Assignments for the week Oct. 10^{th} -- 17^{th}

Problem 1.
Let G and H be two groups. We write $G\times H$ for the set of all pairs $\{(g,h) \mid g\in G,\ h\in H\}$ and we define on $G\times H$ a product by putting $(g,h)(g',h') = (gg',hh')$.a)

a) Show that $G_1 = \{(g,e_H) \mid g\in G\}$ and $H_1 = \{(e_G,h) \mid h\in H\}$ are subgroups of $G\times H$,

b) Show that $G_1\cap H_1 = e$,

c) Show that $G_1H_1 = G\times H$,.

c) Show that any two elements $g_1\in G_1$ and $h_1\in H_1$ commute in $G\times H$.

Problem 2. Show that $\mathbb{Z}_4\times\mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_8$; but is it isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_4$?

The next problem starts with the following
Remark. Let p be a prime number. We have seen that all non-zero elements of the commutative ring $K= \mathbb{Z}_p$ are invertible (because they are represented by numbers $0<a<p$ which cannot have a common divisor $\neq 1$ with p since p is a prime number). Hence K is a field and the techniques of linear algebra apply just like for other fields like $\mathbb{R}$, or $\mathbb{Q}$.

Problem 3. Let A be an additive Abelian group with the property that every non-zero element $a\in A$ has order p (i.e., $pa = 0$ for all $a\in A$). We would like to view A as a vector space over the field K. For this we need to define a product $k\cdot a$, and we attempt to do this by the formula $[n]_p\cdot a = na$, where n is an integer representing $k\in\mathbb{Z}_p$.

a) Why do we have to show that this is well defined ?

b) Prove that it **is** well defined and that it turns A into a vector space over K.

Problem 4. Let p be a prime number.

a) Find all (pairwise non-isomorphic) additive Abelian groups A of order p^2.

b) For each A of part a) find the number of subgroups of A.

Problem 5.

a) Show that the alternating groups A_3 and A_4 are generated by their 3-cycles.

b) show for arbitrary n that A_n is generated by all its 3-cycles.

Problem 6. Prove that if a is an element of the finite group G then $a^{|G|} = e$.