Interest Rates and Discount Rates

Math 346

August 26, 2020

Effective rate of interest

- In this class we will be interested computing the amount of money in a given account and computing measures of how good an account is (for example effective interest rates).
- The amount of money in an account at time t is denoted $A(t)$ and called the **Amount function**.
- In order to compare different accounts, it is often useful to consider how much money is in the account at time t, if the account has \$1.00 at time 0.
- This amount is called the accumulation function and denoted $a(t)$.
- The functions $a(t)$ and $A(t)$ are related by equation:

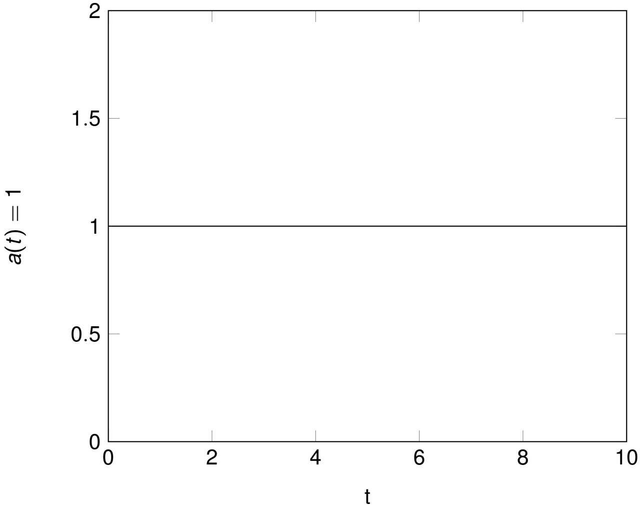
$$a(t) = A(t)/A(0) \quad \text{or} \quad A(t) = A(0)a(t)$$

Definition (Accumulation function, $a(t)$)

The **Accumulation function**, $a(t)$, *is amount of money in a fund at time t if an initial investment of* 1 *is made at time* 0.

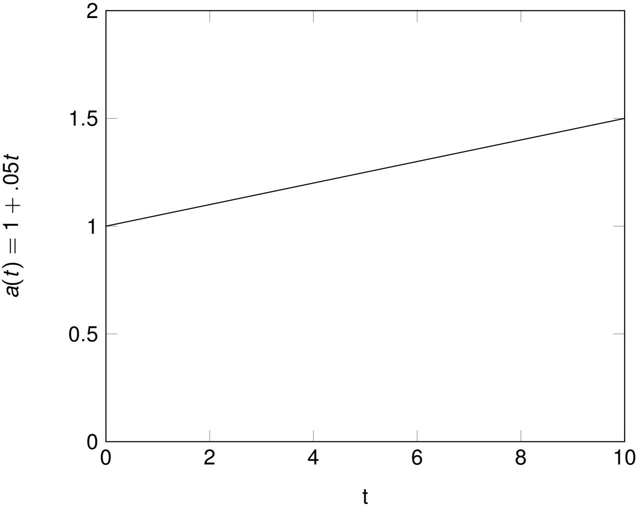
Example 1

The function $a(t) = 1$



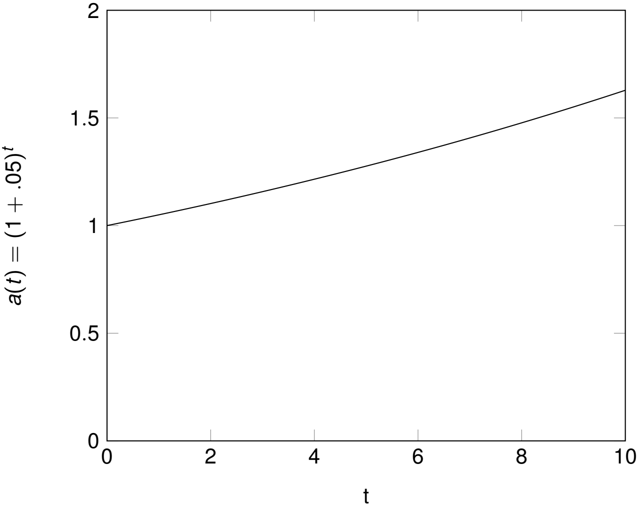
Example 2

The function $a(t) = 1 + .05t$. This is called **Simple Interest**.



