Test 2

October 25th 2012

1. a) In $\mathbb{Z}_{60}$ find – if possible – the multiplicative inverses of [17] and [21].
   b) For an arbitrary prim p>2 find the multiplicative inverse of $[(p-1)/2]$ in $\mathbb{Z}_p$.

2. Show that a group G that contains only the two subgroups {e} and G is finite of order $|G| = p$ a prime number.

3. Prove – as in class – that every subgroup of the additive group $\mathbb{Z}$ is cyclic.

4. Prove that in a group G the following holds
   a) any two conjugate elements have the same order
   b) for arbitrary elements a, b in G ab and ba have the same order.

5. Let G be a group, $H \leqslant G$ a subgroup and $N \leqslant G$ a normal subgroup of G.
   a) prove that $HN = \{hn \mid h \in H, n \in N\}$ is a subgroup of G,
   b) prove that $H \cap N$ is a normal subgroup of H.

6. a) In the symmetric permutation group S_5 find a permutation π with $\pi(1,2,3)\pi^{-1} = (2,3,5)$
   b) Can we choose π in A_5 ? If so – why?

7. By enumerating the corners of a square counter-clockwise we realize the dihedral group D_4 as a subgroup $H \leqslant S_4$.
   a) What are the elements of H ?
   b) Find a complete set of representatives of the cosets xH in S_4.
   c) find the elements of $H \cap A_4$ and the index $|S_4 : (H \cap A_4)|$.

8. Let $\varphi: G \rightarrow H$ be a homomorphism of groups. Prove that if S is a subgroup of H then
   $T = \{a \in G \mid \varphi(a) \in S\}$ is a subgroup of G.

9. Let G be a non-Abelian group of order $|G| = 8$. Show a) that G contains an element $a \in G$ of order 4, b) that the cyclic subgroup gp(a) is normal, c) that a^2 has order 2, and d) the center of G is $Z(G) = \{e, a^2\}$.