Contents

1. Axioms of probability: (1) $P(A) \geq$ ____, (2) $P(S) =$ ___,

2. $P(A \cap B) =$ ______. If A and B are ____________ then $P(A \cap B) = P(A)P(B)$.

3. $P($____$)=1 - P(A)$. $P(A \cup B) = P(A) + P(B)$ ____________.

   If A and B are ___________ then $P(A \cup B) = P(A) + P(B)$.

4. $X \sim bin(n,p)$: $f(i) =$ ____________ if $i \in \{0, ..., n\}$, $\mu =$ _____, $\sigma^2 =$ _____, where $q = 1-p$

5. $X \sim Pois(\lambda)$. $f(i) =$ ____________ if $i \geq 0$. $\mu =$ ______, $\sigma^2 =$ ______

6. $Y = g(X)$. $E(g(X)) = \begin{cases} \sum_y y f_Y(y) & dis \\ \int y f_Y(y) dy & cts \end{cases} = \begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & dis \\ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & cts \end{cases}$, $\mu_Y =$ ____, $\sigma_Y^2 =$ ____

7. The mgf of X is $M(t) =$ _________, $\frac{d^k M(t)}{dt^k}\Big|_{t=0} =$ ____________

8. A cdf $F(t)$ (= _________) satisfying (1) $F(-\infty) =$ ____, and $F(\infty) =$ ____, (2) $F(x+) =$ ____,

   (3) $F(x)$ ____ . Moreover, $F(b) - F(a) = P($________$)$

9. $F(t) = \begin{cases} \sum_{x \leq t} \_\_\_\_\_\_ & dis \\ \\ \int_{-\infty}^{t} \_\_\_\_\_\_\_\_\_\_ & cts \end{cases}$, $f(t) = \begin{cases} \_\_\_\_\_\_\_\_\_\_ & dis \\ \\ \_\_\_\_\_\_ & cts \end{cases}$

10. $E(aX + b) =$ ____________. $Var(aX + b) =$ ____________

11. $X \sim N(\mu, \sigma^2)$. $f(x) =$ ________________, $\frac{X-\mu}{\sigma} \sim$ ____________

12. $X \sim \mathcal{G}(\alpha, \beta)$. $f(x) =$ ____________ if $x >$ _, $\mu =$ _____, $\sigma^2 =$ _____, $\Gamma(\alpha + 1) =$ ______

13. $Exp(\lambda) =$ ____________, $\chi^2(\nu) =$ ____________,

14. $f_X(x) = \begin{cases} \int f(x,y) \_\_\_\_\_ & cts \\ \\ \_\_\_\_\_ f(x,y) & dis \end{cases}$ 15. $E(g(X,Y)) = \begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & cts \\ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & dis \end{cases}$

16. $f_{X|Y}(x|y) =$ ____________, $F_{X|Y}(x|y) = P$ (____________)

17. $E(c) =$ ____, $E(ag(X,Y) + bh(X,Y)) =$ ________________,

18. $Cov(X,Y) =$ ______________, $V(aX + bY) =$ _________________________, $\rho(X,Y) =$ _____

19. $E(X|Y = y) = \begin{cases} \rule{3cm}{0.4pt} & cts \\ \rule{3cm}{0.4pt} & dis \end{cases}$. $E(E(X|Y)) =$ ______, $E(V(X|Y)) + V(E(X|Y)) =$ ______.

20. $U = h(Y)$, where h is _____________ and Y is cts. $f_U(u) = f_Y($______$) \cdot$ ______

21. Let Y_1, ..., Y_n be a random sample of Y. $\overline{Y} =$ ______________, $S^2 = S_Y^2 =$ ______________, $\mu_{\overline{Y}} =$ ______________, and $\sigma^2_{\overline{Y}} =$ ______________.

22. $F_{\overline{Y}}(t)$ ____$\Phi($__________$)$, where $\Phi(t)$ is the cdf of ________.

44. If X_1___X_2.

X_i's $\sim$:	$X_1 + X_2 \sim$:
$G(\alpha_i, \beta)$	__________
$\chi^2(v_i)$	__________
$Pois(\lambda_i)$	__________
$N(\mu_i, \sigma_i^2)$	__________
$bin(n_i, p)$	__________

Typos of the textbook

p.58_{12}. When $n = 1$ and $m = j \ge 1 \to$ When $n = 1$ and $m = j - 1 \ge 0$
p.91_{2-3}. $S_X(x) = e^{-t\theta I(t>0)}$, then the MLE of $S_X(x)$ is $\hat{S}_X(x)$ ......
$\to S_X(t) = e^{-t\theta I(t>0)}$, then the MLE of $S_X(t)$ is $\hat{S}_X(t)$ ......
92.

$$\check{F}(t) = \begin{cases} 0 & \text{if } t < t_1, \\ \hat{F}(t_i) & t \in \{t_1, ..., t_m\}, \\ s\hat{F}(t_i) + (1-s)\hat{F}(t_{i+1}) & \text{if } t = st_i + (1-s)t_{i+1},\ s \in (0,1), \\ & i \in \{1, ..., m-1\}, \\ 1 & \text{if } t > t_m, \end{cases} \tag{1}$$

with d.f. $\check{f}(t) = \frac{n_i}{n(t_i - t_{i-1})}$ if $t \in (t_{i-1}, t_i)$, $i \in \{1, ..., m-1\}$.

$$\to \check{F}(t) = \begin{cases} 0 & \text{if } t < 0 = t_0, \\ \hat{F}(t_i) & t \in \{t_0, ..., t_m\}, \\ s\hat{F}(t_i) + (1-s)\hat{F}(t_{i+1}) & \text{if } t = st_i + (1-s)t_{i+1},\ s \in (0,1),\ \ i \in \{0, ..., m-1\}, \\ 1 & \text{if } t > t_m, \end{cases}$$

with d.f. $\check{f}(t) = \frac{n_i}{n(t_i - t_{i-1})}$ if $t \in (t_{i-1}, t_i)$, $i \in \{1, ..., m\}$.

p.102. 84. $e_{x:\overline{n+m}|} = e_{x:\overline{n}|} + {}_np_x e_{x+m:\overline{m}|}$. $\to e_{x:\overline{n+m}|} = e_{x:\overline{n}|} + {}_np_x e_{x+n:\overline{m}|}$.
p.108.#5. (iv) $s(x) = \frac{8190 - x - x^2}{8190} \to s(x) = \frac{8190 + x - x^2}{8190}$
p.109.#8. $e^{-t\prod_{j=1}^n \frac{1}{\lambda_j}} \to e^{-t\sum_{j=1}^n \frac{1}{\lambda_j}}$
p.109 #11. Change all 25 to 30 in the expressions. That is,
$0.18 = {}_{40|15}q_{10} = {}_{30}p_{10} \cdot {}_{10|15}q_{40} = {}_{30}p_{10}(0.24)$. So, ${}_{30}p_{10} = \frac{0.18}{0.24} = 0.75$.

p.125. #89. (v) ..... $E[(T(x))^2] = \frac{2}{\mu^2}$
$\to$ (v) ..... $E[(T(x))^2] = \frac{2}{\mu^2}$ if μ is not random,

$$= E\left[\frac{2}{\mu^2}\right] = \int_0^\infty \frac{1}{\mu^2} 400\mu e^{-20\mu}\, dt$$
$$= \int_0^\infty \frac{400}{\mu} e^{-20\mu}\, d\mu = \infty,$$
$$\to$$
$$= E\left[\frac{2}{\mu^2}\right] = \int_0^\infty \frac{2}{\mu^2} 400\mu e^{-20\mu}\, d\mu$$
$$= \int_0^\infty \frac{800}{\mu} e^{-20\mu}\, d\mu = \infty,$$

p.112[7]. $= 0.58780194 = 0.0544332806 \to = 0.0544332806$.

p.127[1]. $4e^{-x/4} \to \frac{1}{4}e^{-x/4}$

p.113. #31. $0.8 = \frac{s(30)-s(40)}{s(20)-s(40)}$. $\to 0.8 = \frac{s(40)-s(50)}{s(30)-s(50)}$.
$_{30}p_{10} = \frac{0.9-0.8}{1-0.8} = 0.5$. $\to {}_{20}p_{30} = \frac{0.9-0.8}{1-0.8} = 0.5$.

$$_{10}p_{30}\ {}_{30}p_{40} = {}_{60}p_{10} = {}_{30}p_{10}\ {}_{30}p_{40} \quad \to \quad {}_{10}p_{30}\ {}_{30}p_{40} = {}_{40}p_{30} = {}_{20}p_{30}\ {}_{20}p_{50},$$

$(0.5)_{40}p_{40} = (0.9)(0.03)$ and $_{40}p_{40} = \frac{(0.9)(0.03)}{0.5} = 0.054$.
$\to (0.5)_{20}p_{50} = (0.9)(0.03)$ and $_{20}p_{50} = \frac{(0.9)(0.03)}{0.5} = 0.054$.

p.175 #2. $\frac{\ell_{30}-\ell_{50}}{\ell_{21}} \to \frac{\ell_{30}-\ell_{50}}{\ell_{20}}$

p.182 #28. (ii) $\frac{L_{95}+L_{95}+L_{97}}{\ell_{95}} \to \frac{L_{95}+L_{96}+L_{97}}{\ell_{95}}$

p.186 #44. (iii) $= \frac{\ell_{[40]+1}}{\ell_{[40]}} + \frac{\ell_{[40]+2}}{\ell_{[40]}} + \frac{\ell_{40+3}}{\ell_{[44]}} \to$
$= \frac{\ell_{[40]+1}}{\ell_{[40]}} + \frac{\ell_{[40]+2}}{\ell_{[40]}} + \frac{\ell_{40+3}}{\ell_{[40]}}$

p.187 #45. (iii) $= 2.9787449570368 + \frac{\left(\frac{75578}{77252}\right)^{0.5}-1}{\ln(75578/77252)} = 3.47601648226597 \to$
$= 2.9787449570368 + \frac{77252}{80625}\frac{\left(\frac{75578}{77252}\right)^{0.5}-1}{\ln(75578/77252)} = 3.47601648226597$

p.189 #7. $\mathring{e}_{1:\overline{1.5}|} = \mathring{e}_{1:\overline{1.5}|} + \to \mathring{e}_{1:\overline{1.5}|} = \mathring{e}_{1:\overline{1}|} +$

p.295 #5. (ii) $\mathbb{P}\{K_x \geq 2\} = \mathbb{P}\{T_x > 1\} = \frac{490}{100} = 0.49$. $\to$

$$\mathbb{P}\{K_x \geq 2\} = \mathbb{P}\{T_x > 1\} = \frac{490}{1000} = 0.49.$$

(iii) $E[((80000)^2(Z_{96})^2] = (80000)^2 \cdot {}^2A_{96} = (80000)^2 \sum_{k=1}^{\infty} v^k \frac{\ell_{96+k-1}-\ell_{96+k}}{\ell_{80}} \to E[((80000)^2(Z_{96})^2] = (80000)^2 \cdot {}^2A_{96} = (80000)^2 \sum_{k=1}^{\infty} v^{2k} \frac{\ell_{96+k-1}-\ell_{96+k}}{\ell_{96}}$

Notations in 450: $k, i, n \geq 0$ and $0 \leq m \leq n$. $P(X \geq 0) = 1$ and $x, t > 0$. $S_X(x) = s(x) = \mathbb{P}\{X > x\}$,

1. If $H(0) = 0$ and $\underline{H'} \geq 0$, then $\underline{E(H(X))} = \int_0^\infty s(t)H'(t)\,dt$, *e.g.*, $\underline{E(X)} = \int_0^\infty s(t)\,dt$, $\underline{E(X^p)} = \int_0^\infty s(t)pt^{p-1}\,dt$, $E[X \wedge a] = \int_0^a s(t)\,dt$.
2. If $P(X \in \{0, 1, 2, ...\}) = \underline{1}$ and $\underline{H \uparrow}$, then $E[\underline{H(X)}] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(H(k) - H(k-1))$, $E[\underline{X}] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}$, $E[\underline{X^2}] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(2k-1)$, $E[\min(X, a)] = \sum_{\underline{k=1}}^{a} \mathbb{P}\{X \geq k\}$.
3. $T(x) = T_x = \underline{(X-x)|(X > x)}$, $\underline{{}_tp_x} = S_{T(x)}(t) = \frac{s(x+t)}{s(x)}$, $\underline{{}_tq_x} = F_{T(x)}(t) = \frac{s(x)-s(x+t)}{s(x)}$, $\underline{{}_{s|t}q_x} = \mathbb{P}\{s < T(x) \leq s+t\} = {}_sp_x \cdot {}_tq_{x+s}$, $\underline{{}_{s|}q_x} = {}_{s|1}q_x$, $\underline{p_x} = {}_1p_x$, $\underline{q_x} = {}_1q_x$,
4. $\underline{{}_{m+n}p_x} = {}_mp_x \cdot {}_np_{x+m}$, $\underline{{}_np_x} = p_xp_{x+1}\cdots p_{x+n-1}$, $\underline{{}_{\sum_{j=1}^k n_j}p_x} = {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}$.
5. The force of mortality is $\mu_X(x) = \mu(x) = \mu_x = \underline{\frac{f_X(x)}{S_X(x-)}}$. $\mu_{T(x)}(t) = \underline{\mu_x(t)}$. If X is cts, $\underline{\mu(x)} = -\frac{d}{dx}\ln S_X(x)$, $\underline{S_X(x)} = \exp\left(-\int_0^x \mu(t)\,dt\right)$, $\underline{f_{T(x)}(t)} = {}_tp_x\mu(x+t)$. $\mu_x(t) = \underline{\mu(x+t)}$.
6. $\underline{\mathring{e}_x} = E[T(x)] = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$, $\underline{\mathring{e}_{x:\overline{n}|}} = E[T(x) \wedge n] = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$.
7. The central rate of failure on $(x, x+n]$ is $\underline{{}_nm_x} = \frac{\int_x^{x+n} S_X(t)\mu_X(t)\,dt}{\int_x^{x+n} S_X(t)\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$, $\underline{m_x} = {}_1m_x$, $\underline{{}_na(x)} = E(T(x)|T(x) \leq n) = \frac{\mathring{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$, $\underline{a(x)} = {}_1a(x)$.
8. $\underline{K_x} = \lceil T(x) \rceil$, $\lceil t \rceil = \underline{k}$ if $t \in (k-1, k]$, $K(x) = K_x\underline{-1}$, $\underline{f_{K_x}(k)} = {}_{k-1|}q_x = {}_{k-1}p_x \cdot q_{x+k-1} = (\prod_{j\geq 0}^{k-2} p_{x+j})q_{x+k-1}$
9. $\underline{e_x} = E[K(x)] = p_x(1 + e_{x+1}) = \sum_{k=1}^\infty {}_kp_x = e_{x:\overline{n}|} + {}_np_xe_{x+n}$, $\underline{e_{x:\overline{n}|}} = E(K(x) \wedge n) = \sum_{k=1}^n {}_kp_x$, $\underline{E[(K(x))^2]} = \sum_{k=1}^\infty (2k-1) \cdot {}_kp_x$.
10. The KME $\hat{S}_{pl}(t) = \prod_{t_k \leq t}\underline{(1 - \frac{d_k}{r_k})}$, and $\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \underline{\frac{1}{n}(\hat{S}_{pl}(t))^2} \sum_{k:\ t_k \leq t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}$.

    The Nelson-Aalen estimator: $\tilde{S}_{NA}(t) = \underline{e^{-H(t)}}$, where $H(t) = \sum_{t_k \leq t} \frac{d_k}{r_k}$. $\hat{\sigma}^2_{\tilde{S}_{NA}(t)} = (\tilde{S}_{NA}(t))^2\hat{\sigma}^2_{H(t)}$, where $\hat{\sigma}^2_{H(t)} = \underline{\sum_{t_j \leq t} \frac{(r_j - d_j)d_j}{(r_j-1)r_j^2}}$.
11. $\underline{\ell_x} = \#$ of individuals alive at age x, $\underline{{}_1L_x} = L_x$. ${}_td_x = \underline{\ell_x - \ell_{x+t}}$, $d_x = \underline{{}_1d_x}$, $\underline{{}_tp_x} = \prod_{x \leq k < x+t}(1 - d_k/l_k)$. $\underline{T_x} = \ell_x\mathring{e}_x = \int_0^\infty \ell_{x+t}\,dt = \sum_{k=x}^\infty L_k$ = E(# of years lived beyong age x by the cohort group with l_0 members), $\underline{{}_nL_x} = \ell_x\mathring{e}_{x:\overline{n}|} = T_x - T_{x+n}$. $s(x) = \underline{\frac{\ell_x}{\ell_0}}$, ${}_tp_x = \underline{\frac{\ell_{x+t}}{\ell_x}}$, ($T_x$ in #3 differs from T_x in #11).
12.

Interpolation	ℓ_{x+t}	${}_tp_x$
UDD exponential Balducci		

where _____.

key:

UDD	$(1-t)\ell_x + t\ell_{x+1}$ or $\ell_x - td_x$	$1 - tq_x$
exponential	$(\ell_x)^{1-t}(\ell_{x+1})^t$ or $\ell_xp_x^t$	p_x^t
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}}$	$\frac{p_x}{t+(1-t)p_x}$

, $\underline{t \in [0, 1]}$.

14. **Life Insurance:** $Z = b_{T_x} v_{T_x}$,
Whole life ins: $Z_x = v^{K_x}$, $\overline{Z}_x = v^{T_x}$, $A_x = A_x(v) = E[v^{K_x}] = vq_x + vp_x A_{x+1}$, ${}^2A_x = E[Z_x^2] = A_x(v^2)$.

n–year term : $Z^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n)$, $\overline{Z}^1_{x:\overline{n}|} = v^{T_x} I(T_x \le n)$, $A^1_{x:\overline{n}|} = E[Z^1_{x:\overline{n}|}] = \sum_{k=1}^n v^k f_{K_x}(k) = vq_x + vp_x A^1_{x+1:\overline{n-1}|}$, ${}^2A^1_{x:\overline{n}|} = E((Z^1_{x:\overline{n}|})^2) = A^1_{x:\overline{n}|}(v^2)$.

n–year deferred : ${}_n|Z_x = v^{K_x} I(n < K_x)$, ${}_n|\overline{Z}_x = v^{T_x} I(n < T_x)$, ${}^2{}_n|A_x = {}_n|A_x\ (v^2)$, ${}_n|A_x = E[{}_n|Z_x] = \sum_{k=n+1}^\infty v^k f_{K_x}(k) = v^{n+1} f_{K_x}(n+1) + {}_{n+1}|A_x = vp_x \cdot {}_{n-1}|A_{x+1}$.

n–year pure endowment : $Z_{x:\overline{n}|}^{\ \ 1} = v^n I(n < K_x)$, $\overline{Z}_{x:\overline{n}|}^{\ \ 1} = v^n I(n < T_x)$, $A_{x:\overline{n}|}^{\ \ 1} = {}_nE_x = v^n {}_np_x$, ${}^2A_{x:\overline{n}|}^{\ \ 1} = E((Z_{x:\overline{n}|}^{\ \ 1})^2) = A_{x:\overline{n}|}^{\ \ 1}(v^2)$,

n–year endowment : $Z_{x:\overline{n}|} = v^{K_x \wedge n}$, $\overline{Z}_{x:\overline{n}|} = v^{T_x \wedge n}$, $A_{x:\overline{n}|} = \sum_{k=1}^n v^k f_{K_x}(k) + v^n {}_np_x$, ${}^2A_{x:\overline{n}|} = E((Z_{x:\overline{n}|})^2) = A_{x:\overline{n}|}(v^2)$,

m–year defer n-year term : ${}_m|_nZ_x = v^{K_x} I(m < K_x \le n+m)$, ${}_m|_n\overline{Z}_x = v^{T_x} I(m < T_x \le n+m)$, ${}_m|_nA_x = {}_mE_x A^1_{x+m:\overline{n}|}$, ${}^2{}_m|_nA_{x:\overline{n}|} = E[({}_m|_nZ_x)^2] = \sum_{k=m+1}^{m+n} v^{2k} f_{K_x}(k)$.

$Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x$, $\quad Z^1_{x:\overline{n}|} \cdot {}_n|Z_x = 0$,
$Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, $Z^1_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ 1} = 0$,
${}_n|A_x = {}_nE_x A_{x+n}$, $\quad A_x = A^1_{x:\overline{n}|} + {}_nE_x A_{x+n}$.
15. $(IZ)_x = K_x v^{K_x}$. $(I\overline{Z})_x = T_x v^{T_x}$. $(I\overline{Z})_x = \lceil T_x \rceil v^{T_x}$. $(DZ)^1_{x:\overline{n}|} = (n+1-K_x) v^{K_x} I(K_x \le n)$. $(\overline{DZ})^1_{x:\overline{n}|} = (n - T_x) v^{T_x} I(T_x \le n)$. $(D\overline{Z})^1_{x:\overline{n}|} = \lceil n - T_x \rceil v^{T_x} I(T_x \le n)$.
16. $v = \frac{1}{i+1}$, $\delta = -\ln v$, $d = 1 - v$. $\sum_{k=1}^n kx^{k-1} = (\frac{1-x^{n+1}}{1-x})'_x$

$a_{\overline{n}|} = \sum_{k=1}^n v^k = v(\frac{1-v^n}{1-v})$

17.

due	PV $\quad \ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v}$	APV
whole life	$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = \frac{1-Z_x}{1-v}$	$\ddot{a}_x = \sum_{k=0}^\infty v^k {}_kp_x$
n–y. def.	${}_n|\ddot{Y}_x = \sum_{k\ge n}^{K_x-1} v^k = \frac{v^n - v^{K_x}}{1-v} I(K_x > n)$	${}_n|\ddot{a}_x = \sum_{k=n}^\infty v^k {}_kp_x$
n–y. tem.	$\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{(K_x \wedge n)-1} v^k = \frac{1-Z_{x:\overline{n}|}}{d}$	$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$
n–y. cer.	$\ddot{Y}_{\overline{x:\overline{n}|}} = \sum_{k=0}^{(n \vee K_x)-1} v^k = \ddot{a}_{\overline{n}|} + {}_n|\ddot{Y}_x$	$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_n|\ddot{a}_x$

$i \ne 0$.

immediate: $Y_x = \sum_{k\ge 1}^{K_x-1} v^k = \ddot{Y}_x - 1$, $\quad {}_n|Y_x = \sum_{k>n}^{K_x-1} v^k = {}_{n+1}|\ddot{Y}_x$,
$Y_{x:\overline{n}|} = \sum_{k=1}^{(K_x-1)\wedge n} v^k = \ddot{Y}_{x:\overline{n+1}|} - 1$, $\quad Y_{\overline{x:\overline{n}|}} = \sum_{k=1}^{(K_x-1)\vee n} v^k = \ddot{Y}_{\overline{x:\overline{n+1}|}} - 1$

cts	present value$=\overline{a}_{\overline{n}|} = \underline{\int_0^n v^t dt}$	APV
whole life	$\overline{Y}_x = \int_0^{T_x} v^t dt = \underline{\frac{1-\overline{Z}_x}{\delta}}$	$\overline{a}_x = \underline{\int_0^\infty v^t {}_tp_x dt}$
n–y. def.	${}_n\vert\overline{Y}_x = \int_n^{T_x} v^t dt \underline{I(T_x > n)}$	${}_n\vert\overline{a}_x = \underline{\int_n^\infty v^t {}_tp_x dt}$
n–y. tem.	$\overline{Y}_{x:\overline{n}|} = \int_0^{T_x\wedge n} v^t dt = \underline{\frac{1-v^{T_x\wedge n}}{\delta}}$	$\overline{a}_{x:\overline{n}|} = \underline{\int_0^n v^t {}_tp_x dt}$
n–y. cer.	$\overline{Y}_{\overline{x:\overline{n}|}} = \int_0^{T_x\vee n} v^t dt = \underline{\overline{a}_{\overline{n}|} + {}_n\vert\overline{Y}_x}$	$a_{\overline{x:\overline{n}|}} = \underline{\overline{a}_{\overline{n}|} + {}_n\vert\overline{a}_x}$

18. $\underline{\ddot{Y}_x} = \ddot{Y}_{x:\overline{n}|} + {}_n\vert\ddot{Y}_x$, $\quad E((\ddot{Y}_x)^2)\underline{\neq}\ddot{a}_x(v^2)$, $\ddot{a}_x = \underline{1 + vp_x\ddot{a}_{x+1}}$. $\quad \overline{a}_x = \underline{\overline{a}_{x:\overline{n}|} + v^n {}_np_x\overline{a}_{x+n}}$.
${}_n\vert\ddot{a}_x = {}_nE_x\ddot{a}_{x+n} = \underline{vp_x \cdot {}_{n-1}\vert\ddot{a}_{x+1}}$. $\quad {}_n\vert\overline{a}_x = {}_nE_x \cdot \overline{a}_{x+n} = \underline{vp_x \cdot {}_{n-1}\vert\overline{a}_{x+1}}$.
$\ddot{a}_{x:\overline{n+m}|} = \ddot{a}_{x:\overline{n}|}\underline{+{}_nE_x \cdot \ddot{a}_{x+n:\overline{m}|}}$.

19.

Plan	Loss
Whole life insurance	$\underline{Z_x - P\ddot{Y}_x}$
t–year funded whole life insurance	$\underline{Z_x - P\ddot{Y}_{x:\overline{t}|}}$
n–year term insurance	$\underline{Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}}$
t–year funded n–year term insurance	$\underline{Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}}$
n–year pure endowment insurance	$\underline{Z_{x:\overline{n}|}^{\ \ 1} - P\ddot{Y}_{x:\overline{n}|}}$
t–year funded n–year pure endowment insurance	$\underline{Z_{x:\overline{n}|}^{\ \ 1} - P\ddot{Y}_{x:\overline{t}|}}$
n–year endowment	$\underline{Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}}$
t–year funded n–year endowment insurance	$\underline{Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}}|}$
n–year deferred insurance	$\underline{{}_n\vert Z_x - P\ddot{Y}_x}$
t–year funded n–year deferred insurance	$\underline{{}_n\vert Z_x - P\ddot{Y}_{x:\overline{t}|}}$

Loss in the fully discrete case

13. **Application of Woolhouse's formula:**

$$\overline{a}_x \approx \underline{\ddot{a}_x - \frac{1}{2}} - \frac{1}{12}(\delta + \mu_x) \approx \underline{\ddot{a}_x^{(m)} - \frac{1}{2m}} - \frac{1}{12m^2}(\delta + \mu_x)$$

Notations in 450: $k, i, n \geq 0$ and $0 \leq m \leq n$. $P(X \geq 0) = 1$ and $x, t > 0$. $S_X(x) = s(x) = \mathbb{P}\{X > x\}$,

1. If $H(0) = 0$ and _____ ≥ 0, then __________ $= \int_0^\infty s(t)H'(t)\,dt$, *e.g.*,

   __________ $= \int_0^\infty s(t)\,dt$, __________ $= \int_0^\infty s(t)pt^{p-1}\,dt$, $E[X \wedge a] = \int_0^{\_\_\_} s(t)\,dt$.

2. If $P(X \in \{0, 1, 2, ...\}) =$ _____ and _____, then

   $E[$_____$] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(H(k) - H(k-1))$, $E[$_____$] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}$,

   $E[$_____$] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(2k-1)$, $E[\min(X, a)] = \sum_{\_\_\_}^{\_\_\_} \mathbb{P}\{X \geq k\}$.

3. $T(x) = T_x =$ _______________, ___________ $= S_{T(x)}(t) = \frac{s(x+t)}{s(x)}$, _____ $= F_{T(x)}(t) = \frac{s(x)-s(x+t)}{s(x)}$,

   __________ $= \mathbb{P}\{s < T(x) \leq s+t\} = {}_sp_x \cdot {}_tq_{x+s}$, _____ $= {}_s|_1q_x$, _____ $= {}_1p_x$, _____ $= {}_1q_x$,

4. __________ $= {}_mp_x \cdot {}_np_{x+m}$, __________ $= p_xp_{x+1} \cdots p_{x+n-1}$,

   __________ $= {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}$.

5. The force of mortality is $\mu_X(x) = \mu(x) = \mu_x =$ __________. $\mu_{T(x)}(t) =$ ________

   If X is cts, _____ $= -\frac{d}{dx}\ln S_X(x)$, _____ $= \exp\left(-\int_0^x \mu(t)\,dt\right)$, _____ $= {}_tp_x\mu(x+t)$. $\mu_x(t) =$ _____

6. __________ $= E[T(x)] = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$, __________ $= E[T(x) \wedge n] = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$.

7. The central rate of failure on $(x, x+n]$ is ________ $= \frac{\int_x^{x+n} S_X(t)\mu_X(t)\,dt}{\int_x^{x+n} S_X(t)\,dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$,

   ________ $= {}_1m_x$, ________ $= E(T(x)|T(x) \leq n) = \frac{\mathring{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$, ________ $= {}_1a(x)$.

8. _____ $= \lceil T(x) \rceil$, $\lceil t \rceil =$ _____ if $t \in (k-1, k]$, $K(x) = K_x$_____,

   __________ $= {}_{k-1}|q_x = {}_{k-1}p_x \cdot q_{x+k-1} = (\prod_{j \geq 0}^{k-2} p_{x+j})q_{x+k-1}$

9. ________ $= E[K(x)] = p_x(1 + e_{x+1}) = \sum_{k=1}^\infty {}_kp_x = e_{x:\overline{n}|} + {}_np_xe_{x+n}$,

$\underline{\hspace{2cm}} = E(K(x) \wedge n) = \sum_{k=1}^{n} {}_k p_x$, $\underline{\hspace{2.5cm}} = \sum_{k=1}^{\infty}(2k-1) \cdot {}_k p_x$.

10. The KME $\hat{S}_{pl}(t) = \prod_{t_k \le t}(\underline{\hspace{2cm}})$, and $\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \underline{\hspace{1.5cm}} \sum_{k:\ t_k \le t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}$.

The Nelson-Aalen estimator: $\tilde{S}_{NA}(t) = \underline{\hspace{2.5cm}}$, where $H(t) = \sum_{t_k \le t} \frac{d_k}{r_k}$.

$\hat{\sigma}^2_{\tilde{S}_{NA}(t)} = (\tilde{S}_{NA}(t))^2 \hat{\sigma}^2_{H(t)}$, where $\hat{\sigma}^2_{H(t)} = \underline{\hspace{2.5cm}}$.

11. $\underline{\hspace{2.5cm}}$ = # **of individuals alive** at age x, $\underline{\hspace{2.5cm}} = L_x$,

${}_t d_x = \underline{\hspace{2.5cm}}$, $d_x = \underline{\hspace{2.5cm}}$, $s(x) = \underline{\hspace{2.5cm}}$, ${}_t p_x = \underline{\hspace{1.5cm}}$,

$\underline{\hspace{1.5cm}} = \prod_{x \le k < x+t}(1 - d_k/l_k)$.

$\underline{\hspace{2cm}} = \int_0^{\infty} \ell_{x+t}\, dt = \sum_{k \ge x} L_k$,

= E(# of years lived beyong age x by the cohort group with l_0 members),

$\underline{\hspace{2cm}} = \ell_x \mathring{e}_{x:\overline{n}|} = T_x - T_{x+n}$, ($T_x$ in #3 differs from T_x in #11)

12.

Interpolation	ℓ_{x+t}	${}_t p_x$
UDD		
exponential		
Balducci		

, where $\underline{\hspace{1cm}}$.

