Flat strips, Bowen-Margulis measures, and mixing of the geodesic flow for rank one CAT(0) spaces

by

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TABLE OF CONTENTS

LIST OF APPENDICES

Abstract

Let X be a proper, geodesically complete CAT(0) space under a proper, cocompact, and isometric action; further assume X admits a rank one axis. Patterson's construction gives a family of finite Borel measures, called Patterson-Sullivan measures, on the boundary of X. We use the Patterson-Sullivan measures to construct a finite Borel measure (called the Bowen-Margulis measure) on the space of unit-speed parametrized geodesics of X modulo the action. This measure has full support and is invariant under the geodesic flow.

Although the construction of Bowen-Margulis measures for rank one nonpositively curved manifolds and for CAT(-1) spaces is well-known, the construction for CAT(0) spaces hinges on establishing a new structural result of independent interest about geodesically complete, cocompact rank one CAT(0) spaces: Almost no geodesic, under the Bowen-Margulis measure, bounds a flat strip of any positive width. We also show that almost every point in the boundary of X, under the Patterson-Sullivan measure, is isolated in the Tits metric.

Finally, we identify precisely which geodesically complete, cocompact rank one CAT(0) spaces are mixing. That is, we prove that the Bowen-Margulis measure is mixing under the geodesic flow unless X is a tree with all edge lengths in some discrete subgroup of the reals. This characterization is new, even in the setting of CAT(-1) spaces.

CHAPTER 1

Introduction

CAT(0) spaces are a generalization of nonpositive curvature from Riemannian manifolds to general metric spaces, defined by comparing geodesic triangles with triangles in Euclidean space (see Section 4). Examples of CAT(0) spaces include nonpositively curved Riemannian manifolds, Euclidean buildings, and trees. They share many properties with nonpositively curved Riemannian manifolds. One key difference is that geodesics are not globally determined from a small segment. Much is known about the geometry of CAT(0) spaces (see, e.g., [4] or [9]); however, the ergodic theory of these spaces is less understood, largely due to the lack of natural invariant measures. This thesis presents some results in this direction. One of the main results of this thesis is to construct a generalized Bowen-Margulis measure and precisely characterize mixing of the geodesic flow for this measure in terms of the geometry of the space (Theorem 4). This construction is not an immediate generalization from the manifold setting, but involves establishing two structural results of independent interest for CAT(0) spaces admitting a rank one axis: First, almost every point in ∂X, under the Patterson-Sullivan measure, is isolated in the Tits metric (Theorem 1). Second, almost no geodesic, under the Bowen-Margulis measure, bounds a flat strip of any positive width (Theorem 2).

Although the construction of a Bowen-Margulis measure is now standard in many nonpositively curved settings, its construction in the context of CAT(0) spaces is not an immediate generalization of previous techniques. The main obstacle is the presence of flat strips. In negative curvature (both Riemannian manifolds and CAT(-1) spaces), these strips do not exist. In nonpositively curved Riemannian manifolds, their complement in the unit tangent bundle is a dense open set; furthermore, they are Riemannian submanifolds, which have their own volume form. However, in rank one CAT(0) spaces, a priori it might happen that every geodesic bounds a flat strip; moreover, the strips themselves do not carry a natural Borel measure. Our solution is to construct the Bowen-Margulis measure in two stages, and to prove the necessary structural results between stages.

There is a well-established equivalence between mixing of the Bowen-Margulis measure and arithmeticity of the length spectrum for CAT(-1) spaces (see [15] and [41]). However, the only known (geodesically complete) examples with arithmetic length spectrum are trees. Roblin ([41]) raised the question of what CAT(-1) spaces other than trees could be non-mixing under a proper, non-elementary action. For compact, rank one nonpositively curved Riemannian manifolds, Babillot ([3]) showed that the Bowen-Margulis measure is always mixing. Yet it is an open question whether the Bowen-Margulis measure is always mixing for non-compact negatively curved manifolds with fundamental group that is not virtually cyclic (see [41]). Theorem 4 shows that when the action is cocompact, trees are in fact the only non-mixing (geodesically complete) examples—even in the CAT(0) setting.

We now describe these results and their context in more detail.

1.1 Mixing

Let X be a proper, geodesically complete CAT(0) space and Γ be a group acting properly discontinuously, cocompactly, and by isometries on X. Assume X has a rank one axis—that is, there is a geodesic in X which is translated by some isometry in Γ but does not bound a subspace in X isometric to $\mathbb{R} \times [0, \infty)$. In this thesis, we construct a finite Borel measure, called the Bowen-Margulis measure, on the space of unit-speed parametrized geodesics of X modulo the Γ-action. This measure has full support and is invariant under the geodesic flow. We show (Theorem 4) that the Bowen-Margulis measure is mixing (sometimes called strong mixing), except when X is a tree with all edge lengths in $c\mathbb{Z}$ for some $c > 0$.

Mixing is an important dynamical property strictly stronger than ergodicity. It has been used in a number of circumstances to extract geometric information about a space from the dynamics of the geodesic flow. For example, in his 1970 thesis (see [34]), Margulis used mixing of the geodesic flow on a compact Riemannian manifold of strictly negative curvature to calculate the precise asymptotic growth rate of the number of closed geodesics in such a manifold. Others have used similar techniques for other counting problems in geometry (see, e.g., [20] and [36]). More recently, Kahn and Markovic ([25]) used exponential mixing (i.e., precise estimates on the rate of mixing) to prove Waldhausen's surface subgroup conjecture for 3-manifolds.

Roblin ([41]) showed that for proper CAT(-1) spaces whose Bowen-Margulis measure is not mixing, the set of all translation lengths of hyperbolic isometries (i.e., those isometries that act by translation along some geodesic) in Γ must lie in a discrete subgroup $c\mathbb{Z}$ of $\mathbb{R}$. In this case one says the length spectrum is arithmetic. We remark that Roblin's theorem holds even when the Γ-action is not cocompact but non-elementary (that is, Γ is not virtually cyclic).

If the length spectrum of a proper CAT(-1) space is arithmetic, Roblin concludes that the limit set is totally disconnected. The converse fails, as is easily seen from a tree with edge lengths which are not rationally related. Moreover, ∂X totally disconnected does not imply X is a tree. Ontaneda (see the proof of Proposition 1 in [37]) described proper, geodesically complete CAT(0) spaces that admit a proper, cocompact, isometric action of a free group—hence they are quasi-isometric to trees and, in particular, have totally disconnected boundary—yet are not isometric to trees. Ontaneda's examples are Euclidean 2-complexes, but one can easily adapt the construction to hyperbolic 2-complexes instead. Thus there are proper, cocompact, geodesically complete CAT(-1) spaces with totally disconnected boundary that are not isometric to trees. It is not at all clear, a priori, that one cannot construct such an example where the length spectrum is arithmetic. But our characterization of mixing shows that such non-tree examples cannot be constructed to have arithmetic length spectrum.

Our characterization of mixing also applies when Γ acts non-cocompactly on a proper CAT(-1) space X such that the Bowen-Margulis measure is finite and the Patterson-Sullivan measures have full support on the boundary (see the Remark following Theorem 12.7). In the general case of a non-compact action on a proper CAT(-1) space, however, the problem of characterizing when the length spectrum is arithmetic remains open ([41]). Indeed, even when $\Gamma\backslash X$ is a noncompact Riemannian manifold with sectional curvature ≤ -1 everywhere, it is an open question (see [15]) whether the length spectrum can be arithmetic without Γ being virtally cyclic. The length spectrum is known to be non-arithmetic in a few cases, however (see [15] or [41])—e.g. if Γ contains parabolic elements. For rank one symmetric spaces, the length spectrum was shown to be non-arithmetic by Kim ([29]).

1.2 Previous Constructions of Bowen-Margulis Measures

The Bowen-Margulis measure was first introduced for compact Riemannian manifolds of negative sectional curvature, where Margulis ([34]) and Bowen ([7]) used different methods to construct measures of maximal entropy for the geodesic flow. Bowen ([8]) also proved that the measure of maximal entropy is unique, hence both measures are the same—often called the Bowen-Margulis measure. Sullivan ([42] and [43]) established a third method to obtain this measure, in the case of constant negative curvature. Kaimanovich ([26]) proved that Sullivan's construction extends to all smooth Riemannian manifolds of negative sectional curvature.

Sullivan's method is as follows. First, one uses Patterson's construction ([39]) to obtain a family of finite Borel measures, called Patterson-Sullivan measures, on the boundary of X. Although these measures are not invariant under the action of Γ, they transform in a computable way (see Definition 3.5). Next, one constructs a Γ-invariant Borel measure on the endpoint pairs of geodesics in X. Using this measure, one then constructs a Borel measure on the space SX of unit-speed parametrized geodesics of X (for a Riemannian manifold, SX can be naturally identified with the unit tangent bundle of X). Finally, one shows that there is a well-defined finite Borel quotient measure on $\Gamma\backslash SX$.

Other geometers have used Sullivan's general method to extend the construction of Bowen-Margulis measures to related classes of spaces. Our construction in the class of CAT(0) spaces is in the same vein, and is especially inspired by work in two prior classes of spaces: Knieper's ([30]), where $\Gamma\backslash X$ is a compact Riemannian manifold of nonpositive sectional curvature, and Roblin's ([41]), where X is a proper CAT(-1) space but the Γ-action is not necessarily cocompact.

1.3 Role of the Rank One Axis

Let μ_x be a Patterson-Sullivan measure on the boundary ∂X of X, and let $\mathcal{G}^E \subset \partial X \times \partial X$ be the set of endpoint pairs of geodesics in X. In order to construct Bowen-Margulis measures by Sullivan's method, we must construct a Γ-invariant Borel measure on $\mathcal{G}^E$, and this requires $(\mu_x \times \mu_x)(\mathcal{G}^E) > 0$. If X admits a rank one axis then this condition holds; furthermore, μ_x has full support. On the other hand, if X does not admit a rank one axis, it is unclear whether $(\mu_x \times \mu_x)(\mathcal{G}^E) > 0$.

The existence of a rank one axis in a CAT(0) space forces the group action to exhibit rather strong north-south dynamics (for a precise statement, see Lemma 4.9). This behavior may well be generic for CAT(0) spaces. Indeed, Ballmann and Buyalo ([5]) conjecture that every geodesically complete CAT(0) space under a proper, cocompact, isometric group action that does not admit a rank one axis must either split nontrivially as a product, or be a higher rank symmetric space or Euclidean building. Moreover, this conjecture has been proven (and is called the Rank Rigidity Theorem) in a few important cases, notably for Hadamard manifolds by Ballmann, Brin, Burns, Eberlein, and Spatzier (see [4] and [11]) and for CAT(0) cube complexes by Caprace and Sageev ([12]).

1.4 Flat Strips

Using the Γ-invariant Borel measure μ we construct on $\mathcal{G}^E$, the next step is to produce a flow-invariant Borel measure (the Bowen-Margulis measure) on the generalized unit tangent bundle SX of X, called the space of geodesics of X by Ballmann ([4]).

Our construction of Patterson-Sullivan measures on the boundary follows Patterson closely. Constructing Bowen-Margulis measures, however, is much less straightforward. Knieper ([30]) does it for compact Riemannian manifolds of nonpositive sectional curvature, where "most" geodesics do not bound a flat strip. Likewise, Bourdon ([6]) accomplishes it for CAT(-1) spaces, where no geodesic bounds a flat strip. The novelty in our construction is precisely that of dealing with the possible existence of flat strips for many geodesics in rank one CAT(0) spaces.

More precisely, we first construct a Borel measure m of full support on $\mathcal{G}^E \times \mathbb{R}$ and prove (Proposition 8.3) that it descends to a finite Borel measure m_Γ on the quotient $\Gamma \backslash (\mathcal{G}^E \times \mathbb{R})$. This allows us to prove the following structural result.

Theorem 1 (Theorem 9.1)**.** *Let X be a proper, geodesically complete* CAT(0) *space and* Γ *be a group acting properly discontinuously, cocompactly, and by isometries on X; further assume X admits a rank one axis. Then μ_x-a.e. $\xi \in \partial X$ is isolated in the Tits metric.*

As a corollary, the equivalence classes of higher rank geodesics have zero measure under m. In fact, m-a.e. geodesic bounds no flat strip of any positive width. More precisely, we have the following.

Theorem 2 (Theorem 9.9)**.** *Let X and* Γ *satisfy the assumptions of Theorem 1. The set $\mathcal{Z}^E \subseteq \mathcal{G}^E$ of endpoint pairs of zero-width geodesics has full μ-measure. Thus m-almost no equivalence class of geodesics contains a flat strip of positive width.*

This result brings us back to the situation where "most" geodesics do not bound a flat strip, which allows us to finally define Bowen-Margulis measures (also denoted m and m_Γ) on SX and $\Gamma \backslash SX$.

1.5 Dynamical Results

The classical argument by Hopf ([24]) is readily adapted to prove ergodicity of the geodesic flow.

Theorem 3 (Theorem 9.16)**.** *Let X and* Γ *satisfy the assumptions of Theorem 1. The Bowen-Margulis measure m_Γ is ergodic under the geodesic flow on* $\Gamma \backslash SX$.

Ergodicity, although weaker than mixing, is still a very useful and important property of a dynamical system. In fact, one of Sullivan's motivations to study Patterson-Sullivan measures was to characterize ergodicity of the geodesic flow on hyperbolic manifolds.

Mixing is trickier to prove. When $\Gamma\backslash X$ is a compact Riemannian manifold, Babillot ([3]) showed that m_Γ is mixing on $\Gamma\backslash SX$ (the measure m_Γ having been previously constructed by Knieper). However, it is easy to see that if X is a tree with only integer edge lengths, then m_Γ is not mixing under the geodesic flow; thus one cannot hope to show that m_Γ is mixing for every CAT(0) space. Nevertheless, we prove that every proper, cocompact, geodesically complete CAT(0) space X, where m_Γ is not mixing, is isometric to such a tree, up to uniformly rescaling the metric of X.

Our proof starts by relating mixing to cross-ratios (see Definition 11.3 for the definition of cross-ratios; they are defined on the space of quadrilaterals $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}} \subset (\partial X)^4$, which is defined in Definition 11.1). This part follows Babillot's work ([3]) for Riemanninan manifolds. But CAT(0) spaces allow geodesics to branch, which makes the cross-ratios more subtle; this can be seen in the difference among trees, where some are not mixing. Consequently, in the second part of the proof, we shift focus from the asymptotic behavior of ∂X to the local behavior of the links of points. Additionally, we relate mixing to the length spectrum. This gives us the following characterization:

Theorem 4 (Theorem 12.7)**.** *Let X and Γ satisfy the assumptions of Theorem 1. The following are equivalent:*

1. *The Bowen-Margulis measure m_Γ is not mixing under the geodesic flow on $\Gamma\backslash SX$.*
2. *The length spectrum is arithmetic—that is, the set of all translation lengths of hyperbolic isometries in Γ must lie in some discrete subgroup $c\mathbb{Z}$ of $\mathbb{R}$.*
3. *There is some $c \in \mathbb{R}$ such that every cross-ratio of $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ lies in $c\mathbb{Z}$.*
4. *There is some $c > 0$ such that X is isometric to a tree with all edge lengths in $c\mathbb{Z}$.*

Note that if m_Γ is not mixing, it also fails to be weak mixing because $\Gamma\backslash SX$ factors continuously over a circle for the trees in Theorem 4.

CHAPTER 2

Some Ergodic Theory Review

We begin by reviewing some ergodic theory.

A *measurable space* is a set X, equipped with a σ-algebra $\mathfrak{M}$ of subsets of X. A set $A \subseteq X$ is said to be *measurable* if $A \in \mathfrak{M}$. For measurable spaces X and Y, a map $\psi\colon X \to Y$ is *measurable* if the preimage of every measurable set is measurable.

A *topogolical group* is a group G, endowed with a Hausdorff topology under which the multiplication and inverse maps are continuous. We are mostly interested in the cases G countable and $G = \mathbb{R}$. Note that the Borel σ-algebra of any topological group gives it a natural measurable structure.