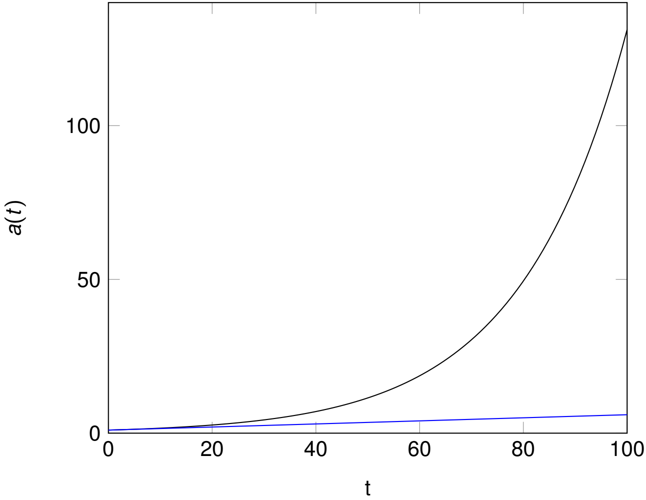
Example 3

The function $a(t) = (1 + .05)^t$. This is called **Compound Interest** and will be the most common example in this class.



Example 3

Comparison of Simple Interest and Compound Interest
$a(t) = (1 + .05)^t$. $a(t) = (1 + .05t)$.



Effective rate of interest

- Over the n^{th} year (for $t = 1, 2, 3, \ldots$), the total amount of **interest** earned is $A(t) - A(t-1)$.
- The total amount interest earned is not a good measure of good a fund is; we should instead the normalized rate of growth:

$$\frac{A(t) - A(t-1)}{A(t-1)} = \frac{a(t) - a(t-1)}{a(t-1)}$$

- The growth rate is called the effective rate of interest and is denoted i_t.

Definition (Effective rate of interest, i_t)

The **Effective rate of interest**, i_t, *ratio*

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

- The accumulation function at time t, $a(t)$, can be computed from $a(t-1)$ and i_t by the formula:

$$a(t) = (1 + i_t)a(t-1)$$

Example

Compute the effective rate of interest if the accumulation function is $a(t) = (1 + .05)^t$.

Solution

$$\begin{aligned}
i_t &= \frac{a(t) - a(t-1)}{a(t-1)} \\
&= \frac{(1+.05)^t - (1+.05)^{t-1}}{(1+.05)^{t-1}} \\
&= \frac{(1+.05)^t}{(1+.05)^{t-1}} - \frac{(1+.05)^{t-1}}{(1+.05)^{t-1}} \\
&= (1+.05) - 1 \\
&= .05
\end{aligned}$$

Note that in this problem the effective rate of interest doesn't depend on the time t.
Note that .05 is replaced by i then $i_t = i$.

Example

Compute the effective rate of interest if the accumulation function is $a(t) = 1 + .05t$.

Solution

$$
\begin{aligned}
i_t &= \frac{a(t) - a(t-1)}{a(t-1)} \\
&= \frac{(1 + .05t) - (1 + .05(t-1))}{1 + .05(t-1)} \\
&= \frac{(1 + .05t) - (1 + .05t - .05)}{1 + .05(t-1)} \\
&= \frac{.05}{1 + .05(t-1)}
\end{aligned}
$$

Note that in this problem the effective rate of interest is decreasing with time.

Effective rate of interest

- Given the effective rate of interest i_t at each time $t = 1, 2, 3, \ldots$ the accumulation function can be computed by inductively using the relationship:

$$a(t) = (1 + i_t)a(t - 1))$$

- Applying it again we get

$$a(t) = (1 + i_t)(1 + i_{t-1})a(t - 2)$$

- After t applications we have

$$a(t) = (1 + i_t)(1 + i_{t-1}) \ldots (1 + i_1)a(0)$$

- Using that $a(0) = 1$ we get

$$a(t) = \prod_{j=1}^{t}(1 + i_j)$$

- If $i_t = i$ is constant, then this simplifies to just

$$(1 + i)^t$$

  This very important special case is compound interest.

Present Value

- The amount of money in a fund at a given time is referred to as the accumulated value (**AV**).
- The reverse question can be asked: If you want to have a certain amount of money at a certain time, how much do you need to invest?
- This quantity is called the Present Value (**PV**).
- To figure out how much money you need today in order to have \$1 at t, we recall if you have \$$k$ today then at time t you'll have \$$a(t)k$. So if you start with $k = 1/a(t)$ dollars, then you'll have \$1 at time t.
- The function $1/a(t)$ is called the **discount function**.
- In the special case that the effective interest rate, i, is constant. The discount function is $\frac{1}{(1+i)^t}$. For notational convenience we define:

$$v = \frac{1}{(1+i)} = (1+i)^{-1}$$

Example

What amount of money do you need today, in order to make a payment of \$100 a year from now, a payment of \$200 2 years from now, and a payment of \$300 3 years from now? Assume an annual effective interest rate of 10%.

