RESEARCH STATEMENT

Russell Ricks

1. Introduction

One way to study a compact, negatively curved Riemannian manifold is by looking at the isometric action of the fundamental group on the universal cover. For instance, Dehn [14] proved that the fundamental group of a hyperbolic surface has solvable word problem, giving an explicit algorithm. The field of geometric group theory studies many types of finitely presented groups in a similar way: using isometric actions on a metric space to study the group's properties.

Two important classes of spaces in geometric group theory are δ-hyperbolic spaces and CAT(0) spaces. Gromov [15] introduced δ-hyperbolic spaces as a generalization of negative curvature to arbitrary metric spaces. Geodesic triangles in δ-hyperbolic spaces satisfy a "δ-thin" condition (that is, they look like triangles in negative curvature, up to bounded error), and δ-hyperbolic spaces share many global properties with negatively curved manifolds. Gromov also popularized the study of CAT(0) spaces, which had been previously studied by Alexandrov [2] under a different name.

CAT(0) spaces are a generalization of nonpositive curvature from Riemannian manifolds to general metric spaces, defined by comparing geodesic triangles with triangles in Euclidean space. Examples of CAT(0) spaces include nonpositively curved Riemannian manifolds, Euclidean and hyperbolic buildings, and trees. In contrast with δ-hyperbolic spaces, which share properties with negatively curved manifolds only at large scales, CAT(0) spaces share many properties with nonpositively curved Riemannian manifolds at all scales. One key difference from the Riemannian setting, however, is that geodesics in CAT(0) spaces are not globally determined from a small segment.

CAT(0) geometry has been successfully applied to other areas. For instance, Agol [1] used CAT(0) cube complexes to prove the virtual fibering and virtual Haken conjectures in 3-manifold theory. Ontaneda [26] used smoothings of the Charney-Davis complex, similarly built from gluing CAT(0) pieces, to construct many new examples of negatively curved Riemannian manifolds.

We are interested in using dynamical methods to study the geometry of CAT(0) spaces. This has been a useful tool for studying the geometry of nonpositively curved Riemannian manifolds. In fact, one of the breakthroughs in understanding the geometry of such manifolds was the application of ergodic theory (i.e., measure-theoretic dynamics) to geometry. This led, for instance, to the proof of rank rigidity for such manifolds. Ergodic theory is also a classical tool for understanding negatively curved manifolds, used for instance in Mostow's original proof [25] of Mostow rigidity for closed, negatively curved manifolds. We hope that the application of these methods to CAT(0) spaces will help us to understand their geometry better in a similar way.

1.1. CAT(0) **Spaces.**

We recall a few properties of CAT(0) spaces (see [4] and [11]).

A CAT(0) space X is a uniquely geodesic metric space such that the distance between a pair of points on any geodesic triangle $\triangle$ in X is less than or equal to the distance between the corresponding pair of points on a Euclidean *comparison triangle*—a triangle $\overline{\triangle}$ in the Euclidean plane with the same edge lengths as $\triangle$. One can similarly define CAT(-1) and CAT(1) spaces by comparison with triangles in the hyperbolic plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, respectively.

Standing Hypothesis. Throughout this document, let Γ be a group acting properly discontinuously, cocompactly, and by isometries on a proper, geodesically complete CAT(0) space X (*proper* meaning closed metric balls in X are compact).

A *geodesic* in X is an isometric embedding $\mathbb{R} \to X$, while a *geodesic ray* is an isometric embedding $[0,\infty) \to X$. A subspace $Y \subset X$ isometric to $\mathbb{R} \times [0,\infty)$ is called a *flat half-plane*; note that half-planes are automatically convex. Call a geodesic in X *rank one* if its image does not bound a flat half-plane in X. We call a $\mathrm{CAT}(0)$ space *rank one* if it contains a rank one geodesic, *higher rank* if not. If a rank one geodesic is the axis of an isometry $\gamma \in \Gamma$, we call it a *rank one axis.* Also, a subspace $Y \subset X$ isometric to $\mathbb{R} \times [0,R]$, $R > 0$, is called a *flat strip of width* R. Write SX for the space of unit-speed parametrized geodesics in X.

