Test III type problems

1. a) Let G be a group, N a normal subgroup of G. Use the Isomorphism Theorem to prove: *"Every subgroup* H *of* G *with the property that* $H \cap N = e$ *is isomorphic to a subgroup of* G/N*"*.
b) Is the converse of a) true ? In particular, is the quotient group G/N always isomorphic to a subgroup of G ?

2. Prove that if a group G has only finitely many subgroups then G is finite.
(**Hint**: Use cyclic subgroups)

3. Let $\varphi: G \to H$ be a homomorphism of finite groups. Prove:
a) $|\varphi(G)|$ divides both $|G|$ and $|H|$,
b) If S is a subgroup of G then $|\varphi(S)|$ divides the g.c.d($|G|$,$|H|$)

4. Give a complete proof of the Theorem: *The polynomial ring* K[X] *over a field* K *is a* p.i.d.

5. In the polynomial ring K[X] of problem 4 consider the ideal I generated by a polynomial f(X) and its quotient ring R = K[X]/I. Which of the polynomials $g(X) \in K[X]$ have the property that the coset g(X) + I is invertible in the ring R/I ?
Is $(X^3 - 1) + I$ invertible in R when $f(X) = X^2 - 1$?
(**Hint**: Remember the analogous case when $R = \mathbb{Z}/m\mathbb{Z}$).

6. a) Let $\pi: \mathbb{Q}[X] \to \mathbb{R}$ be the map which evaluates the polynomial f(X) at $\sqrt{5}$, i.e., $\pi(f(X)) = f(\sqrt{5})$. Show that π is a ring homomorphism, find its kernel I and use the Isomorphism Theorem to express the image $\pi(\mathbb{Q}[X])$ as a quotient ring.

7. Let $f(X) = X^2+X+1 \in \mathbb{Z}_2[X]$, let I be the principal ideal generated by f(X) and $R = \mathbb{Z}_2[X]/I$ its quotient ring.
a) Why is K a field ?
b) How many elements has R ?
c) Find the inverse of the element $((X+1) + I) \in R$.

Same question for other irreducible polynomials (like X^3+X^2+1) in $\mathbb{Z}_2[X]$.