Moderen Algebra Test 3

November 29th 2012

1. Let I be the principal ideal of $\mathbb{Z}$ generated by 12, and R=$\mathbb{Z}$/I its quotient ring.
   a) Is the element 3 + I in R invertible*) ?
   b) Is R an integral domain*) ?

2. a) Compute the gcd(X^5 + 1,X^3 + 1) in $\mathbb{Q}[X]$.
   b) same question in $\mathbb{Z}_2[X]$.

3. a) Prove that if I and J are ideals of the ring R then I∩J is also an ideal of R, and
   b) find a generator for $\mathbb{R}[X](X-1)\cap\mathbb{R}[X](X+1)$.

4. Let I be the principal ideal generated by $X^3 + 1$, and R=$\mathbb{Q}[X]$/I its quotient ring.
   a) Is the element (X^5 + 1) + I in R invertible*) ?
   b) Is R an integral domain*) ?

5. We consider the quotient ring R = $\mathbb{Z}_2[X]/\mathbb{Z}_2[X](X^2 + 1)$.
   a) What is its group of units R* ?
   b) Find the multiplication table of R.

6. Give a complete proof that in the polynomial ring K[X] over a field K every ideal I is principal.

7. Let φ: G→H a homomorphism.
   a) prove that if $n = |H|$ is finite then for every element a∈G we have $a^n \in \ker(\varphi)$.
   b) Prove that for an arbitrary n×n-matrix A over the field $\mathbb{Z}_p$, $\det(A^{p-1})$ is either 0 or 1.

8. Show that if G and H are finite groups whose orders are coprime (i.e., g.c.d.($|G|,|H|$) = 1), then there is only one homomorphism φ: G→H; the homomorphism given by $\varphi(a)=e$ for all a∈G.

9. Use the Isomorphism Theorem for rings to show that $\mathbb{Q}[\pi]\subseteq\mathbb{R}$ (= the smallest subring of $\mathbb{R}$ containing both $\mathbb{Q}$ and the number π) is isomorphic to the polynomial ring $\mathbb{Q}[X]$.
   (Hint: You may use Lindeman's Theorem that π cannot be a root of any polynomial with rational coefficients).

*) Such questions require an argument; a blunt yes/no answer does not count.