$\mathrm{CAT}(0)$ spaces admitting a rank one axis have rich hyperbolic dynamics. On the other hand, Ballmann and Buyalo [6] conjecture that all others are very special:

Conjecture 1 (Rank Rigidity)**.** *If X does not admit a rank one axis, then X is a higher rank symmetric space or Euclidean building, or splits as a nontrivial product of* $\mathrm{CAT}(0)$ *spaces.*

This conjecture is known to hold when X is a nonpositively curved Riemannian manifold (see [4]), and when X is a $\mathrm{CAT}(0)$ cube complex ([12]).

The *visual boundary* ∂X of a $\mathrm{CAT}(0)$ space is the set of equivalence classes of asymptotic geodesic rays. The standard topology is the *visual (or cone) topology* on ∂X.

One can define angles in X and ∂X by comparison with Euclidean space. The *Tits metric* d_T on ∂X is given by taking the infimum of lengths of paths between pairs of points in ∂X under the angle metric $\angle$ on ∂X (the Tits metric may take value $+\infty$); it measures the amount of asymptotic flatness between points in the boundary. The topology induced on ∂X by the Tits metric is finer (usually strictly finer) than the visual topology. We will write $\partial_T X$ for the *Tits boundary* of X—that is, ∂X under the Tits metric d_T.

If ξ, η are the endpoints of a geodesic, then $d_T(\xi,\eta) \geq \pi$. On the other hand, $d_T(\xi,\eta) > \pi$ if and only if ξ, η are joined by a rank one geodesic in X. We remark also that the Tits metric is lower semicontinuous with respect to the visual topology on ∂X.

1.2. **Research Interests.** We would like to better understand the geometry of X and the dynamics of the Γ action. Ergodic properties have aided in the study of negatively curved manifolds; for instance, Margulis [24] used mixing of the Bowen-Margulis measure to count the asymptotic growth rate of the number of closed geodesics of bounded length. However, similar work has only recently started for ergodic properties of $\mathrm{CAT}(0)$ spaces—largely due to the previous lack of a natural invariant measure. In [32], we constructed a natural invariant measure on SX. In [29] and [31], we established some results about higher rank $\mathrm{CAT}(0)$ spaces, making progress toward Rank Rigidity. We would like to push these results further.

2. Past Research

$\mathrm{CAT}(0)$ spaces have different properties depending on whether they are rank one or higher rank. We discuss results for each kind of space in turn.

2.1. **Ergodic Theory of Rank One** $\mathrm{CAT}(0)$ **Spaces.** We summarize our results in [32]. For simplicity, we state the results assuming cocompactness of the Γ-action. However, many of these results actually are shown in [32] without assuming cocompactness—only a non-elementary action by Γ.

Standing Hypothesis. Throughout Section 2.1, we will assume X admits a rank one *axis.*

Patterson's construction [28] gives a family of finite Borel measures, called Patterson-Sullivan measures, on ∂X. We use these measures to construct a finite Borel measure (called the Bowen-Margulis measure) on $\Gamma \backslash SX$, the space of unit-speed parametrized geodesics of X modulo the Γ-action. This measure has full support and is invariant under the geodesic flow.

Bowen-Margulis measures were first introduced for compact, negatively curved Riemannian manifolds, where Margulis ([24]) and Bowen ([9]) used different methods to construct measures of maximal entropy for the geodesic flow. Bowen proved ([10]) both measures are equal; they are now often called the Bowen-Margulis measure. Sullivan ([34] and [35]) established a third method to obtain this measure (in constant negative curvature) using Patterson–Sullivan measures. For CAT(0) spaces, we follow Sullivan's approach to construct a generalized Bowen–Margulis measure, as do Kaimanovich ([18]), Bourdon ([8]), Knieper ([20]), and Roblin ([33]) in other settings.

2.1.1. *Dealing with Flat Strips.* The trickiest part of the construction for CAT(0) spaces is getting from a Borel measure on $\partial X \times \partial X$ to a Borel measure on SX. Here previous constructions of Bowen–Margulis measures (e.g., for rank one, nonpositively curved manifolds, and for CAT(-1) spaces) do not offer a way to deal with the possibility that every geodesic bounds a flat strip. In rank one CAT(0) spaces, however, this is a potentially serious issue.

