Homework Solution.

§**5.4.1.2.** There are 4 observations $(M_i, \delta_i, \mathbf{z}_i)$'s: $(1,1,1)$, $(3,1,2)$, $(4,0,1)$, $(7,1,2)$. Find the Miller estimator of β under the linear regression model. $X_i = \beta z_i + \epsilon_i$. You are able to find the solution explicitly, because there are at most 6 distinct values of $\hat{S}_b(T_i(b))$ (as a function of b) for each fixed i, where $T_i = M_i - bz_i$.

Sol. $T_i(b) = M_i - bz_i \in \{1-b,\ 3-2b,\ (4-b)+,\ 7-2b\}$.

$$H(b) = \int t^2 d\hat{F}_{T(b)}(t) = \sum_{i=1}^n \hat{f}(T_i(b))(T_i(b) - \hat{\alpha}(b))^2,$$

$$\text{where } \hat{f}(T_i(b)) = \frac{\hat{f}_{T(b)}(T_i(b))}{\sum_j \mathbf{1}(T_j(b) = T_i(b))}. \tag{1}$$

$T_i(b) = T_j(b)$ yields $b = -1,\ 2,\ 3,\ 6$.

$b:$	$T_i's$	$\hat{f}(T_i(b))$	$\hat{\alpha}(b) = \sum T_i(b)\hat{f}(T_i)$
$(-\infty,-1)$			$(2+9+21)/8 - b(2+6+6)/8 = 4 - \frac{14}{8}b$
-2	$3, 7, 6+, 11$	$(2,3,0,3)/8$	
-1	$2, 5, 5+, 9$	$(1,1,0,2)/4$	$(1+3+14)/4 - (-1)(1+2+4)/4 = 25/4$
$0 \in (-1,2)$	$1, 3, 4+, 7$	$(1,1,0,2)/4$	$(1+3+14)/4 - b(1+2+4)/4 = \frac{9}{2} - \frac{7}{4}b$
2	$-1, -1, 2+, 3$	$(1,1,0,1)/2$	*modified as follows*
2	$-1, -1, 2+, 3$	$(1,1,0,2)/4$	*(see Eq.(1))*
$2.5 \in (2,3)$	$-1.5, -2, 1.5+, 2$	$(1,1,0,2)/4$	$\frac{9}{2} - \frac{7}{4}b$
3	$-2, -3, 1+, 1$	$(1,1,1,1)/4$	*modified as follows*
3	$-2, -3, 1, 1$	$(1,1,1,1)/4$	$26/4 - 3(9/4) = -3/4$
$4 \in (3,6)$	$-3, -5, 0+, -1$	$(1,1,1,1)/4$	$\frac{15}{4} - 1.5b$
6	$-5, -9, -2+, -5$	$(2,1,1,2)/4$	*modified as*
6	$-5, -9, -2+, -5$	$(1,1,1,1)/4$	$(-31)/4 - 6(9/4) = -85/4$
$7 \in (6,\infty)$	$-6, -11, -3+, -7$	$(1,1,1,1)/4$	$15/4 - 1.5b$

So we just need to consider

$T_i(b) = T_j(b)$ and $\delta_i \neq \delta_j$,

$\Rightarrow b = -1,\ 3.$

$b:$	$T_i's$	$\hat{f}(T_i(b))$	$\hat{\alpha}(b) = \sum T_i(b)\hat{f}(T_i)$
$-2 \in (-\infty,-1)$	$3, 7, 6+, 11$	$(2,3,0,3)/8$	$4 - \frac{14}{8}b$
$0 \in [-1,3)$	$1, 3, 4+, 7$	$(1,1,0,2)/4$	$\frac{9}{2} - \frac{7}{4}b$
$4 \in [3,\infty)$	$-3, -5, 0, -1$	$(1,1,1,1)/4$	$\frac{15}{4} - 1.5b$