13. **Application of Woolhouse's formula:**

$$\overline{a}_x \approx \underline{\hspace{3cm}} - \frac{1}{12}(\delta + \mu(x)) \approx \underline{\hspace{3cm}} - \frac{1}{12m^2}(\delta + \mu(x))$$

14. Life Insurance $Z = \underline{\hspace{2cm}}$

Whole life ins: $Z_x = \underline{\hspace{2cm}}$, $\overline{Z}_x = \underline{\hspace{2cm}}$, $A_x = A_x(v) = \underline{\hspace{2cm}} = \underline{\hspace{2.5cm}}$,

${}^2A_x = E[Z_x^2] = \underline{\hspace{2cm}}$,

n–year term : $Z^1_{x:\overline{n}|} = \underline{\hspace{2cm}}$, $\overline{Z}^1_{x:\overline{n}|} = \underline{\hspace{2cm}}$, $A^1_{x:\overline{n}|} = E[Z^1_{x:\overline{n}|}] = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}$,

${}^2A^1_{x:\overline{n}|} = E((Z^1_{x:\overline{n}|})^2) = \underline{\hspace{2cm}}$,

n–year deferred : $\quad {}_{n}|Z_x = \underline{\qquad\qquad}, \; {}_{n}|\overline{Z}_x = \underline{\qquad\qquad},$

${}^2{}_{n}|A_x = {}_{n}|A_x \; \underline{\qquad}, \; {}_{n}|A_x = E[{}_{n}|Z_x] = \underline{\qquad\quad} \; v^k f_{K_x}(k) = \underline{\qquad\qquad\qquad\qquad} = vp_x \cdot {}_{n-1}|A_{x+1}.$

n–year pure endowment : $\quad Z_{x:\overline{n}|}^{\;\;1} = \underline{\qquad\quad}, \; \overline{Z}_{x:\overline{n}|}^{\;\;1} = \underline{\qquad\qquad},$
$A_{x:\overline{n}|}^{\;\;1} = {}_{n}E_x = \underline{\qquad\qquad}, \; {}^2A_{x:\overline{n}|}^{\;\;1} = E((Z_{x:\overline{n}|}^{\;\;1})^2) = A_{x:\overline{n}|}^{\;\;1}\underline{\qquad\quad},$

n–year endowment : $\quad Z_{x:\overline{n}|} = \underline{\qquad\quad}, \; \overline{Z}_{x:\overline{n}|} = \underline{\qquad\quad}, \; A_{x:\overline{n}|} = \underline{\qquad\qquad\qquad\qquad},$

${}^2A_{x:\overline{n}|} = E((Z_{x:\overline{n}|})^2) = A_{x:\overline{n}|}\underline{\qquad},$
m–year defer n-year term : ${}_{m}|{}_{n}Z_x = \underline{\qquad\qquad\qquad\qquad\qquad}, \; {}_{m}|{}_{n}\overline{Z}_x = \underline{\qquad\qquad\qquad\qquad\qquad},$

${}_{m}|{}_{n}A_x = {}_{m}E_x\underline{\qquad\qquad\qquad}, \quad {}^2{}_{m}|{}_{n}A_{x:\overline{n}|} = E[({}_{m}|{}_{n}Z_x)^2] = \underline{\qquad\qquad} \; v^{2k} f_{K_x}(k),$

$\underline{\qquad\quad} = Z^1_{x:\overline{n}|} + {}_{n}|Z_x, \quad Z^1_{x:\overline{n}|} \cdot {}_{n}|Z_x = \underline{\qquad}, \quad \underline{\qquad\qquad} = Z^1_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\;\;1}, \; Z^1_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\;\;1} = \underline{\qquad},$

$\underline{\qquad\qquad} = {}_{n}E_x A_{x+n}, \quad \underline{\qquad\quad} = A^1_{x:\overline{n}|} + {}_{n}E_x A_{x+n}.$

15. $(IZ)_x = \underline{\qquad}. \; (\overline{IZ})_x = \underline{\qquad}. \; (I\overline{Z})_x = \underline{\qquad}. \; (DZ)^1_{x:\overline{n}|} = \underline{\qquad\qquad\qquad\qquad}.$

$(\overline{DZ})^1_{x:\overline{n}|} = \underline{\qquad\qquad\qquad\qquad}. \; (D\overline{Z})^1_{x:\overline{n}|} = \underline{\qquad\qquad\qquad\qquad}.$

16. $\underline{\qquad} = \frac{1}{i+1}, \; \underline{\qquad} = -\ln v, \; \underline{\qquad} = 1 - v. \; \sum_{k=1}^n kx^{k-1} = \underline{\qquad\quad} \; a_{\overline{n}|} = \sum_{k=1}^n v^k = \underline{\qquad}$

17.

due	PV $\quad \ddot{a}_{\overline{n}|} = \underline{\qquad\quad} = \underline{\qquad\quad}$	APV
whole life	$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = \underline{\qquad\qquad}$	$\ddot{a}_x = \underline{\qquad\qquad}$
n–y. def.	${}_{n}|\ddot{Y}_x = \underline{\qquad\qquad} = \frac{v^n - v^{K_x}}{1-v}\underline{\qquad\qquad}$	${}_{n}|\ddot{a}_x = \underline{\qquad\qquad}$
n–y. temp.	$\ddot{Y}_{x:\overline{n}|} = \underline{\qquad\qquad} = \frac{1-Z_{x:\overline{n}|}}{d}$	$\ddot{a}_{x:\overline{n}|} = \underline{\qquad\qquad}$
n–y. cer.	$\ddot{Y}_{\overline{x:\overline{n}|}} = \underline{\qquad\qquad} = \ddot{a}_{\overline{n}|} + {}_{n}|\ddot{Y}_x$	$\ddot{a}_{\overline{x:\overline{n}|}} = \underline{\qquad\qquad}$

$i \neq 0.$

immediate: $Y_x =$ __________, ${}_n|Y_x =$ __________, $Y_{x:\overline{n}|} =$ ___________, $Y_{\overline{x:\overline{n}|}} =$ ___________

cts	present value=$\overline{a}_{\overline{n}|} =$ ___________	APV
whole life	$\overline{Y}_x = \int_0^{T_x} v^t dt =$ ___________	$\overline{a}_x =$ _______________dt
n–y. def.	${}_n|\overline{Y}_x = \int_n^{T_x} v^t dt$___________	${}_n|\overline{a}_x =$ _______________dt
n–y. temp.	$\overline{Y}_{x:\overline{n}|} = \int_0^{T_x \wedge n} v^t dt =$ ___________	$\overline{a}_{x:\overline{n}|} =$ _______________dt
n–y. cer.	$\overline{Y}_{\overline{x:\overline{n}|}} = \int_0^{n \vee T_x} v^t dt =$ ___________	$a_{\overline{x:\overline{n}|}} =$ _______________

18. ____ $= \ddot{Y}_{x:\overline{n}|} + {}_n|\ddot{Y}_x$. $E((\ddot{Y}_x)^2)$____$\ddot{a}_x(v^2)$. $\ddot{a}_x =$ _______________.

$\overline{a}_x =$ _______________. ${}_n|\ddot{a}_x = {}_nE_x \ddot{a}_{x+n} =$ _______________.

${}_n|\overline{a}_x = {}_nE_x \cdot \overline{a}_{x+n} =$ _______________. $\ddot{a}_{x:\overline{n+m}|} = \ddot{a}_{x:\overline{n}|}$_______________.

19. Loss in the fully discrete case

Plan	Loss	Loss
Whole life insurance	____________	
t–year funded whole life insurance		____________
n–year term insurance	____________	
t–year funded n–year term insurance		____________
n–year pure endowment insurance	____________	
t–year funded n–year pure endowment insurance		____________
n–year endowment	____________	
t–year funded n–year endowment insurance		____________
n–year deferred insurance	____________	
t–year funded n–year deferred insurance		____________

CHAPTER 1
MATH 450, Syllabus

Syllabus for Math 450:

The material will focus on Long-Term Actuarial Mathematics Exam.
(1) Various Life insurance such as whole life, n–year term, n–year deferred, n–year endowment or pure endowment, and m–year deferred n-year term life insurances.
(2) Various Life annuities such as (due, immediate and continuous), whole life, n–year deferred, n–year termporary, n–year certan annuities with level or non-level payments.
(3) Premiums calculation for insurance and annuity using equivalence principle,
(4) Present value random variable (rv) and the random variables involves in (1), (2) and (3), including loss-at-issue r.v. for premiums. Compute their probability, means, variance, percentiles, force of mortality and central rate of failure with changes in mortality and interest, under select and ultimate survival models, parametric model and tabular model, select and ultimate mortality table, or using approximation methods such as UUD, constant force, Woolhouse and Euler, Kaplan-Meier estimator and Nelson-Aalen estimator.

MATH 450, ACUTUARIAL MATHEMATICS I

The course is a preparation for Long-Term Actuarial Mathematics Exam.

MWF 2:20 - 3:20 CW-115

T 1:15 - 2:40 CW-107

Professor: Qiqing Yu

Office: WH 132

Office hours: 3:30-4:30pm (M and Tu),

Textbook: Arcones' Manual For SOA Exam MLC (First Volumn).

(Chapters to be covered: 2-6)

A pdf file with some tables needed in the homework can be downloaded from my website. http://www.math.binghamton.edu/qyu/qyu_personal
e.g, the Illustrative Life Table needed in some of the homework problems.

Exams: 3 tests + final,

Sept. 24 (Tu), Oct. 29 (Tu), Dec. 3 (Tu), Dec. 9 Mon. 3:15pm-5:15pm CW 202 closed book

You can bring a calculator **without the function of installing formulas**.

Quizes: once a week, at random; **this week is on Friday**

Homework: Due Wednesday in class, no late homework.
Homework due this Friday: Do the final exam of 447 in my website **including Part A !**

Grading Policy:

1. 10% hw +10% quiz +45% tests +35% final

2. Correction: If you make correction and hand in with the old exam, **within 3 days** after I return the test **in class**, you can get 40% of the missing grades back. No partial credit for correction.

3. A- = 85 + and C = 60 +.

Student Attendance in Class: The Bulletin states, Students are expected to attend all scheduled classes, laboratories and discussions. Instructors may establish their own attendance criteria for a course. They may establish both the number of absences permitted to receive credit for the course and the number of absences after which the final grade may be adjusted downward. In such cases it is expected that the instructor stipulate such requirements in the syllabus and that the syllabus be made available to students at or near the beginning of classes. In the absence of such statements, instructors have the right to deny a student the privilege of taking the final examination or of receiving credit for the course, or may prescribe other academic penalties **if the student misses more than 25 percent of the total class sessions.** Excessive tardiness may count as absence.

10+10+45*(0.3+0.4*0.7)+35*0.3=56

CHAPTER 2
Survival models

2.1 Survival models.

2.1.1 A short probability review.

Definition 2.1. *Given a set* Ω, *a* **probability** P *on* Ω *is a function defined on the collection of all events (subsets) of* Ω *such that*

(i) $\mathrm{P}(\emptyset) = 0$;

(ii) $\mathrm{P}(\Omega) = 1$;

(iii) If $\{A_n\}_{n=1}^{\infty}$ *are disjoint events, then* $\mathrm{P}\{\cup_{n=1}^{\infty} A_n\} = \sum_{n=1}^{\infty} \mathrm{P}\{A_n\}$.

Ω is called the **sample space**.

Definition 2.2. *A* **random variable** *(r.v.)* X *is a function from* Ω *into* $\mathbb{R}$.

Definition 2.3. *The* **cumulative distribution function** *(cdf) of the r.v.* X *is* $F_X(x) = P\{X \le x\}$, $x \in \mathbb{R}$.

If X is the age at the death (or failure) of a life, then $X > 0$.

Theorem 2.1. F_X *is a cdf iff*

(i) $F_X \uparrow$, *i.e., for each* $x_1 \le x_2$, $F_X(x_1) \le F_X(x_2)$.

(ii) F_X *is right continuous (cts)* $(\lim_{h \to 0+} F_X(x+h) = F_X(x)\ \forall\ x)$

(or $F(x+) = F(x)\ \forall\ x$*).*

(iii) $\lim_{x \to -\infty} F_X(x) = 0$ *and* $\lim_{x \to \infty} F_X(x) = 1$.

For the c.d.f. of an age–at–failure, we only need to define it for $x > 0$ **Why ??**

Theorem 2.2.

Definition 2.4. *A r.v.* X *is called* **discrete**
if there is a countable set $C \subset \mathbb{R}$ *such that* $\mathrm{P}\{X \in C\} = 1$.

Meaning of countable set C ?

Ans. C is either a finite set or $C = \{c_i : i = 1, ..., \infty\}$.

Definition 2.5. *The probability mass function (or frequency function) (pmf) of the discrete r.v.* X *is the function* $p : \mathbb{R} \to \mathbb{R}$ *defined by*
$p(x) = \mathrm{P}\{X = x\},\ x \in \mathbb{R}$.

If X is a discrete r.v. with pmf p and $A \subset \mathbb{R}$, then
$\mathrm{P}\{X \in A\} = \sum_{x \in A} \mathrm{P}\{X = x\} = \sum_{x \in A} p(x)$.

Theorem 2.3. *$p(x) \geq 0 \ \forall \ x$ and $\sum_x p(x) = 1$.*

Definition 2.6. *A r.v. X is called* **cts** *if there exists a nonnegative function f called the probability density function (pdf) of X such that $\forall \ A \subset \mathbb{R}$,*
$\mathrm{P}\{X \in A\} = \int_A f(x)\,dx$ *($= \int_{x\in A} f(x)\,dx = \int I(x \in A) f(x)\,dx$), where $I(x \in A) = 1$ if $x \in A$.*

Theorem 2.4. *$f(x) \geq 0 \ \forall \ x$ and $\int f(x)\,dx = 1$ ($\int f(x)\,dx = \int_{-\infty}^{\infty} f(x)dx$).*

Q: If a r.v. X is positive and cts and $x < 0$, **then** $f_X(x) =$ **??**

Theorem 2.5.

Definition 2.7. *A r.v. X has a* **mixed distribution** *if there are functions $f(\cdot)$ and $p(\cdot)$, and a countable set D such that*
for each $A \subset \mathbb{R}$, $\mathrm{P}\{X \in A\} = \int_A f(x)\,dx + \sum_{x\in A\cap D} p(x)$.

Remark. For convenience, we shall also call both the pmf and pdf the density function (df) hereafter.

A mixed distribution X has two parts: a cts part and a discrete part (together with D). The function f in the previous definition is the cts part of df and the function p is the discrete part of df. Note that
$\int f(x)\,dx + \sum_{x\in D} p(x) = 1$, $f(x) \geq 0$ and $p(x) \geq 0 \ \forall \ x$.
Moreover, we may use $f(x)$ rather than $f(x)$ and $p(x)$.
For the mixed distribution, one can let $f(x) = p(x)$ if $x \in D$, then

$\mathrm{P}\{X \in A\} = \int_A f(x)\,dx + \sum_{x\in A\cap D} f(x)$.

$\int f(x)\,dx + \sum_{x\in D} f(x) = 1$.

Why ?

$\int_A x dx = ?$ if $A = \{0, 1, 2\}$

$$\int_{(0,1)} x dx \neq \int_{[0,1]} x dx \ ?$$

Theorem 2.6. *Let X be a r.v. with a mixed distribution. Then*
$p(x) = F_X(x) - F_X(x-)$ and
$f(x) = F_X'(x)$ if $F'(x)$ exists. **What happens OW ?**

Example 2.1.

Example 2.2.

Example 2.3.

Example 2.4.

Example 2.5. *Find D, f and p if* $F(x) = \begin{cases} 0 & \text{if } \ x < 0, \\ \frac{x+2}{8} & \text{if } 0 \le x < 1, \\ \frac{3x^2+4}{16} & \text{if } 1 \le x < 2, \\ 1 & \text{if } \ 2 \le x. \end{cases}$

Solution: How to find D ?
$F(x) - F(x-) = 0$ except, perhaps at $\{0, 1, 2\}$ **Why ?**
$p(0) = F(0) - F(0-) = \frac{2}{8} - 0 = \frac{1}{4}$,
$p(1) = F(1) - F(1-) = \frac{7}{16} - \frac{3}{8} = \frac{1}{16}$,
$p(2) = F(2) - F(2-) = 1 - \frac{16}{16} = 0$.
$p(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, \\ \frac{1}{16} & \text{if } x = 1. \end{cases}$ $\qquad D = \{0, 1, 2\}$ or $\{0, 1\}$?
$f(x) = F'(x) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 1, \\ \frac{3x}{8} & \text{if } 1 < x < 2. \end{cases}$ $\qquad$ **What happens OW ?**

Why not $0 \le x < 1$?
Is p a pmf ?
Is f a df ?

Abusing notations, one can write $f(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0 \\ \frac{1}{8} & \text{if } 0 < x < 1 \\ \frac{1}{16} & \text{if } x = 1 \\ \frac{3x}{8} & \text{if } 1 < x < 2 \end{cases}$ and $D = \{0, 1\}$.

Example 2.5 (continued). $P(0 \le X \le 1) =?$

Sol. Two ways:
(1) $P(a \le X \le b) = F(b) - F(a-)$
(2) $P(a \le X \le b) = \int_{x \in [a,b]} f(x)dx + \sum_{x \in [a,b] \cap D} f(x)$.

Answer: (1) $P(0 \le X \le 1) = \frac{1+2}{8} - 0$ or $\frac{3+4}{16} - 0$?
(2) $P(0 \le X \le 1) = \int_0^1 \frac{1}{8} dx$ **??**

2.1.2 Survival function.

Definition 2.8. *The* **survival function** *of a r.v. X is $S_X = 1 - F_X$.*

$S_X(x) = \mathbb{P}\{X > x\}$, $x \in \mathbb{R}$. Sometimes denote $S_X(t)$ by $s(t)$.

Most of time, we only consider S_X of an age-at-death X. Then $P(X > 0) = 1$,

$S_X(-1) =$ **??**
We often suppress the phrase "$S_X(t) = 1$ for $t < 0$"
and only define S_X on $[0, \infty)$.

Theorem 2.7. *A function $S_X : (-\infty, \infty) \to \mathbb{R}$ is the survival function of a positive r.v.*

$$X \text{ iff } \begin{cases} (i)\ S_X(x)\ \textbf{?} & (F_X(x)\uparrow) \\ (ii)\ S_X\ \textbf{is ??} & (F_X \text{ is right cts}). \\ (iii)\ S_X(0) = \textbf{??} & (\lim_{x\to-\infty} F_X(x) = 0). \\ (iv)\ \lim\limits_{x\to\infty} S_X(x) = \textbf{??} & (\lim\limits_{x\to\infty} F_X(x) = 1). \end{cases}$$

Example 2.6. *Determine which of the following functions is a survival function of a non-negative r.v.:*

(i) $s(x) = \frac{2}{x+2}$, for $x \geq 0$.
(ii) $s(x) = (1-x)e^{-x}$, for $x \geq 0$.
(iii) $s(x) = \frac{1+\frac{2}{x+2}}{2}$, for $x \geq 0$.
(iv) $s(x) = (1+x)e^{-x}$, for $x \geq 0$.
(v) $s(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \leq x < 90, \\ 0 & \text{for } x \geq 90. \end{cases}$

Solution: (i) s is a survival function **why ??**
$(\frac{2}{x+2})' = \frac{-2}{(x+2)^2} < 0$ on $[0,\infty) => ?$
$\frac{2}{x+2}$ is continuos on $(-2,\infty) => ?$
$s(0) = ?$
$s(\infty) = ?$
(ii) $s(x) = (1-x)e^{-x}$ is not a survival function because
(1) $s(2) = -e^{-2} < 0 = s(\infty)$.
(2) $s'(x) = -e^{-x} - (1-x)e^{-x} = e^{-x}(-2+x) = 1$ if $x = 3$;
$s(x)$ is not a non-increasing function;
Do we need to point out both ?
(iii) $s(x) = \frac{1+\frac{2}{x+2}}{2}$ is not a survival function because
$\lim_{x\to\infty} \frac{1+\frac{2}{x+2}}{2} = \frac{1}{2} \neq 0$.
(iv) s is a survival function (**why ?)**
(v) s is a survival function (**why ?**)

Example 2.7. *Find the density function for the following survival functions:*
(i) $s(x) = (1+x)e^{-x}$, *for* $x \geq 0$.
(ii) $s(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \leq x < 90, \\ 0 & \text{for } 90 \leq x. \end{cases}$
(iii) $s(x) = \frac{2}{x+2}$, *for* $x \geq 0$.

Solution: The df is $\begin{cases} f(x) = -S'(x) & \text{if } S'(x) \text{ exists} \\ p(x) = S(x-) - S(x) & \textit{otherwise.} \end{cases}$ **Why ?**

(i) $f_X(x) = xe^{-x}$, for $x > 0$.

(ii) The df is $f(x) = \begin{cases} \frac{2x}{10{,}000} & \text{for } 0 < x < 90 \\ 1 - \frac{8100}{10^4} & if\ x = 90 \end{cases}$ with $D = \{90\}$.

(iii) $f_X(x) = \frac{2}{(x+2)^2}$, for $x > 0$.

1. **Does it matter to write $f(x)$ v.s. $f_X(x)$?**

2. Notice the difference between two $\{$.

Example 2.8. *Let the survival function of a person be $S_X(x) = \frac{90-x}{90}$, for $0 \le x \le 90$.*

(i) Find the probability that a person dies before reaching 20 years old.

(ii) Find the probability that a person lives more than 60 years.

Solution: $P(a < X \le b) = F(b) - F(a) = S(a) - S(b) = \int_a^b f(x)dx + \sum_{x \in D \cap (a,b]} f(x)$.

(i) $\text{P}\{X < 20\} = 1 - S_X(20-) = 1 - \frac{90-20}{90} = \frac{2}{9}$.

(ii) $\text{P}\{X > 60\} = S_X(60) = \frac{90-60}{90} = \frac{1}{3}$.

2.1.3 Expectation.

Definition 2.9. *If X is discrete r.v. then*
$E[X] = \sum_x x p_X(x)$ *(if the series converges).*

Definition 2.10. *If X is cts r.v. then*
$E[X] = \int x f_X(x)\, dx$ *(if the integral exists).*

Definition 2.11. *If X is a mixed r.v. then*
$E[X] = \sum_x x p_X(x) + \int x f_X(x)\, dx$ *(if the series and the integral is finite).*

$E[X]$ is called the **expectation** of the r.v. X. $E[X]$ is also called, the **expected value** of X, or the **mean** of X.

Definition 2.12. *Given a set $A \subseteq \mathbb{R}$, the* **indicator function** *of A is the function $I(A) =$*
$I(x \in A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Given a r.v. X and a function $g : \mathbb{R} \to \mathbb{R}$, $Y = g(X)$ is another r.v., i.e. by composing the functions $X : S \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we get the r.v. $g(X) : S \to \mathbb{R}$.

Formula: $E(Y) = E(g(X))$

$$= \begin{cases} \sum_x g(x) p_X(x) & \text{if } X \text{ is discrete} \\ \int g(x) f_X(x) dx & \text{if } X \text{ is cts} \\ \sum_x g(x) p_X(x) + \int g(x) f_X(x) dx & \text{if } X \text{ is mixed} \\ (\sum_{x \in D} g(x) f_X(x) + \int g(x) f_X(x) dx & \text{if } X \text{ is mixed)} \end{cases}$$

$$= \begin{cases} \sum_x x p_Y(x) & \text{if } Y \text{ is discrete} \\ \int x f_Y(x) dx & \text{if } Y \text{ is cts} \\ \sum_x x p_Y(x) + \int x f_Y(x) dx & \text{if } Y \text{ is mixed} \\ (\sum_{x \in D} x f_Y(x) + \int x f_Y(x) dx & \text{if } Y \text{ is mixed)} \end{cases}$$

Q: What is the difference between these two ?

Ans. (1) $g(x)$ vs x; (2) f_X vs f_Y.

Often, to find expectations, instead of the density we will use the survival function. We will often use the following theorem:

Theorem 2.8. *Let X be a nonnegative r.v. with survival function s. Let $h : [0, \infty) \to [0, \infty)$ be a function which is integrable in bounded intervals. Let $H(x) = \int_0^x h(t)\, dt$, $x \geq 0$. Then, $E[H(X)] = \int_0^\infty s(t)h(t)\, dt$.*

Proof. $E[H(X)] = E\left[\int_0^\infty I(X > t)h(t)\, dt\right]$ as $H(x) = \int_0^\infty I(x > t)h(t)\, dt$,
$= \int_0^\infty E[I(X > t)]h(t)\, dt$
$= \int_0^\infty s(t)h(t)\, dt.$ ∎

In some sense Theorem 2.8 says

$$E[H(X)] = \int_0^\infty s(t)H'(t)\, dt, \ \underline{\textbf{where } H' \geq 0 \textbf{ and } H(0) = 0}.$$

Th 2.8 applies to functions H where H' exsits on $[0, \infty) \setminus D$, where D is countable.

4 corollaries to be proved later, assuming $P(X \geq 0) = 1$, letting $X \wedge a = \min\{X, a\}$:

where $H' \geq 0$ and $H(0) = 0$

1. $E(X^p) = \int_0^\infty pt^{p-1} S_X(t)dt$, where $p > 0$. $\quad (x^p)' = px^{p-1}$ *and* $x^p\big|_{x=0} = 0$
2. $E(X) = \int_0^\infty S_X(t)dt$. $\quad (x)' = 1$ *and* $x^1\big|_{x=0} = 0$
3. $E(X^2) = \int_0^\infty 2t S_X(t)dt$. $\quad (x^2)' = 2x^1$ *and* $x^2\big|_{x=0} = 0$
4. $E(X \wedge a) = \int_0^a S_X(t)dt$, where $a > 0$. $\quad (x \wedge a)' =?$ *and* $(x \wedge a)\big|_{x=0} = 0$

$$(x \wedge a)' = x' \wedge a'??$$

$$(x \wedge a)' = \begin{cases} x' & \text{if } x \in (0, a) \\ a' & \text{if } x \in (a, \infty) \end{cases} = \begin{cases} 1 & \text{if } x \in (0, a) \\ 0 & \text{if } x \in (a, \infty), \end{cases} \text{ and } (x \wedge a)\big|_{x=0} = 0.$$

Example 2.9. *Let $S_X(x) = s(x) = e^{-x}(x + 1)$, $x \geq 0$. $E(X) =$? $E(X \wedge 10) =$?*

Solution: Three approaches for $E(H(X))$ for a cts r.v. $H(X)$:

$E(H(X)) = \int x f_{H(X)}(x)dx = \int H(x) f_X(x)dx = \int s(x)h(x)dx.$

$f(x) = -s'(x) = -e^{-x}(-1)(x + 1) - e^{-x}(1) = e^{-x}x$ **is it done ?**

How many approaches for $E(X)$?

(i) $E[X] = \int xf(x)\,dx = \int_0^\infty x^2 e^{-x}\,dx = 2\int_0^\infty \frac{x^{3-1}e^{-x}}{\Gamma(3)1^3}dx = 2.$

(ii) $E[X] = \int_0^\infty s(t)\,dt = \int_0^\infty e^{-t}(t+1)\,dt = \int_0^\infty te^{-t}\,dt + \int_0^\infty e^{-t}\,dt$

$= \int_0^\infty \frac{te^{-t}}{\Gamma(2)1^2}\,dt + 1 = 2.$

How many approaches for $E(X \wedge 10)$?

Two convenient approaches for $E(X \wedge 10)$:

$E[\min(X,10)] = \int_0^{10} s(t)dt = \int_0^\infty \min(x,10) f_X(x)\,dx.$

$$
\begin{aligned}
E[\min(X,10)] &= \int_0^{10} s(t)\,dt = \int_0^{10} e^{-t}(t+1)\,dt \\
&= -\int_0^{10} (t+1)\,de^{-t} \\
&= -\left[(t+1)e^{-t}\Big|_0^{10} - \int_0^{10} e^{-t}d(t+1)\right] \\
&= -\left[11e^{-10} - 1 + e^{-t}\Big|_0^{10}\right] \\
&= 2 - 12e^{-10}.
\end{aligned}
$$

$$
\begin{aligned}
E[\min(X,10)] &= \int_0^\infty (t\wedge 10) f(t)\,dt \\
&= \int_0^{10} tf(t)dt + \int_{10}^\infty 10f(t)dt \\
&= \int_0^{10} te^{-t}tdt + \int_{10}^\infty 10e^{-t}tdt = \cdots
\end{aligned}
$$

Corollary 2.1. *Let X be a nonnegative r.v. and $a \geq 0$. Then, $E[\min(X,a)] = \int_0^a S_X(t)\,dt$.*

Proof. Let $H(t) = min\{t,a\}$ for each $t \geq 0$.

$h(t) = H' = I(t \in (0,a))$ if $t \in (0,\infty)\setminus\{a\}$. H' does not exist at $\{0,a\}$. Notice that $H(x) = \min(x,a) = \begin{cases} x & \text{if } x < a, \\ a & \text{if } a \leq x. \end{cases}$ is cts in $[0,\infty)$ and ctsly differentiable in $(0,a)\cup(a,\infty)$, but not at a and 0.

Check the condition in Th 2.8 directly. For $x \geq 0$,

(1) $h(x) = I(x \in (0,a)) \geq 0$ and

(2) $H(x) = \int_0^x h(t)\,dt = \int_0^x I(t \in [0,a])\,dt = \int_0^{\min(x,a)} 1\,dt = \min(x,a).$

By Theorem 2.8, $E[\min(X,a)] = \int_0^a s(t)\,dt.$ ■

Corollary 2.2. *Let X be a nonnegative r.v.. Then, $E[X] = \int_0^\infty S_X(t)\,dt$.*

Corollary 2.3. *Let X be a nonnegative r.v.. Then, $E[X^p] = \int_0^\infty S_X(t)pt^{p-1}\,dt$ if $p > 0$.*

Proof. Let $H(t) = t^p$, for each $t \geq 0$. Hence, $h(t) = H' = pt^{p-1} \geq 0$, and $H(0) = 0$. By Theorem 2.8, $E[X^p] = \int_0^\infty s(t)pt^{p-1}\,dt.$ ■

Question: $E[X^2] =?$

Corollary 2.4. *Let X be a nonnegative r.v. with survival function s. Let $\delta > 0$. Then,* $E[e^{-\delta X}] = 1 - \int_0^\infty \delta e^{-\delta t} s(t)\, dt$.

Proof. Q: Can we try $E[e^{-\delta X}] = \int_0^\infty (e^{-\delta t})' s(t)\, dt$ **??**
Why not ??
$H'(x) = -\delta e^{-\delta x} < 0$ **!!**
We shall show $E[1 - e^{-\delta X}] = \int_0^\infty \delta e^{-\delta t} s(t)\, dt$ **Why ??**
Let $H(t) = 1 - e^{-\delta t}$, then $H(0) = 0$ and $h(t) = H'(t) = \delta e^{-\delta t} > 0$. By Th 2.8,
$E[1 - e^{-\delta X}] = E[H(X)] = \int_0^\infty h(t)s(t)\, dt = \int_0^\infty \delta e^{-\delta t} s(t)\, dt$.
Can we take $H(t) = 2 - e^{-\delta t}$ **?** ∎

The special case of Th. 2.8 for discrete X ($E(H(X)) = \int_0^\infty s(t)H'(t)dt$:

Theorem 2.9. *Let X be a discrete r.v. whose possible values are $\underline{non-negative}$ integers. Let $h : [0,\infty) \to [0,\infty)$ be a function. Let $H(x) = \int_0^x h(t)\, dt$, $x \geq 0$. Then,*
$$E[H(X)] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

It follows from the previous theorem that

$$E[X] = \int_0^\infty s(t)dt = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\} \tag{2.1}$$

$$= \sum_{k=1}^\infty s(k)\textbf{??} \textbf{ or } = \sum_{k=1}^\infty s(k-)\textbf{??} \textbf{ or } = \sum_{k=0}^\infty s(k)\textbf{??}$$

If $X \sim bin(1,p)$, then $E(X) = \int_0^\infty s(t)dt$? $E(X) = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\} = ?$
If $X \sim U(0,1)$, then $E(X) = \int_0^\infty s(t)dt$? $E(X) = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}$ **???**

$$E[X^2] = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(k^2 - (k-1)^2) = \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}(2k-1), \tag{2.2}$$

$$E[\min(X,n)] = \sum_{k=1}^n \mathbb{P}\{X \geq k\},\ n \geq 1. \tag{2.3}$$

Theorem 2.10.

Theorem 2.11.

Theorem 2.12.

Theorem 2.13.

Theorem 2.14.

Theorem 2.15.

Proof of Th 2.9. $E[H(X)] = \int_0^\infty s(t)h(t)\,dt$ (by Theorem 2.8 in page 40)

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} s(t)h(t)\,dt$$

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} \mathbb{P}\{X \geq k\}h(t)\,dt \qquad (s(t) = \mathbb{P}\{X > t\} = \mathbb{P}\{X \geq k\}, \text{ for } k-1 \leq t < k)$$

$(s(k) = P(X \geq k)$?? does it matter ??)

$$= \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} \int_{k-1}^{k} h(t)\,dt$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

□

Example 2.10. *Find $E[X]$ and $E[X^2]$ if*

k	*0*	*1*	*2*
$\mathbb{P}\{X = k\}$	*0.2*	*0.3*	*0.5*

Solution: Two convenient approaches among 4:
(i) using that $E[H(X)] = \sum_{k=0}^{2} H(k)\mathbb{P}\{X = k\}$.
(ii) using (2.1) and (2.2).
(iii) $E(H(X)) = \int_0^\infty H'(x)s(x)dx$.
(iv) $E(H(X)) = \sum_x x f_{H(X)}(x)$.

(i) $E[X] = (0)(0.2) + (1)(0.3) + (2)(0.5) = 1.3$
$E[X^2] = (0)^2(0.2) + (1)^2(0.3) + (2)^2(0.5) = 2.3$.

(ii) We have that $\mathbb{P}\{X \geq 1\} = 0.8$, $\mathbb{P}\{X \geq 2\} = 0.5$, and $\mathbb{P}\{X \geq k\} = 0$, for each $k \geq 3$. Hence,

$$\begin{aligned} E[X] &= \mathbb{P}\{X \geq 1\} + \mathbb{P}\{X \geq 2\} && \textit{by } (2.1) \\ &= 0.8 + 0.5 = 1.3 \\ E[X^2] &= \mathbb{P}\{X \geq 1\}((2)(1) - 1) + \mathbb{P}\{X \geq 2\}((2)(2) - 1) && \textit{by } (2.2) \\ &= 0.8 + 0.5(3) = 2.3. \end{aligned}$$

Theorem 2.16.

Example 2.11.

Example 2.12.

Example 2.13.

2.1.4 Quantiles

Definition 2.13. *Given* $0 < p < 1$, *the* $100p$**–th percentile** *(or* p**–th quantile***) of a r.v.* X *is a value* ξ_p *such that*

$$\mathrm{P}\{X < \xi_p\} \leq p \leq \mathrm{P}\{X \leq \xi_p\}.$$

Definition 2.14. *median* = 0.50*–th quantile.*

Definition 2.15. *The* **first quartile** Q_1 *of a r.v.* X *is the* 25*–th percentile of the r.v.* X.
The **third quartile** Q_3 *of a r.v.* X *is the* 75*–th percentile of the r.v.* X.
The **second quartile** Q_2 = *median.*

3 ways to find ξ_p**.**

1. If X has a cts strictly increasing ($\uparrow\uparrow$) cdf, then solve $F(\xi_p) = p$.
2. Definition. $\xi_p = \xi_p^{**} = \inf\{x : F_X(x) \geq p\}$
   or $\xi_p = \xi_p^{*} = \sup\{x : F_X(x-) \leq p\}$.
3. Theorem 2.17.

Theorem 2.17. *Let* X *be a cts r.v. with range* (a, b)*. Let* $0 < p < 1$*.*
Let $h : (a, b) \to (c, d)$ *be a one–to–one onto function and* $Y = h(X)$*.*
Let ξ_p *be a p–th quantile of* X*.*

A p–th quantile of Y *is* $\zeta_p = h(\xi_p)$ *if* $h \uparrow$*.*

A p–th quantile of Y *is* $\zeta_p = h(\xi_{1-p})$ *if* $h \downarrow$*.*

Theorem 2.18. *The p–th quantile* ξ_p *of a normal r.v. with mean* μ *and variance* σ^2 *is* $\mu + \Phi^{-1}(p)\sigma$*.*

Proof. (i) Let $Z \sim N(0, 1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$.
The cdf Φ of $N(0, 1)$ satisfies $\Phi(t) \uparrow\uparrow$ in t. So
$\Phi^{-1}(p)$ is p-th quantile of $N(0, 1)$.
$\xi_p = h(\Phi^{-1}(p))$, where $h(z) = \mu + \sigma z$ **Where does it come from ?** ∎

Example 2.14. *Let* $Z \sim U(0, 1)$*,* $X = 2Z + 1$ *and* $Y = X^2 + X$*. Find the 70th percentile of* Z*,* X *and* Y*.*

Solution: By Th2.17: h is 1-1 and onto function and
$X = h(Z)$. Let ξ_p be a p–th quantile of Z.
A p–th quantile of X is $\zeta_p = h(\xi_p)$ if $h \uparrow$.
A p–th quantile of X is $\zeta_p = h(\xi_{1-p})$ if $h \downarrow$.

$F_Z(t) = t \ \forall \ t \in (0, 1)$. The 70th percentile of Z is 0.7.
Since $g(z) = 2z + 1 \uparrow\uparrow$ in z, the 70th percentile of Z is $g(0.7) = 2 * 0.7 + 1 = 2.4$.
Since $h(x) = x^2 + x \uparrow\uparrow$, the 70th percentile ζ of Y is $h(2.4) = (2.4)^2 + 2.4 = 8.16$.

Often, we will assume that the individuals do not live more than a certain age. This age ω is called the **terminal age** or **limiting age** of the population. So, $S(t) = 0$, for each $t \geq \omega$.

