Extensions of Slutsky's Theorem in Probability Theory

By Qiqing Yu[a] Guoqing Zheng[b] and Shihong Ding[b]

a Department of Mathematical Sciences, Binghamton University, NY 13902, USA

b South China Agricultural University, China

Corresponding author's email address: qyu@math.binghamton.edu

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Abstract: Slutsky's Theorem has important applications in biostatistics. Several generalizations of Slutsky's Theorem are presented. For instance, we study the limiting distribution of Y_n/X_n when $X_n \to 0$ in distribution. Then the sequence of random varibles tends to an extended random variable.

1. Introduction. We study the generalization of the Slutsky's Theorem in this short note. Slutcky's Theorem is an important theorem in the elementary probability course and plays an important role in deriving the asymptotic distribution of varies estimators. Thus Slutsky's Theorem also has important applications in biostatistics. Let X_n, Y_n and X be random variables and a be a constant. Slutsky's Theorem states as follows.

$$\text{If } Y_n \xrightarrow{D} a \text{ and } X_n \xrightarrow{D} X, \text{ then } Y_n + X_n \xrightarrow{D} a + X \text{ and } Y_n X_n \xrightarrow{D} aX.$$

There are some simple generalizations of the theorem. For instance, it is trivially true that assuming $a \neq 0$,

$$\text{if } Y_n \xrightarrow{D} a \text{ and } X_n \xrightarrow{D} X, \text{ then } X_n/Y_n \xrightarrow{D} X/a. \tag{1}$$

We shall study some non-trivial generalizations.

For instance, if $a = 0$, is the statement (1) also valid under certain assumptions ? Moreover, one may wonder whether another generalization of Slutsky's Theorem is as follows.

$$\text{If } Y_n \xrightarrow{D} a \text{ and } X_n \xrightarrow{D} X, \text{ then } Y_n/X_n \xrightarrow{D} a/X, \tag{2}$$

or $\pm 1/X_n \xrightarrow{D} \pm 1/X$, with a certain modification. A well-known result is as follows.

Proposition 1. (Mann & Wald (1943)). Statement (2) holds if $P(X = 0) = 0$.

These problems are interesting. We show in section 2 that the necessary and sufficient condition for statement (2) holds with $a \neq 0$ is $F_{X_n}(0-) \to F_X(0-)$; and that for statement (1) holds with $a = 0$ and $P(Y_n = 0) \to 1$ is $P(X_n = 0) \to P(X = 0)$; among other results.

2. Main Results. In order to study the possible extensions of statements (1) and (2), we first study some simple examples. Notice that if $a = 0$ or $\{X = 0\} \neq \emptyset$, $W = a/X$ involves $\frac{0}{0}$ or $\frac{a}{0}$. Conventionally, $0/0$ can be defined as 0 or 1. In this note, we define $a/0 = \begin{cases} \infty & \text{if } a > 0 \\ 1 & \text{if } a = 0 \\ -\infty & \text{if } a < 0. \end{cases}$ Then $\{W = \pm\infty\} \neq \emptyset$, and W is called an extended random variable. Moreover, if $P(X = 0) > 0$, then $P(X = \pm\infty) > 0$,

$$\lim_{t \to -\infty} F_W(t) = P(X = 0) > 0 \text{ if } a < 0 \text{ and } \lim_{t \to \infty} F_W(t) = P(X \neq 0) < 1 \text{ if } a > 0.$$

In general, statements (1) and (2) are false, and two counterexamples are as follows.

Example 1. Let $X \sim bin(1, 0.5)$, the Bernoulli distribution, $X_{2n+1} = X$ and $X_{2n} = X - \frac{1}{2n}$, $n \geq 1$. Then $X_n \xrightarrow{D} X$. $F_{1/X}(t) = 0.5\mathbf{1}(t \geq 1)$, where $\mathbf{1}(A)$ is the indicator function of the event A. Notice that $F_{1/X}$ is a degenetate cdf, *i.e.*, $\lim_{t\to\infty} F_{1/X}(t) < F_{1/X}(\infty) = 1$ and $P(1/X = +\infty) = 0.5$. However, $P(1/X_{2k} < 0) = P(X = 0) = 0.5$, $k \geq 1$, and $F_{1/X_n}(0-) = F_{1/X_n}(0) = \begin{cases} 0.5 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$ Thus, $1/X_n$ diverges in distribution. Letting $Y_n = 1$, then $Y_n \xrightarrow{D} a = 1$, but Y_n/X_n diverges in distribution. *i.e.*, statement (2) fails. Moreover, let $Z_n = \frac{1}{n}$. Then $Z_n \to 0$, but both Z_n/X_n and X_n/Z_n diverge in distribution, as $Z_n/X_n = \begin{cases} \frac{1}{n\times 0}\mathbf{1}(X=0) + \frac{\mathbf{1}(X=1)}{n} \to \infty\mathbf{1}(X=0) & \text{if } n \text{ is odd} \\ -\mathbf{1}(X=0) + \frac{\mathbf{1}(X=1)}{n-1} \to -\mathbf{1}(X=0) & \text{if } n \text{ is even.} \end{cases}$

Example 2. Let $X \sim bin(1, 0.5)$, $X_n = X + 1/n$, and $Y_n = c/n$, $n \geq 1$, where $c > 0$. It can be verified that $X_n \xrightarrow{D} X$, $1/X_n \xrightarrow{D} 1/X$ and $Y_n \xrightarrow{D} 0$. Notice that

$$Y_n/X_n = c\mathbf{1}(X=0) + \tfrac{c}{n+1}\mathbf{1}(X=1) \to c\mathbf{1}(X=0),$$

$$X_n/Y_n = \mathbf{1}(X=0)/c + \tfrac{n+1}{c}\mathbf{1}(X=1) \to \mathbf{1}(X=0)/c + \infty\mathbf{1}(X=1).$$

Thus, $c = 1$ iff $Y_n/X_n \xrightarrow{D} 0/X$ iff $X_n/Y_n \xrightarrow{D} X/0$. In other words, if $c \neq 1$, both statements (1) and (2) do not hold.