$$\begin{aligned}
H_1(b) =& \sum_{i=1}^n \hat{f}(T_i(b))(T_i(b) - \hat{\alpha}(b))^2 && b < -1 \\
=& \frac{1}{8}[2(1-b-(4-\frac{14}{8}b))^2 + 3(3-2b-(4-\frac{14}{8}b))^2 + 3(7-2b-(4-\frac{14}{8}b))^2] \\
\geq& H_1(-1) = 7.687 && \textit{minimum at } b = 4. \\
H(b) =& \sum_{i=1}^n \hat{f}(T_i(b))(T_i(b) - \hat{\alpha}(b))^2 && b \in [-1,3) \\
=& \frac{1}{4}[(1-b-\frac{9}{2}+\frac{7}{4}b)^2 + (3-2b-\frac{9}{2}+\frac{7}{4}b)^2 + 2(7-2b-\frac{9}{2}+\frac{7}{4}b)^2] \\
=& \frac{1}{4}[(-\frac{7}{2}+\frac{3}{4}b)^2 + (-\frac{3}{2}-\frac{1}{4}b)^2 + 2(\frac{5}{2}-\frac{1}{4}b)^2] \\
=& \frac{1}{4}[b^2(9/16+1/16+2/16) + b(-21/4+3/4+5/2) + (49/4+9/4+25/2)] \\
=& \frac{1}{4}[(3/4)b^2 - 2b + 108/4] && \textit{if } b \in [-1,3) \\
\geq& H(4/3) = \frac{1}{4}[(4/3) - 8/3 + 108/4] \approx 6.4 && \textit{minimum at } b = 4/3. \\
H_3(b) =& \frac{1}{4}[(1-b-(15/4-1.5*b))^2 + (3-2*b-(15/4-1.5*b))^2 && b \geq 3 \\
\geq& H_3(3) \approx 3.2 && (\text{minimum at b=2.5. } \hat{\beta} = ?)
\end{aligned}$$

$$
\begin{aligned}
H_3(b) =& \sum_{i=1}^n \hat f(T_i(b))(T_i(b)-\hat\alpha(b))^2 && b \geq 3\\
=& \frac{1}{4}[(1-b-(15/4-1.5*b))^2+(3-2*b-(15/4-1.5*b))^2\\
& +(4-b-(15/4-1.5*b))^2+(7-2*b-(15/4-1.5*b))^2 && b\in[3,\infty)\\
\geq & H_3(3)\approx 3.2 && \textit{minimum at } b=2.5.
\end{aligned}
$$

Does the iterative algorithm result in Miller's estimate ?

Thus Miller's estimate is $\hat\beta = 3$.

§**5.4.2.2. Homework**

1. There are 4 observations $(M_i, \delta_i, \mathbf{z}_i)$'s: $(3,1,2)$, $(4,0,1)$, $(1,1,1)$, $(7,1,2)$. Show that there is only one BJE of β under the linear regression model and it is 2.

Remark. A non-iterative algorithm for obtaining all BJE's (for $p=1$):

1. Let b_{ij} be the solution to an equation $T_i(b) = T_j(b)$, where $z_i \neq z_j$ and $\delta_i \neq \delta_j$. Let $q_1 < \cdots < q_m$ be all the distinct solutions b_{ij}'s. Let $q_0 = -\infty$ and $q_{m+1} = \infty$.
2. (Case (1)). For each $h = 0, 1, ..., m$, first compute the PLE $\hat S_b$ for a $b \in (q_h, q_{h+1})$. If $\hat b_h \in (q_h, q_{h+1})$ and $H(\hat b_h) = 0$ (see (2.7)), it is a BJE of β.
3. (Case (2)). Compute $H(q_i-)$, $H(q_i)$ and $H(q_i+)$, $i = 1, ..., m$. If $H(q_i-)H(q_i+) \leq 0$, or $H(q_i-)H(q_i) \leq 0$, or $H(q_i)H(q_i+) \leq 0$, then q_i is a zero-crossing point of H and thus is a BJE of β.

Remark. In computing $H(b)$ is better off to use the following equivalent expression

$$
H(b) = \sum_{i=1}^n (M_i^* - z_i^* b)(z_i - \overline{z}) = \sum_{i=1}^n (T_i^*(b))(z_i - \overline{z}), \tag{2.7}
$$

where $T_i^*(b) = \delta_i T_i(b) + (1-\delta_i)\frac{\sum_{t>T_{(i)}} t\hat f_{T(b)}(t)}{\hat S_{T(b)}(T_i(b))}$. because even though (M_i^*, z_i^*) depends on b, it is constant in b on each interval (q_j, q_{j+1}), but $\hat X_i^*$ is not. Compare to the objective function in Miller's approach:

$$
H(b) = \int t^2 d\hat F_{T(b)}(t) = \sum_{i=1}^n \hat f(T_i(b))(T_i(b)-\hat\alpha(b))^2, \text{ where } \hat f(T_i(b)) = \frac{\hat f_{T(b)}(T_i(b))}{\sum_j \mathbf{1}(T_j(b) = T_i(b))}.
$$

Partition points: $T_1(b) = T_2(b)$ and $T_2(b) = T_4(b)$ yield -1 and 3.

