Typical Test II Type Questions

1. is the following true or false? *"if two elements of G are in the same coset mod H then they have the same order"* (proof or a counter example).

2. a) Let $a \in G$ be an element of finite order m in a group G. Show that the set $S = \{k \in \mathbb{Z} \mid a^k = e\}$ is a subgroup of $(\mathbb{Z},+)$ and used to prove that $a^k = e$ implies that k is a multiple of m.
   b) Let N be a normal subgroup of the group G, and $a \in G$ an element of finite order m in G. Show that the set $S' = \{k \in \mathbb{Z} \mid a^k \in N\}$ is a subgroup of $(\mathbb{Z},+)$ which contains S, and use this to prove that the order of aN in G/N is a divisor of m.

3. Show that the center Z of a group G is a normal subgroup of G, and prove: When G/Z iz cyclic then G is Abelian.

4. Let N, M be two normal subgroups of G.
   a) Is the intersection $N \cap M$ a normal subgroups of G ?
   b) Is the product $NM = \{nm \mid n \in N, m \in M\}$ a subgroup of G ?
   c) Is the product $NM = \{nm \mid n \in N, m \in M\}$ a normal subgroup G ?

5. For two natural numbers n and m consider in the additive group $(\mathbb{Z},+)$ the two subgroups:
   A = the cyclic subgroup generated by m,
   B = the cyclic subgroup generated by n.
   Find generators for A+B and for $A \cap B$.

6. In the symmetric group $G = S_n$ consider the subgroup G_k consisting of all permutations which fix the number k. Find
   a) the index of G_k in G,
   b) representatives for the left cosets of G mod G_k,
   c) representatives for the right cosets of G mod G_k,
   d) the subgroups which are conjugate to G_k

7. Let G be a non-abelian group of order 6.
   a) prove that G contains an element a of order 3,
   b) prove that a and a^2 are the only elements of order 3 in G,
   c) prove that the cyclic subgroup N = gp(a) is normal on G,
   d) pick an element x of G which is not in N and prove that x has order 2.
   e) show that $xa = a^2x$ and $ax = xa^2$,
   f) show that $G = \{e, a, a^2, x, xa, xa^2\}$ and that the multiplication table in terms of this description of the elements is uniquely determine. Hence, up to isomorphism, there is only one non-Abelian group of order 6.