Solution

Since the annual effective interest rate is a constant 10%*, the accumulation function is* $a(t) = (1.1)^t$*, and therefore the discount function is* $(1.1)^{-t}$*.*

- *To compute the present value of* \$100 *a year from now, we evaluate the discount function at* $t = 1$ *and multiply it by 100.*

$$100(1.1)^{-1} = 90.91$$

- *The present value of* \$200*, 2 years from now is*

$$200(1.1)^{-2} = 165.29$$

- *The present value of* \$300*, 3 years from now is*

$$300(1.1)^{-3} = 225.39$$

- *Finally we sum the present values of the 3 payments*

$$90.9 + 165.30 + 225.39 = 481.59$$

So the PV of the 3 payments is 481.59.

Effective rate of discount

- We've used the effective interest rate to compute the value of the a fund in the future.

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} \qquad a(t) = (1 + i_t)a(t-1)$$

- To compute the value of a fund at a time in the past it can be useful to use the **effective discount rate**.

$$d_t = \frac{a(t) - a(t-1)}{a(t)} \qquad a(t-1) = (1 - d_t)a(t)$$

- Mathematically, it is only necessary to use the interest rate, but in some cases it is more natural to use the discount rate.
- The discount rate d and interest rate i are related by:

$$1 - d = v = \frac{1}{1+i}$$

- From this relationship, we can solve for d in terms of i and vice versa

$$i = \frac{d}{1-d} \qquad d = \frac{i}{1+i}$$

Effective rate of discount

- Some other useful relationships

$$d = iv$$

$$i - d = id$$

$$\frac{1}{d} - \frac{1}{i} = 1$$

- Recall we used the interest rate to compute the Present Value of payments in the future with the discount function $v = (1+i)^{-1}$. We can also use the discount rate to compute the accumulation function.

Example

Compute the accumulation function of a fund with an effective discount rate of 5%.

Solution

We know

$$a(t) = (1+i)^t$$

So using the relationship $(1+i) = v^{-1} = (1-d)^{-1}$ *we have:*

$$a(t) = (1+i)^t = (1-d)^{-t}$$

Nominal Rates of Interest

- Nominal interest rates are used to describe interest rates when they are compounded more (or less) often than once per year.
- Nominal rates are divided by the number of times they are compounded per year to give the effective rate over the smaller time period.

Nominal Rates of Interest

Example

A bank offers a nominal annual rate of interest of 8% compounded quarterly. If \$100 is deposited into this fund on January 1, 2020, how much money is in the fund on January 1, 2021?

Solution

After one quarter of a year the fund will have earned $8/4\% = 2\%$. *So on April 1, it will have*

$$\$100(1+.02) = \$102$$

After the next quarter of a year the fund will once again earn $8/4\% = 2\%$. *So on July 1, it will have*

$$\$102(1+.02) = \$100(1+.02)^2 = \$104.04$$

At the end of the year it will have

$$\$100(1+.02)^4 = \$108.243$$

NOTE: We never used 8% in the calculations. In general, nominal interest rates are “fake" rates that we use to compute the actual rates.

Nominal Rates of Interest

- In general, the nominal interest rate is divided by the number of times it is compounded to give effective interest over this smaller time period.
- The notation for the nominal interest rate compounded m times per year is $i^{(m)}$. The effective annual interest of rate of the nominal interest rate $i^{(m)}$ is

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$

- In the last example $m = 4$ and $i^{(4)} = .08$. We computed that $i = 8.243\%$.
- Theses formula make sense and are true for any value of $m > 0$. So we could consider $m = 1/2$, meaning the nominal interest is compounded once every two years.
- We can also solve for $i^{(m)}$ in terms of i:

$$i^{(m)} = \frac{(1+i)^{1/m} - 1}{1/m}$$

Equivalent Rates

- We've seen that an account can be described by an effective interest rate, effective discount rate, nominal interest rate, or nominal discount rate.
- If two different choices produce the same results over a time period they are said to be **equivalent rates**.
- The general idea is to solve an equation of the form

$$\left(1+\frac{i^{(m)}}{m}\right)^m=(1+i) \text{ or } \left(1+\frac{i^{(m)}}{m}\right)^m=(1-d)^{-1} \text{ etc.}$$

Example

Determine the nominal annual rate of interest compounded monthly equivalent to a nominal annual rate of discount of 5% compounded semiannually.