Case (1), root of $H(t)$:

(1) For $b = -2$, $H(t) = 0 => t < -1$?
(2) For $b = 2$, $H(t) = 0 => t \in (-1, 3)$?
(3) For $b = 4$, $H(t) = 0 => t \in (3, \infty)$?

Case (2) zero crossing point of $H(b)$:

$b = -1$. $H(b-)$. Does $H(b)$, $H(b+)$ change signs ?
$b = 3$. $H(b-)$. Does $H(b)$, $H(b+)$ change signs ?

§5.4.2.

2. Under interval censoring, (2.2) can be rewritten as

$$
X_i^* = E(X_i | X_i \in I_i), \text{ where } I_i \text{ is the } i\text{-th observed interval.}
$$

2.a. Verify (2.3) under the mixed case model with continuous random vectors.
2.b. Give corresponding expressions for (2.5).

Sol. Note that our observations are (L_i, R_i, z_i), $i = 1, ..., n$,
or (I_i, z_i), $i = 1, ..., n$,

$X_i \in I_i$ and L_i and R_i are the endpoints of I_i.
Thus

$$X_i^* = E(X_i|X_i \in I_i) = E(X_i|I_i) = E(X_i|L_i, R_i) = \frac{\int_{x\in I_i} x dF_{X_i|z_i}(x)}{\int_{x\in I_i} dF_{X_i|z_i}(x)}.$$

$X_i^* = E(X_i|z_i) = g(z_i)$ where $g(t) = E(X_i|z_i = t)$
and z_i is treated as a random variable. The IC data are (L_i, R_i, z_i), $i = 1, ..., n$, from model $X_i = \beta z_i + \epsilon_i$, where ϵ_is are i.i.d.

$$I_i = \begin{cases} (-\infty, Y_1] & \text{(denoted by } J_0) & \text{if } X_i \le Y_1 \\ (Y_j, Y_{j+1}] & \text{(denoted by } J_j) & \text{if } Y_{j-1} < X_i \le Y_j,\ 1 \le j < N \\ (Y_N, \infty) & \text{(denoted by } J_N) & \text{if } Y_N < X_i. \end{cases}$$

Denote $\mathbf{Y} = (Y_1, Y_2, ...)$. $X \perp (\mathbf{Y}, N)$.

$$\begin{aligned}
&E(X_i^*|z_i) = E\{E(X_i|L_i, R_i)|z_i\} \\
=&\int \frac{\int_{x\in I} x dF_{X_i|z_i}(x)}{\int_{x\in I} dF_{X_i|z_i}(x)} dF_{L_i,R_i}(l, r) \\
&\qquad\qquad \text{(where the endpoints of } I \text{ is } l \text{ and } r\text{, discregarding models)} \\
&\qquad\qquad \text{(under the mixed case IC model hereafter)} \\
=&\sum_{k=1}^{\infty} f_N(k) \int \frac{\int_{x\in I} x dF_{X_i|z_i}(x)}{\int_{x\in I} dF_{X_i|z_i}(x)} dF_{L_i,R_i|N}(l, r|k) \\
=&\sum_{k=1}^{\infty} f_N(k)\{ \int [1(I_i = J_0 = (-\infty, r]) \frac{\int_{x\in I_i} x dF_{X_i|z_i}(x)}{\int_{x\in I_i} dF_{X_i|z_i}(x)} \int_{x\in I_i} dF_{X_i|z_i}(x)] dF_{Y_1|N}(r|k) \\
&+ \sum_{j=2}^{k} \int [1(I_i = J_{j-1} = (l, r]) \frac{\int_{x\in I_i} x dF_{X_i|z_i}(x)}{\int_{x\in I_i} dF_{X_i|z_i}(x)} \int_{x\in I_i} dF_{X_i|z_i}(x)] dF_{Y_{j-1},Y_j|N}(l, r|k) \\
&+ \int [1(I_i = J_k = (l, \infty)) \frac{\int_{x\in I_i} x dF_{X_i|z_i}(x)}{\int_{x\in I_i} dF_{X_i|z_i}(x)} \int_{x\in I_i} dF_{X_i|z_i}(x)] dF_{Y_k|N}(l|k)\} \\
=&\sum_{k=1}^{\infty} f_N(k)\{ \int [1(I_i = J_0 = (-\infty, r]) \int_{x\in I_i} x dF_{X_i|z_i}(x)] dF_{Y_1|N}(r|k) \\
&+ \sum_{j=2}^{k} \int [1(I_i = J_{j-1} = (l, r]) \int_{x\in I_i} x dF_{X_i|z_i}(x)] dF_{Y_{j-1},Y_j|N}(l, r|k) \\
&+ \int [1(I_i = J_k = (l, \infty)) \int_{x\in I_i} x dF_{X_i|z_i}(x)] dF_{Y_k|N}(l|k)\} \\
=&\sum_{k=1}^{\infty} \{E(X_i 1(I_i = J_0)|N = k, z_i) + \sum_{j=1}^{k-1} E(X_i 1(I_i = J_j)|N = k, z_i) \\
&+ E(X_i 1(I_i = J_k)|N = k, z_i)\} f_N(k) \\
=&\sum_{k=1}^{\infty} \{E(X_i|N = k, z_i)\} f_N(k) \\
=&E(X_i|z_i)
\end{aligned}$$