Let G be a topological group acting on a topological space X. The action is *continuous* if the action map $G \times X \to X$, given by $(g, x) \mapsto gx$, is continuous (where $G \times X$ is given the product topology). Note that if G is discrete, this is equivalent to saying the map $x \mapsto gx$ is a homeomorphism for every $g \in G$.

Now suppose G acts on a measurable space X. The action is *measurable* if the action map $G \times X \to X$ is measurable (where $G \times X$ is given the product σ-algebra). Note that if G is discrete, this is equivalent to saying the map $x \mapsto gx$ is measurable for every $g \in G$.

Throughout this chapter, G will be a locally compact, second countable, topological group, acting measurably on a measurable space X. (The relevant σ-algebra on G is its Borel σ-algebra.)

Let μ be a measure on the measurable space X. A set of zero measure is called a *null set*, and the complement of a null set is called *conull*. The *measure class* of μ is the equivalence class of measures with the same null sets as μ.

For a measure μ on X, its *pushforward* by a measurable map $\psi\colon X \to Y$ is the measure $\psi_*\mu$ defined by $(\psi_*\mu)(A) = \mu(\psi^{-1}(A))$ for all measurable $A \subseteq X$. If the group G acts measurably on X and $g_*\mu = \mu$ for all $g \in G$, we say μ is *invariant*, or G *preserves* μ. If G preserves only the measure class of μ, we say μ is *quasi-invariant*.

An invariant measure μ is *ergodic* if every G-invariant measurable set is either null or co-null. If μ is only quasi-invariant, but every G-invariant measurable set is either null or co-null, we call μ *quasi-ergodic*. Ergodicity has various equivalent formulations; the following standard proposition states a few.

Proposition 2.1. *Let μ be a finite invariant measure on X. The following are equivalent:*

1. *μ is ergodic.*

2. *Every invariant measurable function is constant almost everywhere.*

3. *Every invariant L^2 function is constant almost everywhere.*

Moreover, if μ is invariant but not finite, we still have 1 <=> 2.

Proof. The implications (2) $\implies$ (3) and (2) $\implies$ (1) are trivial, and if μ is finite we have (3) $\implies$ (1) because $\chi_A \in L^2(\mu)$ for all measurable $A \subseteq X$. For (1) $\implies$ (2), let $f\colon X \to [-\infty,\infty]$ be measurable, and let R be the supremum of $r \in [-\infty,\infty]$ such that $f^{-1}([-\infty,r])$ is a null set. Then $f^{-1}([-\infty,r])$ is conull for all $r > R$ by 1, so both $f^{-1}([-\infty,r))$ and $f^{-1}((r,\infty])$ are null sets by separability of $[-\infty,\infty]$. This proves (1) $\implies$ (2). $\square$

One can also replace the invariant functions in Proposition 2.1 (including χ_A for invariant sets $A \subseteq X$) by *essentially G-invariant* functions—functions f such that $f(x) = f(gx)$ a.e. for all $g \in G$ (see [45, Lemma 2.2.16]).

A finite invariant measure μ is *mixing* if, for every sequence $g_n \to \infty$ in G and every pair of measurable sets $A, B \subseteq X$, we have $\mu(g_n A \cap B) \to \frac{1}{\mu(X)}\mu(A)\mu(B)$.

Proposition 2.2. *Let μ be an finite invariant measure on X. The following are equivalent:*

1. *μ is mixing.*

2. *$\int(\varphi \circ g_n)\cdot\psi\, d\mu \to \frac{1}{\mu(X)}\int \varphi\, d\mu \cdot \int \psi\, d\mu$ for all $\varphi, \psi \in L^2(\mu)$.*

3. *$\varphi \circ g_n \to 0$ weakly in $L^2(\mu)$ for all $\varphi \in L^2(\mu)$ with $\int \varphi\, d\mu = 0$.*

We now restrict our attention to $\mathbb{R}$-actions, also called *flows*, to state the famous ergodic theorems (see [4, Appendix, Theorem 2.3]).

Definition 2.3 (Ergodic averages)**.** For any measurable function $f\colon X \to \mathbb{R}$, define $A_T f$ by $(A_T f)(x) = \frac{1}{T}\int_0^T f(\phi^t x)dt$.

Theorem 2.4 (Birkhoff's Pointwise Ergodic Theorem)**.** *Let ϕ^t be a measurable flow on X with finite invariant measure μ. For all $f \in L^1$, $A_T(f)$ and $A_{-T}(f)$ converge a.e. to ϕ^t-invariant L^1 functions as $T \to \infty$; moreover, the limits are equal a.e.*

Theorem 2.5 (von Neumann's Mean Ergodic Theorem)**.** *Let ϕ^t be a measurable flow on X with finite invariant measure μ. For all $f \in L^2$, both $A_T(f)$ and $A_{-T}(f)$ converge in L^2 to $\pi_H(f)$, the L^2-projection of f onto the closed subspace $H \subseteq L^2(\mu)$ of ϕ^t-invariant L^2 functions, as $T \to \infty$.*

CHAPTER 3

Patterson's Construction

First we construct Patterson-Sullivan measures on the boundary of fairly general spaces. Their construction is standard (cf. e.g. [39], [42], [31], and [41]), and they have been studied in a variety of contexts; we mention only a few. Patterson ([39]) used them to calculate the Hausdorff dimension of the limit set of a Fuchsian group. Albuquerque ([2]) and Quint ([40]) studied Patterson-Sullivan measures for the boundary of higher rank (nonpositively curved) symmetric spaces. Ledrappier and Wang ([32]) used Patterson-Sullivan measures on the Busemann boundary of compact Riemannian manifolds—without curvature assumptions—to prove a rigidity theorem for the volume growth entropy. Prior to Ledrappier and Wang, Patterson's construction was done on the visual boundary by using the equivalence of the visual and Busemann boundaries in nonpositive curvature. We extend Ledrappier and Wang's approach to any proper metric space.

Standing Hypothesis. In this chapter, let X be a proper metric space, that is, a metric space in which all closed metric balls are compact. Let Γ be an infinite group of isometries acting properly discontinuously on X—that is, for every compact set $K \subseteq X$, there are only finitely many $\gamma \in \Gamma$ such that $K \cap \gamma K$ is nonempty.

Remark. Since X is proper, requiring the Γ-action to be properly discontinuous is equivalent to requiring that the Γ-action be proper—that is, every $x \in X$ has a neighborhood $U \subseteq X$ such that $U \cap \gamma U$ is nonempty for only finitely many $\gamma \in \Gamma$ (see Remark I.8.3(1) of [9]).

For $p, q \in X$, $s \in \mathbb{R}$, the Dirichlet series

$$P(s,p,q) = \sum_{\gamma \in \Gamma} e^{-sd(p,\gamma q)}$$

is called the Poincaré series associated to Γ.

Fix $p, q \in X$. Let $V_t = \{\gamma \in \Gamma \mid d(p,\gamma q) \leq t\}$ and $\delta_\Gamma = \limsup_{t\to\infty} \frac{1}{t} \log|V_t|$. One thinks of $|V_t|$ as measuring the volume of a ball in X centered at p of radius t, and of δ_Γ as the volume growth entropy of $\Gamma\backslash X$ under this measure. If $\Gamma\backslash X$ is a compact smooth Riemannian manifold, δ_Γ is in fact the volume growth entropy (see [30], for instance).

Lemma 3.1. *The Poincaré series $P(s,p,q)$ converges for $s>\delta_\Gamma$ and diverges for $s<\delta_\Gamma$. That is,*

$$\delta_\Gamma = \inf\{s \geq 0 \mid P(s,p,q) < \infty\}.$$

Furthermore, δ_Γ does not depend on choice of p or q.

Proof. If $s > s' > \delta_\Gamma$ then there is some $N > 0$ such that $e^{s't} > |V_t|$ for $t \geq N$. Let $A_k = \{\gamma \in \Gamma \mid k-1 < d(p,\gamma q) \leq k\}$. Then

$$\begin{aligned} P(s,p,q) &= \sum_{\gamma\in\Gamma} e^{-sd(p,\gamma q)} = \sum_{k\in\mathbb{Z}}\sum_{\gamma\in A_k} e^{-sd(p,\gamma q)} \\ &\leq \sum_{k\in\mathbb{Z}}\sum_{\gamma\in A_k} e^{-s(k-1)} = \sum_{k\in\mathbb{Z}} |A_k| e^{-s(k-1)} \\ &= C + \sum_{k>N} |A_k| e^{-s(k-1)}, \end{aligned}$$

for some $C \geq 0$. But $|A_k| \leq |V_k| < e^{s'k}$, so

$$\sum_{k>N} |A_k| e^{-s(k-1)} \leq \sum_{k>N} e^{s'k-sk+s} \leq e^s \sum_{k>N} e^{(s'-s)k} < \infty,$$

and therefore $P(s,p,q) < \infty$.

On the other hand, if $s < s' < \delta_\Gamma$ then there is a sequence $t_n \to \infty$ such that $e^{s't_n} < \left|V_{t_n}\right|$. Hence $\left|V_{t_n}\right| e^{-st_n} > e^{(s'-s)t_n}$, and thus

$$\begin{aligned} P(s,p,q) &= \lim_{t\to\infty} \sum_{\gamma\in V_t} e^{-sd(p,\gamma q)} \geq \lim_{t\to\infty} \sum_{\gamma\in V_t} e^{-st} \\ &= \lim_{t\to\infty} |V_t| e^{-st} = \lim_{t_n\to\infty} \left|V_{t_n}\right| e^{-st_n} \\ &\geq \lim_{t_n\to\infty} e^{(s'-s)t_n} = \infty. \end{aligned}$$

Finally, let $p,q,p',q' \in X$, and let $R = d(p,p') + d(q,q')$. Then

$$\begin{aligned} P(s,p',q') &= \sum_{\gamma\in\Gamma} e^{-sd(p',\gamma q')} \geq \sum_{\gamma\in\Gamma} e^{-s[d(p',p)+d(p,\gamma q)+d(\gamma q,\gamma q')]} \\ &= \sum_{\gamma\in\Gamma} e^{-s[d(p',p)+R]} = e^{-sR} \sum_{\gamma\in\Gamma} e^{-sd(p,\gamma q)} = e^{-sR} P(s,p,q) \end{aligned}$$

and, by symmetric argument, $P(s,p,q) \geq e^{-sR}P(s,p',q')$. Thus

$$e^{-sR}P(s,p,q) \leq P(s,p',q') \leq e^{sR}P(s,p,q),$$

hence $P(s,p',q') < \infty$ if and only if $P(s,p,q) < \infty$. Since $\delta_\Gamma = \inf\{s \geq 0 \mid P(s,p,q) < \infty\}$, we see that δ_Γ is does not depend on p or q. $\square$

We will work only in the case that δ_Γ is finite. This assumption is quite mild, considering the following observation, the proof of which is standard. (Recall that the action is said to be *cocompact* if there is some compact set $K \subseteq X$ such that $\Gamma K = X$.)

Lemma 3.2. *If Γ is finitely generated, then δ_Γ is finite. In particular, if X is connected, and Γ acts cocompactly on X, then δ_Γ is finite.*

Proof. Since Γ is quasi-isometric to its Cayley graph, if Γ is finitely generated, then it has at most exponential volume growth. Furthermore, if X is connected and Γ acts cocompactly on X, then Γ is finitely generated (see [9, I.8.10]). $\square$

Definition 3.3. Let X be a proper metric space. Write $C(X)$ for the space of continuous maps $X \to \mathbb{R}$, equipped with the compact-open topology (which is the topology of uniform convergence on compact subsets). Fix $p \in X$, and let $\iota_p\colon X \to C(X)$ be the embedding given by $x \mapsto [d(\cdot,x) - d(p,x)]$. The *Busemann compactification of* X, denoted $\bar{X}$, is the closure of the image of ι_p in $C(X)$.

If $\xi \in \bar{X}$, then technically ξ is a function $\xi\colon X \to \mathbb{R}$. However, one usually prefers to think of ξ as a point in X (if ξ lies in the image of ι_p) or in the *Busemann boundary*, $\partial X = \bar{X} \smallsetminus X$, of X. Instead of working with the function $\xi\colon X \to \mathbb{R}$, we will work with the Busemann function $b_\xi\colon X \times X \to \mathbb{R}$ given by $b_\xi(x,y) = \xi(x) - \xi(y)$. Note that b_ξ (unlike $\xi\colon X \to \mathbb{R}$) does not depend on choice of $p \in X$.

The Busemann functions b_ξ are 1-Lipschitz in both variables and satisfy the *cocycle property* $b_\xi(x,y) + b_\xi(y,z) = b_\xi(x,z)$. Furthermore, $b_{\gamma\xi}(\gamma x, \gamma y) = b_\xi(x,y)$ for all $\gamma \in \operatorname{Isom} X$.

Lemma 3.4. *Let X be a proper metric space. The space of* 1*-Lipschitz functions $X \to \mathbb{R}$ which take value* 0 *at a fixed point $p \in X$ is compact and metrizable under the compact-open topology. In particular, the Busemann boundary ∂X of X is compact and metrizable.*

Proof. An explicit metric is given by $d(f,g) = \sup_{x \in X} e^{-d(p,x)}|f(x) - g(x)|$. Compactness follows by Ascoli's Theorem (Theorem 47.1 in [35]). $\square$

For a measure μ on X and a measurable map $\gamma\colon X \to X$, we write $\gamma_*\mu$ for the pushforward measure given by $(\gamma_*\mu)(A) = \mu(\gamma^{-1}(A))$ for all measurable $A \subseteq X$.

Definition 3.5. A family $\{\mu_p\}_{p\in X}$ of finite nontrivial Borel measures on ∂X is called a *conformal density* if

1. $\gamma_*\mu_p = \mu_{\gamma p}$ for all $\gamma \in \Gamma$ and $p \in X$, and
2. for all $p, q \in X$, the measures μ_p and μ_q are equivalent with Radon-Nikodym derivative
$$\frac{d\mu_q}{d\mu_p}(\xi) = e^{-\delta_\Gamma b_\xi(q,p)}.$$

Condition (1) is equivalent to requiring that $\mu_p(f\circ\gamma) = \mu_{\gamma p}(f)$ for all $f \in C(\overline{X})$, since the Tietze Extension Theorem allows us to extend a continuous function $f \in C(\partial X)$ to $\overline{f} \in C(\overline{X})$.

Remark. A conformal density, as defined above, is often called a *conformal density of dimension* δ_Γ in the literature.

The *limit set* $\Lambda(\Gamma)$ of Γ is defined to be the subset of ∂X given by

$$\Lambda(\Gamma) = \{\xi \in \partial X \mid \gamma_i x \to \xi \text{ for some } (\gamma_i) \subset \Gamma \text{ and } x \in X\}.$$

The *support* of a Borel measure ν on a topological space Z is the set

$$\mathrm{supp}(\nu) = \{z \in Z \mid \nu(U) > 0 \text{ for every neighborhood } U \text{ of } z \in Z\},$$

and a Borel measure ν is said to have *full support* if $\mathrm{supp}(\nu) = Z$ (i.e., every nonempty open set has positive ν-measure). If $\{\mu_p\}_{p\in X}$ is a conformal density, the support of μ_p does not depend on $p \in X$, so the support of the conformal density is well-defined as $\mathrm{supp}(\mu_p)$ for any $p \in X$.

The proof of Theorem 3.6 uses the idea of weak convergence of measures on $\overline{X}$. This convergence is the same as weak-* convergence in $C(\overline{X})^*$; that is, a sequence of probability measures (ν_n) on $\overline{X}$ converges weakly to a probability measure ν on $\overline{X}$ if and only if $\int f\, d\nu_n \to \int f\, d\nu$ for all continuous functions $f : \overline{X} \to \mathbb{R}$.

Theorem 3.6. *Let* Γ *be an infinite group of isometries acting properly discontinuously on a proper metric space X, and suppose* $\delta_\Gamma < \infty$. *Then the Busemann boundary of X admits a conformal density with support in* $\Lambda(\Gamma)$.

