_{58}^{00} dt \approx 0.04463816 \qquad {}_tp_{58}^{00} = \begin{cases} {}_tp_{58}^{00} & if\ t \in [0,2) \\ 0.7 \times {}_{2-}p_{58}^{00} {}_tp_{60}^{00} & if\ t \in [2,7) \end{cases}
\end{aligned}
$$

$(c)\quad {}_3p_{58}^{'04} = \exp(-\mu_x^{04}(3)) = \exp(-0.01(3)) = e^{-0.03}.$

2. XYZ offers a pension plan with the following lump sum death-in-service benefits, payable immediately on death:

(1) \$10,000 for each full year of service on death in service between ages 63 and 64.

(2) \$15,000 for each full year of service on death in service between ages 64 and 65.

You are given:

(i) Death is assumed to occur half-way through the year of age.

(ii) Decrements for this pension plan follow the Service Table.

(iii) i = 0.05

(iv) XYZ uses the Traditional Unit Credit funding method.

Calculate the normal cost for this benefit for a new employee who is age 50.

x	l_x	w_x	i_x	r_x	d_x	x	l_x	w_x	i_x	r_x	d_x
20	1000000	95104	951	0	237	44	137656	6708	134	0	95
21	903707	85846	859	0	218	45	130719	2586	129	0	100
22	816684	77670	777	0	200	46	127904	2530	127	0	106
23	738038	70190	702	0	184	47	125140	2476	124	0	113
24	666962	63430	634	0	170	48	122428	2422	121	0	121
25	602728	57321	573	0	157	49	119763	2369	118	0	130
26	544677	51800	518	0	145	50	117145	2317	116	0	140
27	492213	46811	468	0	134	51	114572	2266	113	0	151
28	444800	42301	423	0	125	52	112042	2216	111	0	163
29	401951	38226	382	0	117	53	109553	2166	108	0	176
30	363226	34543	345	0	109	54	107102	2118	106	0	190
31	328228	31215	312	0	102	55	104688	2070	103	0	206
32	296599	28201	282	0	96	56	102308	2023	101	0	224
33	268014	25488	255	0	91	57	99960	1976	99	0	243
34	242181	23031	230	0	86	58	97642	1930	96	0	264
35	218834	10665	213	0	83	59	95351	1884	94	0	288
36	207872	10131	203	0	84	60−	93085	0	0	27926	0
37	197455	9623	192	0	84	60+	65160	0	62	6188	210
38	187555	9141	183	0	85	61	58700	0	56	5573	212
39	178147	8682	174	0	86	62	52860	0	50	5018	213
40	169206	8246	165	0	87	63	47579	0	45	4515	214
41	160708	7832	157	0	89	64	42805	0	41	4061	215
42	152631	7438	149	0	90	65−	38488	0	0	38488	0
43	144954	7064	141	0	93						

Table 1. Pension plan service table

2. Sol. The normal cost is the APV of a single year's accrual (see §10.4 (around page 114):

$E(b_{T_x} v^{T_x} I(J_x = 4))$ with T_x at mid-year.

$E(10^4 v^{K_x} I(K_x = 63.5 - x) + 1.5(10^4 v^{K_x} I(K_x = 64.5 - x)$

$= 10^4 v^{63.5-x}({}_{63-x}|q_x^{04}) + 1.5(10^4) v^{64.5-x}({}_{64-x}|q_x^{04})$, where $x = 50$.

$NC = 10^4 \frac{v^{63.5-50} d_{63}^{(d)} + 1.5 v^{64.5-50} d_{64}^{(d)}}{l_{50}} = 23.02383.$

v=1/1.05

10**4*(v**(63.5-50)*214+1.5*v**(64.5-50)*215)/117145

Question: How about replacing assumptions (i) and (ii) by (5) in # 1, *i.e.* $\mu_x^{04} = 0.01$, $x > 0$?

Ans: $NC = E(b_{T_x} v^{T_x} I(J_x = 4)) = 10^4 \int_{13}^{14} v^t f_{50}^{04}(t) dt + 1.5 \times 10^4 \int_{14}^{15} v^t f_{50}^{04}(t) dt$

$= 10^4 \int_{13}^{14} v^t 0.01 {}_tp_{50}^{00} dt + 1.5 \times 10^4 \int_{14}^{15} v^t 0.01 {}_tp_{50}^{00} dt$

${}_tp_{50}^{00} = \frac{S_X(50+t)}{S_X(50)} = 0.7 \exp(-0.031(10) - 0.111(t - 60))$

as $\frac{0.7\exp(-0.111(35)-0.061(45-35)-0.031(50-45+10)-0.111(t-60))}{\exp(-0.111(35)-0.061(45-35)-0.031(50-45))}$
$= 0.7\exp(-0.031(10) - 0.111(t-60))$