Appendix I. Roots of a polynomial of degree 3 or 4.

The roots of \( h(x) = ax^3 + bx^2 + cx + d = 0 \) can be derived as follows. Let \( x = y - \frac{b}{3a} \).

Then \( h(x) = 0 \Rightarrow y^3 + py + q = 0 \), where \( p = \frac{3ac - b^2}{3a^2} \) and \( q = \frac{2b^3 - 9abc + 27a^2d}{27a^4} \). Let \( \Delta = \frac{q^2}{4} + \frac{p^3}{27} \).

1. If \( \Delta < 0 \), there are 3 real roots:

   \[ y_1 = \left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3} + \left( -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3}, \]

   \[ y_2 = \omega_1 \left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3} + \omega_2 \left( -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3}, \]

   \[ y_3 = \omega_2 \left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3} + \omega_1 \left( -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3}, \]

   where \( \omega_1 = -1 + \sqrt{3}i \) and \( \omega_2 = -1 - \sqrt{3}i \).

2. If \( \Delta = 0 \), then \( y_1 = 2(-q/2)^{1/3} \) and \( y_2 = y_3 = (q/2)^{1/3} \).

3. If \( \Delta > 0 \), then \( y_1 = y_2 = y_3 = \left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3} + \left( -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{1/3} \).

The roots of \( x^4 + ax^3 + bx^2 + cx + d = 0 \) can be derived as follows.

The 4 roots can be solved through these 2 quadratic equations in \( x \):

\[ x^2 + \frac{a \pm \sqrt{8y + a^2 - 4b}}{2}x + \left( y \pm \frac{ay - c}{\sqrt{8y + a^2 - 4b}} \right) = 0 \]

where \( y \) is any root of the equation \( y^3 - \frac{b}{2}y^2 + \frac{ac - 4d}{4}y + \frac{d(4b - a^2) - c^2}{8} = 0 \). \( \square \)

Appendix II. Proof of Theorem 4 in Cases (1) and (2). The proof of (1) (sufficiency) is almost identical, except revising \( (\eta_1, \eta_2) \) in the proof there by setting \( \eta_2 = b \) in Case (1) and \( \eta_1 = a \) in Case (2). Thus we only give the proofs that \( \tau_Y > a \) is the necessary condition.

Suppose Case (1) is true. That is \( \theta = a \). Then

\[ f_{Z, \delta}(t, s; \theta) = \left( \frac{1(t \in (\theta, b])}{b - \theta} S_Y(t -) \right)^{\delta} \left[ \frac{1(t \in (\theta, b])(b - t)}{b - \theta} + 1(t < \theta) \right] f_Y(t) \]
Assume $\tau_Y \leq a$. Let $\eta = \theta + c$, where $c = (b - \theta)/2$. By (A.1), $f_{Z,\delta}(t, 1; \theta) = 1_{(t \in [\theta, b])}/(b - \theta) Y \sim 1_{(t \in [\theta, b])}/(b - \theta)$. Thus, $f_{Z,\delta}(t, 1; \eta) = 0 = f_{Z,\delta}(t, 1; \eta)$ (except perhaps at $t = \theta$), as (1) $S_Y(t) = 0 = f_Y(t)$ for $t > \theta \geq \tau_Y$, and (2) $1(t \in [\theta, b]) = 1(t \in [\theta + c, b])$ if $t < \theta$. Moreover,

$$f_{Z,\delta}(t, 0; \theta) = \begin{cases} 1(t < \theta) + 1(t \in [\theta, b])/(b - t) & \text{if } t > \theta \\ 0 & \text{if } t \leq \theta \end{cases} \equiv f_Y(t).$$

(A.2)

Thus, $f_{Z,\delta}(t, 1; \eta) = (1(t < a + c) + 1(t \in [a + c, b])/(b - t)) f_Y(t) \equiv f_Y(t) = f_{Z,\delta}(t, 1; \theta)$ by (A.2). Moreover, $f_{Z,\delta}(t, s; \theta) = (f_{Z,\delta}(t, 1; \theta))^{s} (f_{Z,\delta}(t, 0; \theta))^{1-s} = (f_{Z,\delta}(t, 1; \eta))^{s} (f_{Z,\delta}(t, 0; \eta))^{1-s} = f_{Z,\delta}(t, s; \eta)$, i.e., $f_{Z,\delta}(t, s; \eta) = f_{Z,\delta}(t, s; \theta)$ a.e. and consequently, Eq. (5.2) holds. Thus $\tau_Y > \theta (= a)$ is the necessary identifiability condition in Case 1.