We therefore construct the Bowen–Margulis measure in two stages: first on $(\partial X \times \partial X) \times \mathbb{R}$, and then on SX. The first stage allows us to use ergodic theory to prove the following structural result about geodesically complete, cocompact, rank one CAT(0) spaces:

Theorem 2. *Almost every pair of points in ∂X is joined by a rank one geodesic in X that does not bound a flat strip.*

This result allows us to define the Bowen–Margulis measure on SX.

2.1.2. *Dynamical Results.* Ergodicity follows by the classical Hopf argument [17].

Theorem 3. *The Bowen–Margulis measure on $\Gamma\backslash SX$ is ergodic.*

Finally, we identify precisely when the Bowen–Margulis measure is mixing under the geodesic flow. For compact, rank one, nonpositively curved Riemannian manifolds, it is always mixing by Babillot [3]. However, there are examples of CAT(0) spaces for which the Bowen–Margulis measure is not mixing—trees of integer edge lengths. By considering the local behavior of cross-ratios at the links of points, we prove that such trees are the only such non-mixing examples, up to rescaling. This characterization of mixing is new, even for CAT(-1) spaces. Here is the full theorem.

Theorem 4. *Assume X admits a rank one axis. The following are equivalent:*

1. *The Bowen-Margulis measure is not mixing under the geodesic flow on $\Gamma\backslash SX$.*
2. *The length spectrum is arithmetic—that is, the set of all hyperbolic translation lengths of Γ must lie in some discrete subgroup $c\mathbb{Z}$ of $\mathbb{R}$.*
3. *There is some $c \in \mathbb{R}$ such that every cross-ratio lies in $c\mathbb{Z}$.*
4. *There is some $c > 0$ such that X is isometric to a tree with all edge lengths in $c\mathbb{Z}$.*

We remark that relating mixing to the length spectrum is standard (see [13] and [33]).

2.2. **Higher Rank** CAT(0) **Spaces.** We now discuss some work we have done on higher rank CAT(0) spaces. We begin by clarifying which spaces we are investigating.

We split the Rank Rigidity Conjecture for CAT(0) spaces into three cases:

1. The Tits diameter of ∂X is $> \pi$.
2. The Γ-action on ∂X is minimal.
3. The Tits diameter of ∂X equals π, and Γ does not act minimally on ∂X.

In the first two cases, the Rank Rigidity Conjecture asserts that X has a rank one axis; it is known [4] that any space satisfying both conditions 1 and 2 has a rank one axis. In the third case, the conjecture asserts that X is a higher rank symmetric space or Euclidean building, or a product; by Leeb [22], it would suffice to prove that $\partial_T X$ is a spherical building or join.

Some of the key methods in proving rank rigidity for manifolds (in particular, using the Liouville measure to get recurrence, and proving the Angle Lemma) do not seem to extend to CAT(0) spaces. However, Guralnik and Swenson [16] recently developed a new tool for studying Case 3.

2.2.1. CAT(0) *Boundary Actions.* By Leeb [22], one can prove rigidity in Case 3 by understanding the action of Γ on ∂X. We will call a subset $K \subset \partial_T X$ a *round sphere* if it is isometric to the standard unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ with $n = \dim \partial_T X$; round spheres exist by Kleiner [19].

Guralnik and Swenson [16] construct certain "folding maps," 1-Lipschitz projections of the whole boundary onto a round sphere. This offers a way to study the dynamics of the boundary action by using spherical geometry to describe the folded image of the closed, invariant sets of ∂X. Note that we have many closed, invariant sets by non-minimality of the Γ-action on ∂X in Case 3.

2.2.2. *Results.* We now briefly describe two results obtained by studying closed, invariant sets via spherical geometry.

The first theorem (Theorem 5) extends two previous results that detect splittings of X by looking at radii of closed, invariant sets in ∂X. It was known that X splits as a product $X = Y \times \mathbb{R}$ if there is either (1) a closed, invariant set $A \subset \partial X$ with radius $< \pi/2$ [27, proof of Theorem 23], or (2) a Tits-closed, invariant, π-*convex* set $A \subset \partial X$ with radius $\leq \pi/2$ [7, Proposition 4.2]. In contrast, Theorem 5 allows us to detect general products, not just suspensions. Moreover, the theorem can be extended to hold in the case that X is not geodesically complete; the conclusion is then that a closed, convex, quasi-dense subset Y of X splits as a product.