Remark 1. Examples 1 and 2 indicate that under the assumptions in Slutsky's Theorem,

(1) it is not always true that $1/X_n \xrightarrow{D} 1/X$;

(2) Slutsky's Theorem is not applicable to the sequence of extended random variables Y_n/X_n, unless additional assumptions are imposed.

In Proposition 1, a sufficient condition is given, that is, $P(X = 0) = 0$. It is an interesting problem to find the necessary and sufficient condition for the generalization of Slutsky's Theorem as in Eq. (2). To this end, we first establish two lemmas.

Lemma 1. *Let X be a random variable. Then*

$$F_{1/X}(t) = \begin{cases} F_X(0-) - F_X(s-) & \text{if } t < 0 \\ F_X(0-) & \text{if } t = 0 \\ F_X(0-) + 1 - F_X(s-) & \text{if } t > 0, \end{cases} \quad \text{where } s = 1/t; \tag{3}$$

$$F_{-1/X}(t) = \begin{cases} F_X(-1/t) - F_X(0-) & \text{if } t < 0 \\ 1 - F_X(0-) & \text{if } t = 0 \\ 1 - F_X(0-) + F_X(-1/t) & \text{if } t > 0. \end{cases} \tag{4}$$

Remark 2. By the lemma, $F_{1/X}(-\infty) = 0$ and $P(1/X = \infty) = P(X = 0)$. Moreover, $F_{-1/X}(-\infty) = P(X = 0)$ and $P(-1/X = \infty) = 0$.

Remark 3. If $Y_n = -1$ and statement (2) holds, then $Y_n/X_n \xrightarrow{D} -1/X$. By defining $Y = -X$, one may derive the expression of $F_{-1/X}$ as follows. Letting $s = 1/t$,

$$
\begin{aligned}
F_{-1/X}(t) =& F_{1/Y}(t) \\
=& \begin{cases} F_Y(0-) - F_Y(s-) & \text{if } t < 0 \\ F_Y(0-) & \text{if } t = 0 \\ F_Y(0-) + 1 - F_Y(s-) & \text{if } t > 0 \end{cases} = \begin{cases} F_X(-1/t) - F_X(0) & \text{if } t < 0 \\ 1 - F_X(0) & \text{if } t = 0 \\ 1 - F_X(0) + F_X(-1/t) & \text{if } t > 0, \end{cases}
\end{aligned}
$$

which is false (see Eq. (4)), as $F_X(0) \neq F_X(0-)$, unless $P(X = 0) = 0$. The problem in deriving $F_{-1/X}$ through $Y = -X$ is due to $\frac{1}{-X} = -\infty$ if $X = 0$, but $\frac{1}{Y} = \infty$ if $Y = -X = 0$.

Proof of Lemma 1. It suffices to prove the lemma in these three cases:

(a) $t = 0$, (b) $t \in (-\infty, 0)$ and (c) $t \in (0, \infty)$.

Case (a). If $t = 0$ then

$$
\begin{aligned}
F_{1/X}(0) &= P(1/X \leq 0 \ \&\ X < 0) + P(1/X \leq 0 \ \&\ X = 0) + P(1/X \leq 0 \ \&\ X > 0) \\
&= P(1/X \leq 0 \ \&\ X < 0) = P(X < 0) = F_X(0-), \\
F_{-1/X}(0) &= P(-1/X \leq 0 \ \&\ X > 0) + P(-1/X \leq 0 \ \&\ X = 0) + P(-1/X \leq 0 \ \&\ X < 0) \\
&= P(-1/X \leq 0 \ \&\ X > 0) + P(-1/X \leq 0 \ \&\ X = 0) \\
&= P(X > 0) + P(X = 0) \\
&= 1 - F_X(0-).
\end{aligned}
$$

Case (b). If $t < 0$, then

$$
\begin{aligned}
F_{1/X}(t) &= P(1/X \leq t \ \&\ X < 0) + P(1/X \leq t \ \&\ X = 0) + P(1/X \leq t \ \&\ X > 0) \\
&= P(1/X \leq t \ \&\ X < 0) \\
&= P(1/t \leq X < 0) \\
&= F_X(0-) - F_X(s-), \text{ where } s = 1/t, \\
F_{-1/X}(t) &= P(-1/X \leq t \ \&\ X < 0) + P(-1/X \leq t \ \&\ X = 0) + P(-1/X \leq t \ \&\ X > 0)
\end{aligned}
$$

$$= P(-1/X \le t \ \& \ X = 0) + P(-1/X \le t \ \& \ X > 0)$$

$$= P(X = 0) + P(-1/t \ge X > 0)$$

$$= P(-1/t \ge X \ge 0)$$

$$= F_X(-1/t) - F_X(0-).$$

Case (c). If $t > 0$, then

$$\begin{aligned}
F_{1/X}(t) &= P(1/X \le t \ \& \ X < 0) + P(1/X \le t \ \& \ X > 0) + P(1/X \le t \ \& \ X = 0) \\
&= P(X < 0 \ \& \ 1/X \le t) + P(X > 0 \ \& \ 1/X \le t) \\
&= P(X < 0) + P(X \ge 1/t) \\
&= F_X(0-) + 1 - F_X(s-), \ \text{where } s = 1/t, \\
F_{-1/X}(t) &= P(\frac{-1}{X} \le t \ \& \ X > 0) + P(\frac{-1}{X} \le t \ \& \ X < 0) + P(\frac{-1}{X} \le t \ \& \ X = 0) \\
&= P(X > 0) + P(X < 0 \ \& \ X \le -1/t) + P(X = 0) \\
&= P(X \ge 0) + P(X \le -1/t) \\
&= 1 - F_X(0-) + F_X(-1/t). \ \square
\end{aligned}$$

Lemma 2. Assume that $X_n \xrightarrow{D} X$. Then $\pm 1/X_n \xrightarrow{D} \pm 1/X$ iff $F_{X_n}(0-) \to F_X(0-)$.