Now suppose Case (2) is true. Thus $\theta = b$. Then

$$f_{Z,\delta}(t, s; \theta) = \begin{cases} 1(t \in [a, \theta)) \frac{S_Y(t)}{\theta - a} & \text{if } t > \theta \\ 0 & \text{if } t \leq \theta \end{cases} \equiv f_Y(t).$$

(A.3)

Assume $\tau_Y \leq a$. Let $\eta = \theta + 1$. By (A.3), $f_{Z,\delta}(t, 1; \theta) = \frac{1(t \in [a, \theta])}{\theta - a} Y \sim 1_{(t \in [a, \theta])}/(\theta - a)$. Thus, $f_{Z,\delta}(t, 1; \eta) = 0 = f_{Z,\delta}(t, 1; \eta)$ (except perhaps at $t = a$), as (1) $S_Y(t) = 0 = f_Y(t)$ for $t > a \geq \tau_Y$, and (2) $1(t \in [a, b]) = 0 = 1(t \in [a, b + 1])$ if $t < a$. Moreover,

$$f_{Z,\delta}(t, 0; \theta) = \begin{cases} 1(t < a) + 1(t \in [a, b])/(b - t) & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases} \equiv f_Y(t).$$

(A.4)

Thus, $f_{Z,\delta}(t, 1; \eta) = (1(t < a) + 1(t \in [a, b + 1])/(b + 1 - a)) f_Y(t) \equiv f_Y(t) = f_{Z,\delta}(t, 1; \theta)$ by (A.4). Moreover, $f_{Z,\delta}(t, s; \eta) = (f_{Z,\delta}(t, 1; \theta))^{s} (f_{Z,\delta}(t, 0; \theta))^{1-s} = (f_{Z,\delta}(t, 1; \eta))^{s} (f_{Z,\delta}(t, 0; \eta))^{1-s} = f_{Z,\delta}(t, s; \eta)$, i.e., $f_{Z,\delta}(t, s; \eta) = f_{Z,\delta}(t, s; \theta)$ a.e. and consequently, Eq. (5.2) holds. Thus $\tau_Y > a$ is the necessary identifiability condition in Case 2. □

Appendix III. Proof of Theorem 6. We first consider $\hat{b}$. Abusing notations, we write $a = \hat{a} = Z(1)$. Let $k = \sum_{i=1}^{n} Z_i = c$ (since $\sum_{i=1}^{n} Z_i > c$, as $1(Z = c) = 1(Y \equiv c < X)$).

Then $\frac{d}{db} \ln L(b) = 0$ leads to $H(b) = -\frac{n}{b - a} + \frac{k}{b - c} = 0$. Thus the root of $H(b)$ is

$$\hat{b} = \frac{nc - ka}{n - k} = \frac{(n - k)c + k(c - a)}{n - k} = c + (c - \hat{a}) \frac{1(X > c)}{1 - 1(X > c)} = c + (c - \hat{a}) \frac{1 - \hat{p}}{\hat{p}}$$

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\[ > c = Z_{(n)}. \] By Lemma 1, the MLE is \( \hat{b} = \tilde{b} \). It converges to \( c + (c - a)\frac{b - c}{c - a} = b \) a.s.

Since \( V(\hat{a}) = O(n^{-2}) \) and \( 1(X \leq c) \sim \text{bin}(1, p) \) with \( p = P(X \leq c) \), by the central limit theorem and Slutsky’s Theorem, \( \sqrt{n}(\hat{b} - b) \xrightarrow{D} N(0, \sigma^2) \), where \( \sigma^2 = n(g'(p))^2V(\hat{p}) = \frac{(c - a)^2(1 - p)}{p^3} \) and \( g(p) = c + (c - \hat{a})\frac{1 - p}{\tilde{p}} \) (notice that \( V(\hat{a}) = O(1/n^2) \)).

Of course, \( \hat{a} \) is not asymptotically normally distributed. It is well known that the MLE of \( a \) is \( X_{(1)} \). Let \( X_o \sim U(0, 1) \) and \( T_o = (X_o)_{(n)} \), then \( F_{T_o}(t) = (F_{X_o}(t))^n = t^n1(t \in [0, 1]) + 1(t > 1) \).

\[
E(T_o) = \int_0^1 (1 - t^n)dt = 1 - t^{n+1}/(n + 1)|_0^1 = 1 - \frac{1}{n+1}.
\]

\[
E(T_o^2) = \int_0^1 2t(1 - t^n)dt = [t^2 - 2t^{n+2}/(n + 2)]|_0^1 = 1 - \frac{2}{n+2}.
\]

\[
V(T_o) = 1 - \frac{2}{n+2} - (1 - \frac{1}{n+1})^2 = -\frac{2}{n+2} + \frac{2}{n+1} - (\frac{1}{n+1})^2 = \frac{2}{(n+1)(n+2)} - (\frac{1}{n+1})^2 = \frac{2(n+2) - n - 2}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)}.
\]

If \( X \sim U(a, b) (= U(a, \theta + a) \), then \( X = \theta X_o + a, F_X(t) = (\frac{t - a}{\theta})^n1(t \in (a, b)) + 1(t \geq b) \) and \( S_X(t) = (\frac{b - 1}{\theta})^n1(t \in (a, b)) + 1(t \leq a) \). Moreover, \( V(X_{(n)}) = \theta^2 V(T_o) = \frac{n\theta^2}{(n+1)^2(n+2)}, V(X_{(1)}) = \frac{n\theta^2}{(n+1)^2(n+2)}, \) as \( X_{(1)} = \theta(1 - T_o(n)) + a = \theta - (\theta T_o(n) + a) + 2a = \theta + 2a - X_{(n)} \). \( \Box \)