Proof. Fix $x \in X$. First suppose that $P(\delta_\Gamma, x, x)$ diverges. (In this case, one says that Γ is of *divergence type*.) For $s > \delta_\Gamma$, define the Borel probability measure $\mu_{x,s}$ on X by

$$\mu_{x,s} = \frac{1}{P(s,x,x)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} \delta_{\gamma x},$$

where $\delta_{\gamma x}$ is the Dirac measure based at $\gamma x \in X$. By the Banach-Alaoglu Theorem and the Riesz Representation Theorem, there is a sequence $s_k \searrow \delta_\Gamma$ such that μ_{x,s_k} converges weakly to some Borel probability measure μ_x on $\overline{X}$. Note that $\mathrm{supp}(\mu_x) \subseteq \partial X$, since $\mu_{x,s_k}(f) \to 0$ for all compactly supported $f \in C(X)$; thus it is clear that $\mathrm{supp}(\mu_x) \subseteq \Lambda(\Gamma) = \partial X \cap \overline{\Gamma x}$.

For $p \in X$, define μ_p by setting

$$\mu_p(f) = \int_{\overline{X}} f(\xi) e^{-\delta_\Gamma b_\xi(p,x)} d\mu_x(\xi) \tag{$*$}$$

for all $f \in C(\overline{X})$. We want to show that $\{\mu_p\}_{p \in X}$ is a conformal density. Condition (2) is immediate from $(*)$ and the cocycle property of Busemann functions, so it remains to show that $\gamma_* \mu_p = \mu_{\gamma p}$ for all $\gamma \in \Gamma$ and $p \in X$. But, unraveling the definitions, we have

$$\begin{aligned}
\mu_p(f) &= \int_{\overline{X}} f(\xi) e^{-\delta_\Gamma b_\xi(p,x)} d\mu_x(\xi) \\
&= \lim_{k\to\infty} \int_{\overline{X}} f(\xi) e^{-s_k b_\xi(p,x)} d\mu_{x,s_k}(\xi) \\
&= \lim_{k\to\infty} \int_{X} f(y) e^{-s_k [d(p,y) - d(x,y)]} d\mu_{x,s_k}(y) \\
&= \lim_{k\to\infty} \frac{1}{P(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\gamma x) e^{-s_k[d(p,\gamma x) - d(x,\gamma x)]} e^{-s_k d(x,\gamma x)} \\
&= \lim_{k\to\infty} \frac{1}{P(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\gamma x) e^{-s_k d(p,\gamma x)}.
\end{aligned}$$

Hence for $\alpha \in \Gamma$,

$$\begin{aligned}
\mu_{\alpha p}(f) &= \lim_{k\to\infty} \frac{1}{P(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\gamma x) e^{-s_k d(\alpha p,\gamma x)} \\
&= \lim_{k\to\infty} \frac{1}{P(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\alpha\gamma x) e^{-s_k d(\alpha p,\alpha\gamma x)} \\
&= \lim_{k\to\infty} \frac{1}{P(s_k,x,x)} \sum_{\gamma\in\Gamma} (f\circ\alpha)(\gamma x) e^{-s_k d(p,\gamma x)} = \mu_p(f\circ\alpha),
\end{aligned}$$

which concludes the proof when Γ is of divergence type.

Now suppose that $P(\delta_\Gamma, x, x)$ converges. There is (see e.g. [39, Lemma 3.1]) a continuous, nondecreasing function $h\colon \mathbb{R} \to (0,\infty)$ such that $\frac{h(t+a)}{h(t)} \to 1$ as $t \to \infty$ for all $a \in \mathbb{R}$, and such that the modified Poincaré series

$$\widetilde{P}(s,p,q) = \sum_{\gamma\in\Gamma} e^{-sd(p,\gamma q)} h(d(p,\gamma q))$$

diverges for $\widetilde{P}(\delta_\Gamma, x, x)$. For $s > \delta_\Gamma$, define the Borel probability measure $\mu_{x,s}$ on X by

$$\mu_{x,s} = \frac{1}{\widetilde{P}(s,x,x)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} h(d(x,\gamma x))\delta_{\gamma x},$$

where $\delta_{\gamma x}$ is the Dirac measure based at $\gamma x \in X$. Again there is a sequence $s_k \searrow \delta_\Gamma$ such that μ_{x,s_k} converges weakly to some Borel probability measure μ_x on $\overline{X}$, and it is clear that $\mathrm{supp}(\mu_x) \subseteq \Lambda(\Gamma)$.

As in the case when Γ is of divergence type, define μ_p by $(*)$. Unraveling the definitions, we now have

$$\begin{aligned}
\mu_p(f) &= \int_{\overline{X}} f(\xi) e^{-\delta_\Gamma b_\xi(p,x)} d\mu_x(\xi) \\
&= \lim_{k\to\infty} \int_{\overline{X}} f(\xi) e^{-s_k b_\xi(p,x)} d\mu_{x,s_k}(\xi) \\
&= \lim_{k\to\infty} \int_{X} f(y) e^{-s_k [d(p,y)-d(x,y)]} d\mu_{x,s_k}(y) \\
&= \lim_{k\to\infty} \frac{1}{\widetilde{P}(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\gamma x) e^{-s_k[d(p,\gamma x)-d(x,\gamma x)]} e^{-s_k d(x,\gamma x)} h(d(x,\gamma x)) \\
&= \lim_{k\to\infty} \frac{1}{\widetilde{P}(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\gamma x) e^{-s_k d(p,\gamma x)} h(d(x,\gamma x)).
\end{aligned}$$

Hence for $\alpha \in \Gamma$,

$$\begin{aligned}
\mu_{\alpha p}(f) &= \lim_{k\to\infty} \frac{1}{\widetilde{P}(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\gamma x) e^{-s_k d(\alpha p,\gamma x)} h(d(x,\gamma x)) \\
&= \lim_{k\to\infty} \frac{1}{\widetilde{P}(s_k,x,x)} \sum_{\gamma\in\Gamma} f(\alpha\gamma x) e^{-s_k d(\alpha p,\alpha\gamma x)} h(d(x,\alpha\gamma x)) \\
&= \lim_{k\to\infty} \frac{1}{\widetilde{P}(s_k,x,x)} \sum_{\gamma\in\Gamma} (f\circ\alpha)(\gamma x) e^{-s_k d(p,\gamma x)} h(d(\alpha^{-1}x,\gamma x)) \\
&= \lim_{k\to\infty} \frac{1}{\widetilde{P}(s_k,x,x)} \sum_{\gamma\in\Gamma} (f\circ\alpha)(\gamma x) e^{-s_k d(p,\gamma x)} h(d(x,\gamma x)) \frac{h(d(\alpha^{-1}x,\gamma x))}{h(d(x,\gamma x))}.
\end{aligned}$$

But $\left|d(\alpha^{-1}x,\gamma x) - d(x,\gamma x)\right| \le d(x,\alpha^{-1}x)$, and $\frac{h(t+a)}{h(t)} \to 1$ as $t \to \infty$ for all $a \in \mathbb{R}$ by choice of h. Furthermore, since h is nondecreasing, this convergence occurs uniformly in a for $|a| \le d(x,\alpha^{-1}x)$. Thus for every $\epsilon > 0$, we have $1-\epsilon \le \frac{h(d(\alpha^{-1}x,\gamma x))}{h(d(x,\gamma x))} \le 1+\epsilon$ for all but finitely many $\gamma \in \Gamma$. Hence

$$(1-\epsilon)\mu_p(f\circ\alpha) \le \mu_{\alpha p}(f) \le (1+\epsilon)\mu_p(f\circ\alpha)$$

for all $\epsilon > 0$. This concludes the proof. $\square$

A conformal density constructed as in the proof of Theorem 3.6 is called a *Patterson-Sullivan measure* on ∂X. We do not know that such a conformal density is independent of the many choices we made. However, μ_x is a probability measure by construction.

Convention. Throughout this thesis, μ_x will always refer to a measure from a conformal density $\{\mu_p\}_{p \in X}$ on ∂X.

It would be useful to know that $\operatorname{supp}(\mu_p) = \partial X$ (for some, equivalently every, $p \in X$). If X is a proper rank one CAT(0) space and Γ acts cocompactly, this turns out (Proposition 8.5) to be equivalent to the existence of a rank one axis in X.

CHAPTER 4

Rank of Geodesics in CAT(0) Spaces

A CAT(0) space X is a uniquely geodesic metric space of nonpositive curvature. More precisely, the distance between a pair of points on a geodesic triangle $\triangle$ in X is less than or equal to the distance between the corresponding pair of points on a Euclidean *comparison triangle*—a triangle $\overline{\triangle}$ in the Euclidean plane with the same edge lengths as $\triangle$. The class of CAT(0) spaces generalizes the class of Riemannian manifolds with nonpositive sectional curvature everywhere; it also includes trees, Euclidean and hyperbolic buildings, and CAT(0) cube complexes—besides many other spaces.

We now recall some properties of rank one geodesics in CAT(0) spaces. We assume some familiarity with CAT(0) spaces ([4] and [9] are good references). The results in this chapter are found in the existing literature and generally stated without proof. Proposition 4.10 is not in the literature as stated, but will not surprise the experts.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Further suppose, for simplicity, that $|\partial X| > 2$.

Remark. Since X is geodesically complete, requiring $|\partial X| > 2$ merely forces X not to be isometric to the real line $\mathbb{R}$ or a single point, and Γ to be infinite.

For CAT(0) spaces, the Busemann boundary is canonically homeomorphic to the visual boundary obtained by taking equivalence classes of asymptotic geodesic rays (see [4] or [9]). Thus we will write ∂X for the boundary with this topology, and use either description as convenient.

A *geodesic* in X is an isometric embedding $v\colon \mathbb{R} \to X$. A subspace $Y \subset X$ isometric to $\mathbb{R} \times [0,\infty)$ is called a *flat half-plane*; note that half-planes are automatically convex. Call a geodesic in X *rank one* if its image does not bound a flat half-plane in X. If a rank one geodesic is the axis of an isometry $\gamma \in \Gamma$, we call it a *rank one axis*.

Angles are defined as follows: Let $x \in X$. For $y,z \in X \smallsetminus \{x\}$, the *comparison angle* $\overline{\angle}_x(y,z)$ at x between y and z is the angle at the corresponding point $\overline{x}$ in the Euclidean comparison triangle $\overline{\triangle}$ for the geodesic triangle $\triangle$ in X. If v and w are geodesics in X with $v(0) = w(0) = x$, the *angle* at x between v and w is $\angle_x(v,w) = \lim_{s,t\to 0^+} \overline{\angle}_x(v(s), w(t))$. For $p, q \in \overline{X} \setminus \{x\}$, the *angle* at x between p and q is $\angle_x(p,q) = \angle_x(v,w)$, where v and w are geodesics with $v(0) = w(0) = x$, $v(d(x,p)) = p$, and $v(d(x,q)) = q$.

The next proposition describes the extent to which angles are continuous.

Proposition 4.1 (Proposition II.9.2)**.** *Let X be a complete CAT(0) space.*

1. *For fixed $p \in X$, the function $(x,y) \mapsto \angle_p(x,y)$, which takes values in $[0,\pi]$, is continuous at all points $(x,y) \in \overline{X}\times\overline{X}$ with $x \neq p$ and $y \neq p$.*

2. *The function $(p,x,y) \mapsto \angle_p(x,y)$ is upper semicontinuous at points $(p,x,y) \in X\times\overline{X}\times\overline{X}$ with $x \neq p$ and $y \neq p$.*

Now we define the angle between points of the boundary. For $\xi, \eta \in \partial X$, let $\angle(\xi,\eta) = \sup_{x\in X} \angle_x(\xi,\eta)$. Then $\angle$ defines a complete CAT(1) metric on ∂X (c.f. Section 10); this metric induces a topology on ∂X that is finer (usually strictly finer) than the standard topology. Two additional characterizations of $\angle$ are described in the following proposition.

For any geodesic $v\colon \mathbb{R} \to X$, denote $v^+ = \lim_{t\to+\infty} v(t)$ and $v^- = \lim_{t\to-\infty} v(t)$.

Proposition 4.2 (Proposition II.9.8 in [9])**.** *Let $\xi, \eta \in \partial X$ and $x_0 \in X$. Let $v, w\colon \mathbb{R} \to X$ be geodesics with $v(0) = w(0) = x_0$, $v^+ = \xi$, and $w^+ = \eta$. Then*

1. *$\angle(\xi,\eta)$ is the increasing limit $\lim_{s,t\to\infty} \overline{\angle}_{x_0}(v(s), w(t))$, and*

2. *$\angle(\xi,\eta)$ is the increasing limit $\lim_{t\to\infty} \angle_{v(t)}(\xi,\eta)$.*

The following is an important consequence of Proposition 4.2.

Corollary 4.3 (Corollary II.9.9 in [9])**.** *If for some point $x_0 \in X$ we have $\angle_{x_0}(\xi,\eta) = \angle(\xi,\eta) < \pi$, then the convex hull of the geodesic rays $\sigma\colon [0,\infty) \to X$ and $\tau\colon [0,\infty) \to X$ issuing from x_0 with $\sigma^+ = \xi$ and $\tau^+ = \eta$ is isometric to a sector in the Euclidean plane $\mathbb{R}^2$ bounded by two rays which meet at an angle $\angle(\xi,\eta)$.*

The *Tits metric*, d_T, on ∂X is the path metric induced by $\angle$ (which may take the value $+\infty$). The *Tits boundary* of X is ∂X, equipped with the Tits metric d_T. We now state three important results about the Tits metric; they all follow from Theorem II.4.11 in [4]. The first tells us that d_T and $\angle$ coincide for all short distances.

Lemma 4.4. *If $\angle(\xi,\eta) < \pi$ then $d_T(\xi,\eta) = \angle(\xi,\eta)$.*

Note that if ξ,η are the endpoints of a geodesic, then $d_T(\xi,\eta) \geq \angle(\xi,\eta) = \pi$. Whether the inequality is strict or not depends on whether the geodesic is rank one, as the second lemma describes.

Lemma 4.5. *A pair of points $\xi,\eta \in \partial X$ is joined by a rank one geodesic in X if and only if $d_T(\xi,\eta) > \pi$.*

The third lemma shows that the Tits metric $d_T\colon \partial X \times \partial X \to [0,\infty]$ is actually lower semicontinuous with respect to the visual topology on ∂X.

Lemma 4.6. *Suppose $\xi_k \to \xi$ and $\eta_k \to \eta$ in the visual topology on ∂X. Then*

$$d_T(\xi,\eta) \leq \limsup_k d_T(\xi_k,\eta_k).$$

A subspace $Y \subset X$ isometric to $\mathbb{R} \times [0,R]$ is called a *flat strip of width R*. The next lemma is fundamental to understanding rank one geodesics in CAT(0) spaces. It implies, in particular, that the endpoint pairs of rank one geodesics form an open set in $\partial X \times \partial X$.

Lemma 4.7 (Lemma III.3.1 in [4])**.** *Let $w\colon \mathbb{R} \to X$ be a geodesic which does not bound a flat strip of width $R > 0$. Then there are neighborhoods U and V in $\bar{X}$ of the endpoints of w such that for any $\xi \in U$ and $\eta \in V$, there is a geodesic joining ξ to η. For any such geodesic v, we have $d(v,w(0)) < R$; in particular, v does not bound a flat strip of width $2R$.*

Now we turn to Chen and Eberlein's duality condition from [13]. It is based on Γ-duality of pairs of points in ∂X, introduced by Eberlein in [16].

Definition 4.8. Two points $\xi,\eta \in \partial X$ are called Γ-*dual* if there exists a sequence (γ_n) in Γ such that $\gamma_n x \to \xi$ and $\gamma_n^{-1} x \to \eta$ for some (hence any) $x \in X$. Write $\mathcal{D}(\xi)$ for the set of points in ∂X that are Γ-dual to ξ. We say *Chen and Eberlein's duality condition holds* on ∂X if v^+ and v^- are Γ-dual for every geodesic $v\colon R \to X$.

The next lemma describes the hyperbolic dynamics associated to a rank one axis.