Proof. Notice that t is a continuous point of a cdf $F_X(t)$ iff $P(X = t) = 0$. For each t, letting $s = 1/t$, $S_X = 1 - F_X$, and $S_{X_n} = 1 - F_{X_n}$, it follows from Lemma 1 that

$$F_{1/X_n}(t) = \begin{cases} F_{X_n}(0-) - F_{X_n}(s-) & \text{if } t < 0 \\ F_{X_n}(0-) & \text{if } t = 0 \\ F_{X_n}(0-) + S_{X_n}(s-) & \text{if } t > 0, \end{cases} \quad \text{and } F_{1/X}(t) = \begin{cases} F_X(0-) - F_X(s-) & \text{if } t < 0 \\ F_X(0-) & \text{if } t = 0 \\ F_X(0-) + S_X(s-) & \text{if } t > 0. \end{cases}$$

If $t \ne 0$, then t is a continuous point of $F_{1/X}$ iff $s = 1/t$ is a continuous point of F_X. On the other hand, if $t = 0$ then $s = \infty$, and $P(1/X = 0) = P(X = \pm\infty) = 0$, as X is a random variable. As a consequence, $t = 0$ is a continuous point of $F_{1/X}(t)$. By the assumption that $X_n \xrightarrow{D} X$, and in view of the expressions of $F_{1/X}$ and F_{1/X_n} given above, if t is a continuous point of $F_{1/X}$, then

$$F_{1/X_n}(t) \to F_{1/X}(t) \text{ iff } F_{X_n}(0-) \to F_X(0-).$$

Consequently, $1/X_n \xrightarrow{D} 1/X$ iff $F_{X_n}(0-) \to F_X(0-)$.

By comparing F_{-1/X_n} and $F_{-1/X}$ (see Eq. (4) in Lemma 1), as comparing F_{1/X_n} and $F_{1/X}$ in the previous paragraph, one can prove that $-1/X_n \xrightarrow{D} -1/X$ iff $F_{X_n}(0-) \to F_X(0-)$. We skip the details. □

Corollary. Suppose that $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} a$, then

$Y_n \pm 1/X_n \xrightarrow{D} a \pm 1/X$ iff $F_{X_n}(0-) \to F_X(0-)$.

Proof. Assume that $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} a$. We shall first prove that

$$F_{X_n}(0-) \to F_X(0-) \text{ iff } a \pm 1/X_n \xrightarrow{D} a \pm 1/X \tag{5}$$

It can be shown that

$$F_{a+\frac{1}{X_n}}(t) = \begin{cases} F_{X_n}(0-) - F_{X_n}(s-) & \text{if } t < a \\ F_{X_n}(0-) & \text{if } t = a \\ F_{X_n}(0-) + S_{X_n}(s-) & \text{if } t > a \end{cases} \text{ and } F_{a+\frac{1}{X}}(t) = \begin{cases} F_X(0-) - F_X(s-) & \text{if } t < a \\ F_X(0-) & \text{if } t = a \\ F_X(0-) + S_X(s-) & \text{if } t > a \end{cases}$$

where $s = 1/(t-a)$. If $t \neq a$, then t is a continuous point of $F_{a+1/X}$ iff $s = 1/(t-a)$ is a continuous point of F_X. On the other hand, if $t = a$ then $s = \infty$, and $P(1/X = 0) = P(X = \pm\infty) = 0$. As a consequence, $t = a$ is a continuous point of $F_{1/X}(t)$. By the assumption that $X_n \xrightarrow{D} X$, and in view of the expressions of $F_{a+1/X}$ and F_{a+1/X_n} given above, if t is a continuous point of $F_{a+1/X}$, then

$$F_{a+1/X_n}(t) \to F_{a+1/X}(t) \text{ iff } F_{X_n}(0-) \to F_X(0-).$$

Consequently, $a + 1/X_n \xrightarrow{D} a + 1/X$ iff $F_{X_n}(0-) \to F_X(0-)$. Thus (5) holds.

In order to prove the corollary, in view of (5) it suffices to show that

$$Y_n + 1/X_n \xrightarrow{D} a + 1/X \text{ iff } a + 1/X_n \xrightarrow{D} a + 1/X. \tag{6}$$

Let $y = t$ be a continuous point of $F_{a+1/X}(y)$, then $\forall\, \epsilon > 0$, $\exists\, \eta > 0$ such that

$$|F_{a+1/X}(y) - F_{a+1/X}(t)| < \epsilon \text{ whenever } |y - t| \leq \eta.$$

Let $t-\eta_o$ and $t+\eta_o$ be two continuous points of $F_{a+1/X}$ satisfying $\eta_o \in (0,\eta]$ (as the set of continuous points of $F_{a+1/X}$ is dense). For the given $\epsilon > 0$ above, $\exists\ n_o$ such that $P(|Y_n - a| > \eta_o) < \epsilon$ whenever $n \geq n_o$. We now prove (6).

$$\begin{aligned}(=>).\quad & P(a+\frac{1}{X_n} \leq t, |Y_n - a| \leq \eta_o) = P(Y_n + \frac{1}{X_n} \leq t + (Y_n - a), |Y_n - a| \leq \eta_o) \\ & \in (P(Y_n + \frac{1}{X_n} \leq t-\eta_o, |Y_n - a| \leq \eta_o), P(Y_n + \frac{1}{X_n} \leq t+\eta_o, |Y_n - a| \leq \eta_o)). \quad (7)\end{aligned}$$

Notice that if $n \geq n_o$, then

$$\begin{aligned}&|P(a+\frac{1}{X_n} \leq t, |Y_n - a| \leq \eta_o) - P(a+\frac{1}{X_n} \leq t)| = P(a+\frac{1}{X_n} \leq t, |Y_n - a| > \eta_o) \leq \epsilon, \\ &|P(Y_n+\frac{1}{X_n} \leq t-\eta_o, |Y_n - a| \leq \eta_o) - P(Y_n+\frac{1}{X_n} \leq t-\eta_o)| < \epsilon, \\ &|P(Y_n+\frac{1}{X_n} \leq t+\eta_o, |Y_n - a| \leq \eta_o) - P(Y_n+\frac{1}{X_n} \leq t+\eta_o)| < \epsilon.\end{aligned}$$

These three inequalities yield

$$P(Y_n + \frac{1}{X_n} \leq t-\eta_o) - 2\epsilon \leq P(a+\frac{1}{X_n} \leq t) \leq P(Y_n + \frac{1}{X_n} \leq t+\eta_o) + 2\epsilon. \quad (8)$$

Since $F_{a+1/X}$ is continuous at $t-\eta_o$ and $t+\eta_o$, (8) and (7) yield

$$F_{a+1/X}(t-\eta_o) - 2\epsilon \leq \underline{\lim}_{n\to\infty} F_{a+1/X_n}(t) \leq \overline{\lim}_{n\to\infty} F_{a+1/X_n}(t) \leq F_{a+1/X}(t+\eta_o) + 2\epsilon.$$

Since ϵ is arbitrary and $F_{a+1/X}$ is continuous at t, letting $\eta_o \to 0$ yields $\lim F_{a+1/X_n}(t) = F_{a+1/X}(t)$. That is, $a+1/X_n \xrightarrow{D} a+1/X$.