Example 2.15.

Example 2.16. *Suppose that the age–at–failure r.v. X has density*

$f_X(x) = 5x^4k^{-5}I(0 < x < k)$ *and the expected age–at–failure is 70 years. Find the 4 intervals determined by the 3 quartiles and the terminal age.*

Solution: Let ξ_p be p–th quantile of the age–at–failure.

$\xi_p = ?$ for $p = 0.25$, 0.5, and 0.75.

$$p = F(\xi_p) = \mathrm{P}\{X \leq \xi_p\} = \int_0^{\xi_p} \frac{5x^4}{k^5}\,dx = \frac{x^5}{k^5}\Big|_0^{\xi_p} = \frac{\xi_p^5}{k^5}$$

$\xi_p = kp^{1/5} = ?$ k= ?

$$70 = E[X] = \int_0^k x\frac{5x^4}{k^5}\,dx = \frac{5x^6}{6k^5}\Big|_0^k = \frac{5k}{6} \quad => \quad k = \frac{(70)(6)}{5} = 84.$$

So, $\xi_p = 84p^{\frac{1}{5}}$.

The first quartile of X is $\xi_{0.25} = 84(0.25)^{\frac{1}{5}} \approx 63.66$.

$m = \frac{84}{2^{\frac{1}{5}}} \approx 73.12624732$.

The third quartile of X is $\xi_{0.75} = 84(0.75)^{\frac{1}{5}} \approx 79.30$.

The 4 intervals determined by the 3 quartiles and the terminal age are

[0, 63.66], [63.66, 73.13], [73.13, 79.30], [79.30, 84].

Definition 2.16.

Definition 2.17.

Definition 2.18.

Definition 2.19.

Definition 2.20.

Example 2.17.

Example 2.18. *Let X be a r.v such* $\mathrm{P}\{X = 1\} = \frac{1}{2}$ *and* $\mathrm{P}\{X = 2\} = \frac{1}{2}$. *Find the first quartile Q_1 and median of X.*

Solution. 3 ways to find ξ_p ($F(\xi_p-) \leq p \leq F(\xi_p)$).

1. If X has a cts strictly increasing cdf, then solve $F(\xi_p) = p$.

2. Definition. $\xi_p = \xi_p^* = \sup\{x : F_X(x-) \le p\}$,
   or $\xi_p = \xi_p^{**} = \inf\{x : F_X(x) \ge p\}$.

3. Theorem 2.17. Relation of the quantiles of $g(X)$ and X.

Does Method 1 work here ?
Does Method 2 work here ?
Does Method 3 work here ?

	$F(x-)$ $(= \mathrm{P}\{X < x\})$	$F(x)$ $(= \mathrm{P}\{X \le x\})$
$x \in (-\infty, 1)$	0	0
$x = 1$	0	$\frac{1}{2}$
$x \in (1, 2)$	$\frac{1}{2}$	$\frac{1}{2}$
$x = 2$	$\frac{1}{2}$	1
$x \in (2, \infty)$	1	1
	for ξ_p^*	for ξ_p^{**}
$Q_1 =$	$\xi_{1/4}^* = 1$	$\xi_{1/4}^{**} =$ **??**
$Q_2 =$	$\xi_{1/2}^* =?$	$\xi_{1/2}^{**} =$ **??**

$Q_1 = 1$.
The values of x **that satisfy** $\mathrm{P}\{X < x\} \le \frac{1}{2} \le \mathrm{P}\{X \le x\}$ **??**
Ans: $x \in [1, 2]$.
Thus m is a median of X if and only if $m \in [1, 2]$.

Remark. A median may not be unique.

2.2 Actuarial notation for survival analysis.

Def. In Actuary, denote (x) a life that survives to age x
  (x) is called a life-age-x or a life aged x.
  Let X be the lifetime of the person.
  $T(x)$ or T_x $(= (X - x)|\{X > x\})$ – the future lifetime of (x).
Notice that $T(x)$ is a conditional random variable.
${}_tp_x = S_{T_x}(t)$ – probability that (x) survives t years.
${}_tq_x = F_{T_x}(t)$ – probability that (x) dies within t years.
$p_x = {}_1p_x = P(T(x) > 1)$.
${}_{s|t}q_x = P(s < T(x) \le s + t)$.
${}_{t|}q_x = {}_{t|1}q_x$.
$q_x = {}_{0|1}q_x = {}_1q_x = P(T(x) \le 1)$.

Q: ${}_tp_x + {}_tq_x =$ **??**
  $p_x + q_x =$ **??**

Remark. ${}_tp_x = P(T(x) > t) = P(X - x > t | X > x) = \frac{P(X > x+t)}{P(X > x)} = S_X(x + t)/S_X(x)$.
  $p_x = P(T(x) > 1) = S_X(x + 1)/S_X(x)$. **See Notations 3 & 4 for 450.**
Quiz 447 and 1-4 in 450 next week.

Q: How about the other notations ?

$$
\begin{aligned}
{}_{s|t}q_x &= P(s < T(x) \le s+t) = P(s < X - x \le s+t | X > x) \\
&= \frac{P(x+s < X \le x+s+t)}{P(X > x)} = \frac{S_X(x+s) - S_X(x+s+t)}{S_X(x)} = {}_s p_x - {}_{s+t}\, p_x.
\end{aligned}
$$

$q_x = P(T(x) \le 1) = P(X \le x+1 | X > x) = \frac{S_X(x) - S_X(x+1)}{S_X(x)}$.

Theorem 2.19. *For each $t, s \ge 0$, ${}_{t+s}p_x = {}_t p_x \cdot {}_s p_{x+t}$.*

Proof. ${}_{t+s}p_x = \frac{S_X(x+t+s)}{S_X(x)} = \frac{S_X(x+t)}{S_X(x)} \frac{S_X(x+t+s)}{S_X(x+t)} = {}_t p_x \cdot {}_s p_{x+t}$

Theorem 2.20. ${}_n p_x = p_x p_{x+1} \dots p_{x+n-1} = \prod_{j=0}^{n-1} p_{x+j}$.
For each $t_1, \dots, t_m \ge 0$, ${}_{t_1+\cdots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot {}_{t_3}p_{x+t_1+t_2} \cdots {}_{t_m}p_{x+t_1+\cdots+t_{m-1}}$.

Q: Relation between Theorems 2.19 and 2.20 ??

Def. $X \simeq Y$ means X and Y have the same distribution.

Theorem 2.21. $(T(x) - t)|\{T(x) > t\} \simeq T(x+t)$.

Proof. It suffices to show either their F's, or S's or f's are the same. Use S's here. Recall that $\forall\ y > 0$, $P(T(x) > y) = P(X - x > y | X > y) = \frac{S_X(x+y)}{S_X(x)}$;

$$
P((T(x)-t) > y | T(x) > t) = \frac{P((T(x)-t) > y)}{P(T(x) > t)} = \frac{P(T(x) > y+t)}{P(T(x) > t)} = \frac{S_X(x+t+y)}{S_X(x)} \Big/ \frac{S_X(x+t)}{S_X(x)}
$$

$= \frac{S_X(x+t+y)}{S_X(x+t)} = P(T(x+t) > y)$.
Denote $s = S_X$.

Example 2.19. *Let $s(t) = \frac{85-t}{85}$, $0 \le t \le 85$.*
(i) Calculate ${}_t p_{40}$.
(ii) Calculate the density function of $T(40)$.

Solution: (i) ${}_t p_{40} = P(X > t+40 | X > 40) = \frac{s(40+t)}{s(40)} = \frac{\frac{85-(40+t)}{85}}{\frac{85-40}{85}} = \frac{\frac{45-t}{85}}{\frac{85-40}{85}} = \frac{45-t}{45}$, ????

$t \in [0, 45]$.
${}_t p_{40} = 0$ **if** $t > 45$ **?**
${}_t p_{40} = 0$ **if** $t < 0$ **?**

(ii) The density function of $T(40)$ is

$$
f_{T(40)}(t) = -\frac{d}{dt}{}_t p_{40} = -\frac{d}{dt}\left(\frac{45-t}{45}\right) = \frac{1}{45},\ t \in (0, 45).
$$

$f_{T(40)}(t) = 0$ **if** $t \ge 45$ **?**
$f_{T(40)}(t) = 0$ **if** $t \le 0$ **?**

Notice the difference between the domains of $S_X(t)$ and $f_X(t)$.

Example 2.20. *If* ${}_tp_x = 1 - \frac{t}{90-x}$, $0 \le t \le 90 - x$, *find the probability that a 25–year–old reaches age 80 and the density of* $T(x)$.

Solution: The probability that a 25–year-old reaches age 80 is
${}_{80-25}p_{25} = {}_{55}p_{25} = 1 - \frac{55}{90-25} = \frac{2}{13}$.
The density of $T(x)$ is $-(1 - \frac{t}{90-x})'_x$ **or** $-(1 - \frac{t}{90-x})'_t$ **??**

$$f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x = \frac{1}{90-x}, 0 < t < 90 - x.$$

Example 2.21.

Example 2.22. *Suppose that probability that a 30–year–old reaches age 40 is* 0.95*, the probability that a 40–year–old reaches age 50 is* 0.99*, and the probability that a 50–year–old reaches age 60 is* 0.95*. Find the probability that a 30–year–old reaches age 60.*

Solution: ${}_{60-30}p_{30} = ?$
Given conditions:

${}_{40-30}p_{30}$ – probability that a 30–year–old reaches age 40 is 0.95

${}_{50-40}p_{40}$ – the probability that a 40–year–old reaches age 50 is 0.99, and

${}_{60-50}p_{50}$ – the probability that a 50–year–old reaches age 60 is 0.95.
Formula: ${}_{t_1+\cdots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot {}_{t_3}p_{x+t_1+t_2} \cdots {}_{t_m}p_{x+t_1+\cdots+t_{m-1}}$. (Formula 4).
The probability that a 30–year–old reaches age 60 is

$${}_{30}p_{30} = {}_{10}p_{30} \cdot {}_{10}p_{40} \cdot {}_{10}p_{50} = (0.95)(0.99)(0.95) = 0.893475.$$

Example 2.23.

Example 2.24.

Example 2.25. *Suppose that the survival function of a person is given by*
$S_X(x) = \frac{90-x}{90}$, *for* $0 \le x \le 90$.
Given a married couple with husband aged 40 and wife aged 35,
what is the probability that
the husband will die before age 60 and the wife will survive to age 75?
Here, we assume that their times of death are independent r.v.'s.

Solution: $\underbrace{{}_{60-40}q_{40} \times {}_{75-35}\, p_{35}}_{\textbf{Why??}} = ?$

$$
\begin{aligned}
{}_{60-40}q_{40} &= {}_{20}q_{40} = \frac{s(40)-s(60)}{s(40)} = \frac{\frac{90-40}{90} - \frac{90-60}{90}}{\frac{90-40}{90}} = \frac{20}{50} = \frac{2}{5}. \\
{}_{75-35}p_{35} &= {}_{40}p_{35} = \frac{s(75)}{s(35)} = \frac{\frac{90-75}{90}}{\frac{90-35}{90}} = \frac{15}{55} = \frac{3}{11}. \\
Answer: &\ {}_{60-40}q_{40} \times {}_{75-35}\, p_{35} = \frac{2}{5} \times \frac{3}{11} = \frac{6}{55}.
\end{aligned}
$$

Example 2.26.

Example 2.27.

Example 2.28. *Suppose that* $s(t) = \frac{85-t}{85}$, $0 \le t \le 85$, *find the probability that a 40–year–old will die in less than one year*

Solution: $q_{40} = ?$
Formula: ${}_tq_x = 1 - {}_tp_x$ and ${}_tp_x = \frac{s(x+t)}{s(x)}$.

$$q_{40} = 1 - \frac{s(41)}{s(40)} = 1 - \frac{\frac{85-41}{85}}{\frac{85-40}{85}} = \frac{1}{45}.$$

Example 2.29. *Suppose that:*
(i) The probability that a 30–year–old will die in less than one year is 0.012
(ii) The probability that a 31–year–old will die in less than one year is 0.013
(iii) The probability that a 32–year–old will die in less than one year is 0.014.
Find the probability that a 30–year–old will die in less than three years.

Solution: ${}_3q_{30} = ?$
Given conditions: (i) q_{30}, (ii) q_{31}, (iii) q_{32}.
Formula: ${}_tq_x + {}_tp_x = 1$.
${}_{t_1+\cdots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot {}_{t_3}p_{x+t_1+t_2} \cdots {}_{t_m}p_{x+t_1+\cdots+t_{m-1}}$.

$${}_3p_{30} = p_{30}p_{31}p_{32} = (1-0.012)(1-0.013)(1-0.014) \approx 0.9615.$$

$${}_3q_{30} = 1 - {}_3p_{30} \approx 1 - 0.9615 = 0.038.$$

Theorem 2.22.

Theorem 2.23. ${}_{s|t}q_x = {}_sp_x - {}_{s+t}p_x = {}_{s+t}q_x - {}_sq_x = {}_sp_x \cdot {}_tq_{x+s}$.

Proof. We have that

$$\begin{aligned}
{}_{s|t}q_x =& \mathbb{P}\{s < T(x) \le s+t\}\\
=& F_{T(x)}(s+t) - F_{T(x)}(s) = {}_{s+t}q_x - {}_sq_x,\\
{}_{s|t}q_x =& \mathbb{P}\{s < T(x) \le s+t\}\\
=& S_{T(x)}(s) - S_{T(x)}(s+t) = {}_sp_x - {}_{s+t}p_x,\\
{}_sp_x \cdot {}_tq_{x+s} =& {}_sp_x \cdot (1 - {}_tp_{x+s}) = {}_sp_x - {}_sp_x \cdot {}_tp_{x+s}\\
=& {}_sp_x - {}_{s+t}p_x = {}_{s|t}q_x.
\end{aligned}$$

Example 2.30. *Let $S_X(x) = (\frac{90-x}{90})^2$, $x \in (0, 90)$.*
(i) Find ${}_{s|t}q_x$, where $0 < x, s, t$ and $x + s + t \le 90$.
(ii) Find the probability that a 30–year–old dies between ages 55 and 60.

Solution: (i) ${}_{s|t}q_x = ?$ (ii) ${}_{55-30|5}q_{30} = ?$

$$\begin{aligned}
{}_{s|t}q_x =& P(x + s < X \le x + s + t)|X > x) \qquad P(a < X \le b) = F(b) - F(a) = S(a) - S(b)\\
=& \frac{(\frac{90-(x+s)}{90})^2}{(\frac{90-x}{90})^2} - \frac{(\frac{90-(x+s+t)}{90})^2}{(\frac{90-x}{90})^2} \qquad (a^2 - b^2 = (a+b)(a-b))\\
=& \frac{(180 - 2x - 2s - t)t}{(90-x)^2} \text{...... } \textbf{done ?}
\end{aligned}$$

x, s, $t \ge 0$ and $s + t + x \le 90$.
(ii) ${}_{55-30|60-55}q_{30} = {}_{25|5}q_{30} = \frac{(180-2(30)-2(25)-5)5}{(90-30)^2} = \frac{65\times 5}{60^2} = \frac{13}{12^2} \approx 0.09$.

Theorem 2.24. *For $x \ge 0$, and each positive integer n,*

$${}_nq_x = \sum_{j=1}^{n} {}_{j-1|}q_x = \sum_{j=1}^{n} {}_{j-1}p_x q_{x+j-1}.$$

Proof. ${}_nq_x = \mathrm{P}\{T(x) \le n\} = \sum_{j=1}^n \mathrm{P}\{j - 1 < T(x) \le j\} = \sum_{j=1}^n {}_{j-1|}q_x$
$= \sum_{j=1}^n {}_{j-1}p_x q_{x+j-1}$. ■

Theorem 2.25. $\sum_{j=1}^{\infty} {}_{j-1|}q_x = \sum_{j=0}^{\infty} {}_{j-1}p_x q_{x+j-1} = 1$.

Theorem 2.26. ${}_{t+s|u}q_x = {}_tp_x \cdot {}_{s|u}q_{x+t}$.

Theorem 2.27. ${}_{n+m|}q_x = {}_np_x \cdot {}_{m|}q_{x+n}$.

When $n = 1$ and $m = j - 1 \ge 0$, we get that

$${}_{j|}q_x = p_x \cdot {}_{j-1|}q_{x+1}.$$

Example 2.31.

Definition 2.21.

Definition 2.22.

Example 2.32.

Example 2.33.

Example 2.34.

Definition 2.23.

Theorem 2.28.

Definition 2.24.

Theorem 2.29.

Example 2.35.

Example 2.36.

2.3 Force of mortality

Definition. The **hazard function** of the survival function $S_X(x)$ or the **force of mortality** (denoted by $\mu_X(x)$, $\mu(x)$, μ_x and $\lambda_X(x)$), is defined as

$$\lambda_X(x) = \frac{f_X(x)}{S_X(x-)},$$

where f_X is the pdf or pmf of the r.v. X.

Denote $\mu_x(t) = \mu_{T_x}(t)$ $(\neq \mu_X(t))$.

$\mu_x = \mu_0(x) = \mu_x(0) \neq \mu_x(\cdot)$.

Theorem 2.28. If $X \geq 0$ is cts and has the force of mortality $\mu(\cdot)$, then,
(i) $-\frac{d}{dt}\ln S_X(t) = \frac{f_X(t)}{S_X(t)} = \mu(t)$ and $S_X(t) = \exp\left(-\int_0^t \mu(s)\,ds\right), t \geq 0$.
(ii) $f_X(t) = S_X(t)\mu(t) = \exp\left(-\int_0^t \mu(s)\,ds\right)\mu(t), t \geq 0$.
(iii) $S_{T(x)}(t) = {}_tp_x = \exp\left(-\int_0^t \mu_x(s)\,ds\right), t \geq 0$.
(iv) $\mu_x(t) = \mu(x+t)$
(v) $f_{T(x)}(t) = {}_tp_x\mu_x(t) = \exp(-\int_0^t \mu(x+s)ds)\mu(x+t), t \geq 0$.

Proof (iv) The survival function of $T(x)$ is $S_{T(x)}(t) = \frac{S_X(x+t)}{S_X(x)}$, $t \geq 0$. ,

$$\mu_x(t) = -\frac{d}{dt}\ln S_{T(x)}(t) = -\frac{d}{dt}\ln\left(\frac{S_X(x+t)}{S_X(x)}\right) = \frac{f_X(x+t)}{S_X(x+t)} = \mu(x+t).$$

Hereafter, when we consider the force of mortality, we assume that X is cts.

The force of mortality of a life at time x, $x > 0$, satisfies that $\mu(x) = \lim_{t\to 0} \frac{{}_tq_x}{t}$, as

$$\lim_{t\to 0} \frac{{}_tq_x}{t} = \lim_{t\to 0} \frac{\frac{s(x)-s(x+t)}{s(x)}}{t} = -\lim_{t\to 0} \frac{s(x+t)-s(x)}{ts(x)} = -\frac{s'(x)}{s(x)} = \frac{f(x)}{s(x)} = \mu(x),\ x > 0. \tag{2.4}$$

If t is small, the proportion of people aged x who will die within t years is $\frac{s(x)-s(x+t)}{s(x)} \approx t\mu_x$. For example, if $\mu(x) = 0.06$ and t is $1/12$ (a month), we expect that from each 1,000 individuals with age x, $t\mu(x)10^3 = \frac{60}{12} = 5$ individuals will die within a month.

The force of mortality is the rate of death for lives aged x. For a life aged x, the force of mortality t years later is the force of mortality for a $(x+t)$–year old.

Theorem 2.30. *Let $\mu : [0,\infty] \to \mathbb{R}$ be a function which is cts everywhere except at finitely many points. Then, μ is the force of mortality of an age–at–death r.v.*
iff (1) $\mu(x) \geq 0\ \forall\ x$ and (2) $\int_0^\infty \mu(t)\,dt = \infty$.

Example 2.37. *Suppose that the survival function of a new born is*
$S_X(t) = \frac{85^4-t^4}{85^4}$, for $0 < t < 85$.
(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged 20.

Solution: (i) $\mu(t) = -\frac{d}{dt}\ln S_X(t) = -\frac{d}{dt}\ln\left(\frac{85^4-t^4}{85^4}\right) = -\frac{d}{dt}[\ln(85^4-t^4) - \ln(85^4)] = \frac{4t^3}{85^4-t^4}$, $0 < t < 85$

(ii) $\mu_{20}(t) = \mu(20+t) = \frac{4(20+t)^3}{85^4-(20+t)^4}$, $\quad 0 < t <$ **??**

Example 2.38. *If $\mu(x) = \frac{1}{x+1}$ for $x \geq 0$, find S_X, f_X, $\mu_{T(x)}$, ${}_tp_x$ and $f_{T(x)}$.*

Solution: Note: S_X, f_X, $\mu_{T(x)}$, ${}_tp_x$ and $f_{T(x)}$ are functions, *e.g.*, we can write $S_X(x)$ or $S_X(t)$. **Do we write** $\mu_{T(x)}(x)$, $\mu_{T(x)}(t)$, $\mu_x(x)$ or $\mu_x(t)$?

$$\begin{aligned}
&s(x) = \exp\left(-\int_0^x \mu(t)\,dt\right) = \exp(-\ln(1+x)) = \frac{1}{x+1}, \quad x \geq 0,\\
&\quad as \int_0^x \mu(t)\,dt = \int_0^x \frac{1}{t+1}\,dt = \ln(1+t)\Big|_0^x = \ln(1+x),\\
&f_X(x) = \mu(x)s(x-) = \frac{1}{(x+1)^2}, \quad x \geq 0, \qquad\qquad (\mu(x) = f(x)/s(x-))\\
&{}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\frac{1}{x+t+1}}{\frac{1}{x+1}} = \frac{x+1}{x+t+1}, \quad x,t \geq 0,\\
&f_{T(x)}(t) = {}_tp_x\mu_x(t) = {}_tp_x\mu(x+t) = \frac{x+1}{x+t+1}\frac{1}{x+t+1} = \frac{x+1}{(x+t+1)^2},\ t \geq 0,
\end{aligned}$$

Definition 2.25.

Definition 2.26.

Example 2.39.

Theorem 2.31.

Theorem 2.32.

Example 2.40.

Example 2.41.

Theorem 2.33.

2.4 Expectation of life

Definition 2.27. $\mathring{e}_x = E[T(x)]$ *is called the* **expected future lifetime at age** x *or the* **complete expectation of a life at age** x.
$\mathring{e}_0$ *is also called the* **complete expectation of life at birth**.

Definition 2.28. *The n–year* **temporary** *complete life expectancy is* $\mathring{e}_{x:\overline{n}|} = E(T(x) \wedge n)$, *the expected number of years lived between ages x and $x+n$ by a survivor aged x.* $0 \le \mathring{e}_{x:\overline{n}|} \le n$.

Example 2.42. *An actuary models the lifetime in years of a random selected person as a r.v. X with* $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find:*
(i) $\mathring{e}_0$ *and* $\mathrm{Var}(X)$; *(ii)* $\mathring{e}_{30}$; *(iii)* $\mathring{e}_{30:\overline{10}|}$.

Solution: $\mathrm{Var}(X) = E[X^2] - (E[X])^2$.
Two ways for taking expectation.
(1) $\mathring{e}_x = \int_0^\infty {}_tp_x dt$, where ${}_tp_x = \frac{s(x+t)}{s(x)}$, and (if X is cts) $f_{T(x)}(t) = {}_tp_x \mu_x(t) = \frac{f_X(x+t)}{S_X(x)}$.

$$(2)\ \mathring{e}_x = \begin{cases} \int_0^\infty t f_{T(x)}(t)dt & \text{if } X \text{ is cts} \\ \sum_t t f_{T(x)}(t) & \text{if } X \text{ is discrete,} \\ \int_0^\infty t f_{T(x)}(t)dt + \sum_{t\in D} t f_{T(x)}(t) & \text{if } X \text{ is mixed.} \end{cases}$$

Which way is more convenient here ?

(i) $\mathring{e}_0$ and σ_X^2. Method 1: $\mathring{e}_0 = \displaystyle\int_0^\infty S_X(x)dx = \int_0^{90} \frac{90^6 - x^6}{90^6} dx = x - \frac{x^7}{(7)90^6}\Big|_0^{90} = 77.142857$,

$$E[X^2] = \int_0^\infty 2xS_X(x)dx = \int_0^{90} 2x\frac{90^6 - x^6}{90^6}dx = [x^2 - \frac{x^8}{(4)90^6}]\Big|_0^{90} = 6075,$$

$$\text{Method 2: } f_{T(0)}(x) = f_X(x) = -\frac{d}{dx}S_X(x) = -\frac{d}{dx}\frac{90^6 - x^6}{90^6} = \frac{6x^5}{90^6}, 0 < x < 90.$$

$$\overset{\circ}{e}_0 = \int_0^\infty x f_X(x)dx = \int_0^{90} x\frac{6x^5}{90^6}dx = \frac{6x^7}{(7)90^6}\Big|_0^{90} \approx 77.142857,$$

$$E[X^2] = \int_0^\infty x^2 f_X(x)dx = \int_0^{90} x^2\frac{6x^5}{90^6}dx = \frac{6x^8}{(8)90^6}\Big|_0^{90} = 6075,$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = 6075 - (77.143)^2 \approx 1123.980.$$

Method 1 for $\overset{\circ}{e}_{30}$: $S_{T(30)}(t) = {}_tp_{30} = \frac{S_X(30+t)}{S_X(30)} = \begin{cases} ? & if\ t < 0 \\ \frac{90^6-(30+t)^6}{(90)^6-(30)^6} & \text{if } 0 \le t < 60, \\ ? & \text{if } 60 < t. \end{cases}$

$$\overset{\circ}{e}_{30} = \int_0^\infty {}_tp_{30}\, dt = \int_0^{60} \frac{90^6 - (30+t)^6}{90^6 - (30)^6}\, dt = \frac{90^6 t}{90^6 - (30)^6} - \frac{(30+t)^7}{(7)(90^6 - (30)^6)}\Bigg|_0^{60} = 47.21.$$

(ii) Method 2 for $\overset{\circ}{e}_{30}$: $f_{T(30)}(t) = -\frac{d}{dt}{}_tp_{30} = \frac{f_X(30+t)}{S_X(30)} = \frac{\frac{6(30+t)^5}{90^6}}{\frac{(90)^6-(30)^6}{90^6}} = \frac{6(30+t)^5}{(90)^6-(30)^6}, \quad 0 < t <??.$

$$\overset{\circ}{e}_{30} = \int_0^\infty t f_{T(30)}(t)\, dt = \int_0^{60} t\frac{6(30+t)^5}{(90)^6 - (30)^6}\, dt \qquad (S_X(x) = \tfrac{90^6-x^6}{90^6}, \text{ for } 0 < x < 90)$$

$$= \int_{30}^{90} (s-30)\frac{6s^5}{(90)^6 - (30)^6}\, ds \qquad (\text{change of variables } s = 30+t)$$

$$= \frac{6s^7}{(7)(90^6 - (30)^6)} - \frac{(30)s^6}{(90)^6 - (30)^6}\Bigg|_{30}^{90} = 47.21.$$

(iii) The two methods for $\overset{\circ}{e}_{x:\overline{n}|}$ are stated as follows.

Theorem 2.34. $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n t f_{T(x)}(t)\, dt + n \cdot {}_np_x = \int_0^n {}_tp_x\, dt.$

Method 1. ${}_tp_{30} = \frac{s(30+t)}{s(30)} = \frac{90^6-(30+t)^6}{90^6-30^6}$, for $0 \le t < 60$. So,

$$\overset{\circ}{e}_{30:\overline{10}|} = \int_0^{10} {}_tp_{30}\ dt = \int_0^{10} \frac{90^6 - (30+t)^6}{90^6 - 30^6}\, dt$$

$$= \frac{(90^6)t}{90^6 - 30^6} - \frac{(30+t)^7}{(7)(90^6 - 30^6)}\Bigg|_0^{10} = 9.975520756.$$

Method 2. $Y = T(x) \wedge n$ has a mixed distribution with df. $f_Y(t) = \begin{cases} f_{T(x)}(t) & \text{if } t \in (0,n) \\ {}_np_x & \text{if } t \in D = \{n\}. \end{cases}$

$\overset{\circ}{e}_{30:\overline{10}|} = \int_0^{10} t\frac{6(30+t)^5}{(90)^6-(30)^6}\, dt + 10{}_{10}p_{30} = \cdots.$

Theorem 2.35. *For* $0 < m < n$, $\overset{\circ}{e}_{x:\overline{n}|} = \overset{\circ}{e}_{x:\overline{m}|} + {}_mp_x \overset{\circ}{e}_{x+m:\overline{n-m}|}$

Example 2.43. *You are given that:*
The expected # of years lived between ages 40 & 50 by a 40-year old is 9.7.
The probability that a 40–year old survives to age 50 is 0.98.
The expected # of years lived between ages 50 & 70 by a 50-year old is 19.5.
Find the expected # of years lived between ages 40 and 70 by a 40-year old.

Solution: (i) $\overset{\circ}{e}_{40:\overline{10}|} = 9.7$, (ii) ${}_{10}p_{40} = 0.98$, (iii) $\overset{\circ}{e}_{50:\overline{20}|} = 19.5$. $\overset{\circ}{e}_{40:\overline{30}|} = ?$

$$\overset{\circ}{e}_{40:\overline{30}|} = \overset{\circ}{e}_{40:\overline{10}|} + {}_{10}p_{40}\overset{\circ}{e}_{50:\overline{20}|} = 9.7 + (0.98)(19.5) = 28.81.$$

Example 2.44. *Assume that*

(i) The expected future lifetime of a 40–year old is 45 years.

(ii) The expected future lifetime of a 50–year old is 36 years.

(iii) The probability that a 40–year old survives to age 50 is 0.98.
The expected number of years lived between ages 40 and 50 by a 40-year old ?

Solution: Given: $\overset{\circ}{e}_{40} = 45$; $\overset{\circ}{e}_{50} = 36$; ${}_{50-40}p_{40} = 0.98$. $\overset{\circ}{e}_{40:\overline{10}|}=?$

Formula: $\overset{\circ}{e}_{x:\overline{n}|} = \overset{\circ}{e}_{x:\overline{m}|} + {}_{m}p_x\overset{\circ}{e}_{x+m:\overline{n-m}|}$.

Letting $n \to \infty \Longrightarrow \underbrace{\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{m}|} + {}_{m}p_x\overset{\circ}{e}_{x+m}}_{45=\overset{\circ}{e}_{40:\overline{10}|}+(0.98)(36)}$,

$\overset{\circ}{e}_{40:\overline{10}|} = 45 - (0.98)(36) = 9.72.$

Proof of Theorem 2.35. $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,dt = \int_0^m {}_tp_x\,dt + \int_m^n {}_tp_x\,dt = \overset{\circ}{e}_{x:\overline{m}|} + \int_m^n {}_mp_x \cdot {}_{t-m}p_{x+m}\,dt$
$= \overset{\circ}{e}_{x:\overline{m}|} + \int_0^{n-m} {}_mp_x \cdot {}_tp_{x+m}\,dt = \overset{\circ}{e}_{x:\overline{m}|} + {}_mp_x\overset{\circ}{e}_{x+m:\overline{n-m}|}$

Example 2.45.

Example 2.46.

Theorem 2.36.

Definition 2.29. ${}_nm_x = \frac{\int_x^{x+n} S_X(t)\mu_X(t)\,dt}{\int_x^{x+n} S_X(u)\,du}$ *$(= \int_x^{x+n} \frac{S_X(t)}{\int_x^{x+n} S_X(u)\,du}\mu_X(t)\,dt)$ is called the* **central death rate** *or the* **central rate of failure** *over the age interval x and $x+n$.*

${}_nm_x$ is the weighted average of the force mortality on the interval $[x, x+n]$ using the survival function as a weight *i.e.*, $\frac{S_X(t)}{\int_x^{x+n} S_X(t)dt}$. Denote $m_x = {}_1m_x$.

Theorem 2.37. ${}_nm_x = \frac{S_X(x)-S_X(x+n)}{\int_x^{x+n} S_X(t)\,dt} = \frac{\int_0^n {}_tp_x\mu_{T_x}(t)\,dt}{\int_0^n {}_tp_x\,dt} = \frac{{}_nq_x}{\overset{\circ}{e}_{x:\overline{n}|}}$.

Proof. $S_X(t)\mu_X(t) = f_X(t)$, ${}_tp_x\mu_{T_x}(t) = f_{T(x)}(t) = \frac{f_X(x+t)}{s(x)}$, ${}_tp_x = \frac{s(x+t)}{s(x)}$

$${}_nm_x = \frac{S_X(x) - S_X(x+n)}{\int_x^{x+n} S_X(t)\,dt} = \frac{S_X(x) - S_X(x+n)}{\int_0^n S_X(x+u)\,du} = \frac{1 - {}_np_x}{\int_0^n {}_up_x\,du} = \frac{{}_nq_x}{\overset{\circ}{e}_{x:\overline{n}|}}.$$

$${}_nm_x = \frac{\int_x^{x+n} f_X(t)dt}{\int_x^{x+n} S_X(t)dt} = \frac{\int_0^n f_X(x+u)du}{\int_0^n S_X(x+u)du} = \frac{\int_0^n {}_tp_x \mu_{T_x}(t)\, dt}{\int_0^n {}_tp_x\, dt} \qquad \blacksquare$$

Definition 2.30. *The* **median future lifetime** *of* (x) *is* $m(x)$ *such that*

$$\mathbb{P}\{T(x) < m(x)\} \leq \frac{1}{2} \leq \mathbb{P}\{T(x) \leq m(x)\}.$$

Definition 2.31. ${}_na(x) = E(T(x)|T(x) \leq n)$*, the average future lifetime of those who survive to age* x*, but die within the next* n *years.* $a(x) = {}_1a(x)$*.*

$x + {}_na(x)$ is the mean age at death of those who survive to age x, but die in the next n years.

Theorem 2.38. ${}_na(x) = \frac{\mathring{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$.

