Week 13.

A. Based on the model concluded in §6.9, carry out a simulation study

A.1. on a $2^{7-4}_{III}$ FD,

A.2. on a Multiple-Column foldover,

A.3. on another one similar to the economic alternative to total foldover.

In particular, use the LSE and $\hat{\sigma}^2$ from the data in §6.9, generate new data for A.1, A.2, A.3.

First use one set of the generate data from the simulation go over the analysis as in §6.9. In each of A1, A2 and A3, you need to suggest what is the optimal condition for shorterning the filtration time and whether your suggestion works.

Then carry out simulation study 100 times and do analysis on testing

$H_0$: the simplist model is the one concluded in A.1, A.2 and A.3, respectively. In your summary, you should state clearly

(a) what $H_0$ is in each case,

(b) what the test statistics is,

(c) $\alpha = ?$

(d) rejection region = ?

In the three simulation studies, use only one model.

$H_0$'s are different in each study.

In the summary in each of these simulation studies, you need to discuss,

is $H_0$ true ?

what is $H_1$ ?

what is the true model ?

$P(H_1|H_0) = ?$

$P(H_0|H_1) = ?$

Why do these phenomenona occur ?

Is it the best conclusion we can get ?

Remark. Use only one model in the three studies. Generate the table of contract a,b,c,

D=a*b

E=a*c

F=b*c

G=a*F

y=-10*a*E-8*E-4*a+rnom(8,64,?) # for $2^{7-4}_{III}$ pp Design multiple foldover yourself.

**Sol.**

A. Based on the model concluded in §6.9, carry out a simulation study

A.1. on a $2^{7-4}_{III}$ FD (the model is $Y = 66 - 5.4A - 8.3C - 11.4E + \epsilon$).

A.2. on a Multiple-Column foldover, (the model is $Y = 64 - 8A * E - 10E + \epsilon$).

A.3. on another one similar to the economic alternative to total foldover (the model is $Y = 64 - 8.6A * E - 11E + \epsilon$ with $s^2 \approx 15$ and you may use this model).

In particular, use the notation for defined in my notes for a, b, c, ..., A, B, C, ..., a, b, c, ...

A1. **Choose the round off model** $y1 = -9A * E - 11E + rnorm(8, 64, 3.8)$, where A, B, C, ..., G ($\in \{-1, 1\}$) are the 8 factors, Test

$H_o$: $y1 \sim A + C + E$, v.s. $H_1$: $Y \sim A + B + C + D + E + F$ (less G as it is very small)

A2. foldover 8 data $y2 = -9(-A) * (-E) - 11(-E) + rnorm(8, 64, 3.8)$.

$Y = (y1, y2)$ and define a, e, e*a, b, similar to the codes in the notes

$H_o$: $Y \sim e + I(e * a)$, v.s. $H_1$: $Y \sim e * a * b$

A3. Use the data in A1 and A2.

```
d=c(16,11,13,10)
a=c(A,a[d])
```

```
c=c(C,c[d])
e=c(A*C,ac[d])
d=c(8,3,5,2)          # d+8=c(16,11,13,10)
y=c(y1,y2[d])
```
$H_o$: $Y \sim e + I(e * a)$, v.s. $H_1$: $Y \sim e * a$

**Remark.** In one loop,

first generate 8 data for A1,
then another 8 data for A2 but use both 16 data from A1 and A2;
finally in A3, use 12 data, 8 from A1 and 4 from A2 (c(8,3,5,2)).

Then carry out simulation study 100 times and do analysis on testing in each loop, record rejection rate from A1, A2 and A3.

Notice that $H_0$ is A1 is actually incorrect, explain why.

B. Mimic the lecture notes in section 6.10 to form a $2^4$ FD for ABCD and the 4 duplicated $2^{4-1}$ FD's for I=BCDE.

C. p. 251. 6.3
page 257. 6.4 (in part (b) the three runs refer to 3 out of (15,12,14,9)).
Page 259. Exercise 6.5, 6.6,
Page 275-276 1, 2, 3, 7

**Future revision of the Simulation Study.**

repeat 100 times.
```
v=rep(0,4)
n=8
x=rep(2:5,2)
t=rbinom(n,2,0.5)
y=x+t*t
u=lm(y~factor(x)+t)
v[1]=summary(u)$coef[5,4]
s=summary(u)
qqnorm(s$resid)
qqline(s$resid)
y=x+t+t*t
u=lm(y~factor(x)+t)
v[2]=summary(u)$coef[5,4]
y=2*x+t*t+ rnorm(n,2,0.5)
u=lm(y~factor(x)+t)
v[3]=summary(u)$coef[5,4]
y=2*x+ rnorm(n,2,0.5)
u=lm(y~factor(x)+t)
v[4]=summary(u)$coef[5,4]
```
1. In each of the above summary(u), based on the simulation set-up and results, answer the following questions:

What is the null hypothesis about $t$ ? Is it true ? Is the test valid ?
If we do not reject $H_o$, does $H_o$ is true ?
If the qqnorm suggest NID, does it prove that NID is true ?
$P(H_1|H_0) \approx$ ? $P(H_0|H_1) \approx$ ? (Given "=" whenever possible, see Remark below).