(<=). In a similar manner as in deriving (8), one can show

$$P(a + \frac{1}{X_n} \leq t-\eta_o) - 2\epsilon \leq P(Y_n+\frac{1}{X_n} \leq t) \leq P(a + \frac{1}{X_n} \leq t+\eta_o) + 2\epsilon. \quad (9)$$

Since $F_{a+1/X}$ is continuous at $t-\eta_o$ and $t+\eta_o$, (9) yields

$$F_{a+1/X}(t-\eta_o) - 2\epsilon \leq \underline{\lim}_{n\to\infty} F_{Y_n+1/X_n}(t) \leq \overline{\lim}_{n\to\infty} F_{Y_n+1/X_n}(t) \leq F_{a+1/X}(t+\eta_o) + 2\epsilon.$$

Since ϵ is arbitrary and $F_{a+1/X}$ is continuous at t, $\lim F_{Y_n+1/X_n}(t) = F_{a+1/X}(t)$. □

The next theorem is the main result.

Theorem 1. Assume that $a \neq 0$, $Y_n \xrightarrow{D} a$ and $X_n \xrightarrow{D} X$. Then

$F_{X_n}(0-) \to F_X(0-)$ iff $Y_n/X_n \xrightarrow{D} a/X$.

Proof. Since $a \neq 0$, it yields (a) $a > 0$ or (b) $a < 0$. In view of Remark 3, we shall give the proof separately in these two cases. For simplicity, we put the proof of case (b) in Appendix, and only give the proof of case (a) here.

In case (a), we can define $Y_n^* = Y_n/a$, $X_n^* = X_n/a$ and $X^* = X/a$. Then $Y_n/X_n = Y_n^*/X_n^*$ and $a/X = 1/X^*$. By the given assumptions and Slutsky's theorem, $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} a \neq 0$ iff $X_n^* \xrightarrow{D} X^*$ and $Y_n^* \xrightarrow{D} 1$. Thus, without loss of generality, we can assume $a = 1$, *i.e.*, $Y_n \xrightarrow{D} 1$.

(<=). By Lemma 1, $t = 0$ is a continuous point of $F_{1/X}(t)$ and $F_{1/X}(0) = F_X(0-)$. Thus statement (2) yields $F_{Y_n/X_n}(0) \to F_{1/X}(0) = F_X(0-)$. Consequently, statement (2) also implies that $\forall\ \epsilon > 0$ and $\delta \in (0, 0.1)$, $\exists\ n_o$ such that

$$|F_{Y_n/X_n}(0) - F_X(0-)| < \epsilon \text{ and } P(|Y_n - 1| > \delta) < \epsilon \text{ whenever } n \geq n_o. \tag{10}$$

Verify that

$$\begin{aligned}
\{X_n < 0\} =& \{Y_n/X_n \leq 0, X_n < 0, |Y_n - 1| \leq \delta\} \cup \{X_n < 0, |Y_n - 1| > \delta\} \\
& \cup \{Y_n/X_n > 0, X_n < 0, |Y_n - 1| \leq \delta\} \\
=& \{Y_n/X_n \leq 0, |Y_n - 1| \leq \delta\} \cup \{X_n < 0, |Y_n - 1| > \delta\}; \\
\{Y_n/X_n \leq 0\} =& \{Y_n/X_n \leq 0, |Y_n - 1| \leq \delta\} \cup \{Y_n/X_n \leq 0, |Y_n - 1| > \delta\}.
\end{aligned}$$

$$=> \quad |P(X_n < 0) - P(Y_n/X_n \leq 0)| \leq 2\epsilon \text{ and}$$

$$|F_{X_n}(0-) - F_X(0-)| \leq |P(X_n < 0) - P(Y_n/X_n \leq 0)| + |P(Y_n/X_n \leq 0) - F_X(0-)| \leq 3\epsilon$$

(by (10)), if $n \geq n_o$. Since ϵ is arbitrary, $F_{X_n}(0-) \to F_X(0-)$.

(=>). Now assume that $F_{X_n}(0-) \to F_X(0-)$, $Y_n \xrightarrow{D} 1$ and $X_n \xrightarrow{D} X$. Then $1/X_n \xrightarrow{D} 1/X$ by Lemma 2. It suffices to show the statement as follows.

$$\text{If } Y_n \xrightarrow{D} a = 1 \text{ and } 1/X_n \xrightarrow{D} 1/X, \text{ then } Y_n/X_n \xrightarrow{D} a/X. \tag{11}$$

If we let $Z_n = 1/X_n$, then Eq. (11) looks like Slutsky's theorem. Notice that Slutsky's Theorem is proved under the assumption that Z is a random variable and a is an arbitrary constant. Since $Z = 1/X$ is an extended random variable, and Examples 1 and 2 suggest that the extension of Slutsky's theorem may not be true if $Z = 1/X$ and $a = 0$, we shall prove statement (11) rigorously.