Lemma 4.9 (Lemma III.3.3 in [4])**.** *Let γ be an isometry of X, and suppose $w\colon \mathbb{R} \to X$ is an axis for γ, where w is a geodesic which does not bound a flat half-plane. Then*

1. *For any neighborhood U of w^- and any neighborhood V of w^+ in $\overline{X}$ there exists $n > 0$ such that*
$$\gamma^k(\overline{X} \setminus U) \subset V \quad \textit{and} \quad \gamma^{-k}(\overline{X} \setminus V) \subset U \quad \textit{for all } k \geq n.$$

2. *For any $\xi \in \partial X \setminus \{w^+\}$, there is a geodesic w_ξ from ξ to w^+, and any such geodesic is rank one. Moreover, for $K \subset \partial X \setminus \{w^+\}$ compact, the set of these geodesics is compact (modulo parametrization).*

The next proposition summarizes the situation for rank one CAT(0) spaces (cf. Proposition 7.5 and Proposition 8.5).

Proposition 4.10. *Let Γ be a group acting properly discontinuously, cocompactly, and isometrically on a proper, geodesically complete* CAT(0) *space X. Suppose X contains a rank one geodesic, and that $|\partial X| > 2$. The following are equivalent:*

1. *X has a rank one axis.*

2. *Every pair $\xi, \eta \in \partial X$ is Γ-dual.*

3. *Chen and Eberlein's duality condition holds on ∂X (that is, v^- and v^+ are Γ-dual for every geodesic $v\colon R \to X$).*

4. *Γ acts minimally on ∂X (that is, every $p \in \partial X$ has a dense Γ-orbit).*

5. *Some $\xi \in \partial X$ has infinite Tits distance to every other $\eta \in \partial X$.*

6. *∂X has Tits diameter $\geq \frac{3\pi}{2}$.*

Proof. (3) $\Longrightarrow$ (4) and (3) $\Longrightarrow$ (1) are shown in Ballmann (Theorems III.2.4 and III.3.4, respectively, of [4]). (1) $\Longrightarrow$ (5) is an easy exercise using Lemma 4.9(2), while (5) $\Longrightarrow$ (6) and (2) $\Longrightarrow$ (3) are trivial. (6) $\Longrightarrow$ (1) is shown (with slightly better bounds for any fixed dimension) in Guralnik and Swenson ([22]). (4) $\Longrightarrow$ (2) follows immediately from Corollary 1.6 of Ballmann and Buyalo ([5]).

It remains to prove (1) $\Longrightarrow$ (4). Let p, q be the endpoints of a rank one axis, and let M be a minimal nonempty closed Γ-invariant subset of ∂X. By Lemma 4.9(1), both p, q must lie in M; thus M is the only minimal set. By Corollary 2.1 of Ballmann and Buyalo ([5]), the orbit of p is dense in the boundary. Since $p \in M$, this means the Γ-action is minimal on the boundary. □

A well-known conjecture of Ballmann and Buyalo ([5]) is that, given the hypotheses of Proposition 4.10, all the equivalent conditions in the conclusion hold.

CHAPTER 5

Patterson-Sullivan Measures on CAT(0) Boundaries

We make a few observations about Patterson-Sullivan measures for CAT(0) spaces.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, and by isometries on a proper, geodesically complete CAT(0) space X.

Definition 5.1. Define the *r-shadow of y from x* to be

$$\mathcal{O}_r(x,y) = \{\xi \in \partial X \mid [x,\xi) \cap B(y,r) \neq \varnothing\},$$

where $[x,\xi)$ is the image of the geodesic ray from x to ξ.

Lemma 5.2. *Suppose $x, y \in X$ and $r > 0$. Then*

$$d(x,y) - r \leq b_\xi(x,y) \leq d(x,y)$$

for all $\xi \in \mathcal{O}_r(x,y)$.

Proof. The inequality on the right is just the 1-Lipschitz property of b_ξ. For the one on the left, let $z \in B(y,r)$ be a point on the geodesic ray $[x,\xi)$ from x to ξ. Then $b_\xi(x,y) = b_\xi(x,z) - b_\xi(y,z)$ by the cocycle property of Busemann functions, so $b_\xi(x,y) \geq b_\xi(x,z) - r$ by the 1-Lipschitz property of b_ξ. But $b_\xi(x,z) = d(x,z)$ because z lies on $[x,\xi)$. Thus $b_\xi(x,y) \geq d(x,z) - r$. $\square$

The next lemma is a version of Sullivan's Shadow Lemma.

Lemma 5.3. *For every $r > 0$, there is some $C_r > 0$ such that*

$$\mu_x(\mathcal{O}_r(x,\gamma x)) \leq C_r e^{-\delta_\Gamma d(x,\gamma x)}$$

for all $x \in X$ and $\gamma \in \Gamma$.

Proof. Unraveling the definitions, we have

$$\begin{aligned}\mu_x(\mathcal{O}_r(x,\gamma x)) &= \mu_x(\gamma\,\mathcal{O}_r(\gamma^{-1}x,x))\\ &= \mu_{\gamma^{-1}}(\mathcal{O}_r(\gamma^{-1}x,x))\\ &= \int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\delta_\Gamma b_\xi(\gamma^{-1}x,x)}\,d\mu_x(\xi).\end{aligned}$$

By Lemma 5.2, we obtain

$$\mu_x(\mathcal{O}_r(x,\gamma x)) \le \mu_x(\mathcal{O}_r(\gamma^{-1}x,x))\cdot e^{\delta_\Gamma(r-d(\gamma^{-1}x,x))}.$$

But $d(\gamma^{-1}x,x) = d(x,\gamma x)$, so

$$\mu_x(\mathcal{O}_r(x,\gamma x)) \le \mu_x(X)\cdot e^{\delta_\Gamma(r-d(x,\gamma x))}.$$

Therefore, the lemma holds with $C_r = \mu_x(X)\cdot e^{\delta_\Gamma r}$. □

Call a subspace F of X a *flat* if F is isometric to some Euclidean n-space $\mathbb{R}^n$.

Proposition 5.4. *If $\delta_\Gamma > 0$, then $\mu_x(\partial F) = 0$ for any flat $F \subset X$.*

Proof. Let $F \subset X$ be a flat. Fix $x \in X$; we may assume $x \in F$. By cocompactness of the Γ-action, there is some $R > 0$ such that $\Gamma B(x,R) = X$. Now the spheres

$$S_F(x,r) = \{y \in F \mid d(y,x) = r\}$$

in F based at x may be covered by at most $p(r)$ R-balls in F, for some polynomial function $p\colon \mathbb{R} \to \mathbb{R}$. But the center of each of these balls lies within distance R of some γx in X ($\gamma \in \Gamma$). Thus

$$S_F(x,r) \subset \bigcup_{\gamma\in A_r} B(\gamma x, 2R),$$

where $A_r \subset \Gamma$ has cardinality at most $p(r)$. Now by Lemma 5.3, for every $r > 0$ we have

$$\mu_x(\partial F) \le \sum_{\gamma\in A_r} \mu_x(\mathcal{O}_{2R}(x,\gamma x)) \le \sum_{\gamma\in A_r} C_{2R}\cdot e^{-\delta_\Gamma d(x,\gamma x)} = C_{2R}\cdot e^{-\delta_\Gamma r}\,|A_r|\,.$$

Since $|A_r| \le p(r)$, we therefore have $\mu_x(\partial F) \le C_{2R}\cdot e^{-\delta_\Gamma r}p(r)$ for all $r > 0$. But $e^{-\delta_\Gamma r}p(r) \to 0$ as $r \to +\infty$ because $\delta_\Gamma > 0$ and $p(r)$ is polynomial. Thus $\mu_x(\partial F) = 0$, as required. □

On the other hand, we have the following result.

Lemma 5.5. *If $\delta_\Gamma = 0$, then X is flat—that is, X is isometric to flat Euclidean n-space $\mathbb{R}^n$ for some n.*

Proof. Suppose $\delta_\Gamma = 0$. Then Γ must have subexponential growth, so Γ is amenable. By Adams and Ballmann ([1, Corollary C]), X is flat. $\square$

The previous two results immediately give us the following corollary.

Corollary 5.6. *If X is not flat, then μ_x is not atomic—that is, $\mu_x(\xi) = 0$ for all $\xi \in \partial X$.*

CHAPTER 6

A Weak Product Structure

We now study the space SX of unit-speed parametrized geodesics in X. Much of our work in later chapters depends on a certain product structure on this space, which we will describe shortly.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Assume that $|\partial X| > 2$.

Let SX be the space of unit-speed parametrized geodesics in X, endowed with the compact-open topology, and let $\mathcal{R} \subset SX$ be the space of rank one geodesics in SX. For $v \in SX$ denote $v^+ = \lim_{t\to+\infty} v(t)$ and $v^- = \lim_{t\to-\infty} v(t)$. Let

$$\mathcal{G}^E = \left\{(v^-, v^+) \in \partial X \times \partial X \mid v \in SX\right\}$$

and

$$\mathcal{R}^E = \left\{(v^-, v^+) \in \partial X \times \partial X \mid v \in \mathcal{R}\right\}.$$

Note that $\mathcal{R}^E$ is open in $\mathcal{G}^E$ by Lemma 4.7, and the natural projection $\mathrm{E}\colon SX \to \mathcal{G}^E$ is a continuous surjection with $\mathcal{R} = \mathrm{E}^{-1}(\mathcal{R}^E)$, so $\mathcal{R}$ is open in SX.

There are many metrics on SX (compatible with the compact-open topology) on which the natural Γ-action $\gamma(v) = \gamma \circ v$ is by isometries. For simplicity, we will use the metric on SX given by

$$d(v, w) = \sup_{t\in\mathbb{R}} e^{-|t|} d(v(t), w(t)).$$

Lemma 6.1. *Under the metric given above, SX is a proper metric space, and the Γ-action on X induces a proper, cocompact Γ-action on SX by isometries.*

Proof. Let $\pi\colon SX \to X$ be the footpoint projection $\pi(v) = v(0)$. Clearly π is continuous (1-Lipschitz, even); since X is geodesically complete, π is surjective. We will show that π is a proper map, that is, $\pi^{-1}(K)$ is compact for any compact set $K \subset X$. So let K be a compact set in X. If (v_n) is a sequence in $\pi^{-1}(K)$, then $v_n(0) \in K$ for all n, hence a subsequence $v_{n_k} \to v \in SX$ by the Arzelà-Ascoli Theorem. But $v_{n_k}(0) \to v(0)$ must be in K by compactness of K, hence $v \in \pi^{-1}(K)$. Thus π is a proper map.

Since π is 1-Lipschitz, $\overline{B}(v,r) \subseteq \pi^{-1}(\overline{B}(v(0),r))$ for any $v \in SX$ and $r > 0$; thus SX is proper because X is proper. Since only finitely many $\gamma \in \Gamma$ have $\overline{B}(v(0),r) \cap \gamma\overline{B}(v(0),r) \neq \varnothing$, the same holds for $\pi^{-1}(\overline{B}(v(0),r))$. If $K \subset X$ is compact such that $\Gamma K = X$ then $\pi^{-1}(K)$ is compact by properness of π, and if $w \in SX$ then $\gamma w(0) \in K$ for some $\gamma \in \Gamma$, hence $\Gamma\pi^{-1}(K) = SX$; thus Γ acts cocompactly on SX. □

For $p \in X$, define $\beta_p \colon \partial X \times \partial X \to [-\infty, \infty)$ by $\beta_p(\xi,\eta) = \inf_{x\in X}(b_\xi + b_\eta)(x,p)$.

Lemma 6.2. *For any $\xi,\eta \in \partial X$, $\beta_p(\xi,\eta)$ is finite if and only if $(\xi,\eta) \in \mathcal{G}^E$. Moreover,*

$$\beta_p(\xi,\eta) = (b_\xi + b_\eta)(x,p)$$

if and only if x lies on the image of a geodesic $v \in \mathrm{E}^{-1}(\xi,\eta)$.

Proof. This is shown in the proof of implications (1) $\implies$ (2) and (2) $\implies$ (1) of Proposition II.9.35 in [9]. □

Thus we may (abusing notation slightly) also write $\beta_p \colon SX \to \mathbb{R}$ to mean the map $\beta_p \circ \mathrm{E}$; that is, $\beta_p(v) = \beta_p(v^-,v^+) = (b_{v^-} + b_{v^+})(v(0),p)$.

Lemma 6.3. *For any $p \in X$, the map β_p is continuous on $\mathcal{R}^E$ and upper semicontinuous on $\partial X \times \partial X$.*

Proof. Continuity on $\mathcal{R}^E$ first. Fix $p \in X$, and suppose $(v_n^-,v_n^+) \to (v^-,v^+)$. By Lemma 4.7, we may assume that $d(v_n(0),v(0)) < R$ for some $R > 0$. So by the Arzelà-Ascoli Theorem, we may pass to a subsequence such that $v_n \to u$ for some $u \in SX$. Then u must be parallel to v, hence $\beta_p(u) = \beta_p(v)$. Define $c_w \colon X \to \mathbb{R}$ by $c_w(q) = (b_{w^-} + b_{w^+})(q,p)$. Thus $c_w(w(0)) = \beta_p(w)$ for all $w \in SX$. Since $(v_n^-,v_n^+) \to (v^-,v^+)$, we have $c_{v_n} \to c_v$ uniformly on $\overline{B}(u(0),1)$, and therefore $\{c_{v_n}\} \cup \{c_v\}$ is uniformly equicontinuous on $\overline{B}(u(0),1)$. Thus $v_n(0) \to u(0)$ gives us $c_{v_n}(v_n(0)) \to c_v(u(0))$. But $c_v(u(0)) = c_u(u(0)) = \beta_p(u)$, and $c_{v_n}(v_n(0)) = \beta_p(v_n)$, hence $\beta_p(v_n) \to \beta_p(u)$. Therefore, β_p is continuous on $\mathcal{R}^E$.

Now semicontinuity on $\partial X \times \partial X$. Recall that $\beta_p(\xi,\eta) = \inf_{x\in X}(b_\xi + b_\eta)(x,p)$. Fix $p \in X$, and note that for fixed $x \in X$, the map $(\xi,\eta) \mapsto (b_\xi + b_\eta)(x,p)$ is continuous. But the infimum of a family of continuous functions is upper semicontinuous. □

For $v \in SX$, let P_v be the set of $w \in SX$ parallel to v (we will also write $w \parallel v$). Let X_v be the union of the images of $w \in P_v$. Recall ([4] or [9]) that X_v splits as a canonical product

$Y_v \times \mathbb{R}$, where Y_v is a closed and convex subset of X with $v(0) \in Y_v$. Call X_v the *parallel core* of v and Y_v the *transversal* of X_v at v.

If v is rank one, then Y_v is bounded and therefore has a unique circumcenter (see [4, Proposition I.5.10] or [9, Proposition II.2.7]). Thus we have a canonical *central geodesic* associated to each X_v. Let $\mathcal{R}_C$ denote the subset of central geodesics in $\mathcal{R}$.

Now suppose $v_n \to v \in SX$. Then $b_{v_n^-} \to b_{v^-}$ and $b_{v_n^+} \to b_{v^+}$ by coincidence of the visual and Busemann boundaries. Furthermore, $b_{v_n^-}(v_n(0), x) \to b_{v^-}(v(0), x)$ for all $x \in X$ because $v_n(0) \to v(0)$ while $b_{v_n^-} \to b_{v^-}$ uniformly on $\overline{B}(v, 1)$. This shows the map π_x in the following definition is continuous.

Definition 6.4. Let $\pi_x \colon SX \to \mathcal{G}^E \times \mathbb{R} \subseteq \partial X \times \partial X \times \mathbb{R}$ be the continuous map given by $\pi_x(v) = (v^-, v^+, b_{v^-}(v(0), x))$. Say that a sequence $(v_n) \subset SX$ *converges weakly* to $v \in SX$ if $\pi_x(v_n) \to \pi_x(v)$.

For a sequence that converges in SX, we will sometimes say it *converges strongly* to emphasize that the convergence is not in the weak sense.