Example 2.47. *For the survival function* $S_X(x) = \frac{90^6 - x^6}{(90)^6}$*, for* $0 < x < 90$*.*
Find (1) the median future lifetime of (30)*, (2)* ${}_{10}m_{30}$*, (3)* ${}_{10}a(30)$*.*

Solution: (1) Formula: $p = F(\xi_p) = S(\xi_{1-p})$.
Let $m(30)$ be the median future lifetime of (30). We have that

$$\frac{1}{2} = \mathbb{P}\{T(30) > m(30)\} = \frac{S_X(30 + m(30))}{S_X(30)} = \frac{90^6 - (30 + m(30))^6}{90^6 - 30^6}$$

$=> m(30) = \left(90^6 - \frac{90^6 - 30^6}{2}\right)^{1/6} - 30 \approx 50.19920541.$

(2) Formulas [7]: ${}_nm_x = \frac{S_X(x) - S_X(x+n)}{\int_x^{x+n} S_X(t)\, dt} = \frac{\int_0^n {}_tp_x \mu_x(t)\, dt}{\int_0^n {}_tp_x\, dt} = \frac{{}_nq_x}{\mathring{e}_{x:\overline{n}|}}$.
Try the 3rd (actually 1st is better. **Why?**)
Need (a) ${}_{10}q_{30} = 1 - {}_{10}p_{30} = 1 - \frac{S_X(40)}{S_X(30)} \approx 0.006344$. (b) $\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt \approx 9.976$ (see Ex. 2.42).
For $0 \leq t < 60$, ${}_tp_{30} = \frac{s(30+t)}{s(30)} = \frac{90^6 - (30+t)^6}{90^6 - 30^6}$. So,

${}_{10}m_{30} = \frac{{}_{10}q_{30}}{\mathring{e}_{30:\overline{10}|}} \approx \frac{0.00634}{9.976} \approx 0.000636.$

(3) Formula [7]: ${}_na(x) = \frac{\mathring{e}_{x:\overline{n}|} - n \cdot {}_np_x}{{}_nq_x}$.

$${}_{10}a(30) = \frac{\mathring{e}_{30:\overline{10}|} - 10 \cdot {}_{10}p_{30}}{{}_{10}q_{30}} \approx \frac{9.9755 - (10)(0.9937)}{0.00634430727} \approx 6.1415.$$

Example 2.48.

Example 2.49.

2.5 Future curtate lifetime.

Let $\lceil t \rceil$ be the **least integer greater than or equal to** t. $\lceil t \rceil$ is called the **ceiling** of t.

Notice that $\lceil t \rceil = kI(k-1 < t \le k) = \begin{cases} . & \dots \\ 0 & \text{if } -1 < t \le 0, \\ 1 & \text{if } 0 < t \le 1, \\ 2 & \text{if } 1 < t \le 2, \\ 3 & \text{if } 2 < t \le 3, \\ . & \dots \end{cases}$

Definition 2.32. *Let* $K_x = \lceil T(x) \rceil$.

Definition 2.33. *The* **future curtate lifetime of a life aged** x *is* $K(x) = \lceil T(x) \rceil - 1$.
$e_x = E[K(x)]$ *is the* **curtate life expectation of a life aged** x.
$e_{x:\overline{n}|} = E(n \wedge K(x))$, *the expected number of whole years lived in the interval* $(x, x+n]$ *by* (x).

$$K(x) = K_x - 1 = \begin{cases} 0 & \text{if } 0 < T(x) \le 1, \\ 1 & \text{if } 1 < T(x) \le 2, \\ 2 & \text{if } 2 < T(x) \le 3, \\ . & \dots \end{cases} \quad \text{and } K_x = \begin{cases} 1 & \text{if } 0 < T(x) \le 1, \\ 2 & \text{if } 1 < T(x) \le 2, \\ 3 & \text{if } 2 < T(x) \le 3, \\ . & \dots \end{cases}$$

Q: Which of e_x**,** $\overset{\circ}{e}_x$**,** $E(K_x)$ **is larger ?** $K(x)$, $T(x)$, K_x.

Q: $f_{K_x}(t) = f_{K(x)}(\ )$ **?** $E(K_x) = E(K(x))$**... ?**

Theorem 2.39. $\mathbb{P}\{K_x = k\} = \mathrm{P}\{k-1 < T(x) \le k\} = {}_{k-1}|q_x$
$= {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j=0}^{k-2} p_{x+j}\right) q_{x+k-1} = {}_{k-1}p_x - {}_kp_x = {}_kq_x - {}_{k-1}q_x$
$= \mathrm{P}\{k-1 < X - x \le k | X > x\} = \frac{s(x+k-1)-s(x+k)}{s(x)}$.

Example 2.50. *Suppose* $p_{90} = 0.05, \quad p_{91} = 0.01, \quad p_{92} = 0.001, \quad p_{93} = 0$ *(*$S_X(94) = 0$*). Calculate the probability mass function of* K_{90}.

Solution: Which formula to use in Th 2.39 ?? How about the formulas sheet ?

$$\begin{aligned} Th.2.39\ \mathrm{P}\{K_{90} = 1\} &= q_{90} = 1 - p_{90} = (1 - 0.05) = 0.95, && \textit{sheet} \\ 3rd: \mathrm{P}\{K_{90} = 2\} &= p_{90}q_{91} = (0.05)(1 - 0.01) = 0.0495, && [8] \\ 4th: \mathrm{P}\{K_{90} = 3\} &= p_{90}p_{91}q_{92} = (0.05)(0.01)(1 - 0.001) = 0.0004995, && [8] \\ \mathrm{P}\{K_{90} = 4\} &= p_{90}p_{91}p_{92}q_{93} = (0.05)(0.01)(0.001)(1 - 0) = 0.0000005, \\ \mathrm{P}\{K_{90} = k\} &= 0, \text{ for } k = 5, 6, 7, \dots, \text{ as } p_{93} = P(X > 94 | X > 93) = 0 \end{aligned}$$

Theorem 2.40.

Example 2.51. *Calculate the probability mass function of K_{90} for given*

$$p_{90} = 0.2, \quad {}_2p_{90} = 0.1, \quad {}_3p_{90} = 0.01, \quad {}_4p_{90} = 0.005, \quad {}_5p_{90} = 0.$$

Solution: Which formula in Th 2.39 ??

$$\begin{aligned}
&\mathrm{P}\{K_{90} = 1\} = 1 - p_{90} = 1 - 0.2 = 0.8, \\
5th:\ &\mathrm{P}\{K_{90} = 2\} = p_{90} - {}_2p_{90} = 0.2 - 0.1 = 0.1, \qquad\qquad [8]:\ K(x) = \lceil T(x) \rceil - 1 \\
&\mathrm{P}\{K_{90} = 3\} = {}_2p_{90} - {}_3p_{90} = 0.1 - 0.01 = 0.09, \\
&\mathrm{P}\{K_{90} = 4\} = {}_3p_{90} - {}_4p_{90} = 0.01 - 0.005 = 0.005, \\
&\mathrm{P}\{K_{90} = 5\} = {}_4p_{90} - {}_5p_{90} = 0.005 - 0 = 0.005, \\
&\mathrm{P}\{K_{90} = k\} = 0, \text{ for } k = 6, 7, \ldots, \ \textbf{Why ??}
\end{aligned}$$

Theorem 2.41.

Theorem 2.42.

Theorem 2.43. $e_x = E[K(x)] = \sum_{k=1}^{\infty} k \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} {}_kp_x$ *and*

$$E[(K(x))^2] = \textstyle\sum_{k=1}^{\infty} k^2 \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} (2k-1) \cdot {}_kp_x.$$

Proof. $E[K(x)] = \sum_{k=1}^{\infty} k f_{K(x)}(k) = \sum_{k=1}^{\infty} k \cdot {}_{k|}q_x$ and
$E[(K(x))^2] = \sum_{k=1}^{\infty} k^2 f_{K(x)}(k) = \sum_{k=1}^{\infty} k^2 \cdot {}_{k|}q_x$.

Formula [2]: If $P(X \in \{0, 1, 2, ...\}) = 1$ and $H \uparrow$, then

(1) $E[H(X)] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k-1))$,

(2) $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$,

(3) $E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k-1)$, $\qquad K(x) = k$ if $T(x) \in (k, k+1]$

$$(2) \text{ with } K(x) \text{ for } X \ => E[K(x)] = \sum_{k=1}^{\infty} \mathbb{P}\{K(x) \geq k\} = \sum_{k=1}^{\infty} \mathbb{P}\{T(x) > k\} = \sum_{k=1}^{\infty} {}_kp_x,$$

$$(3) \text{ with } K(x) \text{ for } X \ => E[(K(x))^2] = \sum_{k=1}^{\infty} \mathbb{P}\{K(x) \geq k\}(2k-1) = \sum_{k=1}^{\infty} (2k-1) \cdot {}_kp_x.$$

$$E(T(x)) = \int_0^{\infty} {}_tp_x dt, \qquad E[K(x)] = \sum_{k=1}^{\infty} {}_kp_x$$

$$E(T^2(x)) = \int_0^{\infty} 2t_tp_x dt, \qquad E[(K(x))^2] = \sum_{k=1}^{\infty} (2k-1) \cdot {}_kp_x$$

Remark. Recall $K(x) = \lceil T(x) \rceil - 1$.

What is the meaning of $T(30) = 0.5$?

What is the meaning of $T(30) > 0.5$?

If $T(30) = 0.5$, $K_{30} = ?$ $\qquad K(30) = ?$

Let $H(T(x)) = K(x)$ ($= \lceil T(x) \rceil - 1$, where $K(x) = i$ if $T(x) \in (i, i+1]$, $i \in \{0, 1, 2, ...\}$, then

$$H'(t) \begin{cases} \text{does not exist} & \text{if } t \in \{0, 1, 2, ...\} \\ 0 & \text{if } t \in (i, i+1),\ i \in \{0, 1, 2, ...\}, \end{cases}$$

thus $E(H(T(x))) = \int_0^\infty H'(t)_t p_x dt$ in Formula[1] is not applicable to $H(T(x)) = K(x)$. Otherwise, $E(K(x)) = 0$.

Example 2.52. *Suppose that*

x	*90*	*91*	*92*	*93*	*94*
p_x	*0.2*	*0.1*	*0.05*	*0.01*	*0*

. $e_{90} =$ *?*

Solution: Theorem 2.43: $e_x = E(K(x)) = \sum_{k=1}^\infty k \cdot {}_k|q_x = \sum_{k=1}^\infty {}_k p_x$ (formula [9]). Formula [4]: ${}_2p_x = p_x p_{x+1}$, or ${}_{n+k}p_x = {}_n p_x \cdot {}_k p_{x+n}$, ...

$$e_{90} = \sum_{k=1}^\infty {}_k p_{90} = p_{90} + p_{90} \cdot p_{91} + p_{90} \cdot p_{91} \cdot p_{92} + p_{90} \cdot p_{91} \cdot p_{92} \cdot p_{93} + \cdots$$
$$= (0.2) + (0.2)(0.1) + (0.2)(0.1)(0.05) + (0.2)(0.1)(0.05)(0.01) + 0 \approx 0.22.$$

Example 2.53. *Suppose that* $s(t) = \frac{100-t}{100}$, *for* $0 \le t \le 100$. *Find* $\mathring{e}_x$ *and* e_x, *where* x *is an integer.*

Solution: Formulas:
[1] $\mathring{e}_x = E(T(x)) = \int_0^\infty {}_t p_x \, dt$,
[3] ${}_t p_x = \frac{s(x+t)}{s(x)}$ and
[9] $e_x = E(K(x)) = \sum_{k=1}^\infty {}_k p_x$.
${}_t p_x = \frac{s(x+t)}{s(x)} = \frac{100-x-t}{100-x}$, $0 < x + t \le 100$.

$$\mathring{e}_x = \int_0^{100-x} {}_t p_x \, dt = -\int_0^{100-x} \frac{100-x-t}{100-x} d(100-x-t)$$
$$= -\frac{(100-x-t)^2}{2(100-x)} \Big|_0^{100-x} = \frac{100-x}{2} ...?$$
$$e_x = \sum_{k=1}^\infty {}_k p_x = \sum_{k=1}^{100-x} \frac{100-x-k}{100-x} = \sum_{k=1}^{100-x} [1 - \frac{k}{100-x}]$$
$$= 100 - x - \frac{1}{100-x} \frac{(100-x)(100-x+1)}{2} \qquad (\sum_{k=1}^n k = \frac{n(n+1)}{2})$$
$$= 100 - x - \frac{100-x+1}{2} = \frac{99-x}{2},\ x \in [0, 99] \text{ or } x \in [0, 100) \ ?$$

Definition 2.34.

Definition 2.35.

Definition 2.36.

Theorems 2.44-2.49 below are summarized as formulas as follows.

[9] $e_x = E[K(x)]$, $e_{x:\overline{n}|} = E(K(x) \wedge n)$.
$e_x = p_x(1 + e_{x+1})$, $e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}$,
$e_x = \sum_{k=1}^{\infty} {}_k p_x$, $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x$.
$E[(K(x))^2] = \sum_{k=1}^{\infty} (2k-1) \cdot {}_k p_x$.

Theorem 2.44. *(Iterative formula for e_x)* $e_x = p_x(1 + e_{x+1})$.

Theorem 2.45. *If $p_{x+k} = p_x$, for each integer $k \geq 1$. Then, $e_x = \frac{p_x}{1-p_x}$.*

Theorem 2.46. $e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k \cdot {}_{k|}q_x + n \cdot {}_n p_x = \sum_{k=1}^{n} {}_k p_x$.

Theorem 2.47. $e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}$.

Theorem 2.48. *If $p_{x+k} = p$, for each integer $k \geq 0$. Then, $e_{x:\overline{n}|} = \frac{p - p^{n+1}}{1-p}$.*

Theorem 2.49. $e_{x:\overline{n}|} = p_x \left(1 + e_{x+1:\overline{n-1}|}\right)$.

Example 2.54. *Suppose that $e_x = 30$, $p_x = 0.97$ and $p_{x+1} = 0.95$. $e_{x+2} =$?*

Solution: Method 1. [9.3] or Th2.44: $e_x = p_x(1 + e_{x+1})$ =>
$e_{x+1} = \frac{e_x}{p_x} - 1$ and $e_{x+2} = \frac{e_{x+1}}{p_{x+1}} - 1$. Then
$e_{x+1} = \frac{30}{0.97} - 1 \approx 29.9$ and
$e_{x+2} = \frac{e_{x+1}}{p_{x+1}} - 1 = \frac{29.9}{0.95} - 1 \approx 30.5$.
Method 2. [9.4] or Th2.47: $e_x = e_{x:\overline{2}|} + {}_2 p_x e_{x+2}$ => $\frac{e_x - e_{x:\overline{2}|}}{{}_2 p_x} = e_{x+2}$.

by [9.5] $e_{x:\overline{2}|} = \sum_{k=1}^{2} {}_k p_x$, by [4], ${}_2 p_x = p_x \cdot p_{x+1}$,

$$e_{x:\overline{2}|} = p_x + {}_2 p_x = 0.97 + (0.97)(0.95) = 1.8915.$$

$$30 = e_{x:\overline{2}|} + {}_2 p_x e_{x+2} = 1.8915 + (0.97)(0.95) e_{x+2}.$$

So, $e_{x+2} = \frac{30 - 1.8915}{(0.97)(0.95)} = 30.50298426$.

Example 2.55. *Suppose that*

x	*90*	*91*	*92*	*93*	*94*
p_x	*0.2*	*0.1*	*0.05*	*0.01*	*0*

. $e_{90:\overline{2}|} =$?

Solution: By [9.4] or Th2.46, $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x$. **n= ?**
${}_1 p_x =$? By [4.1] ${}_2 p_x = {}_1 p_x \cdot {}_1 p_{x+1}$,

$$e_{90:\overline{2}|} = p_{90} + p_{90} p_{91} = (0.2) + (0.2)(0.1) = 0.22.$$

Definition 2.37. *Let $S_x = T(x) - K(x)$, the period of time lived through the death interval of an entity aged x.*

S_x is a r.v. taking values in the interval $(0,1]$. Notice that $E[S_x] = \mathring{e}_x - e_x$.

2.6 Selected survival models.

A **select table** is a mortality table for a group of people subject to a special circumstance (disability, retirement, etc). The variable in common of this group of people is called the **concomitant variable**. The probability of surviving from time x, to time $x+t$ for an entity selected at time x is ${}_tp_{[x]}$. Here, the age at selection is denoted by $[x]$. The select survival function is denoted by $S(x;t) = {}_tp_{[x]}$. The force of mortality is $\mu_{[x]+t} = -\frac{d}{dt}\ln S(x;t)$. The expected future life is

$$\mathring{e}_{[x]} = \int_0^\infty S(x;t)\,dt.$$

Example 2.56.

2.7 Common analytical survival models

2.7.1 De Moivre model.

Definition 2.38. *The age–at–death X follows* **De Moivre mortality law** *with terminal age ω, if the distribution of $X \sim U(0,\omega)$.*

Definition 2.39.

Example 2.57.

Example 2.58.

Example 2.59.

Example 2.60.

Example 2.61.

Theorem 2.50.

Theorem 2.51.

Theorem 2.52.

Theorem 2.53.

Theorem 2.54.

Theorem 2.55.

Theorem 2.56.

Theorem 2.57.

Theorem 2.58.

2.7.2 Generalized De Moivre model.

Definition 2.40. *The age–at–death X follows a* **generalized De Moivre mortality** *if* $s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$, *for* $0 \le x \le \omega$, *where* $\alpha > 0$.

If $\alpha = 1$, we have the usual De Moivre law.

Theorem 2.59.

Example 2.62. *The future lifetime of a new born has survival function* $s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$, *for* $0 \le x \le \omega$, *where* $\alpha, \omega > 0$. *Suppose that* $\overset{\circ}{e}_{40} = 8$ *and* $\overset{\circ}{e}_{60} = 4$. *Calculate* α *and* ω.

Solution: Need to compute e_x.
Formulas: $\overset{\circ}{e}_x = \int_0^{\infty} {}_tp_x \, dt$ and ${}_tp_x = \frac{s(x+t)}{s(x)}$.

$$
\begin{aligned}
{}_tp_x &= \frac{s(x+t)}{s(x)} = \frac{(\omega - x - t)^{\alpha}}{(\omega - x)^{\alpha}}, \ 0 \le t \le ? \ (t + x < \omega), \\
\overset{\circ}{e}_x &= \int_0^{\infty} {}_tp_x \, dt = \int_0^{\omega - x} \frac{(\omega - x - t)^{\alpha}}{(\omega - x)^{\alpha}} \, dt = \frac{-(\omega - x - t)^{\alpha+1}}{(\alpha+1)(\omega - x)^{\alpha}} \Bigg|_0^{\omega - x} = \frac{\omega - x}{\alpha + 1}.
\end{aligned}
$$

$8 = \overset{\circ}{e}_{40} = \frac{\omega - 40}{\alpha + 1}$ and
$4 = \overset{\circ}{e}_{60} = \frac{\omega - 60}{\alpha + 1}$.
Dividing one equation over the other, $2 = \frac{\omega - 40}{\omega - 60}$.
So, $(2)(\omega - 60) = \omega - 40$ and $\omega = 80$.
Hence, $8 = \frac{80 - 40}{\alpha + 1} \Rightarrow \alpha = \frac{80 - 40}{8} - 1 = 4$.

2.7.3 Exponential model.

Theorem 2.60. *An exponential r.v. X satisfies, for $x \ge 0$,*

$$
\begin{aligned}
& f_X(x) = \frac{1}{\theta} e^{-x/\theta} = \mu e^{-\mu x}, \quad F_X(x) = 1 - e^{-x/\theta} = 1 - e^{-\mu x}, \quad S_X(x) = e^{-x/\theta} = e^{-\mu x}, \\
& E[X] = \theta \ \textit{and} \ \mathrm{Var}(X) = \theta^2.
\end{aligned}
$$

We write $X \sim Exp(1/\mu)$ or $Exp(\theta)$ and X has constant force of mortality μ. The exponential model is also called the **constant force model**.

Theorem 2.61.

Theorem 2.62. (**Memoryless property of the exponential distribution**) *Let X have an exponential distribution. Then, for each $s, t > 0$,*

$$\mathrm{P}\{T(s) > t\} = \mathrm{P}\{X > s + t \mid X > s\} = \mathrm{P}\{X > t\}.$$

Definition 2.41. *A r.v. X has a* **geometric distribution** *with parameter p, if*

$\mathrm{P}\{X = k\} = (1-p)^k p,\ k = 0, 1, 2\ldots$ *where $0 < p \leq 1$, (# of failures before the 1st success).*

Remark. The geometric distribution in Math 447 is $Y = X + 1$ (# of trials before the 1st success).

Theorem 2.63. (Memoryless property of the geometric distribution) *Let X be a r.v. with a geometric distribution. Then, for each integers $k, n \geq 1$,*

$$\mathrm{P}\{X \geq k + n \mid X \geq k\} = P\{X \geq n\}.$$

Proof. $\vdash$: $\mathrm{P}\{X \geq n\} = (1-p)^n = \mathrm{P}\{X \geq k + n \mid X \geq k\},\ k = 0, 1, 2\ldots$

$\mathrm{P}\{X \geq n\} = \sum_{j=n}^{\infty} \mathrm{P}\{X = j\}$
$= \sum_{j=n}^{\infty} (1-p)^j p \qquad (\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x})(see formula[17])$.
$= p\frac{(1-p)^n}{1-(1-p)}$
$= (1-p)^n$.

$\mathrm{P}\{X \geq k + n \mid X \geq k\}$
$= \frac{\mathrm{P}\{X \geq k+n\}}{\mathrm{P}\{X \geq k\}}$
$= \frac{(1-p)^{k+n}}{(1-p)^k}$
$= (1-p)^n$. Done.

Theorem 2.64. *Let X be a geometric distribution with parameter p. Then, $E[X] = \frac{q}{p}$ and* $\mathrm{Var}(X) = \frac{q}{p^2}$.

Proof. (Math 447): $E(Y) = 1/p$ and $V(Y) = q/p^2$, where $Y = X + 1$.

$Y = \#$ of trials to have a success.

$X = \#$ of trials before a success.

$E(X) = E(Y - 1) = (1/p) - 1 = (1-p)/p = q/p$.

$V(X) = V(Y - 1) = V(Y) = q/p^2$.

Theorem 2.65. *Suppose that for each $k = 1, 2, \ldots,\ p_{x+k} = p_x$. Then, the curtate lifetime $K(x)$ follows a geometric distribution with parameter $p = 1 - p_x$.*

Proof. $\mathbb{P}\{K(x) \geq k\} = {}_k p_x = p_x p_{x+1} \cdots {}_{k-1}p_x = p_x^k$ and

$$\mathbb{P}\{K(x) = k\} = \mathbb{P}\{K(x) \geq k\} - \mathbb{P}\{K(x) \geq k + 1\} = p_x^k - p_x^{k+1} = p_x^k(1 - p_x).$$

Hence, $K(x)$ has a geometric distribution with parameter $p = 1 - p_x$. ∎

Example 2.63.

Example 2.64.

Example 2.65.

Example 2.66. *Suppose that:*
(a) the force of mortality is constant.
(b) the probability that a 30–year–old will survive to age 40 is 0.95.
Calculate:
(i) the expected future lifetime of a 40–year–old.
(ii) the curtate life expectation of a 40–year–old.

Solution: (i) $\overset{\circ}{e}_{40} = E(T(40)) = ?$
(a) $=> T(x) \sim Exp(1/\mu)$. $=> E(T(40)) = 1/\mu = ?$
(b) $=> {}_{10}p_{30} = e^{-10\mu} = 0.95$.

Hence, $\mu = \frac{-\ln(0.95)}{10}$ and
$\overset{\circ}{e}_{40} = \frac{1}{\mu} = \frac{10}{-\ln(0.95)} \approx 195.0$.

(ii) $e_x = \sum_{k=1}^{\infty} {}_kp_x$,
${}_kp_x = e^{-\mu k}$,
$\sum_{k=1}^{n} x^k = x\frac{1-x^n}{1-x}$. (see formula [16]).

$e_x = \frac{e^{-\mu}}{1-e^{-\mu}}$.

Since $e^{-\mu} = (0.95)^{0.1}$, $e_{40} = \frac{1}{e^{\mu}-1} = \frac{1}{(0.95)^{-0.1}-1} \approx 194.5$.

Theorem 2.66. *Suppose that for each* $k = 1, 2, \ldots$, $p_{x+k} = p_x$. *Then,* $e_{x:\overline{n}|} = \frac{p_x(1-p_x^n)}{q_1}$.

Note: It is due to Formula: $\sum_{k=0}^{n} p^k = \frac{1-p^{n+1}}{1-p}$ (see formula [16] and [17]).

Proof. $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x = \sum_{k=1}^{n} p_x^k = \frac{p_x(1-p_x^n)}{q_1}$.

Theorem 2.67.

2.7.4 Gompertz model.

2.7.5 Makeham model.

Makehan (1860) introduced the model $\mu(x) = A + Bc^x$, where $A \geq -B$, $B > 0$ and $c > 1$. Hence,
$S_X(x) = e^{-Ax-m(c^x-1)}$ for $x \geq 0$, where $m = \frac{B}{\ln c}$.
$f_X(x) = s(x)\mu(x) = (A + Bc^x)e^{-Ax-m(c^x-1)}, x \geq 0$.

2.7.6 Weibull model.

2.7.7 Pareto model with parameters α (> 0) and θ (> 0).

$$S_X(x) = \left(\frac{\theta}{x+\theta}\right)^\alpha, \ x > 0;$$
$$f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \ x > 0,$$

2.8 Mixture distributions

Recall that letting X and Y be two r.v.'s,

$$f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y),$$
$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y),$$
$$f_X(x) = \begin{cases} \int f_{X,Y}(x,y)dy & \text{if } Y \text{ is cts.} \\ \sum_y f_{X,Y}(x,y) & \text{if } Y \text{ is dis} \end{cases} = \begin{cases} \int f_{X|Y}(x|y)f_Y(y)dy & \text{if } Y \text{ is cts.} \\ \sum_y f_{X|Y}(x|y)f_Y(y) & \text{if } Y \text{ is dis} \end{cases}$$

F_X is called the marginal cdf of X, and is called the mixture distribution also here.

Question. $f(x|y) = \frac{f(x,y)}{f(y)}$?
Can we say that

X has df $f(x) = e^{-x}$, $x > 0$ and Y has df $f(y) = 1$, $y \in (0,1)$ **??**

How about : X has df $f_X(x) = e^{-x}$, $x > 0$ and Y has df $f_Y(x) = 1$ $x \in (0,1)$?

Theorem 2.68. (**Double expectation theorem for expectations**) $E[E[X|Y]] = E[X]$.

Theorem 2.69. (**Double expectation theorem for variances**)

$$\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}(X|Y)].$$

Example 2.67. *You are given that:*
(a) Men follow a de Moivre model with terminal age 100.
(b) Women follow a de Moivre model with terminal age 110.
(c) 55% of births are male.
(i) Calculate the expectated life of a randomly chosen life.
(ii) Calculate the probability that a newborn survives 80 years.
(iii) Calculate the density of the future lifetime T of a randomly chosen life.

Solution: (i) Let X be the future lifetime of a newborn. $E(X) = ?$

Let $Y = \begin{cases} 1 & \text{if a new born is male,} \\ 0 & \text{if a new born is female.} \end{cases}$ $E[X] = E[E[X|Y]]$

$$E[X] = E[X|Y = 1]\text{P}\{Y = 1\} + E[X|Y = 0]\text{P}\{Y = 0\}.$$

(a) => $X|Y = 1$ follows $U(0, 100)$, So, $E[X|Y = 1] = 50$.
(b) => $X|Y = 0$ follows a $U(0, 110)$. So, $E[X|Y = 0] = 55$. Hence,

$$\begin{aligned} E[X] =& E[X|Y = 1]\text{P}\{Y = 1\} + E[X|Y = 0]\text{P}\{Y = 0\} \\ =& (50)(0.55) + (55)(0.45) = 52.25. \end{aligned}$$

(ii) Letting $Z = I(X > 80)$ ($\sim bin(1, p)$), $\mathrm{P}\{X > 80\} = E(Z) = ?$ $\quad(= \sum_i iP(Z = i))$

$$\begin{aligned}
\mathrm{P}\{X > 80\} =& E(Z) = E(E(Z|Y)) \\
=& E(Z|Y = 1)P(Y = 1) + E(Z|Y = 0)P(Y = 0) \\
=& \mathrm{P}\{X > 80|Y = 1\}\mathrm{P}\{Y = 1\} + \mathrm{P}\{X > 80|Y = 0\}\mathrm{P}\{Y = 0\} \qquad Z|Y \sim bin(1, p_Y) \\
=& \frac{100 - 80}{100}(0.55) + \frac{110 - 80}{110}(0.45) \approx 0.23.
\end{aligned}$$

(ii) $f_X(x) = \sum_j f_{X,Y}(x, j) = \sum_j f_Y(j)f_{X|Y}(x|j) = (0.55)\underbrace{f_{X|Y}(x|1)}_{U(0,100)} + (0.45)\underbrace{f_{X|Y}(x|0)}_{U(0,110)}$

$$= \begin{cases} 0.55\frac{1}{100} + 0.45\frac{1}{110} & \text{if } 0 < x < 100, \\ 0.55 * 0 + 0.45\frac{1}{110} & \text{if } 100 \leq x < 110, \\ 0 & \text{else.} \end{cases}$$

Example 2.68. *The future lifetime $T(x)$ of a live aged x has constant force of mortality μ. μ has a uniform distribution on $(0.01, 0.05)$.*
(i) Calculate $\mathring{e}_x$.
(ii) Calculate $\mathrm{Var}(T(x))$.

Solution: Given conditions: $T(x)|\mu \sim Exp(\beta)$, $\beta = 1/\mu$, and $\mu \sim U(0.01, 0.05)$.
Formula 23: $X \sim \mathcal{G}(\alpha, \beta)$, $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$, $x > 0$,

$E(X) = \alpha\beta$, $\sigma^2 = \alpha\beta^2$, $Exp(\beta) = \mathcal{G}(1, \beta)$.
Thus $f_{T(x)|\mu}(t) = \frac{1}{\beta}\exp(-t/\beta) = \mu e^{-\mu t}$ and $S_{T(x)|\mu}(t) = \exp(-t/\beta) = e^{-\mu t}$.

(i) $E[T(x)|\mu] = \alpha\beta = \beta = \frac{1}{\mu}$. So,

$$\begin{aligned}
E[T(x)] =& E[E[T(x)|\mu]] = E\left[\frac{1}{\mu}\right] = \int_{0.01}^{0.05} \frac{1}{\mu}\frac{1}{0.05 - 0.01}\,d\mu \\
=& \frac{\ln(\mu)}{0.05 - 0.01}\Bigg|_{0.01}^{0.05} = \frac{\ln(0.05/0.01)}{0.05 - 0.01} \approx 40.2.
\end{aligned}$$

(ii) $E[T(x)|\mu] = \frac{1}{\mu}$ and $\mathrm{Var}(T(x)|\mu) = \alpha\beta^2 = \beta^2 = \frac{1}{\mu^2}$. So,

$$\begin{aligned}
\mathrm{Var}(T(x)) =&\mathrm{Var}(E[T(x)|\mu]) + E[\mathrm{Var}(T(x)|\mu)] = \mathrm{Var}\left(\frac{1}{\mu}\right) + E\left[\frac{1}{\mu^2}\right] \\
=&E\left[\frac{1}{\mu^2}\right] - \left(E\left[\frac{1}{\mu}\right]\right)^2 + E\left[\frac{1}{\mu^2}\right] \\
\approx&2E\left[\frac{1}{\mu^2}\right] - (40.2)^2, \\
E\left[\frac{1}{\mu^2}\right] =& \int_{0.01}^{0.05} \mu^{-2}\frac{1}{0.05-0.01}\,d\mu \\
=&\frac{1}{0.05-0.01}\frac{-1}{\mu}\Big|_{0.01}^{0.05} = \frac{1}{0.05-0.01}\left(\frac{1}{0.01} - \frac{1}{0.05}\right) = 2000. \\
=> \quad \mathrm{Var}(T(x)) =&2 * 2000 - (40.2)^2 \approx 2381.1.
\end{aligned}$$

Theorem 2.70.

Theorem 2.71.

2.9 Estimation of the survival function

There are two typical types of estimators of the survival functions:
(1) Parametric estimators such as
the maximum likelihood estimator (MLE),
the method of moment estimator (MME), or
the Bayes estimator, *etc.*.
(2) Non-parametric estimators such as
the non-parametric maximum likelihood estimator (NPMLE) and
the Nelson-Aalen estimator.

Given a parametric distribution form, say $S_X(x;\theta)$, where θ is the parameter, the MLE of θ maximizes
$L(\theta) = \prod_{i=1}^n f_X(X_i;\theta)$ over $\theta \in \Theta$, the parameter space.
Let $\hat\theta$ be the MLE, then the MLE of S_X is $S_X(x;\hat\theta)$. For example,
if X_1, ..., X_n are i.i.d. from the survival function $S_X(t) = e^{-t\theta I(t>0)}$,
then the MLE of $S_X(t)$ is

$$\hat S_X(t) = e^{-tI(t>0)/\overline{X}}$$

as $\hat\theta = 1/\overline{X}$ is the MLE of θ, and is also the MME of θ.

The parametric estimator is based on the assumption that the random sample is from a given parametric distribution, say $f_X(x;\theta)$. Otherwise, we make use of the NPMLE of $F(t)$:

$$\hat F(t) = \frac{1}{n}\sum_{i=1}^n I(X_i \le t).$$

$\hat{F}$ is also called the empirical distribution function (EDF), and $\hat{S}(t) = 1 - \hat{F}(t)$ is the estimator of S_X.

Notice that the EDF is a discrte cdf with the d.f.

$$\hat{f}(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i = t).$$

Remark. It is interesting to notice the following facts:

$\sum_x x\hat{f}(x) = \overline{X}$,

$\sum_x (x - \overline{X})^2 \hat{f}(x) = \overline{X^2} - (\overline{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

A modification of $\hat{F}$ is to smooth it. Letting $t_1 < \cdots < t_m$ be the distinct values of X_i's and
$n_i = \sum_{j=1}^{n} I(X_j = t_i)$,
then $\check{F}(t)$ is a continuous piecewise linear function:

$$\check{F}(t) = \begin{cases} 0 & \text{if } t < 0 = t_0, \\ \hat{F}(t_i) & t \in \{t_0, ..., t_m\}, \\ s\hat{F}(t_i) + (1-s)\hat{F}(t_{i+1}) & \text{if } t = st_i + (1-s)t_{i+1},\ s \in (0,1),\ \ i \in \{0, ..., m-1\}, \\ 1 & \text{if } t > t_m. \end{cases}$$

with d.f. $\check{f}(t) = \frac{n_i}{n(t_i - t_{i-1})}$ if $t \in (t_{i-1}, t_i)$, $i \in \{1, ..., m-1\}$.

Right censored data. If one observed X_1, ..., X_n, which are i.i.d. from F_X, it is called a **complete data** set. Sometimes, one cannot observed each X_i. Instead, one observed (Z_1, δ_1), ..., (Z_n, δ_n), where $Z_i = \min\{X_i, C_i\}$, (X_1, C_1), ..., (X_n, C_n) are i.i.d. from $F_{X,C}$ $(= F_X F_C)$, and $\delta_i = I(X_i \le C_i)$, it is called a right censored (RC) data set. The RC data are often recorded as Z_i (exact) or Z_i+ (right-censored).