Let $y = t$ be a continuous point of $F_{1/X}(y)$, then $\forall\, \epsilon > 0$, $\exists\, \eta > 0$ such that

$$|F_{1/X}(y) - F_{1/X}(t)| < \epsilon \text{ whenever } |y - t| \leq \eta. \tag{12}$$

Let $t-\eta_o$ and $t+\eta_o$ be two continuous points of $F_{1/X}$ satisfying $\eta_o \in (0, \eta]$. Let $g(Y_n) = t/Y_n$. Since $g(x)$ is continuous at $x = 1$, for the given η_o, $\exists\, \delta \in (0, 1/2)$ such that $|t/Y_n - t| \leq \eta_0$ whenever $|Y_n - 1| \leq \delta$. For the given $\epsilon > 0$ above, $\exists\, n_o$ such that $P(|Y_n - 1| > \delta) < \epsilon$ whenever $n \geq n_o$. Thus

$$\begin{aligned} &P(\frac{Y_n}{X_n} \leq t, |Y_n - 1| \leq \delta) = P(\frac{1}{X_n} \leq \frac{t}{Y_n}, |Y_n - 1| \leq \delta) \\ \in&(P(\frac{1}{X_n} \leq t - \eta_o, |Y_n - 1| \leq \delta), P(\frac{1}{X_n} \leq t + \eta_o, |Y_n - 1| \leq \delta)), \end{aligned} \tag{13}$$

if $n \geq n_o$. Notice that

$$P(Y_n/X_n \leq t) = P(1/X_n \leq t/Y_n, |Y_n - 1| \leq \delta) + P(Y_n/X_n \leq t, |Y_n - 1| > \delta),$$

$$P(1/X_n \leq t + \eta_o) = P(1/X_n \leq t + \eta_o, |Y_n - 1| \leq \delta) + P(1/X_n \leq t + \eta_o, |Y_n - 1| > \delta), \tag{14}$$

$$P(1/X_n \leq t - \eta_o) = P(1/X_n \leq t - \eta_o, |Y_n - 1| \leq \delta) + P(1/X_n \leq t - \eta_o, |Y_n - 1| > \delta)]. \tag{15}$$

Since $F_{1/X}$ is continuous at $t - \eta_o$ and $t + \eta_o$,

$$\begin{aligned} F_{1/X}(t) - 2\epsilon &\leq F_{1/X}(t - \eta_o) - \epsilon \text{ (by (12), as } \eta_o \in (0, \eta)) \\ &= \lim P(1/X_n \leq t - \eta_o) - \epsilon \text{ (as } F_{1/X} \text{ is continuous at } t - \eta_o) \\ &\leq \underline{\lim}_{n\to\infty} P(Y_n/X_n \leq t) \text{ (by (13), (15) and } P(Y_n - 1| > \delta) < \epsilon) \\ &\leq \overline{\lim}_{n\to\infty} P(Y_n/X_n \leq t) \\ &\leq \lim P(1/X_n \leq t + \eta_o) + \epsilon \text{ (by (13), (14) and } P(Y_n - 1| > \delta) < \epsilon) \\ &= F_{1/X}(t + \eta_o) + \epsilon \text{ (as } F_{1/X} \text{ is continuous at } t + \eta_o) \\ &\leq F_{1/X}(t) + 2\epsilon \text{ (by (12), as } \eta_o \in (0, \eta)). \end{aligned}$$

Since ϵ is arbitrary, $F_{Y_n/X_n}(t) \to F_{1/X}(t)$ if $F_{1/X}$ is continuous at t. Thus (11) holds. □

In Theorem 1, we impose the condition $a \neq 0$. Notice that in Proposition 1, it allows $a = 0$ but assumes $P(X = 0) = 0$. It follows from $P(X = 0) = 0$ and $X_n \xrightarrow{D} X$ that $F_{X_n}(0-) \to F_X(0-)$. The next two examples illustrate what may happen if $P(X = 0) > 0$ and $a = 0$. The complication is due to $\frac{0}{0}$.

Example 3. Let $X \sim bin(1,p)$, $W \sim U(-1,1)$, $X \perp W$, $X_n = X + \frac{1}{n}$ and $Y_n = W/n$. Then $X_n \to X$ and $Y_n \to 0$. Moreover, $\frac{Y_n}{X_n} = W\mathbf{1}(X = 0) + \frac{W\mathbf{1}(X=1)}{n(1+\frac{1}{n})} \to W\mathbf{1}(X = 0) \neq \frac{0}{X}$. Furthermore , it is also not true that $X_n/Y_n \xrightarrow{D} X/0$, as

$$\frac{X_n}{Y_n} = \frac{\mathbf{1}(X=0)}{W} + \mathbf{1}(X=1)\frac{n+1}{W} \to \frac{\mathbf{1}(X=0)}{W} + \mathbf{1}(X=1)[\infty\mathbf{1}(W \geq 0) - \infty\mathbf{1}(W < 0)] \neq \frac{X}{0}.$$

Example 4. Let $X \sim bin(1,p)$, $W \sim U(-1,1)$, $X \perp W$, $X_n = X + \frac{1}{n}$ and $Y_n = \frac{W}{n}[1+(-1)^n]$. Then $X_n \to X$, $Y_n \to 0$, $F_{X_n}(0-) \to F_X(0-)$, and $1/X_n \xrightarrow{D} 1/X$. Moreover,

$$Y_n/X_n = \begin{cases} 2W[\mathbf{1}(X=0) + \mathbf{1}(X=1)\frac{1}{n(1+\frac{1}{n})}] \to 2W\mathbf{1}(X=0) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Since $P(W\mathbf{1}(X = 0) \neq 0) = 1 - p > 0$, Y_n/X_n diverges in distribution. Moreover, X_n/Y_n diverges too.

In view of Examples 1, 2, 3 and 4, if $a = 0$ then the generalization of Eq. (2) does not relate to $F_{X_n}(0-) \to F_X(0-)$. In particular, $F_{X_n}(0-) \to F_X(0-)$ does not imply Y_n/X_n converges in distribution, vice verse, Y_n/X_n converges in distribution does not imply $F_{X_n}(0-) \to F_X(0-)$.

Theorem 2. Suppose that $P(Y_n = 0) \to 1$ and $X_n \xrightarrow{D} X$. Then

(a) $Y_n/X_n \xrightarrow{D} 0/X$ iff $P(X_n = 0) \to P(X = 0)$;

(b) $X_n/Y_n \xrightarrow{D} X/0$ iff $F_{X_n}(0-) \to F_X(0-)$ and $F_{X_n}(0) \to F_X(0)$;

(c) $X_n/Y_n \xrightarrow{D} X/0$ iff $P(X_n = 0) \to P(X = 0)$.