Note. Weak convergence does not depend on choice of $x \in X$.

Example 6.5. Consider the hyperbolic plane $\mathbb{H}^2$. Cut along a geodesic, and isometrically glue the two halves to the two sides of a flat strip of width 1. Call the resulting space X. A sequence v_n of geodesics in X which converges strongly to one of the geodesics (call it v) bounding the flat strip will also converge weakly to all the geodesics w parallel to v such that $w(0)$ lies on the geodesic segment orthogonal to the image of v. (See Figure 6.1.)

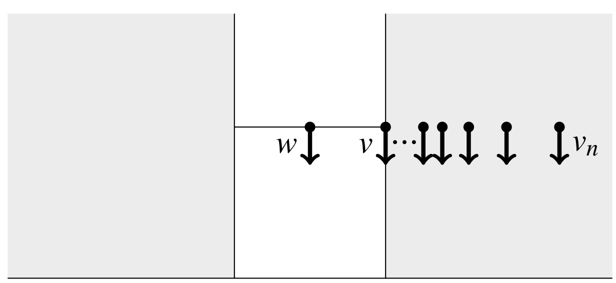

Figure 6.1: The geodesics v_n converge weakly to both v and w, but strongly to v only.

Let us now relate equivalence of geodesics in the product structure to the idea of stable and unstable horospheres, and to the transversals of parallel cores.

Definition 6.6. For $v \in SX$, the *stable horosphere* at v is the set of geodesics

$$H^s(v) = \left\{ w \in SX \mid w^+ = v^+ \text{ and } b_{v^+}(w(0), v(0)) = 0 \right\}.$$

Similarly, the *unstable horosphere* is the set of geodesics

$$H^u(v) = \left\{ w \in SX \mid w^- = v^- \text{ and } b_{v^-}(w(0), v(0)) = 0 \right\}.$$

Proposition 6.7. *For $v, w \in SX$ and $x \in X$, the following are equivalent:*

1. $\pi_x(v) = \pi_x(w)$.
2. $w \in H^u(v)$ *and* $v^+ = w^+$.
3. $w \in H^s(v)$ *and* $v^- = w^-$.
4. $w \in H^s(v) \cap H^u(v)$.
5. $v \parallel w$ *and* $w(0) \in Y_v$.

Proof. We may assume throughout the proof that $v \parallel w$. Since

$$(b_{v^-} + b_{v^+})(v(0), x) = \beta_x(v) = \beta_x(w) = (b_{w^-} + b_{w^+})(w(0), x),$$

we have $b_{v^-}(v(0), x) = b_{w^-}(w(0), x)$ if and only if $b_{v^+}(v(0), x) = b_{w^+}(w(0), x)$; this proves the equivalence of the first four conditions. Recall ([4, Proposition I.5.9], or [9, Theorem II.2.14(2)]) that Y_v is preimage of $v(0)$ in X_v under the orthogonal projection onto the image of v. Now orthogonal projection onto the image of v cannot increase either $b_{v^-}(\cdot, x)$ or $b_{v^+}(\cdot, x)$ by [9, Lemma II.9.36], but $\beta_x(v) = \beta_x(w)$ because $v \parallel w$. So for $w(t_0) \in Y_v$,

$$b_{v^-}(v(0), x) = b_{v^-}(w(t_0), x) = b_{w^-}(w(t_0), x) = t_0 + b_{w^-}(w(0), x).$$

Thus $\pi_x(v) = \pi_x(w)$ if and only if $w(0) \in Y_v$ (note $w(t_0) \in Y_v$ for only one $t_0 \in \mathbb{R}$). This concludes the proof. □

We will write $u \sim v$ if v and w satisfy any of the equivalent conditions in the above proposition. Clearly $\sim$ is an equivalence relation. Note that by Proposition 6.7, this relation does not depend on choice of $x \in X$.

Lemma 6.8. *If $v_n \to v$ weakly and $v \in \mathcal{R}$, then $\{v_n(0)\}$ is bounded in X.*

Proof. Fix $x \in X$. Since $v_n^- \to v^-$, we have $b_{v_n^-} \to b_{v^-}$ uniformly on compact subsets, and so $b_{v_n^-}(v(0), x) \to b_{v^-}(v(0), x)$. On the other hand, we know that $b_{v_n^-}(v_n(0), x) \to b_{v^-}(v(0), x)$ by hypothesis, so

$$\lim_{n\to\infty} b_{v_n^-}(v(0), x) = b_{v^-}(v(0), x) = \lim_{n\to\infty} b_{v_n^-}(v_n(0), x).$$

Hence, by the cocycle property of Busemann functions,

$$\lim_{n\to\infty}\left(b_{v_n^-}(v(0),v_n(0))\right)=\lim_{n\to\infty}\left(b_{v_n^-}(v(0),x)-b_{v_n^-}(v_n(0),x)\right)=0.$$

Now let $R>0$ be large enough so that v does not bound a flat strip in X of width R. By Lemma 4.7, for all sufficiently large n there exist $t_n\in\mathbb{R}$ such that $d(v_n(t_n),v(0))<R$. Thus

$$\begin{aligned}|t_n|=\left|b_{v_n^-}(v_n(t_n),v_n(0))\right|&=\left|b_{v_n^-}(v_n(t_n),v(0))-b_{v_n^-}(v_n(0),v(0))\right|\\&\leq d(v_n(t_n),v(0))+\left|b_{v_n^-}(v_n(0),v(0))\right|<R+1\end{aligned}$$

for all sufficiently large n. In particular,

$$d(v_n(0),v(0))\leq d(v_n(0),v_n(t_n))+d(v_n(t_n),v(0))<|t_n|+R<2R+1.$$

□

Lemma 6.9. *If $v_n\to v$ weakly and $v\in\mathcal{R}$ then a subsequence converges strongly to some $u\sim v$.*

Proof. Fix $x\in X$. By Lemma 6.8, $\{v_n(0)\}$ lies in some compact set in X. Hence by the Arzelà-Ascoli Theorem, passing to a subsequence we may assume that (v_n) converges in SX to some geodesic u. Then $\pi_x(u)=\lim\pi_x(v_n)$ by continuity of π_x, while $\pi_x(v)=\lim\pi_x(v_n)$ by hypothesis, and therefore $u\sim v$. □

Remark. Restricting π_x to $\mathcal{R}_C$ does not automatically give us a homeomorphism from $\mathcal{R}_C$ to $\mathcal{R}^E\times\mathbb{R}$. We get a topology on $\mathcal{R}_C$ at least as course as the subspace topology, though. An explicit example of the failure of π_x to be a homeomorphism is as follows: Take a closed hyperbolic surface, and replace a simple closed geodesic with a flat cylinder of width 1; then there are sequences of geodesics that limit, weakly but not strongly, onto one of the central geodesics in the flat cylinder.

From Lemma 6.9, we see that the continuous map $\pi_x|_{\mathcal{R}}$ is closed (that is, the image of every closed set is closed). Thus $\pi_x|_{\mathcal{R}}$ is a topological quotient map onto $\mathcal{R}^E\times\mathbb{R}$.

Let $g^t\colon SX\to SX$ denote the geodesic flow; that is, $(g^t(v))(s)=v(s+t)$. Note that g^t commutes with Γ. Observe also that the geodesic flow g^t descends to the action on $\mathcal{G}^E\times\mathbb{R}$ given by $g^t(\xi,\eta,s)=(\xi,\eta,s+t)$, hence this is clearly an action by homeomorphisms. Using Lemma 6.9, we obtain the following complementary observation.

Proposition 6.10. *The Γ-action on $\mathcal{R}$ descends to an action on $\mathcal{R}^E\times\mathbb{R}$ by homeomorphisms.*

Proof. Suppose $u \sim v$. Then $\gamma u \sim \gamma v$ follows from the trivial computation

$$b_{\gamma u^-}(\gamma u(0), x) = b_{u^-}(u(0), \gamma^{-1}x) = b_{v^-}(v(0), \gamma^{-1}x) = b_{\gamma v^-}(\gamma v(0), x).$$

Thus γ descends to a continuous map $\mathcal{R}^E \times \mathbb{R} \to \mathcal{R}^E \times \mathbb{R}$ by the universal property of quotient maps. But then γ^{-1} also descends to a continuous map, and therefore Γ acts by homeomorphisms on $\mathcal{R}^E \times \mathbb{R}$. $\square$

We also have the following stronger result.

Proposition 6.11. *The Γ-action on SX descends to an action on $\mathcal{G}^E \times \mathbb{R}$ by homeomorphisms.*

Proof. Fix $x \in X$. First compute

$$\begin{aligned}\pi_x(\gamma v) &= \left(\gamma v^-, \gamma v^+, b_{\gamma v^-}(\gamma v(0), x)\right) \\ &= \left(\gamma v^-, \gamma v^+, b_{v^-}(v(0), \gamma^{-1}x)\right) \\ &= \left(\gamma v^-, \gamma v^+, b_{v^-}(v(0), x) + b_{v^-}(x, \gamma^{-1}x)\right).\end{aligned}$$

Now recall $\xi_n \to \xi$ in ∂X if and only if $b_{\xi_n}(\cdot, p) \to b_\xi(\cdot, p)$ uniformly on compact subsets, for $p \in X$ arbitrary. Hence if $v_n^- \to v^-$ then $b_{v_n^-}(x, \gamma^{-1}x) \to b_{v^-}(x, \gamma^{-1}x)$. So suppose $v_n \to v$ weakly in SX. Then

$$\begin{aligned}\pi_x(\gamma v) &= \left(\gamma v^-, \gamma v^+, b_{v^-}(v(0), x) + b_{v^-}(x, \gamma^{-1}x)\right) \\ &= \lim_{n\to\infty} \left(\gamma v_n^-, \gamma v_n^+, b_{v_n^-}(v_n(0), x) + b_{v_n^-}(x, \gamma^{-1}x)\right) \\ &= \lim_{n\to\infty} \pi_x(\gamma v_n).\end{aligned}$$

Thus γ descends to a continuous map $\mathcal{G}^E \times \mathbb{R} \to \mathcal{G}^E \times \mathbb{R}$. But then γ^{-1} also descends to a continuous map, and therefore Γ acts by homeomorphisms on $\mathcal{G}^E \times \mathbb{R}$. $\square$

Proposition 6.12. *Let* flip: $SX \to SX$ *be the flip given by* $(\operatorname{flip} v)(t) \mapsto v(-t)$*. Then* $\operatorname{flip}|_{\mathcal{R}}$ *descends to a homeomorphism on* $\mathcal{R}^E \times \mathbb{R}$*.*

Proof. Observe that $\operatorname{flip} u \sim \operatorname{flip} v$ whenever $u \sim v$, so $\operatorname{flip}|_{\mathcal{R}}$ descends to the map $\overline{\operatorname{flip}}(\xi, \eta, s) = (\eta, \xi, \beta_x(\xi, \eta) - s)$ on $\mathcal{R}^E \times \mathbb{R}$. By Lemma 6.3, this map $\overline{\operatorname{flip}}$ is continuous. Since it is its own inverse, it is therefore a homeomorphism. $\square$

CHAPTER 7

Recurrence

We now study some of the basic topological properties of the geodesic flow on SX. We want to study these properties both on SX and its weak product structure $\mathcal{G}^E \times \mathbb{R}$ from the previous chapter.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Assume that $|\partial X| > 2$.

Definition 7.1. A geodesic $v \in SX$ is said to Γ-*accumulate* on $w \in SX$ if there exist sequences $t_n \to +\infty$ and $\gamma_n \in \Gamma$ such that $\gamma_n g^{t_n}(v) \to w$ as $n \to \infty$. A geodesic $v \in SX$ called Γ-*recurrent* if it Γ-accumulates on itself.

The definition given above describes *forward* Γ-recurrent geodesics. A *backward* Γ-recurrent geodesic is a geodesic $v \in SX$ such that flip v is forward Γ-recurrent. We will also sometimes use the terms *weakly* and *strongly*, as in Definition 6.4, to specify the convergence in Definition 7.1.

Recurrence is stronger than nonwandering:

Definition 7.2. A geodesic $v \in SX$ is called *nonwandering mod* Γ if there exists a sequence $v_n \in SX$ such that v_n Γ-accumulates on v.

Note that $v \in SX$ is Γ-recurrent if and only if its projection onto $\Gamma\backslash SX$ is recurrent under the geodesic flow g^t_Γ on $\Gamma\backslash SX$. Similarly, $v \in SX$ is nonwandering mod Γ if and only if its projection is nonwandering under the geodesic flow g^t_Γ on $\Gamma\backslash SX$.

Eberlein ([16]) proved the following result for manifolds of nonpositive curvature; it describes duality in ∂X in terms of geodesics.

Lemma 7.3 (Lemma III.1.1 in [4])**.** *If $v, w \in SX$ and $v^+ \in \mathcal{D}(w^-)$, then there exist $(\gamma_n, t_n, v_n) \in \Gamma \times \mathbb{R} \times SX$ such that $v_n \to v$ and $\gamma_n g^{t_n} v_n \to w$.*

Thus Eberlein observed (see [16] and [17]) for manifolds of nonpositive curvature that $v \in SX$ is nonwandering mod Γ if and only if v^- and v^+ are Γ-dual. This fact holds for proper, geodesically complete CAT(0) spaces as well (the one direction is clear, and the other follows from Lemma 7.3; see the discussion preceding Corollary III.1.4 in [4]).

Corollary 7.4. *The geodesic $v \in SX$ is nonwandering mod Γ if and only if v^- and v^+ are Γ-dual.*

We recall the situation for rank one CAT(0) spaces (cf. Proposition 4.10 and Proposition 8.5):

Proposition 7.5. *Let Γ be a group acting properly discontinuously, cocompactly, and isometrically on a proper, geodesically complete* CAT(0) *space X. Suppose X contains a rank one geodesic. The following are equivalent:*

1. *X has a rank one axis.*
2. *The rank one axes of X are weakly dense in $\mathcal{R}$.*
3. *Some rank one geodesic of X is nonwandering mod Γ.*
4. *Every geodesic of X is nonwandering mod Γ.*
5. *The strongly Γ-recurrent geodesics of X are dense in SX.*
6. *Some geodesic of X has a strongly dense orbit mod Γ.*

Proof. (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (3) are immediate. By Proposition 4.10, X has a rank one axis if and only if Chen and Eberlein's duality condition holds, so (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) by Corollaries III.1.4 and II.1.5 of [4], and (1) $\Longrightarrow$ (6) by Theorem III.2.4 of [4]. By Lemma III.3.2 of [4], every rank one geodesic that is nonwandering mod Γ is a weak limit of rank one axes; this proves (3) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (2).

We now prove (6) $\Longrightarrow$ (1). Let $v \in SX$ have dense orbit mod Γ; by Lemma 4.7, $v \in \mathcal{R}$. If (1) fails, then v^+ cannot be isolated in the Tits metric on ∂X by Proposition 4.10. Hence (v^-, v^+) cannot be isolated in $\mathcal{G}^E$, so v must be strongly Γ-recurrent. But we already observed that (1) $\Longleftrightarrow$ (5). $\square$

We will work mainly with Γ-recurrence. The following basic result illustrates the power of Γ-recurrence.

Lemma 7.6. *Let $v \in SX$ be a Γ-recurrent geodesic. Then every $w \in SX$ with $w^+ = v^+$ Γ-accumulates on a geodesic parallel to v.*

Proof. Since v is Γ-recurrent, there exist sequences $t_n \to +\infty$ and $\gamma_n \in \Gamma$ such that $\gamma_n g^{t_n}(v) \to v$. So suppose $w \in SX$ has $w^+ = v^+$. Since $w^+ = v^+$, the function $t \mapsto d(g^t v, g^t w)$ is bounded on $t \geq 0$ by convexity, hence $\{\gamma_n g^{t_n} w(0)\}$ is bounded, and passing to a subsequence we may assume that $\gamma_n g^{t_n}(w) \to u \in SX$. But then

$$d(v(s), u(s)) = \lim_{n\to\infty} d(\gamma_n g^{t_n} v(s), \gamma_n g^{t_n} w(s)) = \lim_{t\to\infty} d(g^t v(s), g^t w(s))$$

is independent of $s \in \mathbb{R}$, and thus u is parallel to v. □

Inspecting the proof, we see that we have actually shown the following.

