Technical Report on

"The MLE Of The Uniform Distribution With Right-censored Data"

Appendix I. Roots of a polynomial of degree 3 or 4.

The roots of $h(x) = ax^3+bx^2+cx+d = 0$ **can be derived as follows.** Let $x = y - \frac{b}{3a}$. Then $h(x) = 0 => y^3 + py + q = 0$, where $p = \frac{3ac-b^2}{3a^2}$ and $q = \frac{2b^3-9abc+27a^2d}{27a^3}$. Let $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$.

(1) If $\Delta < 0$, there are 3 real roots:
$y_1 = \big(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3} + \big(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3}$,
$y_2 = \omega_1\big(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3} + \omega_2\big(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3}$,
$y_3 = \omega_2\big(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3} + \omega_1\big(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3}$, where $\omega_1 = \frac{-1+\sqrt{3}i}{2}$ and $\omega_1 = \frac{-1-\sqrt{3}i}{2}$.

(2) If $\Delta = 0$, then $y_1 = 2(-q/2)^{1/3}$ and $y_2 = y_3 = (q/2)^{1/3}$.

(3) if $\Delta > 0$, then $y_1 = y_2 = y_3 = \big(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3} + \big(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\big)^{1/3}$.

The roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ **can be derived as follows.**

The 4 roots can be solved through these 2 quadratic equations in x:

$$x^2 + \frac{a \pm \sqrt{8y + a^2 - 4b}}{2}x + (y \pm \frac{ay - c}{\sqrt{8y + a^2 - 4b}}) = 0$$

where y is any root of the equation $y^3 - \frac{b}{2}y^2 + \frac{ac-4d}{4}y + \frac{d(4b-a^2)-c^2}{8} = 0$. □

Appendix II. Proof of Theorem 4 in Cases (1) and (2). The proof of (I) (sufficiency) is almost identical, except revising (η_1, η_2) in the proof there by setting $\eta_2 = b$ in Case (1) and $\eta_1 = a$ in Case (2). Thus we only give the proofs that $\tau_Y > a$ is the necessary condition.

Suppose Case (1) is true. That is $\theta = a$. Then

$$f_{Z,\delta}(t, s; \theta) = (\frac{\mathbf{1}(t \in (\theta, b])}{b - \theta}S_Y(t-))^\delta\{[\frac{\mathbf{1}(t \in (\theta, b])(b - t)}{b - \theta} + \mathbf{1}(t < \theta)]f_Y(t)\}^{1-\delta}. \quad (A.1)$$

Assume $\tau_Y \leq a$. Let $\eta = \theta + c$, where $c = (b-\theta)/2$. By (A.1), $f_{Z,\delta}(t,1;\theta) = \frac{\mathbf{1}(t\in(\theta,b])}{b-\theta} S_Y(t-) = 0 = f_{Z,\delta}(t,1;\eta)$ (except perhaps at $t=\theta$), as (1) $S_Y(t-) = 0 = f_Y(t)$ for $t > \theta \geq \tau_Y$, and (2) $\mathbf{1}(t \in [\theta,b]) = 0 = \mathbf{1}(t \in [\theta+c,b])$ if $t < \theta$. Moreover,

$$f_{Z,\delta}(t,0;\theta) = (\mathbf{1}(t<\theta) + \frac{\mathbf{1}(t\in[\theta,b])(b-t)}{b-\theta}) f_Y(t) = \begin{cases} 0 & \text{if } t > \theta \\ f_Y(t) & \text{if } t \leq \theta \end{cases} \equiv f_Y(t). \quad (A.2)$$

$f_{Z,\delta}(t,1;\eta) = (\mathbf{1}(t < a+c) + \frac{\mathbf{1}(t\in[a+c,b])(b-t)}{b-a-c}) f_Y(t) \equiv f_Y(t) = f_{Z,\delta}(t,1;\theta)$ by (A.2). Thus, $f_{Z,\delta}(t,s;\theta) = (f_{Z,\delta}(t,1;\theta))^s (f_{Z,\delta}(t,0;\theta))^{1-s} = (f_{Z,\delta}(t,1;\eta))^s (f_{Z,\delta}(t,0;\eta))^{1-s} = f_{Z,\delta}(t,s;\eta)$, *i.e.*, $f_{Z,\delta}(t,s;\eta) = f_{Z,\delta}(t,s;\theta)$ a.e. and consequently, Eq. (5.2) holds. Thus $\tau_Y > \theta$ $(= a)$ is the necessary identifiability condition in Case 1.