Proof. We first prove statement (a). Notice that

$F_{a/X}(t) = \mathbf{1}(t \geq 0)P(X \neq 0) + \mathbf{1}(t \geq 1)P(X = 0)$ and

$F_{Y_n/X_n}(t) = \mathbf{1}(t \geq 0)P(X_n \neq 0 = Y_n) + \mathbf{1}(t \geq 1)P(X_n = 0 = Y_n) + P(Y_n/X_n \leq t, Y_n \neq 0)$.

Since $0/X \in \{0,1\}$, $P(X=0)+P(X\neq 0)=1 \leq P(X_n=0)+P(X_n\neq 0=Y_n)+P(Y_n\neq 0)$ and $P(Y_n\neq 0)\to 0$, statement (a) is trivially true.

We now prove statement (b). Since $X/a = -\infty\mathbf{1}(X<0)+\infty\mathbf{1}(X>0)+\mathbf{1}(X=0)$,

$$F_{X/a}(t) = P(X<0)+\mathbf{1}(t\geq 1)P(X=0). \tag{16}$$

Since $P(Y_n=0)\to 1$, $\forall\ \epsilon>0$, $\exists\ n_o$ such that $P(Y_n\neq 0)<\epsilon$ whenever $n\geq n_o$. For $n\geq n_o$,

$$\begin{aligned} F_{X_n/Y_n}(t) =& P(X_n/Y_n\leq t, Y_n=0)+P(X_n/Y_n\leq t, Y_n\neq 0)\\ =& P(X_n<0,Y_n=0)+\mathbf{1}(t\geq 1)P(X_n=0,Y_n=0)+P(X_n/Y_n\leq t,Y_n\neq 0). \end{aligned}$$

$|F_{X_n/Y_n}(t) - F_n(t)| < \epsilon$, where

$$F_n(t) = (P(X_n<0,Y_n=0)+\mathbf{1}(t\geq 1)P(X_n=0,Y_n=0)). \tag{17}$$

Since X/a is an extended random variable, and $F_{X/a}(t)$ is continuous at $t\notin\{1,\pm\infty\}$,

$$X_n/Y_n \xrightarrow{D} X/a \text{ iff } F_{X_n/Y_n}(t)\to F_{X/a}(t) \text{ if } t\notin\{1,\pm\infty\}. \tag{18}$$

Since $F_n(t)$ and $F_{X/a}(t)$ are both constant on $(-\infty,1)$ and $[1,\infty)$, respectively, and ϵ is arbitrary, (16) and (18) yield

$$X_n/Y_n \xrightarrow{D} X/a \text{ iff } \lim_{n\to\infty} F_{X_n/Y_n}(0) = F_{X/a}(0) \text{ and } \lim_{n\to\infty} F_{X_n/Y_n}(1) = F_{X/a}(1). \tag{19}$$

By Eq. (17), $|F_{X_n/Y_n}(0)-F_{X_n}(0-)|<\epsilon$, $|F_{X_n/Y_n}(1)-F_{X_n}(0)|<\epsilon$, $F_{X/a}(0)=F_X(0-)$ and $F_{X/a}(1)=F_X(0)$, hence statement (19) yields

$X_n/Y_n \xrightarrow{D} X/a$ iff $\lim_{n\to\infty} F_{X_n}(0-) = F_X(0-)$ and $\lim_{n\to\infty} F_{X_n}(0) = F_X(0)$. This completes the proof of statement (b).

Since $P(X_n=0)=F_{X_n}(0)-F_{X_n}(0-)$ and $P(X=0)=F_X(0)-F_X(0-)$, statement (c) follows from statement (b). □

Notice that it is not necessary to assume $X_n \xrightarrow{D} X$ in Theorem 2. It is assumed in Theorem 2 that $P(Y_n=0)\to 1$, but not in the next theorem.

Theorem 3. Suppose that $Y_n \xrightarrow{D} 0$ and $X_n \xrightarrow{D} X$. Then

(a) $\frac{Y_n}{X_n} \xrightarrow{D} 0/X$ iff $P(|Y_n/X_n - 1| < \delta) \to P(X = 0)\ \forall\ \delta \in (0, 0.1)$;

(b) $\frac{X_n}{Y_n} \xrightarrow{D} X/0$ iff $P(|\frac{X_n}{Y_n} - 1| < \delta) \to P(X = 0)\ \forall\ \delta \in (0, 0.1)$, and $P(\frac{X_n}{Y_n} < 0) \to P(X < 0)$.

Remark 4. Notice that $P(|\frac{Y_n}{X_n} - 1| < \delta) \to P(X = 0)$ and $P(|\frac{X_n}{Y_n} - 1| < \delta) \to P(X = 0)$ are not equivalent, as X_n/Y_n is not continuous at $(X_n, Y_n) = (0, 0)$.

Proof of Theorem 3. We shall give the proof in 3 steps.

Step 1 (preliminary). $\forall\ \epsilon > 0$, $\exists\ s \in (0, 1)$ and $\exists\ n_o$ such that

(i) $F_X(t)$ is continuous at $t \in \{-s, s\}$,

(ii) $P(X \in (-s, 0) \cup (0, s)) < \epsilon$,

(iii) $|P(X_n \in (-s, 0) \cup (0, s)) - P(X \in (-s, 0) \cup (0, s))| < \epsilon$ if $n \geq n_o$ (by (i), as $X_n \xrightarrow{D} X$),

(iv) $P(|Y_n| > \delta) < \epsilon$ if $n \geq n_o$ (as $Y_n \xrightarrow{D} 0$), where $\delta < \epsilon s$.

Consequently,

$$|Y_n/X_n| \leq |Y_n|/s \leq \delta/s < \epsilon\ \forall\ (X_n, Y_n) \in \{X_n \notin (-s, s), |Y_n| \leq \delta\}; \tag{20}$$

$$|X_n/Y_n| \geq s/|Y_n| \geq s/\delta > 1/\epsilon\ \forall\ (X_n, Y_n) \in \{X_n \notin (-s, s), |Y_n| \leq \delta\}. \tag{21}$$

$$P(X_n \in (-s, 0) \cup (0, s)) \leq |P(X_n \in A) - P(X \in A)| + P(X \in A) < 2\epsilon \tag{22}$$

by (ii) and (iii), where $A = (-s, 0) \cup (0, s)$. By (20) and (21),

$$\begin{aligned}
P(|X_n/Y_n| \geq 1/\epsilon) =& P(|Y_n/X_n| \leq \epsilon) && (23)\\
\geq& P(\{|Y_n/X_n| \leq \epsilon\} \cap \{X_n \notin (-s, s), |Y_n| \leq \delta\}) \\
\geq& P(\{X_n \notin (-s, s), |Y_n| \leq \delta\}) && \text{(by (20))}\\
\geq& P(X_n \notin (-s, s)) - P(|Y_n| > \delta) \\
\geq& 1 - P(X_n \in (-s, s)) - \epsilon && (\text{if } n \geq n_o)\\
\to& 1 - P(X \in (-s, s)) - \epsilon && (\text{as } n_o \to \infty)\\
\to& 1 - P(X = 1) - \epsilon \text{ as } s \downarrow 0.
\end{aligned}$$

Step 2 (prove statement (a)).