Lemma 7.7. *Suppose $v, w \in SX$ have $v^+ = w^+$. If v Γ-accumulates on $u \in SX$, then w must Γ-accumulate on a geodesic parallel to u.*

We will need to deal with weak Γ-recurrence, so we revisit Lemma 7.6.

Lemma 7.8. *Let $v \in \mathcal{R}$ be a weakly Γ-recurrent geodesic. Then every $w \in SX$ with $w^+ = v^+$ strongly Γ-accumulates on a geodesic $u \sim v$.*

Proof. By Lemma 6.9, v strongly Γ-accumulates on some $u \sim v$. By Lemma 7.7, w must strongly Γ-accumulate on some $u' \parallel u$. But $g^t u' \sim u$ for some $t \in \mathbb{R}$, so we may assume $u' \sim u$. □

As before, we have actually shown the following.

Lemma 7.9. *Suppose $v, w \in SX$ have $v^+ = w^+$. If v weakly Γ-accumulates on $u \in \mathcal{R}$, then w must strongly Γ-accumulate on a geodesic parallel to u.*

In fact, one has the following more general statement.

Lemma 7.10. *Suppose $v_1, v_2 \in SX$ are asymptotic, and v_1 weakly Γ-accumulates on $w_1 \in \mathcal{R}$. Then there exist $w_2 \in \mathcal{R}$ parallel to w_1 and sequences $t_n \to +\infty$ and $\gamma_n \in \Gamma$ such that $\gamma_n g^{t_n}(v_1) \to u_1 \sim w_1$ and $\gamma_n g^{t_n}(v_2) \to w_2$. Moreover,*

$$d(w_1(s), w_2(s)) = \lim_{t\to\infty} d(v_1(t), v_2(t)) \quad \text{for all } s \in \mathbb{R}.$$

Furthermore, if $v_2 \in H^s(v_1)$ then $w_1 \sim w_2$.

Proof. Fix $x \in X$. Take a sequence (t_n, γ_n) in $\mathbb{R} \times \Gamma$ with $t_n \to +\infty$ such that $\gamma_n g^{t_n}(v_1) \to w_1$ weakly. By Lemma 6.8, $\{\gamma_n g^{t_n} v_1(0)\}$ is bounded. If $v_1^+ = v_2^+$, the function $t \mapsto d(g^t v_1, g^t v_2)$ is bounded on $t \geq 0$ by convexity, and thus $\{\gamma_n g^{t_n} v_2(0)\}$ is bounded. Passing to a subsequence, $\gamma_n g^{t_n}(v_2) \to w_2 \in SX$. But then

$$d(w_1(s), w_2(s)) = \lim_{n\to\infty} d(\gamma_n g^{t_n} v_1(s), \gamma_n g^{t_n} v_2(s)) = \lim_{t\to\infty} d(g^t v_1(s), g^t v_2(s))$$

is independent of $s \in \mathbb{R}$; hence w_2 is parallel to w_1.

Now if $v_2 \in H^s(v_1)$ then

$$\begin{aligned}
b_{w_2^+}(w_2(0), x) &= \lim_{n\to\infty} b_{\gamma_n g^{t_n} v_2^+}(\gamma_n(g^{t_n} v_2(0)), x) \\
&= \lim_{n\to\infty} b_{v_2^+}(v_2(t_n), \gamma_n^{-1} x) \\
&= \lim_{n\to\infty} b_{v_1^+}(v_1(t_n), \gamma_n^{-1} x) \\
&= \lim_{n\to\infty} b_{\gamma_n g^{t_n} v_1^+}(\gamma_n(g^{t_n} v_1(0)), x) \\
&= b_{w_1^+}(w_1(0), x).
\end{aligned}$$

Since we know $w_1 \parallel w_2$, we have $w_1 \sim w_2$ by Proposition 6.7. $\square$

Since convergence preserves distances between all vectors $w' \parallel w \in H^s(v)$, by passing to a subsequence we expect convergence of X_w to an isometric embedding into X_v. This is shown in the following lemma.

Lemma 7.11. *Suppose $w \in SX$ strongly Γ-accumulates on $v \in SX$. Then there are isometric embeddings $X_w \hookrightarrow X_v$ and $Y_w \hookrightarrow Y_v$, each of which maps $w(0) \mapsto v(0)$.*

Proof. Let $(t_n, \gamma_n) \subset \mathbb{R} \times \Gamma$ be a sequence such that $\gamma_n g^{t_n} w \to v$ in SX. Then, in particular, $\gamma_n g^{t_n} w(0) \to v(0)$ in X. So by the Arzelà-Ascoli Theorem, we may pass to a further subsequence such that the natural isometries $Y_w \to \gamma_n Y_{g^{t_n} w}$ converge uniformly to an isometric embedding φ of Y_w into X. Since $\gamma_n g^{t_n}(w) \to v$, the map φ must extend to an isometric embedding of X_w into X_v. But φ must also isometrically embed Y_w into Y_v because $\gamma_n g^{t_n} w(0) \to v(0)$. $\square$

Corollary 7.12. *Let $v \in \mathcal{R}$ be weakly Γ-recurrent. Then for every $w \in SX$ with $w^+ = v^+$, there are isometric embeddings $X_w \hookrightarrow X_v$ and $Y_w \hookrightarrow Y_v$.*

Proof. By Lemma 7.8, w strongly Γ-accumulates on a geodesic $u \sim v$. Since $u \parallel v$, we have $X_u = X_v$ and $Y_u = Y_v$. Now apply Lemma 7.11. $\square$

One immediate consequence of Corollary 7.12 is that if $v \in \mathcal{R}$ is weakly Γ-recurrent with $Y_v = \{v(0)\}$ and $w^+ = v^+$, then $Y_w = \{w(0)\}$. Another is that any $w \in SX$ with $w^+ = v^+$ ($v \in \mathcal{R}$ weakly Γ-recurrent) must have $w \in \mathcal{R}$. But we can do better than this second consequence.

If $v \in \mathcal{R}$ is weakly Γ-recurrent, then $d_T(v^-, v^+) = \infty$. For otherwise, since d_T is a path metric, there is some $\xi \in \partial X$ such that $\pi < d_T(\xi, v^+) < d_T(v^-, v^+)$. By Lemma 4.5, there is a geodesic $w \in \mathrm{E}^{-1}(\xi, v^+)$, and by Lemma 7.8, w weakly Γ-accumulates on v. But d_T is invariant under $\operatorname{Isom} X$, so $d_T(v^-, v^+) \leq d_T(w^-, w^+)$ by Lemma 4.6. This contradicts our choice of w, and the claim follows. In fact, we have shown that if $v \in \mathcal{R}$ is weakly Γ-recurrent, then $d_T(\xi, v^+) \in [0, \pi] \cup \{\infty\}$ for every $\xi \in \partial X$. The next lemma improves on this statement.

Lemma 7.13. *If $v \in \mathcal{R}$ is weakly Γ-recurrent, then v^+ is isolated in the Tits metric—that is, v^+ has infinite Tits distance to every other point in ∂X.*

Proof. Let $v \in \mathcal{R}$ be weakly Γ-recurrent. By Lemma 6.9, there is a sequence (t_n, γ_n) in $\mathbb{R} \times \Gamma$ with $t_n \to +\infty$ and $u \sim v$ such that $\gamma_n g^{t_n}(v) \to u$ strongly; note $u^+ = v^+$. Let $p = u(0)$ and $p_n = v(t_n)$. Suppose $\xi \in \partial X$ has $d_T(\xi, v^+) < \pi$; in particular, $\angle(\xi, v^+) < \pi$. Passing to a subsequence, we may assume $\gamma_n \xi \to \eta \in \partial X$. Clearly $\gamma_n p_n \to p$, hence $\angle_p(\eta, v^+) \geq \limsup_{n\to\infty} \angle_{\gamma_n p_n}(\gamma_n \xi, \gamma_n v^+)$ by upper semicontinuity (Proposition 4.1). But $\angle_{p_n}(\xi, v^+) \to \angle(\xi, v^+)$ because $p_n = v(t_n)$ by Proposition 4.2. And $\gamma_n v^+ \to v^+$, so $\angle(\eta, v^+) \leq \liminf_{n\to\infty} \angle(\gamma_n \xi, \gamma_n v^+)$ by lower semicontinuity (Lemma 4.6). Thus we have

$$\begin{aligned}
\angle_p(\eta, v^+) &\geq \limsup_{n\to\infty} \angle_{\gamma_n p_n}(\gamma_n \xi, \gamma_n v^+) \\
&= \limsup_{n\to\infty} \angle_{p_n}(\xi, v^+) \\
&= \angle(\xi, v^+) \\
&= \liminf_{n\to\infty} \angle(\gamma_n \xi, \gamma_n v^+) \\
&\geq \angle(\eta, v^+).
\end{aligned}$$

But then $\angle_p(\eta, v^+) = \angle(\eta, v^+)$ by definition of $\angle$, and so by Corollary 4.3 there is a flat sector bounded by (p, η, v^+).

Now the points $p_n = v(t_n)$ lie in arbitrarily large balls of a flat half-plane bounded by the image of v. By the Arzelà-Ascoli theorem, $p = \lim \gamma_n p_n$ lies on a full flat half-plane bounded by the image of $u = \lim \gamma_n v$. But this contradicts the fact that $u \sim v \in \mathcal{R}$. $\square$

CHAPTER 8

Bowen-Margulis Measures

We now construct our first Bowen-Margulis measures. In this chapter, we put them on the weak product structure $\mathcal{G}^E \times \mathbb{R}$ and its quotient under Γ. Near the end of Section 9, we will finally be able to define Bowen-Margulis measures on SX and its quotient under Γ.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Assume that $|\partial X| > 2$, and that X admits a rank one geodesic.

Lemma 6.3 allows us to define a Borel measure μ on $\mathcal{G}^E$ by

$$d\mu(\xi,\eta) = e^{-\delta_\Gamma \beta_x(\xi,\eta)} d\mu_x(\xi) d\mu_x(\eta),$$

where $x \in X$ is arbitrary. It follows easily from the definitions that one has

$$d\mu(\xi,\eta) = e^{-\delta_\Gamma \beta_p(\xi,\eta)} d\mu_p(\xi) d\mu_p(\eta)$$

for all $p \in X$. Thus μ does not depend on choice of $x \in X$ and is Γ-invariant. We will, however, need to prove that μ is nontrivial—that is, μ does not give zero measure to every measurable set. We do this in Lemma 8.1, under the hypothesis that $\operatorname{supp}(\mu_x) = \partial X$.

Note that sets of zero μ_x-measure do not depend on choice of $x \in X$, so we may write μ_x-a.e. and $\operatorname{supp}(\mu_x)$ without choosing $x \in X$.

Lemma 8.1. *If* $\operatorname{supp}(\mu_x) = \partial X$, *then* $\mathcal{G}^E \subseteq \operatorname{supp}(\mu)$.

Proof. Recall that $\mathcal{R}^E$ is open in $\partial X \times \partial X$, so $\mathcal{R}^E \subseteq \operatorname{supp}(\mu)$. By Proposition 7.5(6), $\mathcal{R}$ is dense in SX, so $\mathcal{R}^E$ is dense in $\mathcal{G}^E$. Thus $\mathcal{G}^E \subseteq \operatorname{supp}(\mu)$. □

We want to use μ to create a Γ-invariant Borel measure on SX. Potentially, one might do so on $\mathcal{R}_C$, but it is not clear how to ensure that the result would be Borel. We can do so on the related space $\mathcal{G}^E \times \mathbb{R}$, however.

Definition 8.2. Suppose $\operatorname{supp}(\mu_x) = \partial X$. The *Bowen-Margulis measure* m on $\mathcal{G}^E \times \mathbb{R}$ is given by $m = \mu \times \lambda$, where λ is Lebesgue measure on $\mathbb{R}$.

By Proposition 6.11, Γ acts continuously (hence measurably) on $\mathcal{G}^E \times \mathbb{R}$. Thus m is Γ-invariant by Γ-invariance of μ; m is also g^t-invariant. There is a simple way to push the measure m forward modulo the Γ-action, which we describe in Appendix A. However, we still need to show that the resulting measure m_Γ is finite.

Remark. Clearly m is not finite. But neither is μ:

Fact. *μ is not finite.*

Proof. Let $v \in SX$ be a rank one axis transated by $\gamma \in \Gamma$, and let $U \subset \partial X$ be an open neighborhood of v^+ such that $v^- \notin \overline{U}$. By Lemma 4.9, the sequence $(\gamma_k(U \times U))$ of open sets (intersecting $\mathcal{G}^E$) contains a nested subsequence $(\gamma_{k_i}(U \times U))$ with intersection $(v^+, v^+) \in \partial X \times \partial X \smallsetminus \mathcal{G}^E$. Since μ is Γ-invariant, $\mu(U \times U) = 0$ or ∞. But μ has full support, so $\mu(U \times U) = \infty$. □

Proposition 8.3. *Suppose* $\operatorname{supp}(\mu_x) = \partial X$*, and let* $\operatorname{pr}\colon \mathcal{G}^E \times \mathbb{R} \to \Gamma \backslash (\mathcal{G}^E \times \mathbb{R})$ *be the canonical projection. There is a finite Borel measure m_Γ on $\Gamma \backslash (\mathcal{G}^E \times \mathbb{R})$ satisfying both the following properties:*

1. *For all Borel sets $A \subseteq \mathcal{G}^E \times \mathbb{R}$, we have $m_\Gamma(\operatorname{pr}(A)) = 0$ if and only if $m(A) = 0$ if and only if $m(\Gamma A) = 0$. In particular, $\Gamma \backslash (\mathcal{G}^E \times \mathbb{R}) \subseteq \operatorname{supp}(m_\Gamma)$.*
2. *The geodesic flow g_Γ^t on $\Gamma \backslash (\mathcal{G}^E \times \mathbb{R})$, defined by $g_\Gamma^t \circ \operatorname{pr} = \operatorname{pr} \circ g^t$, preserves m_Γ.*

Proof. Proposition A.11 gives us everything except that m_Γ is finite. By Corollary A.12(1), it suffices to show $m(F) < \infty$ for some $F \subseteq \mathcal{G}^E \times \mathbb{R}$ such that $\Gamma F = \mathcal{G}^E \times \mathbb{R}$. Now the Γ-action on SX is cocompact by Lemma 6.1, so there is a compact $K \subset SX$ such that $\Gamma K = SX$. Let $x \in X$ and $F = \pi_x(K)$. Then $\Gamma F = \mathcal{G}^E \times \mathbb{R}$ because $\Gamma K = SX$. We will show $m(F) < \infty$.

Since F is compact by continuity of π_x, we have $F \subseteq \mathcal{G}^E \times [-r, r]$ for some finite $r \geq 0$; thus it suffices to prove $\mu(\mathrm{E}(K)) < \infty$. Let $A = \{v(0) \in X \mid v \in K\}$. By Lemma 6.2, $\beta_x(v) = (b_\xi + b_\eta)(v(0), x)$. Hence

$$\beta_x(K) \subseteq \left\{(b_\xi + b_\eta)(p, x) \mid (\xi, \eta) \in \mathrm{E}(K) \text{ and } p \in A\right\}.$$

So $|\beta_x(K)| \leq 2R$, where R is the diameter of A in X, because the map $p \mapsto b_\zeta(p, x)$ is 1-Lipschitz for all $\zeta \in \partial X$. Thus

$$\mu(\mathrm{E}(K)) = \int_{\mathrm{E}(K)} e^{-\delta_\Gamma \beta_x(\xi,\eta)} d\mu_x(\xi) d\mu_x(\eta) \leq \int_{\mathrm{E}(K)} e^{\delta_\Gamma \cdot 2R} d\mu_x(\xi) d\mu_x(\eta) \leq e^{\delta_\Gamma \cdot 2R}.$$

Hence $\mu(\mathrm{E}(K)) < \infty$, and therefore $m(F) < \infty$. Thus m_Γ is finite. $\square$

Remark. The measure m_Γ is uniquely defined by property (†) of Proposition A.11.