Now suppose Case (2) is true. Thus $\theta = b$. Then

$$f_{Z,\delta}(t,s;\theta) = (\frac{\mathbf{1}(t\in[a,\theta))}{\theta-a} S_Y(t-))^\delta \{[\frac{\mathbf{1}(t\in[a,\theta])(\theta-t)}{\theta-a} + \mathbf{1}(t<a)] f_Y(t)\}^{1-\delta}. \quad (A.3)$$

Assume $\tau_Y \leq a$. Let $\eta = \theta + 1$. By (A.3), $f_{Z,\delta}(t,1;\theta) = \frac{\mathbf{1}(t\in(a,\theta])}{\theta-a} S_Y(t-) = 0 = f_{Z,\delta}(t,1;\eta)$ (except perhaps at $t = a$), as (1) $S_Y(t-) = 0 = f_Y(t)$ for $t > a \geq \tau_Y$, and (2) $\mathbf{1}(t \in [a,b]) = 0 = \mathbf{1}(t \in [a, b+1])$ if $t < a$. Moreover,

$$f_{Z,\delta}(t,0;\theta) = (\mathbf{1}(t<a) + \frac{\mathbf{1}(t\in[a,b])(b-t)}{b-a}) f_Y(t) = \begin{cases} 0 & \text{if } t > a \\ f_Y(t) & \text{if } t \leq a \end{cases} \equiv f_Y(t). \quad (A.4)$$

$f_{Z,\delta}(t,1;\eta) = (\mathbf{1}(t < a) + \frac{\mathbf{1}(t\in[a,b+1])(b+1-t)}{b+1-a}) f_Y(t) \equiv f_Y(t) = f_{Z,\delta}(t,1;\theta)$ by (A.4). Thus, $f_{Z,\delta}(t,s;\eta) = (f_{Z,\delta}(t,1;\theta))^s (f_{Z,\delta}(t,0;\theta))^{1-s} = (f_{Z,\delta}(t,1;\eta))^s (f_{Z,\delta}(t,0;\eta))^{1-s} = f_{Z,\delta}(t,s;\eta)$, *i.e.*, $f_{Z,\delta}(t,s;\eta) = f_{Z,\delta}(t,s;\theta)$ a.e. and consequently, Eq. (5.2) holds. Thus $\tau_Y > a$ is the necessary identifiability condition in Case 2. □

Appendix III. Proof of Theorem 6 . We first consider $\hat{b}$. Abusing notations, we write $a = \hat{a} = Z_{(1)}$. Let $k = \sum_{i=1}^n \mathbf{1}(Z_i = c)$ $(= n\overline{\mathbf{1}(X > c)}$, as $\mathbf{1}(Z = c) = \mathbf{1}(Y \equiv c < X))$. Then $\frac{d}{db}\ln L(b) = 0$ leads to $H(b) = -\frac{n}{b-a} + \frac{k}{b-c} = 0$. Thus the root of $H(b)$ is

$$\tilde{b} = \frac{nc-ka}{n-k} = \frac{(n-k)c + k(c-a)}{n-k} = c + (c-\hat{a})\frac{\overline{\mathbf{1}(X>c)}}{1-\overline{\mathbf{1}(X>c)}} = c + (c-\hat{a})\frac{1-\hat{p}}{\hat{p}}$$

$> c = Z_{(n)}$. By Lemma 1, the MLE is $\hat{b} = \tilde{b}$. It converges to $c + (c-a)\frac{b-c}{c-a} = b$ a.s.. Since $V(\hat{a}) = O(n^{-2})$ and $\mathbf{1}(X \le c) \sim bin(1,p)$ with $p = P(X \le c)$, by the central limit theorem and Slutsky's Theorem, $\sqrt{n}(\hat{b}-b) \xrightarrow{D} N(0,\sigma^2)$, where $\sigma^2 = n(g'(p))^2 V(\hat{p}) = \frac{(c-a)^2(1-p)}{p^3}$ and $g(p) = c + (c-\hat{a})\frac{1-p}{p}$ (notice that $V(\hat{a}) = O(1/n^2)$).

Of course, $\hat{a}$ is not asymptotically normally distributed. It is well known that the MLE of a is $X_{(1)}$. Let $X_o \sim U(0,1)$ and $T_o = (X_o)_{(n)}$, then $F_{T_o}(t) = (F_{X_o}(t))^n = t^n \mathbf{1}(t \in [0,1]) + \mathbf{1}(t > 1)$.

$E(T_o) = \int_0^1 (1-t^n)dt = 1 - t^{n+1}/(n+1)\Big|_0^1 = 1 - \frac{1}{n+1}$.

$E(T_o^2) = \int_0^1 2t(1-t^n)dt = [t^2 - 2t^{n+2}/(n+2)]\Big|_0^1 = 1 - \frac{2}{n+2}$.

$V(T_o) = 1 - \frac{2}{n+2} - (1 - \frac{1}{n+1})^2 = -\frac{2}{n+2} + \frac{2}{n+1} - (\frac{1}{n+1})^2 = \frac{2}{(n+1)(n+2)} - (\frac{1}{n+1})^2 = \frac{2n+2-n-2}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)}$.

If $X \sim U(a,b)$ $(= U(a, \theta + a)$, then $X = \theta X_o + a$, $F_X(t) = (\frac{t-a}{\theta})^n \mathbf{1}\big(t \in (a,b)\big) + \mathbf{1}(t \ge b)$ and $S_X(t) = (\frac{b-1}{\theta})^n \mathbf{1}\big(t \in (a,b)\big) + \mathbf{1}(t \le a)$. Moreover, $V(X_{(n)}) = \theta^2 V(T_o) = \frac{n\theta^2}{(n+1)^2(n+2)}$. $V(X_{(1)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$, as $X_{(1)} = \theta(1 - T_{o(n)}) + a = \theta - (\theta T_{o(n)} + a) + 2a = \theta + 2a - X_{(n)}$. □