Technical Report for "Consistency Of The Semi-parametric MLE Under The Lehmann Family With Right-Censored Data"

Proof. of Corollary 1. Let $k = \inf_n \inf_x f_n(x)$. If $k \geq 0$ then the corollary follows from Lemma 1. Otherwise, let $f_n^-(x) = 0 \wedge f_n(x)$, $f_n^+(x) = 0 \vee f_n(x)$, $f^-(x) = 0 \wedge f(x)$ and $f^+(x) = 0 \vee f(x)$. Then $f_n^+ \to f^+$ and $f_n^- \to f^-$ pointwisely, as $f_n \to f$ in Case (1). Then

$$\varliminf_{n\to\infty} \int f_n d\mu_n = \varliminf_{n\to\infty} \int (f_n^+ + f_n^-) d\mu_n = \varliminf_{n\to\infty} [\int f_n^+ d\mu_n + \int f_n^- d\mu_n]$$

$$\geq \varliminf_{n\to\infty} \int f_n^+ d\mu + \varliminf_{n\to\infty} \int f_n^- d\mu_n$$

$$= \varliminf_{n\to\infty} \int f_n^+ d\mu_n + \int \lim_{n\to\infty} f_n^- d\mu_n \qquad \text{(by statement (2) of Lemma 1, as } |f_n^-(x)| \leq k)$$

$$\geq \int \lim_{n\to\infty} f_n^+ d\mu + \int f^- d\mu \qquad \text{(by statement (1) of Lemma 1, as } f_n^+(x) \text{ is nonnegative)}$$

$$= \int f^+ d\mu + \int f^- d\mu = \int (f^+ + f^-) d\mu = \int f d\mu \quad \textit{i.e.}, \text{ statement (1) holds.}$$

Let $g_n(x) = \inf\{f_k(x): \ k \geq n\}$, then $g_n(x) \to g(x) = \varliminf_{n\to\infty} f_n(x)$. We have

$$\int \varliminf_{n\to\infty} f_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu \leq \varliminf_{n\to\infty} \int g_n \, d\mu_n \text{ (by statement (1)), as } g_n \text{ is bounded below)}$$

$$= \varliminf_{n\to\infty} \int \inf\{f_k: \ k \geq n\} \, d\mu_n$$

$$\leq \varliminf_{n\to\infty} \int f_n \, d\mu_n.$$

Moreover,

$$\int \ln \varliminf_{n\to\infty} f_n \, d\mu = \int \ln \lim_{n\to\infty} g_n \, d\mu$$

$$= \int \lim_{n\to\infty} \ln g_n \, d\mu \qquad \text{(as } \ln x \text{ is continuous in } x)$$

$$\leq \varliminf_{n\to\infty} \int \ln g_n \, d\mu_n \text{ (by statement (1), as } g_n \text{ is bounded below)}$$

$$= \varliminf_{n\to\infty} \int \ln \inf\{f_k: \ k \geq n\} \, d\mu_n$$

$$\leq \varliminf_{n\to\infty} \int \ln f_n \, d\mu_n \text{ (as } \ln x \text{ is increasing in } x), \text{ thus statement (3) holds.}$$

The fourth statement is a modified bounded convergence theorem and we skip its proof. □