Theorem 5. [29] *Suppose $A \subset \partial X$ is a nonempty, closed, invariant set of radius $\leq \pi/2$. Then X splits as a nontrivial product of* CAT(0) *spaces.*

The core idea of the proof is that for any set A on a sphere K, we can decompose K as a spherical join $K = A^{\perp} * A^{\perp\perp}$, and if A has radius $\pi/2$ then $A^{\perp}$ coincides with the set of circumcenters of A. Transferring information about ∂X to K and back is achieved using an extension of Guralnik and Swenson's folding construction to mimic the situation where some cocompact flat has boundary K.

The second theorem is a special case of rank rigidity in Case 3. Ballmann and Brin [5] proved a similar result for piecewise-smooth 2-complexes. However, our setting is much more general. We also hope the methods will generalize better to higher dimensions.

Theorem 6. [31] *Suppose* $\dim(\partial_T X) = 1$ *and* $\operatorname{diam}(\partial_T X) = \pi$. *Then either Γ acts minimally on ∂X, or X is a higher rank symmetric space or Euclidean building, or a nontrivial product.*

The proof of Theorem 6 is summarized as follows: We show that some closed, invariant set A intersects a certain round sphere (circle) K only finitely many times. This highly restricts where geodesics from A can enter K. We show that $\partial_T X$ must therefore consist entirely of circles with highly restricted branching, from which we conclude that $\partial_T X$ is a spherical building or join.

3. Future Research

3.1. **Some Special Cases of Rank Rigidity.** We aim to make progress on some more cases of Case 3 of the Rank Rigidity Conjecture by continuing to investigate the projections of closed invariant sets ∂X onto round spheres via folding maps. The tools from [16] and [27] seem powerful enough to eventually prove the Rank Rigidity Conjecture for a large number of CAT(0) spaces in Case 3 via Leeb's theorem.

One goal is to prove some higher dimensional version of Theorem 6. Currently, we can only describe a few restrictions on where geodesics from a closed, invariant set in ∂X can enter a given round sphere; we have not yet been able to develop the kind of precise statements we did in the one-dimensional case that allowed us to prove Theorem 6.

Another goal is to extend a result of Lytchak [23]:

Theorem 7 (Lytchak). *Suppose $\partial_T X$ contains a proper, nonempty, Tits-closed subset of $\partial_T X$ that contains all its own antipodes. If $\partial_T X$ is geodesically complete, then X is a higher rank symmetric space or Euclidean building, or a nontrivial product.*

This theorem provides a powerful means to establish rigidity, but unfortunately the conditions under which $\partial_T X$ is geodesically complete remain unclear. Under certain conditions (like Theorem 6) we can prove this, but in general the problem looks intractable. However, we can prove Theorem 7 with "geodesically complete" replaced by the condition that for every pair $p, q \in \partial_T X$, we have a sequence of geodesic rays based at p that pass arbitrarily close to q. We hope to prove that $\partial_T X$ satisfies this condition whenever $\operatorname{diam}(\partial_T X) = \pi$ and $\partial_T X$ is not a spherical join.

3.2. **More Ergodic Properties of** CAT(0) **Spaces.** The ergodic properties of the Bowen–Margulis measure for compact, nonpositively curved Riemannian manifolds are very rich. We would like to investigate to what extent they hold for cocompact CAT(0) spaces.

Knieper [21] showed the Bowen–Margulis measure on a compact, nonpositively-curved, rank one Riemannian manifold is the unique measure of maximal entropy. We can prove the Bowen–Margulis measure for a cocompact CAT(0) space, under a virtually torsion-free action, maximizes entropy. We expect Knieper's methods can be adapted to prove the measure of maximal entropy is unique.

We can establish precise asymptotic bounds on the growth rate of the number of closed geodesics for compact locally CAT(0) spaces, as Margulis [24] did for compact, negatively curved manifolds. To be precise, let P_t be the number of parallel classes of (oriented) closed geodesics of length $\leq t$ in a compact locally CAT(0) space X such that the universal cover admits a rank one axis; then $\lim_{t\to\infty} P_t / \frac{e^{ht}}{ht} = 1$, where h is the entropy of the geodesic flow (which equals the critical exponent in the Patterson–Sullivan construction on the universal cover). This result is currently being written up [30]. Where the measure of maximal entropy is unique, we expect to be able to extend this result to the non-cocompact case.

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