2. In each of the lm(), what is the true model, the parameter $\beta$, its true value, the sample mean and SD of the LSE of the 100 replications ?

**Discussion.**

(1) The first study, $H_o$: $\beta_t = 0$, v.s. $H_1$: $\beta_t \neq 0$.

$H_o$ is true, the model is wrong and NID is not valid.
$P(H_o|H_1)$ is not relevent and cannot be obtained by the results of this simulation.
$P(H_1|H_o)$ can be approximated by rejection rate.
$P(H_1|H_o)$ can also be computed exactly as follows.
The test used in summary is $\mathbf{1}(|\hat\beta_t| > t_{0.025}\hat\sigma_{\hat\beta_t})$.
$t$ is a vector of $8 \times 1$,
The 1st coordiante $t_1 \in \{0,1,2\}$, $Y_1 = 2 + t_1^2$, w.p. $\binom{2}{t_1}/8$,
...,
The 4th coordiante $t_4 \in \{0,1,2\}$, $Y_4 = 5 + t_4^2$, w.p. $\binom{2}{t_4}/8$,
The 5st coordiante $t_5 \in \{0,1,2\}$, $Y_5 = 2 + t_5^2$, w.p. $\binom{2}{t_5}/8$,
...,
The 8th coordiante $t_8 \in \{0,1,2\}$, $Y_8 = 5 + t_8^2$, w.p. $\binom{2}{t_8}/8$,
Total of $3^8 = 6561$ choices.
In each case $x = (2,3,4,5)$ and $t = (t_1, ..., t_8)$
$Y_i = \beta_o + \beta_1\mathbf{1}(x=3) + \beta_2\mathbf{1}(x=4) + \beta_3\mathbf{1}(x=5) + t_i^2 - \overline{t^2}$, where $\beta_o = 2 + \overline{t^2}$ $\beta_1 = 1$, $\beta_2 = 2$ and $\beta_3 = 3$, $\beta_4 = 0$.

The design matrix is $X = \begin{pmatrix} 1 & 0 & 0 & 0 & t_1 \\ 1 & 1 & 0 & 0 & t_2 \\ 1 & 0 & 1 & 0 & t_3 \\ 1 & 0 & 0 & 1 & t_4 \\ 1 & 0 & 0 & 0 & t_5 \\ 1 & 1 & 0 & 0 & t_6 \\ 1 & 0 & 1 & 0 & t_7 \\ 1 & 0 & 0 & 1 & t_8 \end{pmatrix}$

$\hat\sigma^2_{\hat\beta_t} = s_{55}$, where $(s_{ij})_{5\times 5} = \hat\sigma^2(X'X)^{-1}$ and $\hat\sigma^2 = \sum_{i=1}^n (Y_i - \hat\beta X_i)^2/(n-5)$ and $n = 8$.

$$P(H_1|H_o) = \sum_{x,t} P(\mathbf{1}(|\hat\beta_t| > t_{0.025}\hat\sigma_{\hat\beta_t}).$$

(2) The second study, $H_o$: $\beta_t = 0$, v.s. $H_1$: $\beta_t \neq 0$.
$H_o$ is false, the model is wrong and NID is not valid.
$P(H_1|H_o)$ can not be approximated by the results of this simulation.
$P(H_o|H_1)$ can be approximated by 1-rejection rate and can also be computed exactly.
The test used in summary is $\mathbf{1}(|\hat\beta_t| > t_{0.025}\hat\sigma_{\hat\beta_t})$.
$t$ is a vector of $8 \times 1$,
The 1st coordiante $t_1 \in \{0,1,2\}$, $Y_1 = 2 + t_1 + t_1^2$, w.p. $\binom{2}{t_1}/8$,
...,
The 4th coordiante $t_4 \in \{0,1,2\}$, $Y_4 = 5 + t_4 + t_4^2$, w.p. $\binom{2}{t_4}/8$,
The 5st coordiante $t_5 \in \{0,1,2\}$, $Y_5 = 2 + t_5 + t_5^2$, w.p. $\binom{2}{t_5}/8$,
...,
The 8th coordiante $t_8 \in \{0,1,2\}$, $Y_8 = 5 + t_8 + t_8^2$, w.p. $\binom{2}{t_8}/8$,
Total of $3^8 = 6561$ choices.
In each case $x = (2,3,4,5)$ and $t = (t_1, ..., t_8)$
$Y_i = \beta_o + \beta_1\mathbf{1}(x=3) + \beta_2\mathbf{1}(x=4) + \beta_3\mathbf{1}(x=5) + t_i^2 - \overline{t^2}$, where $\beta_o = 2 + \overline{t^2}$, $\beta_1 = 1$, $\beta_2 = 2$ and $\beta_3 = 3$, $\beta_4 = 1$.

