# An instructive exercise on sets and functions

Let $f: A \to B$ a function (or a map) from the domain A into the range B. The same letter f is also used when we mean the corresponding function $f: P(A) \to P(B)$, on the power set P(A) of A, defined by $f(X) = \{f(x) \mid x \in X\}$. This "abuse of notation" is harmless as long as we always indicate domain and range together with the symbol f. Similarly for the symbol $f^{-1}$: It may be used for the function $f^{-1}: B \to P(A)$ which assigns to $b \in B$ the fiber $f^{-1}(b) := \{a \in A \mid f(a) = b\}$, or the map $f^{-1}: P(B) \to P(A)$ which assigns to each set $Y \in P(B)$ the set $f^{-1}(Y) := \{a \in A \mid f(a) \in Y\}$, $Y \in P(B)$, or the inverse $f^{-1}: B \to A$ of f when f is a bijection; but this is harmless as long as we always indicate domain and range of the function which we mean.

**Exercise 1.** Prove or disprove

a) $f: A \to B$ is injective if and only if $f: P(A) \to P(B)$ is injective,

b) $f: A \to B$ is surjective if and only if $f: P(A) \to P(B)$ is surjective,

**Exercise 2.**

a) Show that for every $Y \subseteq B$, $f(f^{-1}(Y)) = Y \cap f(A)$,

b) Show that if $f: A \to B$ is surjective then $f \circ f^{-1} = id_{P(B)}$ and $f^{-1}: P(B) \to P(A)$ is injective.

c) Show that if $f^{-1}: P(B) \to P(A)$ is injective then $f: A \to B$ is surjective.

**Exercise 3.**

a) Show that for every $X \subseteq A$, $f^{-1}(f(X))$ is the union of all fibers over the elements in f(X).

b) Show that if $f: A \to B$ is injective then $f^{-1}: P(B) \to P(A)$ is surjective.

c) Show that if $f^{-1}: P(B) \to P(A)$ is surjective then $f: A \to B$ is injective.

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