Assignments for the Thanksgiving Week

1. Show that the ideal of $\mathbb{Z}[X]$ generated by X and 2 is not principal.

2. Find all irreducible Polynomials of degree 3 over $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

3. Find the decomposition of $X^6 + X^3 + 1$ in $\mathbb{Z}_2[X]$ and in $\mathbb{Z}_3[X]$ into a product of primes.

4. In the ring $R = \mathbb{Z}[\sqrt{5}]$ look for units (= elements invertible in R), irreducible elements, prime elements. Try to find a situation which shows that R cannot be a p.i.d.
(Hint: play with the facts $(\sqrt{5}+1)(\sqrt{5} - 1) = 4$, $(\sqrt{5}+2)(\sqrt{5} - 2) = 1$, $(\sqrt{5}+3)(\sqrt{5} - 3) = 4$)

5. Construct fields with 8, and 9 elements as quotient rings of $\mathbb{Z}_2[X]$ or $\mathbb{Z}_3[X]$ modulo the ideal an irreducible polynomial f(X) of degree 3 and 2, respectively. Find the addition and the multiplication table of these fields.

   Example: The field with 4 elements is obtained as $\mathbf{F} = \mathbb{Z}_2[X]/\mathbb{Z}_2[X](1+X+X^2)$.
   Division with remainder shows that every polynomial f(X) of $\mathbb{Z}_2[X]$ can be written as $f(X) = g(X)(1+X+X^2)+r(X)$, where r(X) is either 0 or of degree at most 1. Moreover two different polynomials of degree at most 1 cannot be in the same coset mod the ideal $I = \mathbb{Z}_2[X](1+X+X^2)$, hence they form a complete set of representatives mod I.
   We thus have four elements in $\mathbf{F} = \{0 = I, 1 = 1+I, a = X+I, b=a+1=(1+X)+I\}$ The + and mult-tables are easy to write down if one keeps in mind, that 2a = 0, and $1+a+a^2 = 0$.

+	0	1	a	a+1
0	0	1	a	a+1
1	1	0	a+1	a
a	a	a+1	0	1
a+1	a+1	a	1	0

mult	1	a	a+1
1	1	a	a+1
a	a	a+1	1
a+1	a+1	1	a

Happy Thanksgiving