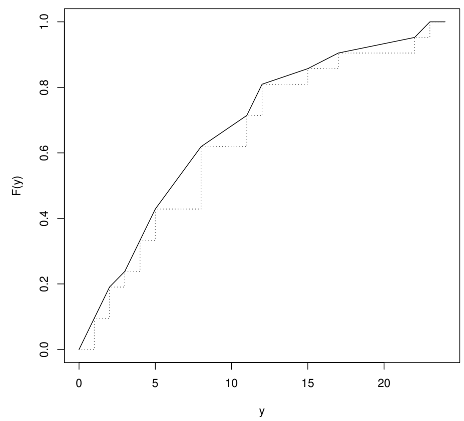
Example 2.69. *Gehan, 1965 recorded times of remission of leukaemia patients (time period that the cancer does not become worsen). Some were treated with drug 6-mercaptopurine (6-MP), the others were serving as a control.*

Table 1.1 (Cox and Oakes (1984) (pages 7,8)). Time of remission (weeks).

Group 0 (6-MP): 6+, 6, 6, 6, 7, 9+, 10+, 10, 11+, 13, 16, 17+, 19+, 20+, 22, 23, 25+, 32+, 32+, 34+, 35+ (m=21),

Group 1 (control): 1, 1, 2, 2, 3, 4, 4, 5, 5, 8, 8, 8, 8, 11, 11, 12, 12, 15, 17, 22, 23 (n=21). Derive $\hat{F}$ and $\check{F}$ using the control group data.

Sol. $\hat{F}(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le t)$



$$\hat{F}(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{2}{21} & \text{if } t \in [1,2) \\ \frac{4}{21} & \text{if } t \in [2,3) \\ \frac{5}{21} & \text{if } t \in [3,4) \\ \frac{7}{21} & \text{if } t \in [4,5) \\ \frac{9}{21} & \text{if } t \in [5,8) \\ \frac{13}{21} & \text{if } t \in [8,11) \\ \frac{15}{21} & \text{if } t \in [11,12) \\ \frac{17}{21} & \text{if } t \in [12,15) \\ \frac{18}{21} & \text{if } t \in [15,17) \\ \frac{19}{21} & \text{if } t \in [17,22) \\ \frac{20}{21} & \text{if } t \in [22,23) \\ 1 & \text{if } t \geq 23 \end{cases}$$

```
X=c(0,1, 2, 3, 4, 5, 8, 11, 12, 15, 17, 22, 23,24)
y=c(0,2, 4, 5, 7, 9, 13, 15, 17, 18, 19, 20, 21,21)/21
plot(X,y,type="s",lty=3, xlab="y", ylab="F(y)")
lines(X,y,type="l",lty=1)
```

The curve of $\hat{F}(t)$ is a step function with jumps at $\{1, 2, 3, 4, 5, 8, 11, 12, 15, 17, 22, 23\}$.

The curve of $\check{F}(t)$ is a piecewise linear curve of $\hat{F}$.

There are two common non-parametric estimators of a survival function with right-censored data. One is the NPMLE, which is also called the **Kaplan-Meier estimator** (KME) or the **product-limit-estimator** (PLE):

$$\hat{S}_{pl}(t) = \prod_{t_k \le t} (1 - \frac{d_k}{r_k}),$$

where $t_1 < \cdots < t_m$ are distinct values of Z_i's with $\delta_i = 1$,
d_k is the number of person died at time t_k, and
r_k is the number of person at risk at time t_k $(= \sum_{i=1}^n I(Z_i \ge t_k))$.
An estimator of $\sigma^2_{\hat{S}_{pl}(t)}$ is

$$\hat{\sigma}^2_{\hat{S}_{pl}(t)} = \frac{1}{n}(\hat{S}_{pl}(t))^2 \sum_{k:\ t_k \le t} \frac{\hat{f}_{pl}(t_k)}{\hat{S}_Z(t_k-)\hat{S}_{pl}(t_k)}.$$

A 95% confidence interval (CI) of $S_X(t)$ is $\hat{S}_{pl}(t) \pm 1.96\hat{\sigma}_{\hat{S}_{pl}(t)}$.

The other estimator is the **Nelson-Aalen estimator**:

$$\tilde{S}_{NA}(t) = e^{-\sum_{t_k \le t} \frac{d_k}{r_k}}$$

Its variance can be estimated by $\hat{\sigma}^2_{\tilde{S}_{NA}(t)} = (\tilde{S}_{NA}(t))^2 \hat{\sigma}^2_{H(t)}$, where
$\hat{\sigma}^2_{H(t)} = \sum_{t_j \le t} \frac{(r_j - d_j)d_j}{(r_j-1)r_j^2}$.
A 95% CI of $S_X(t)$ is $\tilde{S}_{NA}(t) \pm 1.96\hat{\sigma}_{\tilde{S}_{NA}(t)}$.

Questions: $\hat{S}_{pl}(0) = ?$ why ? $\tilde{S}_{NA}(0) = ?$ why ?

Example 2.70. *A follow-up study on a five-year insurance policies is summarized in the the next table. In the table,*

i	x_i	u_i	i	x_i	u_i
1	–	0.1	16	4.8	–
2	–	0.5	17	–	4.8
3	–	0.8	18	–	4.8
4	0.8	–	19 – 30	–	5
5	–	1.8	31	–	5
6	–	1.8	32	–	5
7	–	2.1	33	4.1	–
8	–	2.5	34	3.1	–
9	–	2.8	35	–	3.9
10	2.9	–	36	–	5
11	2.9	–	37	–	4.8
12	–	3.9	38	4.0	–
13	4.0	–	39	–	5
14	–	4.0	40	–	5
15	–	4.1			

(1) i is the policy number, 1-40;

(2) x_i is the duration at which the insured was observed to die. Those who didn't die has "-" in that column;

(3) u_i is the last duration at which those who did not die were observed.

Compute the KME and the Nelson-Aalen estimator of the survival function.

Solution. First try to understand the data, in terms of (Z_i, δ_i).

i	1	2	3	4	5	6	...
Z_i	0.1	0.5	0.8	0.8	1.8	1.8	...
δ_i	0	0	0	1	0	0	...

Then rearrange the data as follows.

i	x_i	u_i	i	x_i	u_i
1	–	0.1	16	4.8	–
2	–	0.5	17	–	4.8
3	–	0.8	18	–	4.8
4	0.8	–	19 – 30	–	5
5	–	1.8	31	–	5
6	–	1.8	32	–	5
7	–	2.1	33	4.1	–
8	–	2.5	34	3.1	–
9	–	2.8	35	–	3.9
10	2.9	–	36	–	5
11	2.9	–	37	–	4.8
12	–	3.9	38	4.0	–
13	4.0	–	39	–	5
14	–	4.0	40	–	5
15	–	4.1			

is rearranged as

i	x_i	u_i	i	x_i	u_i
1	–	0.1	37	–	4.8
2	–	0.5	17	–	4.8
3	–	0.8	18	–	4.8
4	0.8	–	19 – 30	–	5
5	–	1.8	31	–	5
6	–	1.8	32	–	5
7	–	2.1			
8	–	2.5			
9	–	2.8			
10	2.9	–	36	–	5
11	2.9	–			
34	3.1	–			
12	–	3.9			
35	–	3.9			
38	4.0	–			
13	4.0	–	39	–	5
14	–	4.0	40	–	5
15	–	4.1			
33	4.1	–			
16	4.8	–			

time	0.1	0.5	0.8	1.8	2.1	2.5	2.8	2.9	3.1	3.9	4.0	4.1	4.8	5.0
# of events	1	1	2	2	1	1	1	2	1	2	3	2	4	17
# of deaths	0	0	1	0	0	0	0	2	1	0	2	1	1	0

The two estimators are $\hat{S}_{pl}(t) = \prod_{t_k \le t}(1 - \frac{d_k}{r_k})$ and $\tilde{S}_{NA}(t) = \exp(-\sum_{t_k \le t} \frac{d_k}{r_k})$,

where $t_1 < \cdots < t_m$ are distinct values of Z_i's with $\delta_i = 1$,

d_k is the number of person died at time t_k, and

r_k is the number of person at risk at time t_k $(= \sum_{i=1}^n I(Z_i \ge t_k))$.

(t_i, d_i, r_i) are given as folllows.
$$\begin{array}{ccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 \\ t_i & 0.8 & 2.9 & 3.1 & 4.0 & 4.1 & 4.8 \\ d_i & 1 & 2 & 1 & 2 & 1 & 1 \\ r_i & 38 & 31 & 29 & 26 & 23 & 21 \end{array}$$

The following table calculates the KME.

Table 1. Calculation of PLE or KME		
Survival Time	$(1-\frac{d_k}{r_k})$	$\hat{S}_{pl}(t_k)$
0.8	37/38	37/38
2.9	29/31	(37/38)(29/31)
3.1	28/29	(37/38)(29/31)(28/29)
4	24/26	(37/38)(29/31)(28/29)(24/26)
4.1	22/23	(37/38)(29/31)(28/29)(24/26)(22/23)
4.8	20/21	(37/38)(29/31)(28/29)(24/26)(22/23)(20/21)

$$\hat{S}_{pl}(t) = \prod_{k\le t}(1-\frac{d_k}{r_k}) = \begin{cases} 1 & \text{if } t < 0.8, \\ \frac{37}{38}\ (\approx 0.97) & \text{if } t \in [0.8, 2.9), \\ \frac{37*29}{38*31}\ (\approx 0.91) & \text{if } t \in [2.9, 3.1), \\ \frac{37*28}{38*31}\ (\approx 0.88) & \text{if } t \in [3.1, 4), \\ ... & ... \\ ... & \text{if } t \in [4.8, \infty). \end{cases}$$

$\hat{S}_{pl}(100) = 0$?

$$\tilde{S}_{NA}(t) = e^{-\sum_{k\le t}\frac{d_k}{r_k}} = \begin{cases} 1 & \text{if } t < 0.8, \\ \exp(-1/38) & \text{if } t \in [0.8, 2.9), \\ \exp(-(\frac{1}{38}+\frac{2}{31})) & \text{if } t \in [2.9, 3.1), \\ \exp(-(\frac{1}{38}+\frac{2}{31}+\frac{1}{29})) & \text{if } t \in [3.1, 4), \\ ... & ... \\ ... & \text{if } t \in [4.8, \infty). \end{cases}$$

$\hat{S}_{NA}(100) = 0$?

CHAPTER 3

Life Tables

3.1 Life tables

Definition 3.1.
ℓ_x *denotes the* **number of individuals alive** *at age* x*, where* $x \geq 0$*.*
ℓ_x *is also called the* **number living** *or the* **number of lives** *at age* x*.*
ℓ_0 *is called the* **radix** *of a life table.*
A **life table** *(see Example 3.1) is a display of* ℓ_k*, for each* $k = 0, 1, 2, \ldots$
${}_t d_x$ *denotes the number of people which died between ages in* $[x, x+t)$*.*

Note: In the life table, one may assume $P(X = t) = 0$.
Based on life table, one can estimate $F_X(t)$ by the EDF
$\hat{F}(t) = \frac{1}{n}\sum_{i=1}^n I(X_i \leq t) = 1 - \frac{l_{x+t}}{l_x}$ for $x,\ t = 0, 1, 2, \ldots$

$$\textbf{main formula: } \hat{S}_X(x) = \frac{\ell_x}{\ell_0},\ \textit{and } {}_t p_x = \frac{\ell_{x+t}}{\ell_x},$$

$$\textbf{secondary: } {}_t q_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} = \frac{{}_t d_x}{\ell_x},\ {}_t d_x = \ell_x - \ell_{x+t}, \tag{3.1}$$

$$p_x = \frac{\ell_{x+1}}{\ell_x},\ q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x},\ {}_{n|m}q_x = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x} = \frac{{}_m d_{x+n}}{\ell_x}.$$

Example 3.1. *Complete the entries in the table:*

Age	ℓ_x	d_x	p_x	q_x
0	100000	.	.	.
1	97523	.	.	.
2	94123	.	.	.
3	91174	.	.	.
4	87234	.	.	.
5	85938	–	–	–

$p_2 =$ *?*

Solution:

Age x	ℓ_x	$d_x = \ell_x - \ell_{x+1}$	$p_x = \ell_{x+1}/\ell_x$	$q_x = 1 - p_x = d_x/\ell_x$
0	100000	2477	0.97523	0.02477
1	97523	3400	0.96514	0.03486
2	94123	2949	0.96867	0.03133

Example 3.2. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

(i) Calculate d_x *for* $x = 80, 81, \ldots, 86$*.*
(ii) Calculate the d.f. of the curtate life $K(80)$ *and the time interval of death* K_{80}*.*
(iii) Calculate e_{80}*,* $\mathrm{Var}(K(80))$*, and* $e_{80:\overline{3}|}$

Solution: (i) $d_x = ?$

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
$d_x = \ell_x - \ell_{x+1}$	33	56	54	45	34	28	0

(ii) $f_{K(x)} = ?$ $K(x) = K_x - 1$, $K_x = \lceil T(x) \rceil$ (see [8]).
$f_{K(80)}(k) = P\{k < T(x) \leq k+1\} = {}_kp_x - {}_{k+1}p_x = \frac{\ell_{x+k} - \ell_{x+k+1}}{\ell_x} = \frac{d_{x+k}}{\ell_x}$ (see [11]).

$$f_{K_x}(k+1) = f_{K(x)}(k) = \begin{cases} \frac{33}{250} & \text{if } k = 0, \\ \frac{56}{250} & \text{if } k = 1, \\ \frac{54}{250} & \text{if } k = 2, \\ \frac{45}{250} & \text{if } k = 3, \\ \frac{34}{250} & \text{if } k = 4, \\ \frac{28}{250} & \text{if } k = 5, \\ 0 & \text{else.} \end{cases}$$

(iii) $e_{80} = E[K(80)] = ?$ $V(K(80)) = E[(K(80))^2] - e_{80}^2 = ?$
Two ways for $E(K(x))$ or $E((K(x))^2)$:
(1) $E(X) = \sum_x x p_X(x)$ or $E(g(X)) = \sum_x g(x) f_X(x)$ **What is X here ??**
(2) $E(K(x)) = \sum_{k=1}^{\infty} {}_kp_x$ and $E((K(x))^2) = \sum_{k=1}^{\infty}(2k-1)_kp_x$ (${}_kp_x = \frac{\ell_{x+k}}{\ell_x}$ (see [11])).

$$\begin{aligned}
(1) \quad e_{80} =& E[K(80)] = \sum_k k f_{K(x)}(k) \\
=& (0)\frac{33}{250} + (1)\frac{56}{250} + (2)\frac{54}{250} + (3)\frac{45}{250} + (4)\frac{34}{250} + (5)\frac{28}{250} = 2.3, \\
E[(K(80))^2] =& (0)^2\frac{33}{250} + (1)^2\frac{56}{250} + (2)^2\frac{54}{250} + (3)^2\frac{45}{250} + (4)^2\frac{34}{250} + (5)^2\frac{28}{250} = 7.684. \\
(2) \quad e_{80} =& \sum_{k=1}^{\infty} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217}{250} + \frac{161}{250} + \frac{107}{250} + \frac{62}{250} + \frac{28}{250} = 2.3, \\
E[(K(80))^2] =& \sum_{k=1}^{\infty} (2k-1)\frac{\ell_{x+k}}{\ell_x} \\
=& (1)\frac{217}{250} + (3)\frac{161}{250} + (5)\frac{107}{250} + (7)\frac{62}{250} + (9)\frac{28}{250} = 7.684, \\
\mathrm{Var}(K(80)) =& 7.684 - (2.3)^2 = 2.394.
\end{aligned}$$

Note: Without (ii), method (2) is faster.

Two ways for $e_{x:\overline{n}|} = E(K(x) \wedge n)$:
(1) $E(g(X)) = \sum_x g(x) f_X(x)$ **Q:** $X =$ **?** $g(X)$ **??**
(2) $E(K(x) \wedge n) = \sum_{k=1}^{n} {}_kp_x = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x}$.

$E(g(X)) = \sum_{k=0}^{n} k f_X(k) + \sum_{k>n} n f_X(k) = \sum_{k=0}^{n} k f_{K(x)}(k) + \sum_{k>n} n f_{K(x)}(k)$.

$e_{80:\overline{3}|} = \sum_{k=1}^{3} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217}{250} + \frac{161}{250} + \frac{107}{250} \approx 1.94$.

Theorem 3.1.

Example 3.3. *Using the life table in page 250, find:*
(i) ℓ_{10}.
(ii) d_{35}.
(iii) ${}_5d_{35}$.
(iv) P(a newborn will die before reaching 50 years).
(v) P(a newborn will live more than 60 years).
(vi) P(a newborn will die when his age is between 45 and 65 years old).
(vii) P(a 25–year old will die before reaching 50 years).
(viii) P(a 25–year old will live more than 60 years).
(ix) The probability that a 25–year old will die when his age is between 50 and 65 years old.

Solution: (i) $\ell_{10} = 99129$.

(ii) $d_{35} = \ell_{35} - \ell_{36} = 97250 - 97126 = 124$.

(iii) ${}_5d_{35} = \ell_{35} - \ell_{40} = 97250 - 96517 = 733$.

(iv) $P(T(0) < 50) = 1 - s(50) = 1 - \frac{\ell_{50}}{\ell_0} = 1 - \frac{93735}{100000} = 0.06265$.

(v) $P(T(0) > 60) = s(60) = \frac{\ell_{60}}{\ell_0} = \frac{88038}{100000} = 0.88038$.

(vi) $P(45 < T(0) \le 65) = s(45) - s(65) = \frac{\ell_{45}-\ell_{65}}{\ell_0} = \frac{95406-83114}{100000} = 0.12292$.

(vii) ${}_{25}q_{25} = 1 - {}_{25}p_{25} = 1 - \frac{\ell_{50}}{\ell_{25}} = \frac{98246-93735}{98246} = 0.04591535533$.

(viii) ${}_{35}p_{25} = \frac{\ell_{60}}{\ell_{25}} = \frac{88038}{98246} = 0.896097551$.

(ix) ${}_{25|15}q_{25} = {}_{25}p_{25} - {}_{40}p_{25} = \frac{\ell_{50}-\ell_{65}}{\ell_{25}} = \frac{93735-83114}{98246} = 0.1081061824$.

Example 3.4.

Definition 3.2.

Definition 3.3.

Definition 3.4.

Theorem 3.2.

Theorem 3.3.

3.2 Mathematical models

One may skip to section 3.4.

3.3 Deterministic survivorship group and stochastic survivorship group

There are two interpretation about what a life table is.

deterministic survivorship group assumes

(i) The initial group consists of ℓ_0 lives at zero.

(ii) The group is closed. We are able to track all the initial lives and we do not add any individuals to the group.

(iii) ℓ_x denotes the number of individuals alive at time x.

According to the deterministic model, the proportion of people who die at a certain age is given by a life table.

Usually, it is very difficult to track an initial group of lives for a long time. We should expect life expectancies to change over time. A life table using data from people born 100 years ago is not very useful to determine the death rates of the current population.

Often life tables are constructed first estimating the survival function $s(\cdot)$ and assuming that the number of alive individuals follow the same survival function. If this happen, we have a **random survivorship group**, which assumes:

1. ℓ_0 individuals alive at time zero. Let $X_1, \dots, X_{\ell_0}$ be the age–at–death random variables of these individuals.

2. $X_1, \dots, X_{\ell_0}$ are i.i.d. r.v. with survival function $s(\cdot)$ ($s(t) = P(X_i > t)$).

3. The number of individuals alive at time x is the r.v. $\mathcal{L}(x)$.
$\mathcal{L}(x) = \sum_{j=1}^{\ell_0} I(X_j > x)$. $\mathcal{L}(x) \sim bin(\ell_0, s(x))$.
$E[\mathcal{L}(x)] = \ell_0 s(x)$ and $\mathrm{Var}(\mathcal{L}(x)) = \ell_0 s(x)(1 - s(x))$.
In a life table ℓ_x is $\ell_0 s(x)$ rounded up.

Both the deterministic survivorship group and the random survivorship group allow to use past data to predict future lifetimes of a group of individuals.

Some of the previous formulas have an intuitive interpretation using the group determinist approach to life tables.

Consider $e_x = \sum_{k=1}^{\infty} {}_k p_x$.

The number of survivors at time x is ℓ_x. The average complete years lived by the ℓ_x survivors at time x is e_x. So, the total number of complete future years lived by the ℓ_x survivors at time x is $\ell_x e_x$. ℓ_{x+k} is the number of the ℓ_x survivors at time x who live the k–th year, i.e. the period of time $(x + k - 1, x + k]$. Hence, $\sum_{k=1}^{\infty} \ell_{x+k}$ is the total number of complete future years lived by the ℓ_x survivors at time x. Hence,
$\ell_x e_x = \sum_{k=1}^{\infty} \ell_{x+k}$ and
$e_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x} = \sum_{k=1}^{\infty} {}_k p_x$.

Consider $e_x = p_x(1 + e_{x+1})$. The number of lives aged x is ℓ_x. The complete number of years lived by all lives aged x is $\ell_x e_x$. From these ℓ_x lives, during the first year $\ell_x - \ell_{x+1}$ lives die and do not live a complete year. From these ℓ_x lives, ℓ_{x+1} lives survive one year and live one year plus some complete of years after time $x + 1$. The complete number of years lived by all lives aged $x + 1$ is $\ell_{x+1} e_{x+1}$. Hence, $\ell_x e_x = \ell_{x+1} + \ell_{x+1} e_{x+1} = \ell_{x+1}(1 + e_{x+1})$, which implies that $e_x = p_x(1 + e_{x+1})$.

Consider $e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}$. The number of lives aged x is ℓ_x. We have that the complete number of years lived by all lives aged x is $\ell_x e_x$. $\ell_x e_x$ is the complete number of years lived by all lives aged x between times x and $x + n$ plus the complete number of years lived by all lives aged x after time $x+n$. The complete number of years lived by all lives aged x between times x and $x+n$ is $\ell_x e_{x:\overline{n}|}$. The lives aged x who live complete years after time $x+n$ are the ones that survive time $x+n$. The average complete years lived by each of the ℓ_{x+n} survivors at time $x+n$ is e_{x+n}. Hence, the complete number of years lived by all lives aged x after time $x+n$ is $\ell_{x+n}e_{x+n}$. Therefore, $\ell_x e_x = \ell_x e_{x:\overline{n}|} + \ell_{x+n}e_{x+n}$, which implies $e_x = e_{x:\overline{n}|} + {}_np_x e_{x+n}$.

The rest of the section can be ignored!!

Theorem 3.4.

Corollary 3.1.

Theorem 3.5.

Example 3.5.

Theorem 3.6.

Corollary 3.2.

Theorem 3.7.

Example 3.6.

Example 3.7.

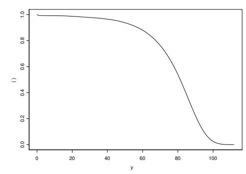


Figure 3.1: Survival function of the age–at–death from Table 7.1.

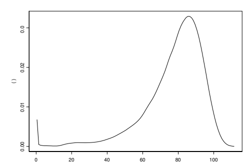


Figure 3.2: Density of the age–at–death from Table 7.1.

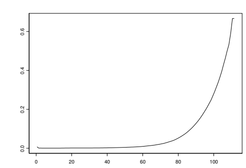


Figure 3.3: d_x, Probability of dying between ages x and $x + 1$.

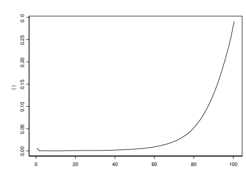


Figure 3.4: Force of mortality at age x from Table 7.1.

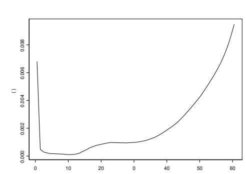


Figure 3.5: Close-up of the force of mortality between ages 0 and 60.

3.4 Continuous computations.

Assuming X is cts, knowing ℓ_x for each real number $x \geq 0$, we can get

$$\begin{array}{ll}
\textbf{main} & \textit{secondary} \\
s(x) = \dfrac{\ell_x}{\ell_0}, & f(x) = -\dfrac{d}{dx}\dfrac{\ell_x}{\ell_0},\ \mu(x) = -\dfrac{d}{dx}(\ln(\ell_x)) = -\dfrac{\ell'_x}{\ell_x}, \\
{}_tp_x = \dfrac{\ell_{x+t}}{\ell_x}, & {}_tq_x = \dfrac{\ell_x - \ell_{x+t}}{\ell_x},\ f_{T(x)}(t) = -\dfrac{d}{dt}\dfrac{\ell_{x+t}}{\ell_x},\ \mu_{x+t} = -\dfrac{d}{dt}\ln\dfrac{\ell_{x+t}}{\ell_x}, \\
& \mathring{e}_0 = \displaystyle\int_0^\infty \frac{\ell_x}{\ell_0}dx,\ \mathring{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x}dt,\ \mathring{e}_{x:\overline{n}|} = \int_0^n \frac{\ell_{x+t}}{\ell_x}dt, \\
& {}_nm_x = \dfrac{\int_0^n \ell_{x+t}\mu_x(t)\,dt}{\int_0^n \ell_{x+t}\,dt}.
\end{array}$$

However, from the life table, we only know ℓ_x at integers.

Q: How to get ℓ_x for $x \in [0, \infty)$?

Ans: linear interpolation or non-linear interpolation.

Linear interpolation of $F(x)$ is just $\check{F}(x)$ (see Eq. (1)), where $(t_1, ..., t_m) = (0, 1, 2, ...,$ terminal age).

Using linear interpolation, Figure 3.1-3.5 show the graphs of the survival function of the age–at–death using Table 7.1 (see page 250), as well as the density, d_x and $\mu(x)$.

3.5 Interpolating life tables

Life tables only show the values of ℓ_x whenever x is a nonnegative integer. In many computations, we need to know ℓ_x for each $x \geq 0$.

Let $f(x) = \ell_x$. Suppose that $x_1 < x_2 < \cdots < x_n$ and $f(x_i)$'s are known but not other $f(x)$. We can estimate the values of $f(x)$ for $x \in (x_1, x_n)$ by linear function or nonlinear function:

(1) **Linear interpolation.** Straight line equation: $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$f(x) = f(x_j) + \frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j}(x - x_j)$

$= (1 - \frac{x-x_j}{x_{j+1}-x_j})f(x_j) + \frac{x-x_j}{x_{j+1}-x_j}f(x_{j+1}),\ x \in (x_j, x_{j+1}).$

If $x_{j+1} - x_j = 1$ and $x = x_j + t$, then

$f(x_j + t) = (1 - t)f(x_j) + tf(x_{j+1}),\ t \in (0, 1).$

(2) **Nonlinear interpolation**. $f(x)$ is a curve passing through x_j's.

3.5.1 Uniform distribution of deaths is to assume

$$\ell_{j+t} = (1-t)\ell_j + t\ell_{j+1} = \ell_j - t \cdot d_j\ \underbrace{0 \leq t \leq 1,\quad j \in \{0, 1, 2, ...\}}_{important}. \tag{3.2}$$

We say that X **is uniform on** $(j, j+1)$ or say a **uniform distribution of deaths (UDD)** or say **linear interpolation for the number of living**.

Q: How to compute the following quantities under UDD ?

$_tp_x = S_{T(x)}(t)$
$p_x = {}_1p_x = S_{T(x)}(1)$
$_{s|t}q_x = P(s < T(x) \le s+t)$
$_{t|}q_x = {}_{t|1}q_x$,
$_tq_x = F_{T(x)}(t)$.

Ans: Key formula: $_tp_x = \frac{l_{x+t}}{l_x}$ and Eq. (3.2).

Theorem 3.8. *Under form (3.2), $\forall$ $x = 0$, 1, 2, ... and $\forall$ $t \in [0,1]$:*
(i) $_tp_x = 1 - tq_x$.
(ii) $_tq_x = tq_x$.
(iii) $f_{T(x)}(t) = q_x$.
(iv) $\mu_{x+t} = \frac{q_x}{1-tq_x}$

Remark. 1. No need to memorize Th. 3.8, they can be derived easily.

Proof. (i) $_tq_x = \frac{l_x - l_{x+t}}{l_x}$ and $q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}$. **Notice the assumption:** $t \in [0,1]$.
By (3.2), $S_{T(x)}(t) = {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_x - t\cdot d_x}{\ell_x} = 1 - t\frac{d_x}{\ell_x} = 1 - tq_x$.
(ii) $F_{T(x)}(t) = {}_tq_x = 1 - {}_tp_x = tq_x$.
(iii) $f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x = q_x$.
(iv) $\mu_{x+t} = \frac{f_{T(x)}(t)}{S_{T(x)}(t)} = \frac{q_x}{1-tq_x}$. ∎

Q: $S_{T(x)}(t) = 1 - tq_x$ **for each** t **and each** x **?**

Example 3.8. *Using the life table in page 250 and assuming UDD, find: (i)* $_{0.5}p_{35}$ *and (ii)* $_{1.4}p_{35.3}$.

Solution: Formulas: $_tp_x = \frac{s(x+t)}{s(x)} = \frac{\ell_{x+t}}{\ell_x}$, $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$, $t \in [0,1]$.
(i) $_{0.5}p_{35} = \frac{\ell_{35.5}}{\ell_{35}}$. $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$, $(x,t) = ?$
$\ell_{x+t} = \ell_{35.5} = (0.5)\ell_{35} + (0.5)\ell_{36} = (0.5)(97250) + (0.5)(97126) = 97188$,
$_{0.5}p_{35} = \frac{\ell_{35.5}}{\ell_{35}} = \frac{97188}{97250} \approx 0.9994$.
(ii) $_{1.4}p_{35.3} = \frac{\ell_{36.7}}{\ell_{35.3}}$.
$\ell_{35.3} = \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$, $(x,t) =$ **???**
$\ell_{35.3} = \ell_{x+t} = (0.7)\ell_{35} + (0.3)\ell_{36} = (0.7)(97250) + (0.3)(97126) = 97212.8$,
$\ell_{36.7} = \ell_{x+t} = (0.3)\ell_{36} + (0.7)\ell_{37} = (0.3)(97126) + (0.7)(96993) = 97032.9$,
$_{1.4}p_{35.3} = \frac{\ell_{36.7}}{\ell_{35.3}} = \frac{97032.9}{97212.8} \approx 0.9981$.

Example 3.9. *(see Example 3.2 in page 73) Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assume linear interpolation.

(A) Calculate the complete expected life at 80.

(B) Calculate $\mathring{e}_{80:\overline{3}|}$.

(C) Calculate ${}_3m_{80}$ *(central death rate).*

Solution:
(A) $E(T(x)) = \int t f_{T(x)}(t)dt = \int_0^\infty {}_tp_x dt = ?$
(B) $\mathring{e}_{80:\overline{3}|} = \int (t \wedge n) f_{T(x)}(t)dt = \int_0^n {}_tp_x dt = ?$
(C) ${}_3m_{80} = \frac{{}_3q_{80}}{\mathring{e}_{80:\overline{3}|}} = ?$

Need formula ${}_tp_x$ or $f_{T(x)}$.

${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$ and $f_{T(x)}(t) = (-{}_tp_x)'_t$

Need formula ℓ_{x+t}.

$\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1} = \ell_x - t \cdot d_x$, $0 \le t \le 1$.

So step (i) ℓ_{x+t}, step (ii) ${}_tp_x$, step (iii) $f_{T(x)}(t)$, step (iv), do A, B, C.

Recall

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

(i) Formula: $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1} = \ell_x - t \cdot d_x$, $0 \le t \le 1$.

Can we say that the answer is $\ell_{80+t} = \ell_{80} - 33t$, $t > 0$?

$\ell_{x+t} = \ell_x - t \cdot d_x$, $0 \le t \le 1$,

$\ell_{80+t} = \ell_{80+k+(t-k)} = \ell_{80+k} - (t-k) \cdot d_{80+k}$, $k \le t < k+1$,

Try $t \in (0,2)$ **first !!** then $t \ge 2$.

$$\ell_{80+t} = \begin{cases} 250 - 33t & \text{if } 0 \le t < 1 \\ 217 - 56(t-1) & \text{if } 1 \le t < 2 \\ 161 - 54(t-2) & \text{if } 2 \le t < 3 \\ 107 - 45(t-3) & \text{if } 3 \le t < 4 \\ 62 - 34(t-4) & \text{if } 4 \le t < 5 \\ 28 - 28(t-5) & \text{if } 5 \le t \le 6 \end{cases} \Rightarrow \text{(ii) } {}_tp_{80} = \begin{cases} \frac{250-33t}{250} & \text{if } 0 \le t < 1, \\ \frac{217-56(t-1)}{250} & \text{if } 1 \le t < 2, \\ \frac{161-54(t-2)}{250} & \text{if } 2 \le t < 3, \\ \frac{107-45(t-3)}{250} & \text{if } 3 \le t < 4, \\ \frac{62-34(t-4)}{250} & \text{if } 4 \le t < 5, \\ \frac{28-28(t-5)}{250} & \text{if } 5 \le t \le 6. \end{cases}$$

(iii) Using that $f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt}$ if the derivative exists,

$${}_tp_{80} = \begin{cases} \frac{250-33t}{250} & \text{if } 0 \le t < 1, \\ \frac{217-56(t-1)}{250} & \text{if } 1 \le t < 2, \\ \frac{161-54(t-2)}{250} & \text{if } 2 \le t < 3, \\ \frac{107-45(t-3)}{250} & \text{if } 3 \le t < 4, \\ \frac{62-34(t-4)}{250} & \text{if } 4 \le t < 5, \\ \frac{28-28(t-5)}{250} & \text{if } 5 \le t \le 6. \end{cases} \Rightarrow f_{T(80)}(t) = \begin{cases} \frac{33}{250} & \text{if } 0 \le t < 1 \\ \frac{56}{250} & \text{if } 1 \le t < 2 \\ \frac{54}{250} & \text{if } 2 \le t < 3 \\ \frac{45}{250} & \text{if } 3 \le t < 4 \\ \frac{34}{250} & \text{if } 4 \le t < 5 \\ \frac{28}{250} & \text{if } 5 \le t \le 6 \end{cases}$$

$$f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt} = \mathbf{??}$$

Notice that the derivative of ${}_tp_{80}$ does not exist at $1, 2, \dots, 5$. But, the density can be defined arbitrarily at finitely many points.

Now compute

(A) Calculate the complete expected life at 80.

(B) Calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

(C) Calculate ${}_3m_{80}$.