The measure m_Γ from Proposition 8.3 is called the *Bowen-Margulis measure* on $\Gamma\backslash(\mathcal{G}^E \times \mathbb{R})$. The following lemma is a simple consequence of Poincaré recurrence.

Lemma 8.4. *Suppose* $\operatorname{supp}(\mu_x) = \partial X$. *Let W be the set of $w \in SX$ such that w and* $\operatorname{flip} w$ *are both weakly Γ-recurrent. Then $\mu(\mathrm{E}(SX \smallsetminus W)) = 0$.*

Proof. Note m_Γ is a finite g^t_Γ-invariant measure on $\Gamma\backslash(\mathcal{G}^E \times \mathbb{R})$, which has a countable basis. So by Poincaré recurrence, the set W_Γ of forward and backward recurrent points in $\Gamma\backslash(\mathcal{G}^E \times \mathbb{R})$ has full m_Γ-measure. Now W is Γ-invariant and projects down to W_Γ in $\Gamma\backslash(\mathcal{G}^E \times \mathbb{R})$, so $m((\mathcal{G}^E \times \mathbb{R}) \smallsetminus \pi_x(W)) = 0$ by Proposition 8.3(1). The result follows from g^t-invariance of W. $\square$

We conclude this chapter by extending Proposition 4.10 and Proposition 7.5.

Proposition 8.5. *Let Γ be a group acting properly discontinuously, cocompactly, and isometrically on a proper, geodesically complete* CAT(0) *space X. Suppose X contains a rank one geodesic. The following are equivalent:*

1. *X has a rank one axis.*
2. $\operatorname{supp}(\mu_x) = \partial X$.
3. $(\mu_x \times \mu_x)(\mathcal{R}^E) > 0$.
4. *Some rank one geodesic of X is weakly Γ-recurrent.*

Proof. (2) $\Longrightarrow$ (3) is clear because $\mathcal{R}^E$ is open; (3) $\Longrightarrow$ (4) is a corollary of Lemma 8.4. For (1) $\Longrightarrow$ (2), recall (Proposition 4.10) that the Γ-action on ∂X is minimal if X has a rank one axis; the claim follows immediately.

We now prove (4) $\Longrightarrow$ (1). Suppose $v \in \mathcal{R}$ is weakly Γ-recurrent; we may assume $v \in \mathcal{R}_C$. By Lemma 6.9, we may find $\gamma_n g^{t_n}(v) \to u \sim v$, and the natural isometries $Y_v \to \gamma_n Y_{g^{t_n} v}$ converge uniformly (on compact subsets) to an isometric embedding φ of Y_v into $Y_u = Y_v$. But $v(0)$ is the centroid of Y_v, and that is isometry-invariant, so we must have $u = v$. Thus v is strongly Γ-recurrent, and therefore nonwandering mod Γ. Therefore, X has a rank one axis by Proposition 7.5. $\square$

CHAPTER 9

Properties of Bowen-Margulis Measures

We now are in a position to prove some important properties about the Bowen-Margulis measures we constructed on $\mathcal{G}^E \times \mathbb{R}$ and $\Gamma\backslash(\mathcal{G}^E \times \mathbb{R})$. In Theorem 9.1, we use the Bowen-Margulis measures to obtain a structural result about the Patterson-Sullivan measures. Then (Theorem 9.9) we prove a structural result about SX. This theorem allows us to finally define Bowen-Margulis measures on SX and $\Gamma\backslash SX$. We end the chapter by showing that the geodesic flow is ergodic with respect to the Bowen-Margulis measure on $\Gamma\backslash SX$.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Assume that $|\partial X| > 2$, and that X admits a rank one axis (not just geodesic).

By Lemma 8.4, we have weak recurrence almost everywhere. Our next theorem uses Lemma 7.13 to capitalize on the prevalence of recurrence.

Theorem 9.1 (Theorem 1)**.** *Let X be a proper, geodesically complete* CAT(0) *space and* Γ *be a group acting properly discontinuously, cocompactly, and by isometries on X; further assume X admits a rank one axis. Then μ_x-a.e. $\xi \in \partial X$ is isolated in the Tits metric.*

Proof. Let Ω be the set of Tits-isolated points in ∂X, and let $\xi \in \partial X$. Find $v \in \mathcal{R}$ the axis of a rank one geodesic; we may assume $v^- \neq \xi$. Then by Lemma 4.9, there is a geodesic $w \in \mathcal{R}$ with $(w^-, w^+) = (v^-, \xi)$. By Lemma 4.7, we have an open product neighborhood $U \times V$ of (v^-, ξ) in $\mathcal{R}^E$.

Let W be the set of weakly Γ-recurrent geodesics in SX. Then $\mu((U \times V) \smallsetminus \mathrm{E}(W)) = 0$ by Lemma 8.4. So by Fubini's theorem, there exists $W^+ \subseteq V$ such that $\mu_x(V \smallsetminus W^+) = 0$, and $\mu_x(\{\zeta \in U \mid (\zeta, \eta) \notin \mathrm{E}(W)\}) = 0$ for every $\eta \in W^+$. Now by Lemma 7.13, if $v \in \mathcal{R}$ is weakly Γ-recurrent, then v^+ is Tits-isolated. Hence $W^+ \subseteq \Omega$.

Thus we have shown that every $\xi \in \partial X$ has a neighborhood V such that $\mu_x(V \smallsetminus \Omega) = 0$. The theorem follows by compactness of ∂X. $\square$

Corollary 9.2. $(\mu_x \times \mu_x)(\partial X \times \partial X \smallsetminus \mathcal{R}^E) = 0$.

Proof. Let $\xi \in \partial X$ be Tits-isolated. Then $(\xi, \eta) \in \mathcal{R}^E$ for all $\eta \in \partial X \smallsetminus \{\xi\}$. Since $\mu_x(\{\xi\}) = 0$ by Corollary 5.6, we see that μ_x-a.e. $\eta \in \partial X$ has $(\xi, \eta) \in \mathcal{R}^E$. The result follows from Theorem 9.1 and Fubini's theorem. □

Observe that μ and $\mu_x \times \mu_x$ are in the same measure class (that is, each is absolutely continuous with respect to the other), so $\mu(\mathcal{G}^E \smallsetminus \mathcal{R}^E) = 0$. Thus almost no geodesic in X (with respect to the Bowen-Margulis measure m on $\mathcal{G}^E \times \mathbb{R}$) bounds a flat half-plane.

Our next goal (Theorem 9.9) is to show that almost no geodesic in X bounds a flat strip of any width—that is, $\operatorname{diam} Y_v = 0$ for almost every geodesic v. We will need a few lemmas, the first of which describes the upper semicontinuity property of the map $v \mapsto Y_v$ from SX into the space of closed subsets of X (with the Hausdorff metric).

Lemma 9.3. *If a sequence $(v_n) \subset SX$ converges to $v \in \mathcal{R}$ then some subsequence of (Y_{v_n}) converges, in the Hausdorff metric, to a closed subset A of Y_v.*

Proof. Let R be the diameter of Y_v. By Lemma 6.1, the closed ball B in SX about v of radius $2R$ is compact, so the space $\mathcal{C}B$ of closed subsets of B is compact under the Hausdorff metric. For $w \in SX$, let $P'_w = \{u \in SX \mid u \sim w\}$. Eventually every P'_{v_n} lies in B, so some subsequence $(P'_{v_{n_k}})$ converges in $\mathcal{C}B$. But every limit point of $w_n \in P'_{v_n}$ must lie in P'_v, thus $(Y_{v_{n_k}})$ converges, in the Hausdorff metric, to a closed subset A of Y_v. □

The next lemma follows easily from Lemma 7.3.

Lemma 9.4. *Suppose $\psi \colon \mathcal{R}^E \to S$ is a Γ-invariant function from $\mathcal{R}^E$ to a set S such that ψ is constant μ-a.e. on every product neighborhood $U \times V \subseteq \mathcal{R}^E$. Then ψ is constant μ-a.e. on $\mathcal{R}^E$.*

Proof. Suppose $U_0 \times V_0$ is a nonempty product neighborhood in $\mathcal{R}^E$, and let Ω_0 be a subset of $U_0 \times V_0$ with $\mu((U_0 \times V_0) \smallsetminus \Omega_0) = 0$ such that ψ is constant on Ω_0. Fix $v_0 \in \mathrm{E}^{-1}(\Omega_0)$. We will show that $\psi(c) = \psi(\mathrm{E}(v_0))$ for μ-a.e. $c \in \mathcal{R}^E$.

Let $U_1 \times V_1$ be another nonempty product neighborhood in $\mathcal{R}^E$, and let Ω_1 be a subset of $U_1 \times V_1$ with $\mu((U_1 \times V_1) \smallsetminus \Omega_1) = 0$ such that ψ is constant on Ω_1. Let $v_1 \in \mathrm{E}^{-1}(\Omega_1)$. Since X has a rank one axis, by Lemma 7.3 we may find $(\gamma_n, t_n, w_n) \in \Gamma \times \mathbb{R} \times SX$ such that $w_n \to v_1$ and $\gamma_n g^{t_n} w_n \to v_0$. Thus $\gamma_n(U_1 \times V_1)$ has nonempty intersection with $U_0 \times V_0$ for some n. Since both sets are open and μ has full support, the intersection has positive measure. Hence $\mu(\Omega_0 \cap \gamma_n \Omega_1) > 0$, and therefore we may find $c \in (\Omega_0 \cap \gamma_n \Omega_1)$. Then $\psi(c) = \psi(\mathrm{E}(v_0))$ on the one hand because $c \in \Omega_0$, but $\psi(c) = \psi(\mathrm{E}(v_1))$ on the other hand because $c \in \gamma_n \Omega_1$ and ψ is Γ-invariant.

Thus we have shown that for every product neighborhood $U_1 \times V_1$ in $\mathcal{R}^E$, μ-a.e. $c \in U_1 \times V_1$ has $\psi(c) = \psi(\mathrm{E}(v_0))$. But $\partial X \times \partial X$ is a compact metric space and therefore second countable; thus the open set $\mathcal{R}^E$ is covered by countably many product neighborhoods. So by removing a set of measure zero from each, we have $\psi(c) = \psi(\mathrm{E}(v_0))$ for μ-a.e. $c \in \mathcal{R}^E$. □

Remark. The function ψ in Lemma 9.4 is not required to be measurable. It suffices for ψ to be constant on a set of full measure.

Now we combine Fubini's theorem with Lemma 9.4.

Lemma 9.5. *Suppose $\psi \colon \mathcal{R}^E \to S$ is a Γ-invariant map from $\mathcal{R}^E$ to a set S. If Ω is a set of full μ-measure in $\mathcal{R}^E$ such that $\psi((a,b)) = \psi((a,d)) = \psi((c,d))$ for any $(a,b),(a,d),(c,d) \in \Omega$, then ψ is constant μ-a.e. on $\mathcal{R}^E$.*

Proof. Let $U \times V$ be a product neighborhood in $\mathcal{R}^E$. By Fubini's theorem, there exists a subset A of U such that $\mu_x(U \setminus A) = 0$ and every $a \in A$ has $(a,b) \in \Omega$ for μ_x-a.e. $b \in V$. Let $(a,b) \in (A \times V) \cap \Omega$; by choice of A there is some $B \subseteq V$ such that $\mu_x(V \setminus B) = 0$ and $\{a\} \times B \subset \Omega$. So take any $(c,d) \in (A \times B) \cap \Omega$; then $(a,d) \in (A \times B) \cap \Omega$ by choice of B, so $(c,d),(a,d),(a,b) \in \Omega$. Hence $\psi((c,d)) = \psi((a,d)) = \psi((a,b))$ by hypothesis. Thus ψ is constant across $(A \times B) \cap \Omega$, which has full measure in $U \times V$, and Lemma 9.4 finishes the proof. □

Corollary 9.6. *Suppose $\psi \colon \mathcal{R}^E \to S$ is a map from $\mathcal{R}^E$ to a set S such that ψ is invariant under both Γ and* flip*. If Ω is a* flip*-invariant set of full measure in $\mathcal{R}^E$ such that $\psi((a,b)) = \psi((c,b))$ for any $(a,b),(c,b) \in \Omega$, then ψ is constant μ-a.e. on $\mathcal{R}^E$.*

We note here the following basic result about compact metric spaces.

Lemma 9.7 (Theorem 1.6.14 in [10])**.** *A compact metric space cannot be isometric to a proper subset of itself.*

Lemma 9.8. *The isometry type of Y_v is the same for μ-a.e. $(v^-,v^+) \in \mathcal{R}^E$.*

Proof. Let W be the set of $w \in \mathcal{R}$ such that w and $\operatorname{flip} w$ are both weakly Γ-recurrent. If $u, v \in W$ have $u^+ = v^+$, then by Corollary 7.12, we have isometric embeddings between the compact metric spaces Y_u and Y_v, thus Y_u and Y_v are isometric by Lemma 9.7. Since Y_v is constant across P_v, we may therefore apply Corollary 9.6 to the map ψ taking $c \in \mathcal{R}^E$ to the isometry type of Y_c, with $\Omega = \mathrm{E}(W)$ by Lemma 8.4. □

Let $\mathcal{Z} = \{v \in SX \mid \mathrm{diam}(Y_v) = 0\}$, the set of zero-width geodesics. Let $\mathcal{Z}^E = \mathrm{E}(\mathcal{Z})$, the set of $(\xi,\eta) \in \mathcal{G}^E$ such that no $v \in SX$ with $(v^-, v^+) = (\xi,\eta)$ bounds a flat strip of positive width. By semicontinuity of the map $v \mapsto Y_v$ (Lemma 9.3), the width function $v \mapsto \mathrm{diam}(Y_v)$ is semicontinuous on SX. Thus $\mathcal{Z}^E \subseteq \mathcal{G}^E$ is Borel measurable.

Theorem 9.9 (Theorem 2)**.** *Let X and Γ satisfy the assumptions of Theorem 1. The set $\mathcal{Z}^E \subseteq \mathcal{G}^E$ of endpoint pairs of zero-width geodesics has full μ-measure. Thus m-almost no equivalence class of geodesics contains a flat strip of positive width.*

Proof. Let $\mathcal{S} \subseteq \mathcal{R}$ be the preimage under E of the a.e.-set in $\mathcal{R}^E$ from Lemma 9.8; then $\pi_x(\mathcal{S})$ has full m-measure. Since $\mathcal{S}$ is weakly dense in $\mathcal{R}$, by Lemma 6.9 and semicontinuity of the map $v \mapsto Y_v$ there is an isometric embedding $Y_u \hookrightarrow Y_v$ for every $u \in \mathcal{S}$ and $v \in \mathcal{R}$. Thus it suffices to show $\mathcal{Z} \neq \varnothing$, for then $\mathcal{S} \subseteq \mathcal{Z}$, and $\mathcal{Z}^E$ will have full μ-measure.

Recall (Proposition 8.5) that since X has a rank one axis, there is some $w \in SX$ with dense orbit in SX mod Γ; we claim $w \in \mathcal{Z}$. Now by Lemma 7.11, every $v \sim w$ induces an isometric embedding $Y_w \hookrightarrow Y_v$ that maps $w(0) \mapsto v(0)$. Since $Y_v = Y_w$, this map is an isometry by Lemma 9.7, and therefore $\mathrm{Isom}(Y_w)$ acts transitively on Y_w. But the circumcenter of Y_w is an isometry invariant, hence Y_w must be a single point. This proves the claim, and the theorem follows. □

It follows that $(\mu_x \times \mu_x)(\partial X \times \partial X \smallsetminus \mathcal{Z}^E) = 0$ (cf. Corollary 9.2).