(=>) Since $F_{a/X}(t) = \mathbf{1}(t \geq 1)P(X = 0) + \mathbf{1}(t \geq 0)P(X \neq 0)$,

$F_{Y_n/X_n}(1+\delta) - F_{Y_n/X_n}(1-\delta) \to P(X = 0)$ for the continuous points $x = 1 \pm \delta$ of $F_X(x)$ that satisfying $x \to 1$. It follows $P(|Y_n/X_n - 1| < \delta) \to P(X = 0)$ if $\delta \in (0, 0.1)$.

(<=). Since $P(|Y_n/X_n - 1| < \delta) \to P(X = 0)$ if $\delta \approx 0+$, it follows from (23) that $\lim_{n\to\infty}[P(|Y_n/X_n| \leq \epsilon) + P(|Y_n/X_n - 1| < \delta)] \geq 1 - P(X = 0) + P(X = 0) - \epsilon \ \forall \ \epsilon > 0$ and $\forall \ \delta \in (0, 0.1)$. That is, $Y_n/X_n \xrightarrow{D} 0/X$, as $0/X \in \{0, 1\}$.

Step 3 (prove statement (b)).

(=>) Since $F_{X/a}(t) = \mathbf{1}(t \geq 1)P(X = 0) + P(X < 0)$,

$F_{X_n/Y_n}(1+t) - F_{X_n/Y_n}(1-t) \to P(X = 0)$ for the continuous points $1 \pm t$ of F_X that satisfying $t \downarrow 0$. It yields $P(|Y_n/X_n - 1| < \delta) \to P(X = 0)$ if $\delta \in (0, 0.1)$.

Moreover, $P(X_n/Y_n < 0) = F_{X_n/Y_n}(0-) \to F_{X/a}(0-) = P(X < 0)$.

(<=). Since $P(X_n/Y_n - 1| < \delta) \to P(X = 0)$ if $\delta \in (0, 0.1)$, it follows from (23) that $\lim[P(|X_n/Y_n| \geq 1/\epsilon) + P(|X_n/Y_n - 1| < \delta)] \geq P(X \neq 0) + P(X = 0) - \epsilon \ \forall \ \epsilon > 0$. Since ϵ is arbitrary, $P(|X_n/Y_n| > M) \to P(X \neq 0) \ \forall \ M > 2$ and $P(|X_n/Y_n - 1| < \delta) \to P(X = 0)$. Moreover, $P(X_n/Y_n \in (-\infty, 1) \cup (1, \infty)) \to 0$. If $P(X_n/Y_n < 0) \to P(X < 0)$, then $P(X_n/Y_n > 2) \to P(X > 0)$ and $F_{X_n/Y_n}(t) \to P(X < 0) + \mathbf{1}(t \geq 1)P(X = 0)$ if $t \neq 1$. □

Corollary. Suppose that $X_n \xrightarrow{D} X$, $Y_n \xrightarrow{D} a = 0$ and $P(Y_n \geq 0) \to 1$. Then $X_n/Y_n \xrightarrow{D} X/a$ iff $P(|X_n/Y_n) - 1| < \delta) \to P(X = 0) \ \forall \ \delta \in (0, 0.1)$.

Notice that Theorem 2 can also be viewed as a corollary of Theorem 3. It seems that Theorem 3 can be further modified to study $Y_n/X_n \xrightarrow{D} Z\mathbf{1}(X = 0)$ and $X_n/Y_n \xrightarrow{D} Z\mathbf{1}(X = 0) - \mathbf{1}(X \neq 0)$ where Z depends on $\{Y_n/X_n\}_{n\geq 1}$, rather than on a/X alone, if Y_n/X_n does converge in distribution.

Reference.

Mann, H. B.; Wald, A. (1943). "On Stochastic Limit and Order Relationships". Annals of Mathematical Statistics. 14 (3): 217-226.

Appendix. We give the proof of case (b) in Theorem 1 here. In case (b), $a<0$. Define $Y_n^* = -Y_n/a$, $X_n^* = -X_n/a$ and $X^* = -X/a$. Then $Y_n/X_n = Y_n^*/X_n^*$ and $a/X = -1/X^*$. By the given assumptions and Slutsky's theorem, $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} a < 0$ iff $X_n^* \xrightarrow{D} X^*$ and $Y_n^* \xrightarrow{D} -1$. Thus, without loss of generality, we can assume $a=-1$, *i.e.*, $Y_n \xrightarrow{D} -1$. By Lemma 2, it suffices to prove that $\frac{Y_n}{X_n} \xrightarrow{D} -\frac{1}{X}$ iff $\frac{1}{X_n} \xrightarrow{D} \frac{1}{X}$.