(A) 2 ways: $\begin{cases} \text{(a) } \overset{\circ}{e}_x = \int_0^\infty t f_{T(80)}(t)dt. \ (f_{T(80)}(t) \text{ was found above}), \\ \text{(b) } \overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt. \ ({}_tp_x \text{ was found above}). \end{cases}$

$$
\begin{aligned}
(a)\quad \overset{\circ}{e}_{80} =& \int_0^\infty t f_{T(80)}(t)dt \\
=&\Big(\int_0^1 t\frac{33}{250} + \int_1^2 t\frac{56}{250} + \int_2^3 t\frac{54}{250} + \int_3^4 t\frac{45}{250} + \int_4^5 t\frac{34}{250} + \int_5^6 t\frac{28}{250}\Big)dt \\
=&\frac{1}{2}\frac{33}{250} + \frac{3}{2}\frac{56}{250} + \frac{5}{2}\frac{54}{250} + \frac{7}{2}\frac{45}{250} + \frac{9}{2}\frac{34}{250} + \frac{11}{2}\frac{28}{250} = 2.8.
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad \overset{\circ}{e}_{80} =& \int_0^\infty {}_tp_{80}dt = \int_0^1 \frac{250-33t}{250}\,dt + \int_1^2 \frac{217-56(t-1)}{250}\,dt + \int_2^3 \frac{161-54(t-2)}{250}\,dt \\
&+ \int_3^4 \frac{107-45(t-3)}{250}\,dt + \int_4^5 \frac{62-34(t-4)}{250}\,dt + \int_5^6 \frac{28-28(t-5)}{250}\,dt \\
=& \left(\frac{250t}{250} - \frac{33t^2}{(2)(250)}\right)\Bigg|_0^1 + \left(\frac{217t}{250} - \frac{56(t-1)^2}{(2)(250)}\right)\Bigg|_1^2 + \left(\frac{161t}{250} - \frac{54(t-2)^2}{(2)(250)}\right)\Bigg|_2^3 \\
&+ \left(\frac{107t}{250} - \frac{45(t-3)^2}{(2)(250)}\right)\Bigg|_3^4 + \left(\frac{62t}{250} - \frac{34(t-4)^2}{(2)(250)}\right)\Bigg|_4^5 + \left(\frac{28t}{250} - \frac{28(t-5)^2}{(2)(250)}\right)\Bigg|_5^6 \\
=&2.8
\end{aligned}
$$

$$
(B)\quad \overset{\circ}{e}_{80:\overline{3}|} = \int_0^3 {}_tp_{80}dt = \int_0^1 \frac{250-33t}{250}\,dt + \int_1^2 \frac{217-56(t-1)}{250}\,dt + \int_2^3 \frac{161-54(t-2)}{250}\,dt = 2.226.
$$

(C) ${}_3m_{80} = \frac{{}_3q_{80}}{\overset{\circ}{e}_{80:\overline{3}|}} = \frac{\frac{\ell_{80}-\ell_{83}}{\ell_{80}}}{\overset{\circ}{e}_{80:\overline{3}|}} = \frac{\frac{250-107}{250}}{2.226} = 0.257.$

Another way for (B):

$$
\begin{aligned}
\text{(B) } \overset{\circ}{e}_{80:\overline{3}|} &= \textstyle\int_0^3 {}_tp_{80}dt \\
&= \textstyle\int_0^1 {}_tp_{80}dt + \int_1^2 {}_tp_{80}dt + \int_2^3 {}_tp_{80}dt \\
&= \textstyle\int_0^1 \frac{\ell_{80+t}}{\ell_{80}}dt + \int_1^2 \frac{\ell_{80+t}}{\ell_{80}}dt + \int_2^3 \frac{\ell_{80+t}}{\ell_{80}}dt \\
&= \textstyle\int_0^1 \frac{\ell_{80+t}}{\ell_{80}}dt + \int_0^1 \frac{\ell_{81+t}}{\ell_{80}}dt + \int_0^1 \frac{\ell_{82+t}}{\ell_{80}}dt \\
&= \textstyle\int_0^1 \frac{\ell_{80}-t(\ell_{80}-\ell_{81})}{\ell_{80}}dt + \int_0^1 \frac{\ell_{81}-t(\ell_{81}-\ell_{82})}{\ell_{80}}dt + \int_0^1 \frac{\ell_{82}-t(\ell_{82}-\ell_{83})}{\ell_{80}}dt \\
&= \frac{\ell_{80}-\frac{1}{2}(\ell_{80}-\ell_{81})}{\ell_{80}} + \frac{\ell_{81}-\frac{1}{2}(\ell_{81}-\ell_{82})}{\ell_{80}} + \frac{\ell_{82}-\frac{1}{2}(\ell_{82}-\ell_{83})}{\ell_{80}} \\
&= \frac{\frac{250+217}{2} + \frac{217+161}{2} + \frac{161+107}{2}}{250} = 2.226.
\end{aligned}
$$

Example 3.10.

Example 3.11. *Under the assumption of uniform distribution of deaths, find the average number of years lived between x and $x+1$ by those who die between those ages.*

Solution: The average number of years lived between x and $x+1$ by those who die in $(x, x+1)$

$$E(X-x|X \in (x, x+1]) = E(T(x)|T(x) \in (0,1]).$$

Is it 0.5 under UDD ?
Recall $E(X-x|X \in (x, x+1)) = E(X-x|Y=1) = \int_x^{x+1}(t-x)f_{X|Y}(t|1)dt$, where
$Y = I(X \in (x, x+1))$ and $f_{X|Y}(t|1) = \frac{f_X(t)}{P(X \in (x,x+1)}$.
$f_{X|Y}(t|1) = \frac{f_{X,Y}(t,1)}{f_Y(1)} = \frac{P(X=t, X \in (x,x+1))}{P(X \in (x,x+1))} = \frac{P(X=t)}{P(X \in (x,x+1))} = \frac{f_X(t)}{P(X \in (x,x+1)}$ if X is discrete.
$f_X(t) = -\frac{d}{dt} {}_t p_x = -\frac{d}{dt}[\ell_x - (t-x)(\ell_x - \ell_{x+1})]/\ell_0 = d_x/\ell_0$ for $t \in (x, x+1]$.
$E(X-x|X \in (x, x+1])$
$= \int_x^{x+1}(t-x)\frac{f_X(t)}{\int_x^{x+1} f_X(u)du}dt$
$= \int_x^{x+1}(t-x)\frac{d_x/\ell_0}{\int_x^{x+1}(d_x/\ell_0)du}dt = \int_x^{x+1}(t-x)dt = (\frac{(t-x)^2}{2})|_x^{x+1} = \frac{1}{2}$.

Theorem 3.9. *Given $t \in [k, k+1)$, where $k \geq 0$, under UDD,*
(i) $s(t) = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.
(ii) $f_X(t)(= {}_k|q_0) = \frac{d_k}{\ell_0}$.
(iii) $f_{T(x)}(t)\ (= {}_k|q_x) = \frac{d_{x+k}}{\ell_x}$.
(iv) $\mu(t) = \frac{d_k}{\ell_k - (t-k)d_k}$.

Notice that the theorem expresses the notations in terms of ℓ_x or d_x. You should derive them yourself in doing the homework, instead of using them directly.

Proof. (i) By (3.2), $\ell_t = \ell_{k+t-k} = \ell_k - (t-k) \cdot d_k$.
Hence, $s(t) = \frac{\ell_t}{\ell_0} = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.
(ii) $f_X(t) = -\frac{d}{dt}s(t) = \frac{d_k}{\ell_0}$.
(iii) $f_{T(x)}(t) = \frac{f_X(x+t)}{s(x)} = \frac{d_{x+k}}{\ell_x}$.
(iv) $\mu(t) = \frac{f_X(t)}{s(t)} = \frac{d_k/\ell_0}{\ell_k/\ell_0 - (t-k)d_k/\ell_0}$. ∎

Insurance companies need to know the total years for all their clients.

Definition 3.5. *Denote $T_x = \ell_x \mathring{e}_x$, the expected number of years lived beyond age x by* **the cohort group with ℓ_0 members***.*

Notice that $T_x \overset{def}{=} \ell_x \mathring{e}_x$ is different from $T_x \overset{def}{=} T(x)$ in Chapter 2.

Definition 3.6. *Denote ${}_nL_x = \ell_x \mathring{e}_{x:\overline{n}|}$, the expected number of years lived between age x and age $x+n$* **by the ℓ_x survivors at age** *x. Denote $L_x = {}_1L_x$.*

Theorem 3.10. *Under a linear form for the number of living,*
(i) $L_x = \ell_x - \frac{d_x}{2} = \ell_{x+1} + \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2}$.
(ii) $\mathring{e}_{x:\overline{1}|} = \frac{1+p_x}{2}$.

(iii) $T_x = \frac{\ell_x}{2} + \sum_{k=x+1}^{\infty} \ell_k$.
(iv) $m_x = \frac{q_x}{1-\frac{q_x}{2}}$ *(central death rate over* $(x, x+1)$*).*
(v) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.
(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1} \frac{\frac{\ell_k+\ell_{k+1}}{2}}{\ell_x}$.

They express the notations in terms of ℓ_x. One needs to learn how to derive them rather memorize the theorem.

Proof. (i) Formulas: ${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|}$, $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt$, ${}_tp_x = \ell_{x+t}/\ell_x$,
$\ell_{x+t} = \ell_x + t(\ell_x - \ell_{x+1})$, t ?

$$L_x = \int_0^1 \ell_x {}_tp_x \, dt = \int_0^1 \ell_{x+t} \, dt = \int_0^1 (\ell_x - t \cdot d_x) \, dt = \ell_x - \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2} = \ell_{x+1} + \frac{d_x}{2}.$$

(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{L_x}{\ell_x}$ **(Why ??)** $= \frac{\ell_x+\ell_{x+1}}{2\ell_x}$ (?) $= \frac{1+p_x}{2}$. **Why ?**

(iii) Formulas: $T_x = \ell_x \overset{\circ}{e}_x$ and (i) in the Th., $\vdash$: $T_x = \sum_{k=x}^{\infty} L_k$
$T_x = {}_\infty L_x = \ell_x \int_0^\infty {}_tp_x dt = \int_0^\infty \ell_{x+t} dt$,
$L_x = \ell_x \overset{\circ}{e}_{x:\overline{1}|} = \ell_x \int_0^1 {}_tp_x dt = \int_0^1 \ell_{x+t} dt$,
$\sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty} \int_0^1 \ell_{k+t} dt = \sum_{k=x}^{\infty} \int_k^{k+1} \ell_u du = \int_x^\infty \ell_u du = \int_0^\infty \ell_{x+t} dt$

$$T_x = \sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty} \left(\frac{\ell_k + \ell_{k+1}}{2} \right) = \frac{\ell_x + \ell_{x+1}}{2} + \frac{\ell_{x+1} + \ell_{x+2}}{2} + \cdots = \frac{\ell_x}{2} + \sum_{k=x+1}^{\infty} \ell_k.$$

(iv) $m_x = \int_x^{x+1} \frac{S_X(t)}{\int_x^{x+1} S_X(u) du} \mu_X(t) dt = q_x / \overset{\circ}{e}_{x:\overline{1}|}$ (by Formula (7))
$= \frac{q_x}{(1+p_x)/2} = \frac{q_x}{(1+1-q_x)/2} = \frac{q_x}{1-\frac{q_x}{2}}$.

(v) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x}$ **Why ??** $= \frac{1}{2} + \sum_{k=x+1}^{\infty} \frac{\ell_k}{\ell_x} = \frac{1}{2} + \sum_{k=1}^{\infty} {}_kp_x = \frac{1}{2} + e_x$.

(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \frac{{}_nL_k}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x} = \sum_{k=x}^{x+n-1} \frac{\frac{\ell_k+\ell_{k+1}}{2}}{\ell_x}$ by (1) in the Th. ∎

Recall that $S_x = T(x) - K(x)$, where $K(x)$ is the curtate duration.

Theorem 3.11. *Under UDD, for each* x*,* $K(x)$ *and* S_x *are independent r.v.'s and* S_x *has a distribution uniform on* $(0, 1)$*.*

Corollary 3.3. *Under the assumption of uniform distribution of deaths:*
(i) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.
(ii) $\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \frac{1}{12}$.

Proof. (i) Since $T(x) = K(x) + S_x$,
$\overset{\circ}{e}_x = E[T(x)] = E[K(x)] + E[S_x] = e_x + \frac{1}{2}$.
(ii) Since $T(x) = K(x) + S_x$ and $K(x)$ and S_x are independent,
$\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \mathrm{Var}(S_x) = \mathrm{Var}(K(x)) + \frac{1}{12}$. ∎

Theorem 3.12.

3.5.2 Exponential interpolation.

Exponential interpolation is a non-linear interpolation:

$$\ln\ell_{x+t} = (1-t)\ln\ell_x + t\ln\ell_{x+1} \qquad (\text{ v.s. } l_{x+t} = (1-t)l_x + tl_{x+1}.)$$

Q: How to remember where to put $(1-t)$?
Ans: $\ell_{x+t}|_{t=0} = \ell_x$, $\ell_{x+t}|_{t=1} = \ell_{x+1}$.

$$\ell_{x+t} = \ell_x p_x^t = \ell_x \left(\frac{\ell_{x+1}}{\ell_x}\right)^t = (\ell_x)^{1-t}(\ell_{x+1})^t \text{ for } t \in [0,1] \text{ and } x = 0, 1, \ldots \tag{3.3}$$

Example 3.12. *Using the life table in page 250 and exponential interpolation, find: (i) ${}_{0.75}p_{80}$ (ii) ${}_{2.25}p_{80}$.*

Solution: (i) Formulas: ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\ell_{x+t}}{\ell_x}$, $\ell_{x+t} = \ell_x(\frac{\ell_{x+1}}{\ell_x})^t$, $t \in (0,1]$. Thus
${}_tp_x = \frac{\ell_x(\frac{\ell_{x+1}}{\ell_x})^t}{\ell_x} = (\frac{\ell_{x+1}}{\ell_x})^t$, $t \in (0,1]$.

$${}_{0.75}p_{80} = \underbrace{\frac{\ell_{x+t}}{\ell_x}}_{\mathbf{t=?}} = \left(\frac{\ell_{81}}{\ell_{80}}\right)^{0.75} = \left(\frac{50987}{53925}\right)^{0.75} = 0.958852885.$$

(ii) Two ways: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, or ${}_{t+s}p_x = {}_tp_x \cdot {}_sp_{x+t}$.

$${}_{2.25}p_{80} = \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_{82.25}}{\ell_{80}} = \frac{\ell_x^{1-t}\ell_{x+1}^t}{\ell_{80}} = \frac{\ell_{82}^{0.75}\ell_{83}^{0.25}}{\ell_{80}} = \frac{(47940)^{0.75}(44803)^{0.25}}{53925} \approx 0.8450.$$

$${}_{2.25}p_{80} = {}_2p_{80} \cdot {}_{0.25}p_{82} = \frac{\ell_{82}}{\ell_{80}} \cdot (\frac{\ell_{83}}{\ell_{82}})^{0.25} = \frac{\ell_{82}^{0.75}\ell_{83}^{0.25}}{\ell_{80}} =\approx 0.8450$$

Example 3.13.

Example 3.14. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Using exponential interpolation, calculate
(i) the complete expected life at 80;
(ii) $\overset{\circ}{e}_{80:\overline{3}|}$;
(iii) the density function of the future life $T(80)$;
(iv) $\mu(80+t)$, $0 \le t \le 6$.

Solution: (i) Formula: $\overset{\circ}{e}_x = \int t f_{T(x)}(t)dt = \int_0^\infty {}_tp_x dt$,
${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$.
$f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt}$, **which one to choose ?**
Since $\ell_{x+t} = \ell_x \left(\frac{\ell_{x+1}}{\ell_x}\right)^t$, $0 \le t \le 1$,
can we say $\ell_{80+t} = \ell_{80}\left(\frac{\ell_{80+1}}{\ell_{80}}\right)^t$, $t \ge 0$ **?**

If $t \in [k, k+1)$, then $\ell_{80+t} = \ell_{80+k+(t-k)} = \ell_{80+k}\left(\frac{\ell_{80+k+1}}{\ell_{80+k}}\right)^{t-k}$.

$$\ell_{80+t} = \begin{cases} 250\left(\frac{217}{250}\right)^t & \text{if } 0 \le t < 1, \\ 217\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t < 2, \\ 161\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t < 3, \\ 107\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t < 4, \\ 62\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 < t \le 6. \end{cases} \quad {}_tp_{80} = \frac{\ell_{x+t}}{\ell_x} = \begin{cases} \frac{250}{250}\left(\frac{217}{250}\right)^t & \text{if } 0 \le t < 1, \\ \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t < 2, \\ \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t < 3, \\ \frac{107}{250}\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t < 4, \\ \frac{62}{250}\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 < t \le 6. \end{cases}$$

$$\begin{aligned}
\mathring{e}_x &= \int_0^\infty {}_tp_x dt = \int_0^1 \frac{250}{250}\left(\frac{217}{250}\right)^t dt + \int_1^2 \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} dt \\
&+ \int_2^3 \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} dt + \int_3^4 \frac{107}{250}\left(\frac{62}{107}\right)^{t-3} dt + \int_4^5 \frac{62}{250}\left(\frac{28}{62}\right)^{t-4} dt \\
&= \left(\frac{250}{250}\left(\frac{217}{250}\right)^t \frac{1}{\ln\left(\frac{217}{250}\right)}\right)\Bigg|_0^1 \qquad \text{(since } (a^x)' = a^x \ln a) \\
&+ \left(\frac{217}{250}\left(\frac{161}{217}\right)^{t-1} \frac{1}{\ln\left(\frac{161}{217}\right)}\right)\Bigg|_1^2 + \left(\frac{161}{250}\left(\frac{107}{161}\right)^{t-2} \frac{1}{\ln\left(\frac{107}{161}\right)}\right)\Bigg|_2^3 \\
&+ \left(\frac{107}{250}\left(\frac{62}{107}\right)^{t-3} \frac{1}{\ln\left(\frac{62}{107}\right)}\right)\Bigg|_3^4 + \left(\frac{62}{250}\left(\frac{28}{62}\right)^{t-4} \frac{1}{\ln\left(\frac{28}{62}\right)}\right)\Bigg|_4^5 \\
&= \frac{217-250}{250\ln\frac{217}{250}} + \frac{161-217}{250\ln\frac{161}{217}} + \frac{107-161}{250\ln\frac{107}{161}} + \frac{62-107}{250\ln\frac{62}{107}} + \frac{28-62}{250\ln\frac{28}{62}} \approx 2.71.
\end{aligned}$$

The last expression is actually a simpler formula $\mathring{e}_x = \sum_{k=x}^\infty \frac{d_k}{-\ell_x \ln p_k}$, where $d_x = \ell_x - \ell_{x+1}$ and $p_x = \frac{\ell_{x+1}}{\ell_x}$.

(ii) There are two ways: (1) $\mathring{e}_{80:\overline{3}|} = \int_0^n {}_tp_x dt$.
(2) $\mathring{e}_{80:\overline{3}|} = \int_0^\infty (x \wedge n) f_{T(80)}(x) dx = \int_1^3 x f_{T(80)}(x) dx + 3 \cdot {}_3p_{80}$ **Which way?**

$$\begin{aligned}
\mathring{e}_{80:\overline{3}|} &= \int_0^3 {}_tp_x dt = \int_0^1 \frac{250}{250}\left(\frac{217}{250}\right)^t dt + \int_1^2 \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} dt + \int_2^3 \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} dt \\
&= \frac{217-250}{250\ln\left(\frac{217}{250}\right)} + \frac{161-217}{250\ln\left(\frac{161}{217}\right)} + \frac{107-161}{250\ln\left(\frac{107}{161}\right)} \approx 2.21.
\end{aligned}$$

The last expression is actually a simpler formula
$\mathring{e}_{80:\overline{3}|} = \sum_{k=80}^{82} \frac{d_k}{-\ell_{80} \ln p_k}$.

(iii) $f_{T(80)}(t) = -\frac{d({}_tp_{80})}{dt}$, (iv) $\mu(80+t) = \mu_{T(80)}(t) = -\frac{d(\ln({}_tp_{80}))}{dt} = \frac{f_{T(80)}(t)}{{}_tp_{80}}$,

$$f_{T(80)}(t)=\begin{cases}\frac{250}{250}\left(\frac{217}{250}\right)^t\left(-\ln\left(\frac{217}{250}\right)\right) & \text{if } 0\le t<1,\\ \frac{217}{250}\left(\frac{161}{217}\right)^{t-1}\left(-\ln\left(\frac{161}{217}\right)\right) & \text{if } 1\le t<2,\\ \frac{161}{250}\left(\frac{107}{161}\right)^{t-2}\left(-\ln\left(\frac{107}{161}\right)\right) & \text{if } 2\le t<3,\\ \frac{107}{250}\left(\frac{62}{107}\right)^{t-3}\left(-\ln\left(\frac{62}{107}\right)\right) & \text{if } 3\le t<4,\\ \frac{62}{250}\left(\frac{28}{62}\right)^{t-4}\left(-\ln\left(\frac{28}{62}\right)\right) & \text{if } 4\le t\le 5,\\ 0 & \text{if } 5<t\le 6.\end{cases}\quad \mu(80+t)=\begin{cases}-\ln\left(\frac{217}{250}\right) & \text{if } 0\le t\le 1,\\ -\ln\left(\frac{161}{217}\right) & \text{if } 1<t\le 2,\\ -\ln\left(\frac{107}{161}\right) & \text{if } 2<t\le 3,\\ -\ln\left(\frac{62}{107}\right) & \text{if } 3<t\le 4,\\ -\ln\left(\frac{28}{62}\right) & \text{if } 4<t\le 5,\\ \frac{0}{0} & \text{if } 5<t\le 6.\end{cases}$$

Notice that the derivative of $\ln({}_tp_{80})$ does not exist at $1,2,\dots,5$. But, the density and force of mortality can be defined arbitrarily at finitely many points.

Theorem 3.13. *Under an exponential form for the number of living, for each nonnegative integer x and each $t\in(0,1)$:*

(i) $S_{T(x)}(t)={}_tp_x=p_x^t$.

(ii) ${}_tq_x=1-(1-q_x)^t$.

(iii) $f_{T(x)}(t)=-p_x^t\ln p_x$.

(iv) $\mu_{x+t}=-\ln p_x$.

Proof. (i) ${}_tp_x=\frac{\ell_{x+t}}{\ell_x}=\ell_x\frac{(\frac{\ell_{x+1}}{\ell_x})^t}{\ell_x}=p_x^t$.

(ii) ${}_tq_x=1-{}_tp_x=1-p_x^t=1-(1-q_x)^t$.

(iii) $f_{T(x)}(t)=-\frac{d}{dt}{}_tp_x=-p_x^t\ln p_x$.

(iv) $\mu_{x+t}=\mu_x(t)=-\frac{d}{dt}\ln({}_tp_x)=\frac{f_{T(x)}}{{}_tp_x}=-\ln p_x$. ■

By Theorem 3.13 (iv), the force of mortality is a constant in the interval $(x,x+1)$. Hence, the form obtained using exponential interpolation is also called the **constant force of mortality form of the number of living**.

Theorem 3.14. *Under an exponential form for ℓ_{x+t},*

(i) $L_x=\frac{d_x}{-\ln p_x}$.

(ii) $\overset{\circ}{e}_{x:\overline{1}|}=\frac{q_x}{-\ln p_x}$.

(iii) $T_x=\sum_{k=x}^{\infty}\frac{d_k}{-\ln p_k}$.

(iv) $m_x=-\ln p_x$.

(v) $\overset{\circ}{e}_x=\sum_{k=x}^{\infty}\frac{d_k}{-\ell_x\ln p_k}$.

(vi) $\overset{\circ}{e}_{x:\overline{n}|}=\sum_{k=x}^{x+n-1}\frac{d_k}{-\ell_x\ln p_k}$.

Proof. Formulas: $L_x=\ell_x\overset{\circ}{e}_{x:\overline{1}|}$, $\overset{\circ}{e}_{x:\overline{n}|}=\int_0^n {}_tp_x dt$, ${}_tp_x=\frac{\ell_{x+t}}{\ell_x}=p_x^t$.

(i) $L_x=\int_0^1\ell_xp_x^t\,dt=\frac{\ell_xp_x^t}{\ln p_x}\Big|_0^1=\frac{\ell_x(p_x-1)}{\ln p_x}=\frac{\ell_xq_x}{-\ln p_x}=\frac{d_x}{-\ln p_x}$.

(ii) $\overset{\circ}{e}_{x:\overline{1}|}=\frac{L_x}{\ell_x}=\frac{q_x}{-\ln p_x}$.

(iii) $T_x = \sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty} \frac{d_k}{-\ln p_k}$.

(iv) **Formula:** ${}_n m_x = \frac{\int_x^{x+n} s(t)\mu(t)dt}{\int_0^{x+n} s(u)du} = \frac{{}_n q_x}{\mathring{e}_{x:\overline{n}|}}$, $L_x = \ell_x \mathring{e}_{x:\overline{1}|}$,

$m_x = \frac{q_x}{\mathring{e}_{x:\overline{1}|}} = \frac{d_x}{L_x}$?? $\quad = -\ln p_x$ by (i).

(v) $\mathring{e}_x = \frac{T_x}{\ell_x} = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \ln p_k}$ by (iii).

(vi) $\mathring{e}_{x:\overline{n}|} = \frac{{}_n L_k}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x} = \sum_{k=x}^{x+n-1} \frac{d_k}{-\ell_x \ln p_k}$ by (iii). ∎

Theorem 3.15. *Given $t \ge 0$, let k be the nonnegative integer such that $k \le t < k+1$. Under exponential interpolation:*

(i) $s(t) = {}_k p_0 \cdot p_k^{t-k}$.

(ii) $f_X(t) = {}_k p_0 \cdot p_k^{t-k}(-\ln p_k)$.

(iii) $f_{T(x)}(t) = {}_k p_x \cdot p_{x+k}^{t-k}(-\ln p_{x+k})$.

Proof. (i) By (3.3), for each integer x and each $0 \le t \le 1$,

$$s(x+t) = \frac{\ell_{x+t}}{\ell_0} = \frac{\ell_x^{1-t}\ell_{x+1}^t}{\ell_0} = \frac{\ell_x}{\ell_0}\left(\frac{\ell_{x+1}}{\ell_x}\right)^t = {}_x p_0 \cdot p_x^t.$$

Hence, for $t \ge 0$ and $k \le t < k+1$,

$$s(t) = s(k+t-k) = {}_k p_0 \cdot p_k^{t-k}.$$

The proofs for (ii) and (iii) can be skipped. ∎

3.5.3 Harmonic interpolation assumes

$$\frac{1}{\ell_{x+t}} = (1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}, \qquad t \in [0,1]$$

$$\text{(recall } \textit{linear}: \quad \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}, \qquad t \in [0,1],$$

$$\textit{Exp}: \quad \ln\ell_{x+t} = (1-t)\ln\ell_x + t\ln\ell_{x+1}, \qquad t \in [0,1]).$$

(3.4) $$\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}} = \frac{1}{\frac{1}{\ell_x} + t\left(\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}\right)} \quad t \in [0,1].$$

A function of the form $\frac{1}{a+bx}$ is called a hyperbolic function.
Harmonic interpolation of the number of living is also called the
hyperbolic form of the number of living or
it satisfies the **Balducci assumption**.

Example 3.15. *Using the life table in page 250 and harmonic interpolation, find: (i) ${}_{0.75}p_{80}$ (ii) ${}_{2.25}p_{80}$.*

Solution: (i) Formulas: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$, and $\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$ (Eq. (3.4))

$$=> \quad {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{1}{(1-t)+t\frac{\ell_x}{\ell_{x+1}}} = \frac{1}{(1-t)+t\frac{1}{p_x}} = \frac{p_x}{t+(1-t)p_x}. \tag{3.5}$$

$${}_{0.75}p_{80} = \frac{1}{0.25+0.75\frac{\ell_{80}}{\ell_{81}}} = \frac{1}{0.25+0.75\frac{53925}{50987}} \approx 0.95857.$$

(ii) Formulas: ${}_{t+s}p_x = {}_tp_x \cdot {}_sp_{x+t}$, ${}_tp_x = \frac{p_x}{t+(1-t)p_x}$ (see (3.5)), with $(t,s,x) = (2, 0.25, 80)$; or

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x} \text{ and (3.4)}$$

$${}_{2.25}p_{80} = \frac{\ell_{82.25}}{\ell_{80}} = \frac{\frac{1}{(1-0.25)\frac{1}{\ell_{82}}+0.25\frac{1}{\ell_{83}}}}{\ell_{80}} = \frac{1}{(1-0.25)\frac{53925}{47940}+0.25\frac{53925}{44803}} \approx 0.8737.$$

Example 3.16.

Example 3.17. *Consider the life table*

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

Assuming harmonic interpolation calculate $\mathring{e}_{80:\overline{3}|}$.

Solution: Two ways:
(1) $\mathring{e}_{80:\overline{3}|} = \int (t \wedge n) f_{T(x)}(t)dt$.
(2) $\mathring{e}_{80:\overline{3}|} = \int_0^n {}_tp_x dt$

Which is better here ?

Formulas: ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$. $\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$, $0 \le t \le 1$,
$\ell_{x+t} = \frac{1}{(1-(t-k))\frac{1}{\ell_{x+k}}+(t-k)\frac{1}{\ell_{x+k+1}}}$, $k \le t \le k+1$,

$$\ell_{80+t} = \begin{cases} \frac{1}{(1-t)\frac{1}{250}+t\frac{1}{217}} & \text{if } 0 \le t < 1, \\ \frac{1}{(1-(t-1))\frac{1}{217}+(t-1)\frac{1}{161}} & \text{if } 1 \le t < 2, \\ \frac{1}{(1-(t-2))\frac{1}{161}+(t-2)\frac{1}{107}} & \text{if } 2 \le t < 3, \end{cases}$$

$${}_tp_{80} = \begin{cases} \frac{1}{(1-t)\frac{250}{250}+t\frac{250}{217}} & \text{if } 0 \le t < 1, \\ \frac{1}{(1-(t-1))\frac{250}{217}+(t-1)\frac{250}{161}} & \text{if } 1 \le t < 2, \\ \frac{1}{(1-(t-2))\frac{250}{161}+(t-2)\frac{250}{107}} & \text{if } 2 \le t < 3, \end{cases} = \begin{cases} \frac{1}{\frac{250}{250}+t(\frac{250}{217}-\frac{250}{250})} & \text{if } 0 \le t < 1, \\ \frac{1}{\frac{250}{217}+(t-1)(\frac{250}{161}-\frac{250}{217})} & \text{if } 1 \le t < 2, \\ \frac{1}{\frac{250}{161}+(t-2)(\frac{250}{107}-\frac{250}{161})} & \text{if } 2 \le t < 3, \end{cases}$$

$$
\begin{aligned}
\overset{\circ}{e}_{80:\overline{3}|} =& \int_0^n {}_tp_x dt = \int_0^1 \frac{1}{\frac{250}{250} + t(\frac{250}{217} - \frac{250}{250})} dt \\
&+ \int_1^2 \frac{1}{\frac{250}{217} + (t-1)(\frac{250}{161} - \frac{250}{217})} dt + \int_2^3 \frac{1}{\frac{250}{161} + (t-2)(\frac{250}{107} - \frac{250}{161})} dt \\
=& \frac{\ln\left(\frac{250}{250} + t(\frac{250}{217} - \frac{250}{250})\right)}{\frac{250}{217} - \frac{250}{250}} \Bigg|_0^1 \quad \textbf{Why ?} \qquad \left(\int \frac{1}{x} dx = \ln x,\ \int \frac{1}{a+bx} dx = \frac{\ln(a+bx)}{b}\right) \\
&+ \frac{\ln\left(\frac{250}{217} + (t-1)(\frac{250}{161} - \frac{250}{217})\right)}{\frac{250}{161} - \frac{250}{217}} \Bigg|_1^2 + \frac{\ln\left(\frac{250}{161} + (t-2)(\frac{250}{107} - \frac{250}{161})\right)}{\frac{250}{107} - \frac{250}{161}} \Bigg|_2^3 \\
=& \frac{\ln\left(\frac{250}{217}\right) - \ln\left(\frac{250}{250}\right)}{\frac{250}{217} - \frac{250}{250}} + \frac{\ln\left(\frac{250}{161}\right) - \ln\left(\frac{250}{217}\right)}{\frac{250}{161} - \frac{250}{217}} + \frac{\ln\left(\frac{250}{107}\right) - \ln\left(\frac{250}{161}\right)}{\frac{250}{107} - \frac{250}{161}} \\
=& \frac{\ln\left(\frac{250}{217}/\frac{250}{250}\right)}{\frac{(250)(250-217)}{(250)(217)}} + \frac{\ln\left(\frac{250}{161}/\frac{250}{217}\right)}{\frac{(250)(217-161)}{(217)(161)}} + \frac{\ln\left(\frac{250}{107}/\frac{250}{161}\right)}{\frac{(250)(161-107)}{(161)(107)}} \\
=& \frac{\ln\left(\frac{250}{217}\right)}{\frac{(250)(250-217)}{(250)(217)}} + \frac{\ln\left(\frac{217}{161}\right)}{\frac{(250)(217-161)}{(217)(161)}} + \frac{\ln\left(\frac{161}{107}\right)}{\frac{(250)(161-107)}{(161)(107)}} = 2.197149575.
\end{aligned}
$$

The last expression is actually $\overset{\circ}{e}_{80:\overline{3}|} = \sum_{k=80}^{82} \frac{-\ell_{k+1} \ln p_k}{\ell_{80} q_k}$.

Similarly, $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \ln p_k}{\ell_x q_k}$.

Theorem 3.16. *Under the Balducci assumption for ℓ_{x+t} and $0 \le t \le 1$,*
(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$.
(ii) ${}_tq_x = \frac{tq_x}{1-(1-t)q_x}$.
(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$.
(iv) $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$.

Proof. (i) By (3.4), ${}_tp_x = \dfrac{p_x}{t+(1-t)p_x} = \dfrac{1-q_x}{t+(1-t)(1-q_x)} = \dfrac{1-q_x}{1-(1-t)q_x}$.

(ii) ${}_tq_x = 1 - {}_tp_x = 1 - \frac{1-q_x}{1-(1-t)q_x} = \frac{1-(1-t)q_x-(1-q_x)}{1-(1-t)q_x} = \frac{tq_x}{1-(1-t)q_x}$.
(iii) We have that

$$
\begin{aligned}
\mu_{x+t} =& -\frac{d}{dt} \ln {}_tp_x = -\frac{d}{dt} \ln \frac{p_x}{t+(1-t)p_x} = \frac{d}{dt} \ln(t+(1-t)p_x) \\
=& \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{t+(1-t)(1-q_x)} \\
=& \frac{q_x}{1-(1-t)q_x}.
\end{aligned}
$$

(iv) $f_{T(x)}(t) = {}_tp_x \mu_{x+t} = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$.