Solution

We should first determine the accumulation of function of each rate after 1 year. For the interest rate compounded monthly we have

$$\left(1+\frac{i^{(12)}}{12}\right)^{12}$$

For the discount rate compounded semiannually we have

$$\left(1-\frac{d^{(2)}}{2}\right)^{-2}=\left(1-\frac{.05}{2}\right)^{-2}=1.05194$$

Then we solve for $i^{(12)}$

$$\left(1+\frac{i^{(12)}}{12}\right)^{12}=1.05194$$

$$i^{(12)}=0.0507$$

Force of Interest

- So far we've just considered the effective interest rate over time intervals.
- This is fine if we only check the value of an account at the end of the year (or end of whatever time interval), but we might want more information about value and the interest rate of the account at any time.
- Recall the annual effective interest rate is

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

- More generally, the effective interest rate at time t over at interval of length h is

$$\frac{a(t) - a(t-h)}{a(t-h)}$$

- We would like, to know the interest rate at time t, but we can't just take h to 0, because this expression would just converge to 0.
- So we consider

$$\lim_{h \to 0} \frac{a(t) - a(t-h)}{ha(t-h)} = \frac{1}{a(t)} \frac{d}{dt} a(t)$$

  This is called the **Force of Interest**, and is denoted δ_t.

Definition

The **Force of Interest** is

$$\delta_t = \frac{1}{a(t)} \frac{d}{dt} a(t)$$

Force of Interest

- Note we can also compute the Force of Interest from the amount function, $A(t)$.

$$\delta_t = \frac{1}{a(t)} \frac{d}{dt} a(t)$$

- It will be useful to note that

$$\delta_t = \frac{d}{dt} \ln(a(t))$$

- Then we can solve for $a(t)$

$$a(t) = e^{\int_0^t \delta_r dr}$$

Force of Interest

Example

Compute force of interest if $a(t) = 1 + t^2/10$.

Solution

We first compute the derivative

$$\frac{d}{dt}a(t) = \frac{d}{dt}1 + t^2/10 = t/5$$

Then substitute into the formula

$$\delta_t = \frac{1}{a(t)}\frac{d}{dt}a(t) = \frac{t/5}{1 + t^2/10} = \frac{t}{5 + t^2/2}$$

Example

Compute the accumulation function if $\delta_t = .1$.

Solution

We first compute the integral

$$\int_0^t \delta_r dr = \int_0^t .1 dr = .1t$$

Then substitute into the formula

$$a(t) = e^{\int_0^t \delta_r dr} = e^{.1t}$$

Note that we can solve for i in the formula

$$(1+i)^t = e^{.1t} = \left(e^{.1}\right)^t$$

$$i = e^{.1} - 1$$

Conversely, if i was a given constant we could compute δ to get

$$\delta = \ln(1+i)$$

Force of Interest

- A Force of interest of $\ln(1+i)$ is equivalent to an annual effective interest rate of i.
- A computation can show that
$$\lim_{m\to\infty} i^{(m)} = \delta$$
So δ can be thought of the nominal interest rate compounded infinitely many times.

Force of Interest Trap

An account earns interest with Force of Interest δ_t. At time $t = 3$, \$1 is placed in the account. How much money is in the account at time $t = 6$?

For t between 0 and 3, the account is empty. Then for $t > 3$, the account has

$$1e^{\int_3^t \delta_r dr}$$

So at $t = 6$ we have

$$e^{\int_3^6 \delta_r dr} = e^{\int_0^6 \delta_r dr - \int_0^3 \delta_r dr} = e^{\int_0^6 \delta_r dr} e^{-\int_0^3 \delta_r dr} = \frac{e^{\int_0^6 \delta_r dr}}{e^{\int_0^3 \delta_r dr}} = \frac{a(6)}{a(3)}$$

This can be thought of computing the PV of the \$1 at time 0 and then letting that amount grow for the entire time by the Force of Interest δ_t.

Force of Interest Trap

Example

An account begins with 100 at time 0. At time 3, an addition deposit of 200 is made. The account gains interest with a Force of Interest $\delta_t = \frac{t^2}{100}$. How much is in the account at time $t = 6$?

Solution

The accumulation function is

$$a(t) = e^{\int_0^t \frac{r^2}{100}\,dr} = e^{\frac{t^3}{300}}$$

At time $t = 6$ the account has

$$100e^{\frac{6^3}{300}} + 200\frac{e^{\frac{6^3}{300}}}{e^{\frac{3^3}{300}}} = 200e^{63/100} + 100e^{18/25} = 580.965$$

Note at time $t = 3$ the account has

$$100e^{\frac{3^3}{300}} + 200$$

Multiplying this number by $\frac{e^{\frac{6^3}{300}}}{e^{\frac{3^3}{300}}}$ *would give the same answer.*