(<=). Let $y=t$ be a continuous point of $F_{-1/X}(y)$, then $\forall\ \epsilon>0$, $\exists\ \eta>0$ such that

$$|F_{-1/X}(y) - F_{-1/X}(t)| < \epsilon \text{ whenever } |y-t| \le \eta. \tag{a0}$$

Let $t-\eta_o$ and $t+\eta_o$ be two continuous points of $F_{-1/X}$ satisfying $\eta_o \in (0,\eta]$. Let $g(Y_n) = -t/Y_n$. Since $g(x)$ is continuous at $x=-1$, for the given η_o, $\exists\ \delta \in (0,1/2)$ such that $|-t/Y_n - t| \le \eta_0$ whenever $|Y_n+1| \le \delta$. For the given $\epsilon>0$ above, $\exists\ n_o$ such that $P(|Y_n+1|>\delta) < \epsilon$ whenever $n \ge n_o$. Thus

$$\begin{aligned} &P(\frac{Y_n}{X_n} \le t, |Y_n+1| \le \delta) = P(-\frac{1}{X_n} \le -\frac{t}{Y_n}, |Y_n+1|\le\delta) \\ \in&(P(-\frac{1}{X_n} \le t-\eta_o, |Y_n+1|\le\delta), P(-\frac{1}{X_n} \le t+\eta_o, |Y_n+1|\le\delta)), \end{aligned} \tag{a1}$$

if $n \ge n_o$. Notice that

$$|P(Y_n/X_n \le t) - P(-1/X_n \le -t/Y_n, |Y_n+1|\le\delta)| \le P(Y_n/X_n \le t, |Y_n+1|>\delta) < \epsilon,$$

$$|P(-1/X_n \le t+\eta_o) - P(-1/X_n \le t+\eta_o, |Y_n+1|\le\delta)| < \epsilon, \tag{a2}$$

$$|P(-1/X_n \le t-\eta_o) - P(-1/X_n \le t-\eta_o, |Y_n+1|\le\delta)| < \epsilon. \tag{a3}$$

$$\begin{aligned} F_{-1/X}(t) - 2\epsilon &\le F_{-1/X}(t-\eta_o) - \epsilon \text{ (by (a0), as } \eta_o \in (0,\eta)) \\ &= \lim P(-1/X_n \le t-\eta_o) - \epsilon \text{ (as } F_{-1/X} \text{ is continuous at } t-\eta_o) \\ &\le \underline{\lim}_{n\to\infty} P(Y_n/X_n \le t) \text{ (by (a1), (a3) and } P(Y_n+1|>\delta)<\epsilon) \\ &\le \overline{\lim}_{n\to\infty} P(Y_n/X_n \le t) \\ &\le \lim P(-1/X_n \le t+\eta_o) + \epsilon \text{ (by (a1), (a2) and } P(Y_n+1|>\delta)<\epsilon) \\ &= F_{-1/X}(t+\eta_o) + \epsilon \text{ (as } F_{-1/X} \text{ is continuous at } t+\eta_o) \\ &\le F_{-1/X}(t) + 2\epsilon \text{ (by (a0), as } \eta_o \in (0,\eta)). \end{aligned}$$

Since ϵ is arbitrary, $F_{Y_n/X_n}(t) \to F_{-1/X}(t)$ if $F_{-1/X}$ is continuous at t,thus, $\frac{Y_n}{X_n} \xrightarrow{D} -\frac{1}{X}$.

(=>). Let $y = t$ be a continuous point of $F_{-1/X}(y)$, then $\forall\ \epsilon > 0$, $\exists\ \eta > 0$ such that

$$|F_{-1/X}(y) - F_{-1/X}(t)| < \epsilon \text{ whenever } |y - t| \le \eta. \tag{a4}$$

Let $t - \eta_o$ and $t + \eta_o$ be two continuous points of $F_{-1/X}$ satisfying $\eta_o \in (0, \eta]$. Letting $g(Y_n) = -Y_n * t$, since $g(x)$ is continuous at $x = -1$, for the given η_o, $\exists\ \delta \in (0, 1/2)$ such that $| - Y_n * t - t| \le \eta_0$ whenever $|Y_n + 1| \le \delta$. For the given $\epsilon > 0$ above, $\exists\ n_o$ such that $P(|Y_n + 1| > \delta) < \epsilon$ whenever $n \ge n_o$. Thus

$$\begin{aligned} &P(-\frac{1}{X_n} \le t, |Y_n + 1| \le \delta) = P(\frac{Y_n}{X_n} \le -Y_n * t, |Y_n + 1| \le \delta) \\ \in &(P(\frac{Y_n}{X_n} \le t - \eta_o, |Y_n + 1| \le \delta), P(\frac{Y_n}{X_n} \le t + \eta_o, |Y_n + 1| \le \delta)), \end{aligned} \tag{a5}$$

if $n \ge n_o$. Notice that

$$|P(-1/X_n \le t) - P(Y_n/X_n \le -Y_n * t, |Y_n + 1| \le \delta)| \le P(-1/X_n \le t, |Y_n + 1| > \delta) < \epsilon,$$

$$|P(Y_n/X_n \le t + \eta_o) - P(Y_n/X_n \le t + \eta_o, |Y_n + 1| \le \delta)| < \epsilon, \tag{a6}$$

$$|P(Y_n/X_n \le t - \eta_o) - P(Y_n/X_n \le t - \eta_o, |Y_n + 1| \le \delta)| < \epsilon, \tag{a7}$$

Since $F_{-1/X}$ is continuous at $t - \eta_o$ and $t + \eta_o$,

$$\begin{aligned} F_{-1/X}(t) - 2\epsilon &\le F_{-1/X}(t - \eta_o) - \epsilon \text{ (by (a4), as } \eta_o \in (0, \eta)) \\ &= \lim P(Y_n/X_n \le t - \eta_o) - \epsilon \text{ (as } F_{-1/X} \text{ is continuous at } t - \eta_o) \\ &\le \underline{\lim}_{n\to\infty} P(-1/X_n \le t) \text{ (by (a5), (a7) and } P(Y_n + 1| > \delta) < \epsilon) \\ &\le \overline{\lim}_{n\to\infty} P(-1/X_n \le t) \\ &\le \lim P(Y_n/X_n \le t + \eta_o) + \epsilon \text{ (by (a5), (a6) and } P(Y_n + 1| > \delta) < \epsilon) \\ &= F_{-1/X}(t + \eta_o) + \epsilon \text{ (as } F_{-1/X} \text{ is continuous at } t + \eta_o) \\ &\le F_{-1/X}(t) + 2\epsilon \text{ (by (a4), as } \eta_o \in (0, \eta)). \end{aligned}$$

Since ϵ is arbitrary, $F_{-1/X_n}(t) \to F_{-1/X}(t)$ if $F_{-1/X}$ is continuous at t,thus, $-\frac{1}{X_n} \xrightarrow{D} -\frac{1}{X}$. $\square$