Theorem 3.17. *Under the Balducci assumption,* ${}_{1-t}q_{x+t} = (1-t)q_x$.

Proof We have that

$$
\begin{aligned}
{}_{1-t}q_{x+t} &= \frac{s(x+t)-s(x+1)}{s(x+t)} = 1 - \frac{p_x}{{}_tp_x} \quad \textbf{Why}??\\
&=1 - p_x\frac{t+(1-t)p_x}{p_x} = 1-(t+(1-t)p_x) = 1-t-(1-t)p_x = (1-t)q_x.
\end{aligned}
$$

Theorem 3.18. *Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under the Balducci assumption,*

(i) $s(t) = \frac{\ell_k}{\ell_0}\frac{p_k}{1-(1-t+k)(1-p_k)}$.

(ii) $f_X(t) = \frac{\ell_k}{\ell_0}\frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}$.

(iii) $f_{T(x)}(t) = \frac{\ell_{x+k}}{\ell_x}\frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2} = {}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}$.

Proof (i) By (3.4), for each integer x and each $0 \leq t \leq 1$,

$$
\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}} \tag{3.4}
$$

$$
=> \quad s(x+t) = \frac{\ell_{x+t}}{\ell_0} = \frac{\ell_x}{\ell_0}\frac{1}{(1-t)+t\frac{\ell_x}{\ell_{x+1}}} = \frac{\ell_x}{\ell_0}\frac{p_x}{(1-t)p_x+t} = \frac{\ell_x}{\ell_0}\frac{p_x}{1-(1-t)(1-p_x)}.
$$

Hence, for $t \geq 0$ and $k \leq t < k+1$,

$$
s(t) = s(k + \underbrace{t-k}_{new\ t}) = \frac{\ell_k}{\ell_0}\frac{p_k}{1-(1-t+k)(1-p_k)}.
$$

$$
(ii) \qquad f_X(t) = f_X(k+(t-k)) = -\frac{d}{dt}s(t) = \frac{\ell_k}{\ell_0}\frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}.
$$

$$
\begin{aligned}
(iii) \qquad f_{T(x)}(t) &= -\frac{d}{dt}{}_tp_x = -\frac{d}{dt}\frac{s(x+t)}{s(x)} = -\frac{d}{dt}\frac{\frac{\ell_{x+k}}{\ell_0}\frac{p_{x+k}}{1-(1-t+k)(1-p_{x+k})}}{\frac{\ell_x}{\ell_0}}\\
&={}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}.
\end{aligned}
$$

Theorem 3.19. *Under the Balducci assumption for ℓ_{x+t},*

(i) $L_x = \frac{-\ell_{x+1}\ln p_x}{q_x}$.

(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{-p_x \ln p_x}{q_x}$.

(iii) $T_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1}\ln p_k}{q_k}$.

(iv) $m_x = \frac{q_x^2}{-p_x\ln p_x}$.

(v) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1}\ln p_k}{\ell_x q_k}$.

(vi) $\overset{\circ}{e}_{x:\overline{n}|} = \sum_{k=x}^{x+n-1} \frac{-\ell_{k+1}\ln p_k}{\ell_x q_k}$.

Proof. (i) ${}_nL_x$ is the expected number of years lived between age x and age $x+n$ **by the** ℓ_x **survivors at age** x.

$$\begin{aligned}
L_x =&\ell_x E(T(x)\wedge 1) = \ell_x \int_0^1 {}_tp_x dt = \int_0^1 \ell_{x+t}\, dt = \int_0^1 \frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}}\, dt \\
=&\int_0^1 \frac{1}{\frac{1}{\ell_x} + t(\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x})}\, dt \qquad\qquad \int \frac{1}{a+bt} dt = \ln(a+bt)/b \\
=&\frac{\ln\left(\frac{1}{\ell_x} + t(\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x})\right)}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}}\Bigg|_0^1 = \frac{\ln\frac{1}{\ell_{x+1}} - \ln\frac{1}{\ell_x}}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}} = \frac{\ell_x\ell_{x+1}\ln\frac{\ell_x}{\ell_{x+1}}}{\ell_x - \ell_{x+1}} = \frac{-\ell_{x+1}\ln p_x}{q_x}.
\end{aligned}$$

(ii) $\overset{\circ}{e}_{x:\overline{1}|} = \frac{L_x}{\ell_x} = \frac{\frac{-\ell_{x+1}\ln p_x}{q_x}}{\ell_x} = \frac{-p_x \ln p_x}{q_x}$.
(iii) $T_x = E(\#$ of years lived beyong age x by the cohort group with l_0 members$) = \sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty} \frac{-\ell_{k+1}\ln p_k}{q_k}$.
(iv) $m_x = \frac{{}_nq_x}{\overset{\circ}{e}_{x:\overline{n}|}}\Big|_{n=1} = \frac{q_x^2}{-p_x \ln p_x}$ by (ii).
(v) $T_x = \ell_x \overset{\circ}{e}_x \Rightarrow \overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty} \frac{-\ell_{k+1}\ln p_k}{\ell_x q_k}$.
(vi) ${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|} \Rightarrow \overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x} = \sum_{k=x}^{x+n-1} L_k = \sum_{k=x}^{x+n-1} \frac{-\ell_{k+1}\ln p_k}{\ell_x q_k}$.

3.5.4 Review of interpolations. For the previous interpolations, it suffices to remember the following table

Interpolation	ℓ_{x+t}	${}_tp_x$	L_x	$\overset{\circ}{e}_{x:\overline{1}\vert}$
linear	$\ell_x + t(\ell_{x+1} - \ell_x)$ $(1-t)\ell_x + t\ell_{x+1}$	$1 - tq_x$	$\frac{\ell_x+\ell_{x+1}}{2}$	$\frac{1+p_x}{2}$
exponential	$\ell_x p_x^t$ $\ln\ell_{x+t} = (1-t)\ln\ell_x + t\ln\ell_{x+1}$	p_x^t	$\frac{d_x}{-\ln p_x}$	$\frac{q_x}{-\log p_x}$
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$ $\frac{1}{\ell_{x+t}} = (1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}$	$\frac{p_x}{t+(1-t)p_x}$	$\frac{-\ell_{x+1}\ln p_x}{q_x}$	$\frac{-p_x \log p_x}{q_x}$

From ${}_tp_x$, we can get

$${}_tq_x = 1 - {}_tp_x,\ f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x,\ \mu_{x+t} = -\frac{d}{dt}\ln {}_tp_x, \qquad 0 \le t \le 1.$$

For the exponential and Balducci assumptions, it is more convenient to know how to derive L_x and $\overset{\circ}{e}_{x:\overline{1}|}$ than to trying remember the corresponding formulas.

3.6 Select and ultimate tables

A **select table** is a mortality table for a group of people subject to a special circumstance (disability, retirement, etc.). Usually, the cohort of people is given by a certain age.
Suppose that we start with $\ell_{[x]}$ lives of a certain cohort at time x.

The number of survivors at time t is denoted by $\ell_{[x]+t}$.
$\ell_{[x+t]}$ is # of lives at $x+t$ for <u>another cohort</u>.

$$_np_{[x]+t} = \frac{\ell_{[x]+t+n}}{\ell_{[x]+t}}.$$

Notice that $_nq_{[x]+t} = 1 - {_np_{[x]+t}}$, $p_{[x]+t} = {_1p_{[x]+t}}$, $q_{[x]+t} = {_1q_{[x]+t}}$ and

$$p_{[x]}p_{[x]+1}\cdots p_{[x]+n-1} = \frac{\ell_{[x]+1}}{\ell_{[x]}}\frac{\ell_{[x]+2}}{\ell_{[x]+1}}\dots, \frac{\ell_{[x]+n}}{\ell_{[x]+n-1}} = \frac{\ell_{[x]+n}}{\ell_{[x]}} = {_np_{[x]}}.$$

A select table of three cohorts:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$
43	958	823	768
44	854	738	701
45	723	687	667

$\ell_{[43]+2}$= ? $\ell_{[44]+1}$= ? $\ell_{[45]}$= ?
They all related to age 45.

A **select and ultimate table** displays the number of living using a select table for a certain number of years and a standard life table when the elapsed time is bigger than this number of years. The number of years such that the select table is used is called the **select period**. A life table which does not use the select period is called an **ultimate table**.

Suppose that the selection period is m.
$\ell_{[x]}$ —- # of living of a certain cohort selected at time x.
$\ell_{[x]+t}$ —- # of their survivors at time $x+t$.
ℓ_x —- # of living at time x for the ultimate table.
In a select and ultimate table, $\ell_{[x]+k} = \ell_{x+k}$, for each $k \geq m$.
Suppose that select period is three years. Then, a select and ultimate life table has the form

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
1	$\ell_{[1]}$	$\ell_{[1]+1}$	$\ell_{[1]+2}$	ℓ_4	4
2	$\ell_{[2]}$	$\ell_{[2]+1}$	$\ell_{[2]+2}$	ℓ_5	5
3	$\ell_{[3]}$	$\ell_{[3]+1}$	$\ell_{[3]+2}$	ℓ_6	6

$\ell_{[1]+4}$= ? $\ell_{[2]+3}$= ? $\ell_{[3]+2}$= ?

An ultimate table:

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Example 3.18. *Consider the following select table:*

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$
43	*958*	*823*	*768*
44	*854*	*738*	*701*
45	*723*	*687*	*667*

Complete the tables

x	$p_{[x]}$	$p_{[x]+1}$
43		
44		
45		

and

x	$q_{[x]}$	$q_{[x]+1}$
43		
44		
45		

Solution: Formulas: $p_{[x]+t} = \frac{\ell_{[x]+t+1}}{\ell_{[x]+t}}$ and $q_{[x]+t} = 1 - \frac{\ell_{[x]+t+1}}{\ell_{[x]+t}}$.

x	$p_{[x]}$	$p_{[x]+1}$
43	$\frac{823}{958}$	$\frac{768}{823}$
44	$\frac{738}{854}$	$\frac{701}{738}$
45	$\frac{687}{723}$	$\frac{667}{687}$

and

x	$q_{[x]}$	$q_{[x]+1}$
43	$1-\frac{823}{958}$	$1-\frac{768}{823}$
44	$1-\frac{738}{854}$	$1-\frac{701}{738}$
45	$1-\frac{667}{687}$	$1-\frac{667}{687}$

Example 3.19. *Consider the following select table:*

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$
35	*0.013*	*0.012*	*0.011*
36	*0.010*	*0.011*	*0.009*

Complete the table

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	$\ell_{[x]+3}$
35				
36				

where $\ell_{[35]} = 1000$ *and* $\ell_{[36]} = 950$.

Solution: Formulas: $1 - q_{[x]+t} = p_{[x]+t} = \frac{\ell_{[x]+t+1}}{\ell_{[x]+t}}$. Hence $\ell_{[x]+t+1} = \ell_{[x]+t}p_{[x]+t} = \ell_{[x]+t}(1 - q_{[x]+t})$.

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	$\ell_{[x]+3}$
		$\rightarrow$	$\rightarrow$	$\rightarrow$
35	1000	987	975.156	964.429284
36	950	940.5	930.1545	921.7831095

x	$p_{[x]}$	$p_{[x]+1}$	$p_{[x]+2}$
35	1- 0.013	1-0.012	1- 0.011
36	1-0.010	1- 0.011	1-0.009

$$\ell_{[35]+1} = \ell_{[35]}p_{[35]} = (1000)(1 - 0.013) = 987,$$
$$\ell_{[35]+2} = \ell_{[35]+1}p_{[35]+1} = (987)(1 - 0.012) = 975.156,$$
$$\ell_{[35]+3} = \ell_{[35]+2}p_{[35]+2} = (975.156)(1 - 0.011) = 964.429284,$$
$$\ell_{[36]+1} = \ell_{[36]}p_{[36]} = (950)(1 - 0.01) = 940.5,$$
$$\ell_{[36]+2} = \ell_{[36]+1}p_{[36]+1} = (940.5)(1 - 0.011) = 930.1545,$$
$$\ell_{[36]+3} = \ell_{[36]+2}p_{[36]+2} = (930.1545)(1 - 0.009) = 921.7831095.$$

Example 3.20. *You are given the following entries extracted from a 2–year select–and–ultimate mortality table:*

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	1235	1124	1039	47
46	1135	1025	978	48
47	1012	996	965	49

(i) Complete the table

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45				47
46				48
47			$-$	49

(ii) Find ${}_2p_{[47]}$, ${}_2p_{[46]+1}$ *and* ${}_2p_{47}$.

Solution: (i) Formula: $$q_x = 1 - \frac{\ell_{x+1}}{\ell_x}.$$

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	$\rightarrow$	$\rightarrow$	$\downarrow$	47
46	$\rightarrow$	$\rightarrow$	$\downarrow$	48
47	$\rightarrow$	$\rightarrow$	$-$	49

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	$1-\frac{1124}{1235}$	$1-\frac{1039}{1124}$	$1-\frac{978}{1039}$	47
46	$1-\frac{1025}{1135}$	$1-\frac{978}{1025}$	$1-\frac{965}{978}$	48
47	$1-\frac{996}{1012}$	$1-\frac{965}{996}$	$-$	49

(ii) Find ${}_2p_{[47]}$, ${}_2p_{[46]+1}$ and ${}_2p_{47}$. **Exercise**

Formula: ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\ell_{x+t}}{\ell_x}$.

$$
\begin{aligned}
{}_2p_{[47]} &= \frac{\ell_{49}}{\ell_{[47]}} = \frac{965}{1012} = 0.9535573123 \\
{}_2p_{[46]+1} &= \frac{\ell_{49}}{\ell_{[46]+1}} = \frac{965}{1025} = 0.9414634146 \\
{}_2p_{47} &= \frac{\ell_{49}}{\ell_{47}} = \frac{965}{1039} = 0.9287776708
\end{aligned}
$$

Example 3.21. *You are given the following entries extracted from a 2–year select–and–ultimate mortality table:*

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	0.009	0.008	0.007	47
46	0.008	0.006	0.005	48
47	0.004	0.003	$-$	49

Complete the table

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	10000			47
46				48
47				49

Solution: Formula: $s(x) = \frac{\ell_x}{\ell_0} => \ell_x = \ell_0 s(x)$.
$p_x = \ell_{x+1}/\ell_x => \ell_{x+1} = \ell_x p_x = \ell_x(1-q_x)$. Flow:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	10000	$\rightarrow$	$\rightarrow$	47
46	$\leftarrow$	$\leftarrow$	$\downarrow$	48
47	$\leftarrow$	$\leftarrow$	$\downarrow$	49

$$\ell_{[45]+1} = \ell_{[45]}p_{[45]} = (10000)(1-0.009) = 9910,$$
$$\ell_{47} = \ell_{[45]+1}p_{[45]+1} = 9910(1-0.008) = 9830.72,$$
$$\ell_{48} = \ell_{47}p_{47} = 9830.72(1-0.007) = 9761.90496,$$
$$\ell_{[46]+1} = \frac{\ell_{48}}{p_{[46]+1}} = \frac{9761.90496}{(1-0.006)} = 9820.82994,$$
$$\ell_{[46]} = \frac{\ell_{[46]+1}}{p_{[46]}} = \frac{9820.82994}{(1-0.008)} = 9900.030181,$$
$$\ell_{49} = \ell_{48}p_{48} = 9761.90496(1-0.005) = 9713.095435,$$
$$\ell_{[47]+1} = \frac{\ell_{49}}{p_{[47]+1}} = \frac{9713.095435}{(1-0.003)} = 9742.322402,$$
$$\ell_{[47]} = \frac{\ell_{[47]+1}}{p_{[47]}} = \frac{9742.322402}{(1-0.004)} = 9810.969669.$$

Hence,

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
45	10000	9910	9830.72	47
46	9900.030181	9820.82994	9761.90496	48
47	9781.448195	9742.322402	9713.095435	49

Example 3.22. *You are given the following entries extracted from a 3–year select mortality table:*

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
40	96489	96319	96084	95906	43
41	96312	96164	95998	95667	44
42	96157	95954	95265	95406	45
43	95895	95480	95243	95122	46
44	98743	96812	95012	94813	47
45	97239	95123	94753	94479	48

Compute *(a)* $e_{[44]:\overline{4}|}$, *(b)* $e_{[42]+2:\overline{4}|}$, *(c)* $e_{44:\overline{4}|}$.

Solution: Formulas: $e_{[x]:\overline{n}|} = \sum_{k=1}^{n} {}_kp_{[x]}$ and ${}_kp_{[x]} = \frac{\ell_{[x]+k}}{\ell_{[x]}}$.

(a) $e_{[44]:\overline{4}|} = p_{[44]} + {}_2p_{[44]} + {}_3p_{[44]} + {}_4p_{[44]} = \frac{\ell_{[44]+1}}{\ell_{[44]}} + \frac{\ell_{[44]+2}}{\ell_{[44]}} + \frac{\ell_{44+3}}{\ell_{[44]}} + \frac{\ell_{44+4}}{\ell_{[44]}}$.

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
44	98743	96812	95012	94813	47
45				94479	48

$$e_{[44]:\overline{4}|} = \frac{96812+95012+94813+94479}{98743} = 3.859676129.$$

(b) $e_{[42]+2:\overline{4}|} = p_{[42]+2} + {}_2p_{[42]+2} + {}_3p_{[42]+2} + {}_4p_{[42]+2} = \frac{\ell_{42+3}}{\ell_{[42]+2}} + \frac{\ell_{42+4}}{\ell_{[42]+2}} + \frac{\ell_{42+5}}{\ell_{[42]+2}} + \frac{\ell_{42+6}}{\ell_{[42]+2}}$.

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
42			95265	95406	45
43				95122	46
44				94813	47
45				94479	48

$$e_{[42]+2:\overline{4}|} = \frac{95406}{95265} + \frac{95122}{95265} + \frac{94813}{95265} + \frac{94479}{95265} = 3.986983677.$$

(c) $e_{44:\overline{4}|} = p_{44} + {}_2p_{44} + {}_3p_{44} + {}_4p_{44} = \frac{\ell_{44+1}}{\ell_{44}} + \frac{\ell_{44+2}}{\ell_{44}} + \frac{\ell_{44+3}}{\ell_{44}} + \frac{\ell_{44+4}}{\ell_{44}}$.

ℓ_{x+3}	$x+3$
95667	44
95406	45
95122	46
94813	47
94479	48

$e_{44:\overline{4}|} = \frac{95406}{95667} + \frac{95122}{95667} + \frac{94813}{95667} + \frac{94479}{95667} = 3.970230069.$

ℓ_x = # **of individuals alive** at age x.

${}_td_x = \ell_x - \ell_{x+t}$ = # of individuals which died in $(x, x+t]$.

$d_x = {}_1d_x = \ell_x - \ell_{x+1}$.

$T_x = \ell_x \mathring{e}_x = \int_0^\infty \ell_{x+t}\,dt$ **($\neq T_x$ in other sections)**

= E(# of years lived beyong age x by the cohort group with l_0 members).

${}_nL_x = \ell_x \mathring{e}_{x:\overline{n}|} = T_x - T_{x+n}$.
$s(x) = \frac{\ell_x}{\ell_0}$,

Estimators based on life table:

$$F_X(x) = \frac{\ell_0 - \ell_x}{\ell_0},$$
$${}_tp_x = \frac{\ell_{x+t}}{\ell_x},\ {}_tq_x = \frac{{}_td_x}{\ell_x},\ q_x = \frac{d_x}{\ell_x},\ {}_{n|m}q_x = \frac{{}_md_{x+n}}{\ell_x}.$$
$$\mu(x) = -\frac{d}{dx}\log(\ell_x),$$
$$\overset{\circ}{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x}dt,\ \overset{\circ}{e}_{x:\overline{n}|} = \int_0^n \frac{\ell_{x+t}}{\ell_x}dt,$$
$$e_x = \sum_{k=1}^\infty \frac{\ell_{x+k}}{\ell_x},\ e_{x:\overline{n}|} = \sum_{k=1}^n \frac{\ell_{x+k}}{\ell_x}$$
$$\overset{\circ}{e}_x = E[T(x)] = \frac{T_x}{\ell_x},\ E[(T(x))^2] = \frac{2\int_x^\infty T_y\,dy}{\ell_x}.$$
$${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = L_x + L_{x+1} + \cdots + L_{x+n-1},\ L_x = {}_1L_x$$
$$T_x = \sum_{k=x}^\infty L_k,\ {}_nm_x = \frac{{}_nd_x}{{}_nL_x},\ m_x = \frac{d_x}{L_x},$$
$$\overset{\circ}{e}_x = \frac{\sum_{k=x}^\infty L_k}{\ell_x},\ \overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}.$$

Interpolation	ℓ_{x+t}	${}_tp_x$	L_x
UDD	$\ell_x + t(\ell_{x+1} - \ell_x)$	$1 - tq_x$	$\frac{\ell_x+\ell_{x+1}}{2}$
exponential	$\ell_x p_x^t = (\ell_x)^{1-t}(\ell_{x+1})^t$	p_x^t	$\frac{d_x}{-\log p_x}$
Balducci	$\frac{1}{(1-t)\frac{1}{\ell_x}+t\frac{1}{\ell_{x+1}}}$	$\frac{p_x}{t+(1-t)p_x}$	$\frac{-\ell_{x+1}\log p_x}{q_x}$

where $t \in [0,1]$

UDD : $\mu_{x+t} = \frac{q_x}{1-tq_x},$, $m_x = \frac{q_x}{1-\frac{q_x}{2}}$, $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.

exponential : $\mu_{x+t} = -\log p_x$, $m_x = -\log p_x$.

Balducci : $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2}$, $m_x = \frac{q_x^2}{-p_x\log p_x}$.

#5 (#35, Exam M, Fall 2005) An actuary for a medical device manufacturer initially models the failure time for a particular device with an exponential distribution with mean 4 years. This distribution is replaced with a spliced model whose density function:
(i) is uniform over $[0, 3]$
(ii) is proportional to the initial modeled density function after 3 years
(iii) is continuous
Calculate the probability of failure in the first 3 years under the revised distribution.
(A) 0.43 (B) 0.45 (C) 0.47 (D) 0.49 (E) 0.51

Solution. (A) Since the density of the exponential with mean four is $4e^{-x/4}$, the density has the form

$$f(x) = \begin{cases} ae^{-x/4} & \text{if } 3 \le x \ \ (from\ (ii)), \\ ae^{-3/4} & \text{if } 0 \le x < 3 \ \ (from\ (i)\ and\ (iii)), \end{cases}$$

where we have used that f is continuous. Since

$$1 = \int_0^\infty f(x)\,dx = \int_0^3 ae^{-3/4}\,dx + \int_3^\infty ae^{-x/4}\,dx = 3ae^{-3/4} + a4e^{-3/4} = 7ae^{-3/4},$$

$a = \frac{e^{3/4}}{7}$. Hence,

$$f(x) = \begin{cases} \frac{1}{7} & \text{if } 0 \le x < 3, \\ \frac{e^{3/4}}{7}e^{-x/4} & \text{if } 3 \le x, \end{cases}$$

and $P\{X \le 3\} = \frac{3}{7} = 0.4285714286$.

Change of problem. An actuary for a medical device manufacturer initially models the failure time for a particular device with an uniform distribution with mean 4 years. This distribution is replaced with a spliced model whose density function:
(i) is exponential over $[0, 3]$,
(ii) is proportional to the initial modeled density function after 3 years,
(iii) is continuous.
Calculate the probability of failure in the first 3 years under the revised distribution.

$\int f(t)dt = 1$ and $\int tf(t)dt = 4$.

CHAPTER 4

Life Insurance

4.1 Introduction to life insurance.

In this chapter, we will consider a cashflow of **contingent payments**, i.e. the payments depend on uncertain events modeled as a random variable. A **contingent cashflow** is a cashflow whose payments are uncertain.

Example 4.1. *On January 1, 2000, John entered a whole life insurance contract.*
This contract pays a death benefit of \$50,000 at the end of the year of death.
On June 13, 2009, John died. The annual effective rate of interest is 6%.
Calculate the **present value** *of the benefit payment at the time of the issue of this contract.*

Solution: Time of death: 6/13/2009. Time of payment: 1/1/2010.
Time of present value: 1/1/2000. Present value= bv^t
The present value on 1/1/2000, of the paid benefit payment is $(50,000)(1.06)^{-10} \approx 27,919.79$.

Definition 4.1. *The mean of the present value at the time of purchase of a cashflow is called its* **actuarial present value (APV)** *of the cashflow of payments,*
its **expected present value**, *or*
its **net single premium**.

Recall that v_t is the t–year discount factor,
the force of interest is $\delta_t = -\frac{d}{dt}\ln v_t$ (similar to $\mu(t) = -\frac{d}{dt}\ln S_X(t)$)
$v_t = e^{-\int_0^t \delta_s\, ds}$ (similar to $S_X(t) = e^{-\int_0^t \mu(x)dx}$).
Under compound interest: $v = (1+i)^{-1} = 1-d = e^{-\delta}$.
i is the annual effective rate of interest,
v is the annual discount factor,
d is the annual discount rate,
$v_t = v^t = (1+i)^{-t}$,
$\delta_t = \delta = \ln(1+i) = -\ln v$.
Two important formulas:
$\sum_{i=1}^n t^i = t\frac{1-t^n}{1-t}$,
$\sum_{i=0}^n t^i = \frac{1-t^{n+1}}{1-t}$,
Special cases: $1+t = \frac{1-t^2}{1-t}$, $1+t+t^2 = \frac{1-t^3}{1-t}$.

Example 4.2. *John pays for its electric bill at the end of each month. John estimates that its electric bill $X_j \sim U[100,300]$. Assume that John is going to pay his bill precisely at the end of each month. Find the APV of the total amount which John will pay in electricity in the next 12 months if $i = 6\%$.*

Solution: $X_1, \ldots, X_{12}$ are the amounts in John's electric bill for the next 12 months. So the total amount is $\sum_{j=1}^{12} X_j$,
but their present values is $Z = \sum_{j=1}^{12} v^{t_j} X_j = \sum_{j=1}^{12} v^{j/12} X_j$, where $v = 1/(1+i) = 1/1.06$.

$$\text{The APV of } Z \text{ is} \quad E(Z) = E\left[\sum_{j=1}^{12} v^{j/12} X_j\right] = E[X_1] \sum_{j=1}^{12} (v^{1/12})^j$$
$$= \frac{100+300}{2} p \frac{1-p^{12}}{1-p}\Big|_{p=v^{1/12}} \approx 2325.76$$

In this chapter, we consider an insurance policy on a certain entity.
Most of the times, the considered entity is a live aged (x).
Let T be the age–at–death of this entity.
The policyholder receives a payment at a certain time in the future.
Both the amount of the payment and the payment date depend on T.
Let b_t be the benefit payment made when failure happens at time t.
Let v_t be the discount factor when failure happens at time t.
The present value of the benefit payment is $b_T v_T$ and is denoted by

$$\begin{cases} \overline{Z} = b_T v_T & \text{if } T \text{ is cts,} \\ Z = b_T v_T & \text{if } T \text{ is discrete.} \end{cases} \tag{4.1}$$

$v_t = v^t$ if the benefit payment is made at the time of death and compound interest is assumed.

In this section, we will see different insurance policies. Each policy has a different (b_t, v_t), $t \geq 0$. The theory in this section applies to life insurance as well as insurance related with the time at failure of inanimate objects.

Example 4.3.

Example 4.4. *An insurance guarantees a payment at the time of failure of a machine.*
(i) The age–at–failure T of this machine satisfies $T \sim U(0, 40)$.
(ii) $i = 7\%$.
(iii) The payment is $b_t = (20000)(1.04)^t$.
Find the mean and the SD of the present value random variable for this insurance.

Solution: The present value random variable of the payment benefit is

$$\overline{Z} = b_T v_T = (20000)(1.04)^T (1.07)^{-T} = (20000)\left(\frac{1.04}{1.07}\right)^T \overset{def}{=} g(T).$$

Possible formulas: $E(\overline{Z}) = \int t \underbrace{f_{\overline{Z}}(t)}_{=?} dt = \int \underbrace{S_{\overline{Z}}(t)}_{=?} dt = \int g(t) \underbrace{f_T(t)}_{=?} dt$ **Which to choose ?**

$E(g(T)) = \int g(t)f_T(t)dt = \int b_t v_t f_T(t)dt$, where $f_T(t) = \frac{1}{40},\ 0 \le t \le 40$.

$$E[\overline{Z}] = \int_0^{40} (20000)\left(\frac{1.04}{1.07}\right)^t \frac{1}{40}\,dt = \frac{(20000)\left(\frac{1.04}{1.07}\right)^t}{40\ln(1.04/1.07)}\Bigg|_0^{40} \quad (\int a^t dt = \frac{a^t}{\ln a} + c)$$
$$= \frac{(20000)((\frac{1.04}{1.07})^{40} - 1)}{40\ln(1.04/1.07)} \approx 11945.07,$$
$$E[\overline{Z}^2] = \int_0^{40} (20000)^2 \left(\frac{1.04}{1.07}\right)^{2t} \frac{1}{40}\,dt = \frac{(20000)^2\left((\frac{1.04}{1.07})^2\right)^t}{40\ln((1.04/1.07)^2)}\Bigg|_0^{40} = 157748208.7,$$
$$\sigma_{\overline{Z}} = \sqrt{\mathrm{Var}(\overline{Z})} = \sqrt{157748208.7 - (11945.07)^2} \approx 3881.19.$$

Example 4.5. *A four–year warranty on a digital television will pay \$400$(5-k)$ if the television breaks during the k–th year, $k = 1, \ldots, 4$.*
The payment will be paid at the end of the year.
The effective annual discount rate is $d = 4\%$.
The survival function $s(x) = \frac{1000}{(x+10)^3}$, $x \ge 0$.
Find the actuarial present value of this warranty benefit.

Solution: T is the time to break of the TV.
$(b_t, v_t) = (400(5-k), v^k) = (400(5-k), (1-d)^k)$ if $k = \lceil t \rceil$ (*i.e.*, $t \in (k-1, k]$), $k \in \{1, 2, 3, 4\}$.
Define $K = \lceil T \rceil$.
In this case the present value of the benefit payment is

$$Z = b_T v_T = 400(5 - \lceil T \rceil)v^{\lceil T \rceil} = 400(5-K)v^K. \tag{4.2}$$

Formulas: $E(Z) = \sum t \underbrace{f_Z(t)}_{=?} = \int \underbrace{S_Z(t)}_{=?}\,dt = \int g(t) \underbrace{f_T(t)}_{=?}\,dt = \sum g(t) \underbrace{f_K(t)}_{=?}$. **Which to choose?**

The actuarial present value of the warranty benefit is

$$E(Z) = \sum_{t=1}^{4} g(t)f_K(t) = \sum_{k=1}^{4} b_k v^k \mathrm{P}\{k-1 < T \le k\}$$
$$= \sum_{k=1}^{4} 400(5-k)v^k(s(k-1) - s(k)) \qquad s(x) = \frac{10^3}{(x+10)^3}$$
$$= 400(5-1)(0.96)\left(\frac{1000}{(10)^3} - \frac{1000}{(1+10)^3}\right)$$
$$+ 400(5-2)(0.96)^2\left(\frac{1000}{(1+10)^3} - \frac{1000}{(2+10)^3}\right)$$
$$+ 400(5-3)(0.96)^3\left(\frac{1000}{(2+10)^3} - \frac{1000}{(3+10)^3}\right)$$
$$+ 400(5-4)(0.96)^4\left(\frac{1000}{(3+10)^3} - \frac{1000}{(4+10)^3}\right)$$
$$\approx 712.14.$$

4.2 Payments paid at the end of the year of death.

4.2.1 Whole life insurance.

Definition 4.2. *A policy is called a* **whole life policy** *if it pays a fixed amount, called the* **face value** *or* **death benefit**, *after the death of the policyholder.*

The payment in a whole life insurance can be paid at different times. In this section, we consider the situation when the face value is paid at the end of the year of death.

An insurer offering life insurance takes a liability. It is of interest to know the amount of this liability.

Definition 4.3. *The present value at time of issue of the death benefit payment of a* unit *whole life insurance payable at the end of the year of the death is denoted by Z_x. Its APV is denoted by A_x, also called the* **premium**. *The APV of a contingent contract is called the* **net single premium** *($\neq A_x$).*

$Z_x = v^{K_x}$, where K_x is the time interval of death of (x).
The future lifetime T_x of (x) satisfies that $T_x \in (K_x - 1, K_x]$.
$A_x = E[Z_x] = E[v^{K_x}]$ and $v^{\omega - x} \leq A_x \leq v$, where
ω is the terminal age of the population.
If x and ω are integers, Z_x is a discrete random variable taking the values $v, v^2, \ldots, v^{\omega-x}$.
The model $Z = b_T v_T$ in (4.1) applies with $b_t = b$ and $v_t = v^{\lceil t \rceil}$.
The model $Z = b_K v_K$ in (4.2) applies with $b_k = b$ and $v_k = v^k$, $k = 1, 2, \ldots$

The insurer would like that a policy holder will die as late as possible. In this way, the present value of the death benefit is as low as possible.
Z_x is a random variable. An insurer may estimate Z_x using its expected value.
The APV of a whole life insurance with payment b is bA_x.

Example 4.6. *Let $i = 5\%$ and*

k	*1*	*2*	*3*
$\mathrm{P}\{K_x = k\}$	*0.2*	*0.3*	*0.5*

(1) Find f_{Z_x}, (2) Find A_x (i.e., $E(Z_x)$) and (3) $\mathrm{Var}(Z_x)$.

Solution: (1) $f_{Z_x} = ?$ $Z_x = v^{K_x}$.
If $Z_x = g(K_x)$, $P(Z_x = z) = \sum_{k:g(k)=z} P(K_x = k)$. **Q:** $g =$ **??**

k	1	2	3
$\mathrm{P}\{K_x = k\}$	0.2	0.3	0.5
$Z_x = v^{K_x} = t$	v^1	v^2	v^3
t	1.05^{-1}	1.05^{-2}	1.05^{-3}
$f_{Z_x}(t)$	?	?	?

t	1.05^{-1}	1.05^{-2}	1.05^{-3}
$f_{Z_x}(t)$	0.2	0.3	0.5

(2) and (3) Formula: $A_x = E(Z_x) = \sum_z z f_{Z_x}(z) = E(v^{K_x}) = \sum_k v^k f_{K_x}(k)$.