The design matrix is $X = \begin{pmatrix} 1 & 0 & 0 & 0 & t_1 \\ 1 & 1 & 0 & 0 & t_2 \\ 1 & 0 & 1 & 0 & t_3 \\ 1 & 0 & 0 & 1 & t_4 \\ 1 & 0 & 0 & 0 & t_5 \\ 1 & 1 & 0 & 0 & t_6 \\ 1 & 0 & 1 & 0 & t_7 \\ 1 & 0 & 0 & 1 & t_8 \end{pmatrix}$

$\hat{\sigma}^2_{\hat{\beta}_t} = s_{55}$, where $(s_{ij})_{5\times 5} = \hat{\sigma}^2(X'X)^{-1}$ and $\hat{\sigma}^2 = \sum_{i=1}^n (Y_i - \hat{\beta}X_i)^2/(n-5)$ and $n = 8$.

$$P(H_o|H_1) = \sum_{x,t} P(\mathbf{1}(|\hat{\beta}_t| \le t_{0.025}\hat{\sigma}_{\hat{\beta}_t}).$$

**Important question :** $P(H_1|H_o) = 1 - P(H_o|H_1)$ ?
A typical counterexample is under NID assumption $N(\mu, 1)$,

$H_o$: $\mu = 0$ v.s. $H_1$: $\mu > 0$, with the test $\mathbf{1}(\frac{\overline{X}}{1/\sqrt{n}} > z_\alpha)$.

$P(H_1|H_o) = \alpha$,

$P(H_o|H_1) = \Phi(\frac{z_\alpha/\sqrt{n}-\mu}{1/\sqrt{n}}) \to 0 < \alpha$ if $n \to \infty$ and $\mu > 0$.

Week 11. Excercise 6.4. (b) Can you see how the problem could be resolved with just three runs (foldover) ?

**Sol.** Recall $2^{7-4}_{III}$ design yields

| *run #* | *a* | *b* | *c* | *ab* | *ac* | *bc* | *abc* | *y* |
|---|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | |
| 1 | − | − | − | + | + | + | − | 68.4 |
| 2 | + | − | − | − | − | + | + | 77.7 |
| 3 | − | + | − | − | + | − | + | 66.4 |
| 4 | + | + | − | + | − | − | − | 81.0 |
| 5 | − | − | + | + | − | − | + | 78.6 |
| 6 | + | − | + | − | + | − | − | 41.2 |
| 7 | − | + | + | − | − | + | − | 68.7 |
| 8 | + | + | + | + | + | + | + | 38.7 |

The estimates suggest that the causes are factors A, C and E. To further investigate, another 8 runs foldover were made.

| *run #* | $-a$ | $-b$ | $-c$ | $-ab$ | $-ac$ | $-bc$ | $-abc$ | *y* |
|---|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | |
| 9 | + | + | + | − | − | − | + | 66.7 |
| 10 | − | + | + | + | + | − | − | 65.0 |
| 11 | + | − | + | + | − | + | − | 86.4 |
| 12 | − | − | + | − | + | + | + | 61.9 |
| 13 | + | + | − | − | + | + | − | 47.8 |
| 14 | − | + | − | + | − | + | + | 59.0 |
| 15 | + | − | − | + | + | − | + | 42.6 |
| 16 | − | − | − | − | − | − | − | 67.6 |

It suggests that the main causes are E and AE, moreover C is a noise.

We can consider $2^3$ FD with factor A, C and E. with 4 additional runs rather than 8 in all foldover.

| *run #* | *a* | *b* | *c* | *ab* | *ac* | *bc* | *abc* | *y* |
|---|---|---|---|---|---|---|---|---|
| | *A* | | *C* | | *E* | | | |
| 16 | − | | − | | − | | | 67.6 |
| 2 | + | | − | | − | | | 77.7 |
| 4 | + | | − | | − | | | 81.0 |
| 5 | − | | + | | − | | | 78.6 |
| 7 | − | | + | | − | | | 68.7 |
| 11 | + | | + | | − | | | 86.4 |
| 1 | − | | − | | + | | | 68.4 |
| 3 | − | | − | | + | | | 66.4 |
| 13 | + | | − | | + | | | 47.8 |
| 10 | − | | + | | + | | | 65.0 |
| 6 | + | | + | | + | | | 41.2 |
| 8 | + | | + | | + | | | 38.7 |

Since the first 8 runs of $2^{7-4}$ design suggest that the causes are factors A, C and E (AC). Choosing 3 out of runs 16,11,10,8, we can estimate 7 effects rather than 8 effects, as A, C and E are vector of $7 \times 1$. They can generate at most 7 linearly independent vectors. By

lm(y A*C*E)

It seems that it is fine to delete any one. The output and code are in 556.out