It suffices to show
$E(X^*|Z = z) = E(X|Z = z)$,
where $X^* = E(X|X \in I, Z = z)$, Z is a random variable and $I = (L, R)]$.
Note that $X^* = \sum_{j=0}^{N} X^* 1(I = J_j)$.

$$E(X^*|Z = z) = E\{E(X|X \in I)|Z = z\}$$

$$
\begin{aligned}
=&E\{E[E(X|X\in I)|N,\mathbf{Y}]|Z=z\}\\
=&E\{E[\sum_{j=0}^{N}E(X|X\in I=J_j)P(I=J_j|N,\mathbf{Y})|N,\mathbf{Y}]|Z=z\}\\
=&E\{E[\sum_{j=0}^{N}E\big(X|X\in J_j\big)P(I=J_j|N,\mathbf{Y})|N,\mathbf{Y}]|Z=z\}\\
=&E\{E[\sum_{j=0}^{N}E\big(X|I=J_j\big)P(I=J_j|N,\mathbf{Y})|N,\mathbf{Y}]|Z=z\}\\
&\qquad\qquad\qquad\qquad\qquad(\text{ the event } \{I=J_j\}=\{X\in J_j\})\\
=&E\{E[E\big(\sum_{j=0}^{N}E(X1(X\in I=J_j)|N,\mathbf{Y}\big)|N,\mathbf{Y}]|Z=z\}\\
=&E\{E[E\big(X|N,\mathbf{Y}\big)|N,\mathbf{Y}]|Z=z\}\\
=&E\{E[X|N,\mathbf{Y}]|Z=z\}\\
=&E\{X|Z=z\}
\end{aligned}
$$

A simple proof. Since $X^*=E(X|X\in I,Z=z)=E(X|(L,R),Z=z)$,
$E(X^*|Z=z)=E(E(X|(L,R),Z=z))=E(X|Z=z)$.

2b Note that (2.5) is an estimate of X_i^*. Verify that

$$F_{X_i|z_i}(t)=P(X_i\le t|Z=z_i)=P(X_i-\beta z_i\le t-\beta z_i|Z=z_i)$$

$$=P(\epsilon\le t-\beta z_i)=F_\epsilon(t-\beta z_i)$$

X_i^* can be estimated by

$$\hat{X}_i^*=\frac{\int_{x\in I_i}xd\hat{F}_{X_i|z_i}(x)}{\int_{x\in I_i}d\hat{F}_{X_i|z_i}(x)}=\frac{\int_{x\in I_i}x\hat{f}_{X_i|z_i}(x)dx}{\int_{x\in I_i}d\hat{F}_{X_i|z_i}(x)}\ ???$$

where $\hat{F}_{X_i|z_i}(x)=\hat{F}_{\hat{\beta}}(x-\hat{\beta}z_i)$ and $\hat{F}_b$ is the GMLE of F_ϵ based on (L_i-bz_i,R_i-bz_i)s and $\hat{\beta}$ is a consistent estimator of β.