$A_x = (1.05)^{-1}(0.2) + (1.05)^{-2}(0.3) + (1.05)^{-3}(0.5) = 0.8945038333$

$E[Z_x^2] = (1.05)^{-2}(0.2) + (1.05)^{-4}(0.3) + (1.05)^{-6}(0.5) = 0.8013243364,$

$\mathrm{Var}(Z_x) = 0.8013243364 - (0.8945038333)^2 = 0.001187228612.$

Example 4.7.

Example 4.8.

Formula: $Z_x = v^{K_x}$ and $A_x = A_x(v) = E(v^{K_x})$.
Notation: ${}^mA_x = E(Z_x^m)$ $(= E(v^{mK_x}) = \sum_k v^{mk} f_{K_x}(k))$.
${}^2A_x = E(Z_x^2)$ $(= E(v^{2K_x}) = \sum_k v^{2k} f_{K_x}(k) = A_x(v^2))$.

Recall under compound interest:
i is the annual effective rate of interest,
$v = (1+i)^{-1} = 1 - d = e^{-\delta}$ is the annual discount factor and
d is the annual discount rate.
$\delta = \ln(1+i) = -\ln v$, the force of interest.

Formula: ${}^2A_x = A_x(v^2)$ corresponds to change v in $A_x = A_x(v)$ to v^2.
If $A_x(v)$ also makes use of notations δ, i or d, then

(i) the force of interest is changed from δ into $\delta' = 2\delta$;

(ii) the interest rate is changed from i into $i' = i(2+i)$.

(iii) the discount rate is changed from d into $d' = d(2-d)$.
However, it is easier to remember by changing (i, d, δ) to v in the expression.

Example 4.9. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.
(i) Find the actuarial present value of this life insurance.
(ii) Suppose that 250 80–year old individuals enter this insurance contract and they die according to the deterministic group approach. Suppose that the insurer makes an account to paid for death benefits. Each insuree pays the APV of this insurance contract to enter this contract. At issue, the insurer deposits all net benefit premiums collected into this account. This account earns interest at annual effective rate of 6.5%. Make a table with:
The balance in this account before making deposits/withdrawals at time k;
The total amount of deposits made at time k;
The total amount of withdrawals made at time k;
The balance in this account after making deposits/withdrawals at time $k \in \{0, 1, \ldots, 6\}$.
(iii) Find the probability that the APV of the life insurance is adequate to cover this insurance.
(iv) An insurance company offers this life insurance to 250 80–year old individuals. How much should each policyholder pay so that the insurer has a probability of 1% that the present value of these 250 policies exceed the total premiums received?

Solution: (i) Letting $Z = bZ_x$ $(b = 50000)$, $A = E(Z) = ?$

Formula: $Z_x = v^{K_x}$, $A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k {}_{k-1}|q_x$.

$$_{k-1}|q_x = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1} \quad [8]$$

$$= {}_{k-1}p_x - {}_kp_x = {}_kq_x - {}_{k-1}q_x.$$ **Which to choose ?**

$_{k-1}|q_x = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = \frac{d_{x+k-1}}{\ell_x}$ =>

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

$$A_{80} = \sum_{k=1}^{\infty} v^k \frac{d_{80+k-1}}{\ell_{80}} = v^1\frac{d_{80}}{\ell_{80}} + \cdots + v^6\frac{d_{85}}{\ell_{80}} + v^7\frac{d_{86}}{\ell_{80}} + 0$$
$$= (1.065)^{-1}\frac{33}{250} + (1.065)^{-2}\frac{56}{250} + (1.065)^{-3}\frac{54}{250} + (1.065)^{-4}\frac{45}{250} + (1.065)^{-5}\frac{34}{250}$$
$$+ (1.065)^{-6}\frac{28}{250} + 0 \approx 0.8162.$$

Hence, the APV for this insurance is $A = bA_{80} = (50000)(0.8162) \approx 40810$.

(ii) Initial balance in the account before making any deposit/withdrawals is 0.
The total amount of deposits made at time 0 is $D_0 = 250A \approx 10202376.46$.

No other deposits are made into this account ($D_k = 0$, $k \geq 1$).
Death benefits are paid at times $k = 1, 2, \ldots, 6$.
The total amount of death benefits paid at time k is $(50000)d_{79+k}$.
Let B_k be the balance in this account at time k before making deposits/withdrawals. Let E_k be the balance in this account at time k after making deposits/withdrawals.

k	Balance before deposits/withdrawals $B_k = (1+i)E_{k-1}$	Deposits D_k	Withdrawals $W_k = (50000)d_{79+k}$	Balance after deposits/withdrawals $E_k = B_k + D_k - W_k$
0	0.000	10,202,376.46	0	10,202,376.460
1	10,865,530.927	0.00	1,650,000	9,215,530.927
2	9,814,540.437	0.00	2,800,000	7,014,540.437
3	7,470,485.566	0.00	2,700,000	4,770,485.566
4	5,080,567.127	0.00	2,250,000	2,830,567.127
5	3,014,553.991	0.00	1,700,000	1,314,553.991
6	1,400,000.000	0.00	1,400,000	0.000

(iii) $A \approx 40809.51$ in (i) is adequate if $A > 50000Z_x$,

<=> $40809.51 > (50000)(1.065)^{-K_x}$,

<=> $K_x > \frac{-\ln(40809.50583/50000)}{\ln(1.065)} \approx 3.225$.

The probability that the APV is adequate is $P\{K_x > 3.225\}$ = **??**

$P\{K_x > 3.225\} = P\{K_x > 3\}$ (**Why ??**)

$= {}_3p_x = \frac{\ell_{x+3}}{\ell_x} = \frac{107}{250} = 0.428$. **What is its implication ?**

$P(Z < E(Z)) < 0.5$, mean<median, A is far from enough for NSP.
(iv) The insurer offers a whole life insurance to n lives aged x with a benefit payment of b paid at the end of the year of death and a price P (for purchasing the insurance).
Let $Z_{x,1}, \dots, Z_{x,n}$ be the present values per unit of their benefit payments.
Let $W = b\sum_{j=1}^n Z_{x,j}/n$. By the CLT,

$$\mathrm{P}(W \le P) \approx \Phi(\tfrac{P-E(W)}{\sigma_W}) = 0.99, \text{ where } \Phi \text{ is the cdf of } N(0,1),$$

$\Rightarrow P = E(W) + z_{0.01}\sigma_W$, where $E(W) = E\left[b\sum_{j=1}^n Z_{x,j}/n\right] = bA_x$ $(\mu_{\overline{Y}} = \mu)$, and

$\sigma_W^2 = \mathrm{Var}(W) = \mathrm{Var}\left(b\sum_{j=1}^n Z_{x,j}/n\right) = b^2V(Z_x)/n$ $\qquad (\sigma_{\overline{Y}}^2 = \sigma^2/n)$.
$= b^2({}^2A_x - A_x^2)/n$

$$\begin{aligned}{}^2A_x =& E(Z_x^2) = A_x(v^2) = \sum_{k=1}^{\infty} v^{2k}\frac{d_{80+k-1}}{\ell_{80}} = (1.065)^{-2}\frac{33}{250} + (1.065)^{-4}\frac{56}{250} \\ &+ (1.065)^{-6}\frac{54}{250} + (1.065)^{-8}\frac{45}{250} + (1.065)^{-10}\frac{34}{250} + (1.065)^{-12}\frac{28}{250} \approx 0.672\end{aligned}$$

Hence, $\sigma_W^2 = 50000^2(0.6723 - 0.8162^2)/250$.

Each policyholder should pay

$$E(W) + z_{0.01}\sigma_W = 40809.50 + 2.326\sqrt{50000^2(0.6723 - 0.8162^2)/250} \approx 41387.92. \quad \square$$

The following is the table of A_{40}, using the life table in page 602. If $i \ge 0$, $0 < A_x \le 1$.

i	2%	3%	4%	5%	6%	7%	8%
A_{40}	0.4658	0.3286	0.2373	0.1754	0.1326	0.1026	0.0812

$A_x \downarrow$ in i.

Table 7.2 (see page 603) shows A_x and ${}^2A_x$ using the life table for the total population of United States in 2004 and $i = 6\%$.

Example 4.10. *An insurer issues a whole life insurance to 100 lives age* 40 *which pays \$20000 at the end of the year of their death.* $i = 0.06$. *Mortality follows the life table for the USA population in 2004 (see pages 250–253). The insurer has a fund with an amount of \$300,000 of dollars to paid for these 100 life insurances. Calculate the probability that this fund is not enough to cover the payments of these 100 life insurances.*

Solution: Let $Z = \sum_{j=1}^{100}(20000)Z_{40,j}$. $Z_{40,j}$'s are i.i.d. from $Z_{40} = v^{K_{40}}$.
$\mathrm{P}\{Z > 300000\} = ?$
Formula: $P(\overline{Y} \le t) \approx \Phi(\frac{t-\mu_{\overline{Y}}}{\sigma_{\overline{Y}}}) \Rightarrow P(n\overline{Y} \le nt) \approx \Phi(\frac{nt-n\mu_{\overline{Y}}}{n\sigma_{\overline{Y}}}) \Rightarrow P(Z \le x) \approx \Phi(\frac{x-\mu_Z}{\sigma_Z})$.
From the life table, $A_{40} = 0.13264232$ and ${}^2A_{40} = 0.03648695$. We have that

$$\begin{aligned}&E[Z] = nbE(Z_{40}) = (100)(20000)(0.13264232) = 265284.64,\\ &\mathrm{Var}(Z_{40}) = 0.03648695 - (0.13264232)^2 = 0.01889296495,\\ &\mathrm{Var}(Z) = nb^2V(Z_{40}) = 100(20000)^2(0.01889296495) = 755718598.\end{aligned}$$

The probability that the fund is not enough to cover the payments is

$$\mathrm{P}\{Z > 300000\} \approx 1 - \Phi(\frac{300000 - 265284.64}{\sqrt{755718598}}) \approx 1 - \Phi(1.26) \approx 0.1038.$$

Example 4.11. *If the mortality of (x) is given by*

k	*0*	*1*	*2*	*3*	*4*
p_{x+k}	0.05	0.01	0.005	0.001	0

calculate A_x if $i = 7.5\%$.

Solution: $A_x = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k f_{K_x}(k)$, where $v = \frac{1}{1+i}$. $f_{K_x}(k) =$

$$_{k-1|}q_x = \frac{s(x+k-1) - s(x+k)}{s(x)} = \frac{\ell_{x+k-1} - \ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1} \quad (1)$$

Which to choose ?

$$\begin{aligned}
A_x &= \sum_{k=1}^{\infty} v^k \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1} \text{ \textbf{(what happens to} } k=1 \text{ \textbf{?)}} \\
&= vq_x + v^2 p_x q_{x+1} + v^3 p_x p_{x+1} q_{x+2} + v^4 p_x p_{x+1} p_{x+2} q_{x+3} + v^5 p_x p_{x+1} p_{x+2} p_{x+3} q_{x+4} \\
&\quad + v^6 p_x p_{x+1} p_{x+2} p_{x+3} \underbrace{p_{x+4}}_{=0} q_{x+5} + \cdots \\
&= (1.075)^{-1}(0.95) + (1.075)^{-2}(0.05)(0.99) + (1.075)^{-3}(0.05)(0.01)(0.995) \\
&\quad + (1.075)^{-4}(0.05)(0.01)(0.005)(0.999) + (1.075)^{-5}(0.05)(0.01)(0.005)(0.001)(1) \approx 0.9270.
\end{aligned}$$

Example 4.12. *Rose is 40 years old. She buys a whole life policy insurance which will pay \$200000 at the end of the year of her death. Suppose that the de Moivre model holds with terminal age* 120*. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* 10%*.*

Solution: $Z = bZ_x$, $Z_x = v^{K_x}$, $b = 200,000$. $\sigma_Z = b\sigma_{Z_x}$. $\sigma^2_{Z_x} = {}^2A_x - A_x^2$. $A_x(v) = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k {}_{k-1|}q_x$. **Which in Eq. (1) to choose for** $_{k-1|}q_x$ **?**
Under de Moivre's model with terminal age ω, if $\omega - x$ is a positive integer,

$$A_x(v) = \sum_{k=1}^{\omega-x} v^k \frac{s(x+k-1) - s(x+k)}{\underbrace{s(x)}_{=(w-x)/w}} = \sum_{k=1}^{\omega-x} v^k \frac{1}{\omega - x} = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}.$$

$$^mA_x = (v^m)\frac{1-(v^m)^{w-x}}{1-(v^m)}\frac{1}{w-x}. \quad \textbf{Why??}$$

$^mA_x(v) = A_x(v^m)$.
$A_{40} = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}\Big|_{v=1/1.1,\ w=120,\ x=40} = 0.124939$ and
$^2A_{40} = v\frac{1-v^{w-x}}{1-v}\frac{1}{w-x}\Big|_{v=(1/1.1)^2,\ w=120,\ x=40} = 0.0595238$.
The actuarial present value of this life insurance is $bA_{40} \approx 24987.80$ ($b = 200000$).
The standard deviation of the present value of this life insurance is

$$b({}^2A_x - (A_x)^2)^{1/2} \approx 41911.36.$$

Recall that the present of a unit annuity–immediate (in Math 346) is
$a_{\overline{n}|i} = \sum_{k=1}^n v^k = v\frac{1-v^n}{1-v} = \frac{1-v^n}{i}$ as $v = \frac{1}{1+i}$.

Theorem 4.1.

Theorem 4.2.

Example 4.13. *Jess and Jane buy a whole life policy insurance on the day of their birthdays. Both policies will pay $50000 at the end of the year of death. Jess is 45 years old and the net single premium of her insurance is $25000. Jane is 44 years old and the net single premium of her insurance is $23702. Suppose that* $i = 0.06$. *Find the probability that a 44–year old will die within one year.*

Solution: $bA_{45} = 25000$ and $bA_{44} = 23702$, with $b = 50000$.
$A_{44} = 23702/50000$, $A_{45} = 25000/50000 = 1/2$ and $i = 0.06$. $q_{44} = {}_{0|1}q_{44}$?

$$A_x = \sum_{k=1}^{\infty} v^k{}_{k-1}|q_x = {}_{1-1|1}q_x v^1 + \sum_{k=2}^{\infty} v^k{}_{k-1}|q_x = \overbrace{q_x}^{?} v + \overbrace{\sum_{k=2}^{\infty} v^k{}_{k-1}|q_x}^{=?A_{x+1}} = A_x.$$

$$\begin{aligned} A_x &= \textstyle\sum_{k=1}^{\infty} v^k{}_{k-1}|q_x && x = ? \\ &= vq_x + \textstyle\sum_{k=2}^{\infty} v^k{}_{k-1}|q_x \\ &= vq_x + \textstyle\sum_{k=2}^{\infty} v^k p_x \cdot {}_{k-2}|q_{x+1} && \text{Formula [8]: } {}_k|q_x = p_x \cdot {}_{k-1}|q_{x+1} \\ &= vq_x + vp_x \textstyle\sum_{k-1=1}^{\infty} v^{k-1}{}_{(k-1)-1}|q_{x+1} \\ &= vq_x + vp_x \textstyle\sum_{j=1}^{\infty} v^j{}_{j-1}|q_{x+1} && j = k-1 \\ &= vq_x + vp_x A_{x+1} && \text{(which is Theorem 4.3 or formula [14]).} \end{aligned}$$

$A_x = vq_x + v(1 - q_x)A_{x+1} = q_x v(1 - A_{x+1}) + vA_{x+1} =>$

$$q_{44} = \left.\frac{A_{44} - vA_{45}}{v(1 - A_{45})}\right|_{v=\frac{1}{1.06}, A_{44}\approx 0.47, A_{45}=0.5} \approx 0.0049648$$

Theorem 4.3. *(*Iterative formula for the APV of a discrete whole life insurance*) For each* $x > 0$, $A_x = vq_x + vp_x A_{x+1}$.

Definition 4.4.

Example 4.14.

Example 4.15.

Example 4.16. *Jane is 30 years old. She buys a whole life policy insurance which will pay \$20000 at the end of the year of her death. Suppose that $\underline{p_x = 0.9, \text{ for each } x \geq 0}$, and $i = 5\%$. Find the APV of this life insurance and its variance.*

Solution: $Z = bZ_x$. The APV $A = 20000 A_x = ?$
Formula: $A_x = \sum_{k=1}^{\infty} v^k f_{K_x}(k) = \sum_{k=1}^{\infty} v^k {}_{k-1}|q_x$, and

$$f_{K_x}(k) = \frac{s(x+k-1) - s(x+k)}{s(x)} = \frac{\ell_{x+k-1} - \ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$$

Which to choose ?

$$\begin{aligned}
A_x =& \sum_{k=1}^{\infty} v^k p_x^{k-1}(1-p_x) \textbf{ Why ?} \\
=& \frac{1-p_x}{p_x} \sum_{k=1}^{\infty} (vp_x)^k = \frac{1-p_x}{p_x} \frac{t(1-(t)^{\infty})}{1-t}\Big|_{t=vp_x} = \frac{1-p_x}{p_x} \frac{vp_x(1-(vp_x)^{\infty})}{1-vp_x} = \frac{1-p_x}{p_x} \frac{vp_x}{1-vp_x} \\
=& \frac{(1-p_x)v}{1-vp_x} = \frac{1-p_x}{\frac{1}{v} - p_x} = \frac{1-p_x}{1+i-p_x} = \frac{q_x}{q_x+i} = A_x \qquad \text{(which is Th 4.4)} \\
{}^2A_x =& A_x(v^2) = \frac{1-p_x}{\frac{1}{v^2} - p_x} = \frac{q_x}{q_x + (1+i)^2 - 1}
\end{aligned}$$

Q: Which of the last two expressions you prefer ?

Since $b = 20000$, the actuarial present value is

$$\begin{aligned}
\text{APV} =& bA_{30} = (20000)\frac{1-p_x}{\frac{1}{v} - p_x}\Big|_{p_x=0.9, v=1/1.05} \approx 13333. \\
{}^2A_{30} =& A_x(v^2) = \frac{1-p_x}{\frac{1}{v} - p_x}\Big|_{p_x=0.9, v=1/1.05^2} \approx 0.493827. \\
\sigma_Z^2 =& b^2({}^2A_x - (A_x)^2) = b^2({}^2A_x) - (bA_x)^2 = 19761991.
\end{aligned}$$

Theorem 4.4. *Suppose that for each $k = 1, 2, \ldots$, $p_{x+k} = p_x$. Then,*

$$A_x = \frac{1-p_x}{\frac{1}{v} - p_x} = \frac{q_x}{q_x+i} \text{ and } {}^mA_x = \frac{1-p_x}{\frac{1}{v^m} - p_x} = \frac{q_x}{q_x + (1+i)^m - 1} \tag{1}$$

In particular, ${}^2A_x = \frac{(1-p_x)}{\frac{1}{v^2} - p_x} = \frac{q_x}{q_x + i(2+i)}$. It is better to derive Eq.(1) rather than memorizing it, as it is valid only for $p_{x+k} = p$.

Corollary 4.1.

Example 4.17. *An actuary models the future lifetime of (30) as follows. The actuary classifies lives according with health into 3 groups: good health, average health, and poor health. The probabilities of belonging to a given group are given by the following table. Individuals for the same group have the same constant force of mortality. The force of mortality for each group is given in the following table*

Group	*good health*	*average health*	*poor health*
Probability	*0.1*	*0.3*	*0.6*
Force of mortality	*0.01*	*0.05*	*0.1*

(3)

The annual effective rate of interest is $i = 7.5\%$. Calculate A_x and $\mathrm{Var}(Z_x)$.

Solution: Eq. (1) yields ${}^mA_x = \frac{1-p_x}{\frac{1}{v^m}-p_x}$. **Can we use it directly here ?**
We introduce a new r.v. Y to denote the health status of an insuree.

$$Y = \begin{cases} 2 & \text{if an insuree is in good health,} \\ 1 & \text{if an insuree is in average health,} \\ 0 & \text{if an insuree is in poor health,} \end{cases}$$

j	2	1	0
$f_Y(j)$	0.1	0.3	0.6
μ_j	0.01	0.05	0.1

Using the double expectation theorem,
${}^mA_x = E[Z_x^m] = E[E[Z_x^m|Y]] = \sum_{j=0}^{2} f_Y(j)E[Z_x^m|Y=j]$, $m = ?$

$$E[Z_x|Y=2] = A_x^{\text{good}} = \frac{1-p_{x,2}}{\frac{1}{v^1}-p_{x,2}} = \frac{1-e^{-0.01}}{1.075-e^{-0.01}} = 0.11712945 \quad \textit{by } (3),$$

$$\text{[5]: For constant force of mortality } \mu_x = c,\ p_x = e^{-\int_0^1 \mu_x dx} = e^{-c}$$

$$E[Z_x^2|Y=2] = {}^2A_x^{\text{good}} = \frac{1-p_{x,2}}{\frac{1}{v^2}-p_{x,2}} = \frac{1-e^{-0.01}}{(1.075)^2-e^{-0.01}} = 0.06009455691.$$

$$E[Z_x|Y=1] = A_x^{\text{average}} = \frac{1-p_{x,1}}{\frac{1}{v^1}-p_{x,1}} = \frac{1-e^{-0.05}}{1.075-e^{-0.05}} = 0.3940401449,$$

$$E[Z_x^2|Y=1] = {}^2A_x^{\text{average}} = \frac{1-e^{-0.05}}{(1.075)^2-e^{-0.05}} = 0.2386087633.$$

$$E[Z_x|Y=0] = A_x^{\text{poor}} = \frac{1-e^{-0.1}}{1.075-e^{-0.1}} = 0.5592450518,$$

$$E[Z_x^2|Y=0] = {}^2A_x^{\text{poor}} = \frac{1-e^{-0.1}}{(1.075)^2-e^{-0.1}} = 0.3794549205.$$

$$\begin{aligned} A_x =& E[Z_x] = E[E[Z_x|Y]] \\ =& E[Z_x|Y=2]\mathrm{P}\{Y=2\} + E[Z_x|Y=1]\mathrm{P}\{Y=1\} + E[Z_x|Y=0]\mathrm{P}\{Y=0\} \\ \approx& (0.1)(0.1171) + (0.3)(0.3940) + (0.6)(0.5592) \approx 0.3053, \\ {}^2A_x =& E[Z_x^2] = E[E[Z_x^2|Y]] \\ =& E[Z_x^2|Y=2]\mathrm{P}\{Y=2\} + E[Z_x^2|Y=1]\mathrm{P}\{Y=1\} + E[Z_x^2|Y=0]\mathrm{P}\{Y=0\} \\ \approx& (0.1)(0.0601) + (0.3)(0.2386) + (0.6)(0.3795) \approx 0.4480, \end{aligned}$$

$$\mathrm{Var}(Z_x) = {}^2A_x - (A_x)^2 \approx 0.448 - (0.3053)^2 \approx 0.0886.$$

4.2.2 n–year term life insurance.

Definition 4.5. *The* n**–th term life insurance policy** *(or n–year term life insurance): It pays a face value b if $T(x) \le n$ (the insured dies within n years of the issue of the policy).*

Definition 4.6. *The present value and the APV of an n–year term life insurance policy which pays a* **unit** *face value at the end of the year of the death is denoted by $Z^1_{x:\overline{n}|}$ and $A^1_{x:\overline{n}|}$, respectively ($A^1_{x:\overline{n}|} = A^1_{x:\overline{n}|}(v) = E[Z^1_{x:\overline{n}|}]$).*

Definition 4.7. $Z^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n) = \begin{cases} v^{K_x} & \text{if } K_x \le n, \\ 0 & \text{if } K_x > n. \end{cases}$

The model $Z = b_k v_k$ applies with $b_k = bI(K_x \le n)$ and $v_k = v^k$, $k \ge 1$.
3 types of problems:
(1) $A = bA^1_{x:\overline{n}|} = ?$
(2) $\sigma^2 = b^2 V(Z^1_{x:\overline{n}|}) = ?$
(3) p, or n, or $P(Z < p)$? (such that $P(Z < p) = \Phi(\frac{p-A}{\sigma_Z}) > 0.99$ or 0.95.)

Theorem 4.5. ${}^mA^1_{x:\overline{n}|} = A^1_{x:\overline{n}|}(v^m) = E[(Z^1_{x:\overline{n}|})^m] = \sum_{k=1}^n (v^m)^k \cdot {}_{k-1}|q_x$.

Example 4.18. *Let $i = 0.05$, $q_x = 0.05$ and $q_{x+1} = 0.02$. Find $A^1_{x:\overline{2}|}$ and* $\mathrm{Var}(Z^1_{x:\overline{2}|})$.

Solution: Formulas: $A^1_{x:\overline{n}|} = E(v^{K_x} I(K_x \le n)) = \sum_{k=1}^n v^k \cdot f_{K_x}(k) = v f_{K_x}(1) + v^2 f_{K_x}(2)$.
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **Which one ?**

$$\begin{aligned}
A^1_{x:\overline{2}|}(v) =& v q_x + v^2 p_x q_{x+1} \quad \textbf{why } ? \\
=&(1.05)^{-1}(0.05) + (1.05)^{-2}(1-0.05)(0.02) = 0.06485260771, \\
{}^2A^1_{x:\overline{2}|} =& A^1_{x:\overline{2}|}(v^2) = v^2 q_x + v^4 p_x q_{x+1} \\
=&(1.05)^{-2}(0.05) + (1.05)^{-4}(1-0.05)(0.02) = 0.06098282094, \\
\mathrm{Var}(Z^1_{x:\overline{2}|}) =& {}^2A^1_{x:\overline{2}|} - \left(A^1_{x:\overline{2}|}\right)^2 \approx 0.06098 - (0.06485)^2 \approx 0.0568.
\end{aligned}$$

Example 4.19. *Consider the life table*

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year term life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the APV and the standard deviation of the present value of this life insurance.

(ii) Find the probability that the APV of this life insurance is adequate to cover it.

(iii) Find the probability that the present value of this life insurance exceeds one standard deviation to its APV.

Solution: (i) The APV of this life insurance $Z = bv^{K_x} I(K_x \le n)$ is $A = (50000)A^1_{80:\overline{3}|}$

Formula: $A^1_{x:\overline{n}|}(v) = \sum_{k=1}^n v^k \cdot f_{K_x}(k)$, $\sigma_Z = 50000\sqrt{{}^2A_{80:\overline{3}|} - A^2_{80:\overline{3}|}}$ and
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\geq 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **Which one ?**

$$
\begin{aligned}
A^1_{80:\overline{3}|} =& \sum_{k=1}^{3} v^k \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}} \\
=&(1.065)^{-1}\frac{250-217}{250} + (1.065)^{-2}\frac{217-161}{250} + (1.065)^{-3}\frac{161-107}{250} = 0.5002507 \\
{}^2A^1_{80:\overline{3}|} =& A^1_{80:\overline{3}|}(v^2) = \sum_{k=1}^{3} v^{2k} \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}} \\
=&(1.065)^{-2}\frac{250-217}{250} + (1.065)^{-4}\frac{217-161}{250} + (1.065)^{-6}\frac{161-107}{250} = 0.4385316, \\
A =&(50000)A^1_{80:\overline{3}|} = 25012.53726. \\
\sigma_Z =& 50000\sqrt{0.4385316 - 0.5002507^2} = 21695.66542.
\end{aligned}
$$

(ii) The probability that 25012.53726 is adequate is

$$
\begin{aligned}
P(B) =& \mathrm{P}\{(50000)Z^1_{80:\overline{3}|} \leq 25012.53726\} = \mathrm{P}\{Z^1_{80:\overline{3}|} \leq 25012.53726/50000\} \\
=&\mathrm{P}\{(1.065)^{-K_x} I(K_x \leq 3) \leq 25012.53726/50000\} \\
=&\mathrm{P}\{B, K_x \leq 3\} + \mathrm{P}\{B, K_x > 3\} \qquad B = \{(1.065)^{-K_x} I(K_x \leq 3) \leq 25012.53726/50000\} \\
=&\mathrm{P}\{(1.065)^{-K_x} \leq 25012.53726/50000, K_x \leq 3\} + \mathrm{P}\{0 \leq 25012.53726/50000, K_x > 3\}\mathbf{??} \\
=&\mathrm{P}\{-K_x \ln(1.065) \leq \ln(25012.53726/50000), K_x \leq 3\} + \mathrm{P}\{K_x > 3\} \\
=&\mathrm{P}\{K_x \geq -\frac{\ln(25012.53726/50000)}{\ln(1.065)}, K_x \leq 3\} + \mathrm{P}\{K_x > 3\} \\
=&\mathrm{P}\{K_x \geq 10.999, K_x \leq 3\} + \mathrm{P}\{K_x > 3\} \\
=&\mathrm{P}\{T_x > 3\} = \frac{\ell_{x+3}}{\ell_x}\Big|_{x=80} = \frac{107}{250} = 0.428.
\end{aligned}
$$

(iii) $\mathrm{P}(Z > A + \sigma_Z) = \mathrm{P}\{(50000)Z^1_{80:\overline{3}|} > 25012.53726 + 21695.66542\}$

$$
\begin{aligned}
=&\mathrm{P}\{(50000)(1.065)^{-K_x} I(K_x \leq 3) > 46708.20268\} \\
=&\mathrm{P}\{(50000)(1.065)^{-K_x} > 46708.20268, K_x \leq 3\} + \mathrm{P}\{0 > 46708.20268, K_x > 3\} \\
=&\mathrm{P}\{K_x < -\frac{\ln(46708.20268/50000)}{\ln(1.065)}, K_x \leq 3\} \\
=&\mathrm{P}\{K_x < 1.081435921, K_x \leq 3\} \\
=&\mathrm{P}\{K_x \leq 1\} = \mathbf{??} \\
=&P(T(x) \leq 1) \\
=&\frac{d_x}{\ell_x} = \frac{250-217}{250} = 0.132.
\end{aligned}
$$

Example 4.20. *Suppose that* $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$. *Find* $A^1_{x:\overline{10}|}$ *and* $\mathrm{Var}(Z^1_{x:\overline{10}|})$.

Solution: Formulae: ${}_tp_x = e^{-\int_0^t \mu(x)dx} = e^{-\mu t} => p_x = e^{\mu}$, $A^1_{x:\overline{n}|}(v) = \sum_{k=1}^n v^k \cdot f_{K_x}(k)$, and $f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **Which one ?** Either the 1st one, or the last two yields

$$
\begin{aligned}
A^1_{x:\overline{10}|}(v) =& \sum_{k=1}^{10} v^k f_{K_x}(k) = \sum_{k=1}^{10} v^k (p_x)^{k-1} q_x \textbf{ (Why ??)} \\
=& \sum_{k=1}^{10} (vp_x)^k p_x^{-1}(1-p_x) \\
=& p_x^{-1}(1-p_x)\frac{vp_x(1-(vp_x)^{10})}{1-vp_x} \\
=& (1-p_x)\frac{v(1-(vp_x)^{10})}{1-vp_x}\Big|_{p_x = e^{-0.03} \text{ and } v = e^{-0.04}} \qquad => Th.4.6 \\
=& 0.2114417945, \\
{}^2A^1_{x:\overline{10}|}(v) =& A^1_{x:\overline{10}|}(v^2) \\
=& (1-p_x)\frac{v^2(1-(v^2p_x)^{10})}{1-v^2p_x}\Big|_{p_x = e^{-0.03} \text{ and } v = e^{-0.04}} \\
=& 0.1747285636, \\
\mathrm{Var}(Z^1_{x:\overline{10}|}) =& {}^2A^1_{x:\overline{10}|} - {A^1_{x:\overline{10}|}}^2 \\
=& 0.1747285636 - (0.2114417945)^2 = 0.1300209311.
\end{aligned}
$$

Example 4.21. *An insurer offers a 20–year term life insurance of \$100,000 to independent lives age* 45. $i = 7.5\%$. *Mortality follows de Moivre model with terminal age* 110. *The insurer has a fund with* 10^6 *dollars to pay for these insurances. Using the normal approximation, calculate the maximum number of policies the insurer can cover so that the probability that the aggregate present value for the issued policies exceeds the amount in the fund is less than 0.01.*

Solution: Let n be the number of policies that the insurer can cover. The present value for the aggregate n insurances is $Z = \sum_{j=1}^n (100000) Z^1_{45:\overline{20}|}$. We need determine n so that

$P(Z > 10^6) \approx \Phi(\frac{10^6 - E(Z)}{\sigma_Z}) = \Phi(z_{0.01}) = 0.01, => \frac{10^6-E(Z)}{\sigma_Z} = z_{0.01} =>$
$10^6 = \underbrace{E(Z)}_{n10^5 A^1_{x:\overline{20}|}(v)} + z_{0.01} \times \underbrace{\sigma_Z}_{\sqrt{n(10^5)^2(A^1_{x:\overline{20}|}(v^2)-(A^1_{x:\overline{20}|}(v))^2]}}$, or $an + b\sqrt{n} + c = 0$.
$A^1_{x:\overline{20}|} = \sum_{k=1}^{20} v^k f_{K_x}(k)$.
$f_{K_x}(k) = \frac{s(x+k-1)-s(x+k)}{s(x)} = \frac{\ell_{x+k-1}-\ell_{x+k}}{\ell_x} = {}_{k-1}p_x \cdot q_{x+k-1} = \left(\prod_{j\ge 0}^{k-2} p_{x+j}\right) q_{x+k-1}$ **Which one ?**

$s(x) = 1 - x/w = \frac{w-x}{w}$ and $f_{K_x}(k) = \frac{w-x-k-(w-x-k-1)}{w-x}$

$$A^1_{x:\overline{20}|} = \sum_{k=1}^{20} v^k {}_{k-1|}q_x = \sum_{k=1}^{20} v^k \frac{1}{\omega - x} = \frac{a_{\overline{20}|}}{\omega - x} = \frac{1}{w-x}\frac{v(1-v^n)}{1-v}. \tag{1}$$

$$E(Z) = n10^5 A^1_{45:\overline{20}|} = n10^5 \frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075} = 15683.83286n,$$

$$\mathrm{Var}(Z) = n(10^5)^2 \left(\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075^2} - \left(\frac{v(1-v^{20})}{(1-v)65}\Big|_{v=1/1.075}\right)^2\right) = 687801161.6n.$$

$$\begin{aligned} 10^6 =& E(Z) + z_{0.01}\sigma_Z \\ 10^6 =& 15683.83286n + (2.3263479)\sqrt{687801161.6n} \\ =>& \quad 15683.83286n + 61010.71512\sqrt{n} - 10^6 = 0. \end{aligned}$$

$\sqrt{n} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. The positive solution of $\sqrt{n}$ is $\sqrt{n} = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\sqrt{n} \approx 6.27$. So, $n = 39.35601235$. The maximum number of policies that the insurer can cover is 39.