Let $\mathcal{S} \subseteq \mathcal{Z}$ be as in Theorem 9.9, with $\pi_x(\mathcal{S})$ having full m-measure in $\mathcal{G}^E \times \mathbb{R}$. Note that every $v \in \mathcal{S}$ is weakly forward and backward Γ-recurrent by construction. Furthermore, $\mathcal{S}$ is g^t-invariant, and we may assume that $\mathcal{S}$ is invariant under Γ.

Lemma 9.10. *If $v_n \to v$ weakly, and $v \in \mathcal{Z}$, then $v_n \to v$ strongly.*

Proof. Let $v \in \mathcal{Z}$, and suppose $v_n \to v$. Take an arbitrary subsequence of (v_n). By Lemma 6.9, there is a further subsequence that converges strongly to some $u \sim v$. By Theorem 9.9, $u = v$. Thus we have shown that every subsequence of (v_n) contains a further subsequence that converges strongly to v. Therefore, $v_n \to v$ strongly. □

Corollary 9.11. *Every $v \in \mathcal{S}$ is strongly forward and backward Γ-recurrent.*

Corollary 9.12. *The restriction of π_x to $\mathcal{Z}$ is a homeomorphism onto its image.*

Proof. Fix $x \in X$. By definition, $\pi_x|_{\mathcal{Z}}$ is injective, hence bijective onto its image. Since π_x is continuous (that is, every strongly convergent sequence in $\mathcal{Z}$ is weakly convergent), it remains to observe that $\pi_x|_{\mathcal{Z}}^{-1}$ is continuous (that is, every weakly convergent sequence in $\mathcal{Z}$ is strongly convergent) by Lemma 9.10. □

Definition 9.13. By Corollary 9.12, $\pi_x|_{\mathcal{Z}}$ maps Borel sets to Borel sets, hence we may view m as a g^t- and Γ-invariant Borel measure on SX by setting $m(A) = m(\pi_x(A \cap \mathcal{Z}))$ for any Borel set $A \subseteq SX$. We will write m for this measure on SX, and m_Γ for the corresponding finite Borel measure on $\Gamma \backslash SX$.

Proposition 9.14. *The Bowen-Margulis measure m on SX has full support.*

Proof. For clarity, we write m_{down} for the measure m on $\mathcal{G}^E \times \mathbb{R}$ and m_{up} for the measure m on SX defined by $m_{up}(A) = m_{down}(\pi_x(A \cap \mathcal{Z}))$ for all Borel sets $A \subseteq SX$. Our goal is to show that $\operatorname{supp}(m_{up}) = SX$.

Recall (Proposition 8.5) that since X has a rank one axis, there is some $w_0 \in SX$ with dense orbit in SX mod Γ. By upper semicontinuity of the width function $v \mapsto \operatorname{diam}(Y_v)$ on SX, we know $w_0 \in \mathcal{Z}$. Since the orbit of $w_0 \in \mathcal{Z}$ is dense in SX mod Γ, it follows that $SX = \overline{\mathcal{Z}}$.

We claim that $\overline{\mathcal{Z}} \subseteq \operatorname{supp}(m_{up})$. Since $\operatorname{supp}(m_{up})$ is closed, it suffices to show that $\mathcal{Z} \subseteq \operatorname{supp}(m_{up})$. So let $v \in \mathcal{Z}$ and let $U \subseteq SX$ be an open set containing v. Then $U \cap \mathcal{Z}$ is open in $\mathcal{Z}$ by definition, so $\pi_x(U \cap \mathcal{Z})$ is open in $\pi_x(\mathcal{Z})$ because $\pi_x|_{\mathcal{Z}}$ is a homeomorphism. This means $\pi_x(U \cap \mathcal{Z}) = V \cap \pi_x(\mathcal{Z})$ for some open set V of $\mathcal{G}^E \times \mathbb{R}$. But $\pi_x(v) \in V$, so V is nonempty. Recall (Lemma 8.1) that m_{down} has full support, so $m_{down}(V) > 0$. But $\pi_x(\mathcal{Z})$ has full measure in $\mathcal{G}^E \times \mathbb{R}$, and thus

$$m_{up}(U) = m_{down}(\pi_x(U \cap \mathcal{Z})) = m_{down}(V \cap \pi_x(\mathcal{Z})) = m_{down}(V) > 0.$$

Hence $v \in \operatorname{supp}(m_{up})$, as claimed.

We have now shown $SX \subseteq \overline{\mathcal{Z}} \subseteq \operatorname{supp}(m_{up})$. Thus $\operatorname{supp}(m_{up}) = SX$. □

Note that $\mathcal{S}$ and $\mathcal{Z}$ both have full m-measure in SX, but $\mathcal{S}$ has additional strong recurrence properties. For this reason, we will use $\mathcal{S}$ in the future.

In Proposition 7.5, we mentioned if X has some rank one axis, then the rank one axes of X are weakly dense in $\mathcal{R}$. We can now improve that result.

Corollary 9.15. *The rank one axes of X are strongly dense in SX.*

Proof. By Proposition 9.14, we know that the zero-width geodesics are dense in SX, hence it suffices to prove that every $v \in \mathcal{R}$ with $\operatorname{diam}(Y_v) = 0$ is a strong limit of rank one axes. By Proposition 7.5, the rank one axes of X are weakly dense in $\mathcal{R}$, so we have a sequence (v_n) of rank one axes such that $v_n \to v$ weakly. By Lemma 6.9, some subsequence of (v_n) converges strongly to some $u \sim v$. But $u = v$ because $\operatorname{diam}(Y_v) = 0$, thus v is a strong limit of rank one axes. □

We now come to our third main theorem. For a group G acting measurably on a space Z, a G-invariant measure ν on Z is *ergodic* under the action of G if every G-invariant measurable set $A \subseteq Z$ has either $\nu(A) = 0$ or $\nu(Z \setminus A) = 0$. If G preserves only the measure class of ν, and every G-invariant set has either zero or full ν-measure, ν is called *quasi-ergodic*.

Theorem 9.16 (Theorem 3)**.** *Let X and Γ satisfy the assumptions of Theorem 1. The Bowen-Margulis measure m_Γ is ergodic under the geodesic flow on $\Gamma\backslash SX$.*

Proof. We use the classical argument by Hopf ([24]) to show ergodicity. The goal is to show that every g_Γ^t-invariant $L^2(m_\Gamma)$ function is constant a.e. Let $H \subseteq L^2(m_\Gamma)$ be the closed subspace of g^t-invariant functions. Since the subspace $C(\Gamma\backslash SX)$ of continuous functions on $\Gamma\backslash SX$ is dense in $L^2(m_\Gamma)$, the L^2-projection π_H onto H maps $C(\Gamma\backslash SX)$ to a dense subspace of H. Thus it suffices to show that $\pi_H(f)$ is constant a.e., for every continuous $f\colon \Gamma\backslash SX \to \mathbb{R}$.

Let $f\colon \Gamma\backslash SX \to \mathbb{R}$ be a continuous function, and let $A_T(f)$ be the ergodic average $(A_T f)(v) = \frac{1}{T}\int_0^T f(g^t v)dt$. Let $f^+ = \lim_{T\to\infty} A_T(f)$ and $f^- = \lim_{T\to\infty} A_{-T}(f)$. Lift the maps f, f^+, f^- to $F, F^+, F^-\colon SX \to \mathbb{R}$, respectively, by precomposing with the canonical projection $SX \to \Gamma\backslash SX$. By von Neumann's mean ergodic theorem (see [19, Theorem 8.19], for example), f^+ and f^- exist and equal $\pi_H(f)$ m_Γ-a.e. Hence we may find a Γ-invariant Borel subset Ω of SX with $m(SX \setminus \Omega) = 0$ such that $F^+(v) = F^-(v)$ for every $v \in \Omega$. We may assume $\Omega \subseteq \mathcal{S}$, so every $v \in \Omega$ is forward and backward Γ-recurrent. We may also assume Ω is g^t-invariant because f^+ is g^t-invariant.

Now suppose $v \in \mathcal{S}$ and $w \in SX$ with $w^+ = v^+$. By Lemma 7.8, there are sequences $t_n \to +\infty$ and $\gamma_n \in \Gamma$ such that $\gamma_n g^{t_n}(w) \to v$. Write $w_n = \gamma_n g^{t_n}(w)$. Let $\epsilon > 0$ be given; by uniform continuity of F, there is some $\delta > 0$ such that $|F(a) - F(b)| < \epsilon$ whenever $a, b \in SX$ have $d(a,b) < \delta$. Find $N > 0$ such that $d(w_n, v) < \delta$ for all $n \geq N$. Since $w^+ = v^+$, by convexity we have $d(g^t(w_n), g^t(v)) \leq d(w_n, v) < \delta$ for all $n \geq N$ and $t \geq 0$. So by choice of δ, we have $\left|F(g^t w_n) - F(g^t v)\right| < \epsilon$ for all $n \geq N$ and $t \geq 0$. Thus

$$\limsup_{T\to\infty} |(A_T F)(w_n) - (A_T F)(v)| \leq \epsilon$$

for all $n \geq N$. But $g^t \gamma_n = \gamma_n g^t$, so

$$|(A_T F)(w) - (A_T F)(w_n)| = \frac{1}{T}\left|\int_0^T F(g^t w)\,dt - \int_0^T F(g^{t+t_n} w)\,dt\right|$$

by Γ-invariance of F; then

$$|(A_T F)(w) - (A_T F)(w_n)| = \frac{1}{T}\left|\int_0^T F(g^t w)\,dt - \int_{t_n}^{T+t_n} F(g^t w)\,dt\right|$$
$$\leq \frac{1}{T}\cdot 2\,|t_n|\cdot \sup_{u\in SX} |F(u)|$$

for any given n. Hence

$$\limsup_{T\to\infty} |(A_T F)(w) - (A_T F)(w_n)| = 0$$

for all n, and thus

$$\limsup_{T\to\infty} |(A_T F)(w) - (A_T F)(v)| \leq \epsilon.$$

But $\epsilon > 0$ was arbitrary, so $F^+(w) = F^+(v)$.

Thus, for every $v \in \mathcal{S}$, we have shown that $F^+(w) = F^+(v)$ for all w with $w^+ = v^+$. By similar argument, $F^-(w) = F^-(v)$ for all w with $w^- = v^-$. But $F^+ = F^-$ on $\Omega \subseteq \mathcal{S}$, so we may apply Lemma 9.5 with $\psi(v^-, v^+) = F^+(v)$. □

Corollary 9.17. *If $f\colon SX \to \mathbb{R}$ is a measurable function that is both Γ- and g^t-invariant, then f is constant m-a.e.*

Proof. By Γ-invariance, f descends to a measurable map $f_\Gamma\colon \Gamma\backslash SX \to \mathbb{R}$. By g^t-invariance of f, Theorem 9.16 forces f_Γ to be constant m_Γ-a.e. Thus f must be constant m-a.e. by Proposition 8.3 (1). □

It follows that the diagonal action of Γ on $(\mathcal{G}^E, \mu)$ is ergodic. Since μ and $\mu_x \times \mu_x$ are in the same measure class (see Corollary 9.2), the diagonal action of Γ on $(\partial X \times \partial X, \mu_x \times \mu_x)$ is quasi-ergodic. It follows that the Γ-action on $(\partial X, \mu_x)$ is also quasi-ergodic.

CHAPTER 10

On Links

It is convenient here to recall a few properties of links in CAT(κ) spaces. CAT(κ) spaces, like CAT(0) spaces, satisfy a triangle comparison requirement for small triangles, but the comparison is to a triangle in a complete, simply connected manifold of constant curvature κ. One may always put $\kappa = 0$ in this chapter, which will be the only case we use later in this thesis.

We will begin with the definition of a link, and then give a proof of Proposition 10.6. Lytchak ([33]) states a version of this result when Y is CAT(1) and compact, but we need to allow Y to be proper in place of compact.

Definition 10.1. Let Y be a CAT(κ) space and $p \in Y$. Write Σ_p for the space of geodesic germs in Y issuing from p, equipped with the metric $\angle_p$, and write $\mathrm{Lk}(p)$ (called the *link of p* or the *link of Y at p*) for the completion of Σ_p (cf. [9] or [33]).

By Nikolaev's theorem (Theorem II.3.19 in [9]), the link of $p \in Y$ is CAT(1). Note also that if Y is proper and geodesically complete, Σ_p is already complete.

Definition 10.2. Let Y be a CAT(κ) space and $p \in Y$. The *tangent cone at p*, denoted T_pY, is the Euclidean cone on $\mathrm{Lk}(p)$, the link of p.

Lemma 10.3. *Let Y be compact and the Gromov-Hausdorff limit of a sequence (Y_i) of compact metric spaces. Then any sequence of isometric embeddings $\sigma_i\colon [0,1] \to Y_i$ has a subsequence that converges to an isometric embedding $\sigma\colon [0,1] \to Y$.*

Proof. A subsequence of the spaces Y_i, together with Y, may be isometrically embedded into a single compact metric space. Apply the Arzelà-Ascoli theorem. □

We recall the following result from [9], used there to prove Nikolaev's theorem. Here M_κ^2 denotes the complete, simply connected, Riemannian manifold of constant curvature κ and dimension 2.

Lemma 10.4 (Lemma II.3.20 in [9]). *For each $\kappa > 0$ there is a function $C\colon [0, D_\kappa) \to \mathbb{R}$ such that $\lim_{r\to 0} C(r) = 1$ and for all $p \in M^2_\kappa$ and all $x, y \in B(p, r)$*

$$d(\epsilon x, \epsilon y) \leq \epsilon C(r) d(x, y),$$

where ϵx denotes the point distance $\epsilon d(p, x)$ from p on the geodesic $[p, x]$.

One may find another proof of the following lemma in [10, Theorem 9.1.48].

Lemma 10.5. *Let (Y, d) be a proper, geodesically complete* CAT(κ) *space, $\kappa \in \mathbb{R}$, and let $p \in Y$. Fix $r > 0$, and for $t \in (0, 1]$, let (Y_t, d_t) be the compact metric space $(\overline{B}_Y(p, rt), \frac{1}{t} d)$. Let (Y_0, d_0) be the closed ball of radius r about the cone point $\bar{p}$ in the tangent cone $T_p Y$ at p. Then $Y_t \to Y_0$ in the Gromov-Hausdorff metric as $t \to 0$.*

Proof. We may assume $r > 0$ is sufficiently small that Y_1 is uniquely geodesic. For each $y \in Y$, let $\sigma_y\colon [0, 1] \to Y$ be the constant-speed geodesic with $\sigma(0) = p$ and $\sigma(1) = y$. For $t \in (0, 1]$, let $\rho_t\colon Y_1 \to Y_t$ be the map $\rho_t(y) = \sigma_y(t)$. Let $\rho_0\colon Y_1 \to Y_0$ be the map sending y to the point of $T_p X$ that is distance $d(p, y)$ from the cone point $\bar{p}$ and (for $y \neq p$) in the direction of the germ of σ_y in the link.

For $t \in [0, 1]$ and $y, z \in Y_1$, let $f_t(y, z) = d_t(\rho_t(y), \rho_t(z))$. Note that for $y, z \in Y_1$ fixed, $f_t(y, z) \to f_0(y, z)$ as $t \to 0$. Now if $\kappa \leq 0$, then convexity of the metric on Y gives us $f_s \leq f_t$ for all $s \leq t$. If $\kappa > 0$, it follows from Lemma 10.4 that $f_s \leq C(t) \cdot f_t$ for all $s \leq t$, where $C(t) \to 1$ as $t \to 0$. Thus $f_t \to f_0$ uniformly as $t \to 0$. Each ρ_t is surjective, so this proves the lemma. □

Proposition 10.6. *For any proper* CAT(κ) *space Y, $\kappa \in \mathbb{R}$, the following are equivalent:*

1. *Y is geodesically complete.*
2. *For every point $p \in Y$, the tangent cone $T_p Y$ at p is geodesically complete.*
3. *For every point $p \in Y$, the link* Lk(p) *of p is geodesically complete and has at least two points.*
4. *For every point $p \in Y$, every point in the link* Lk(p) *of p has at least one antipode—that is, for every $\alpha \in \mathrm{Lk}(p)$, there is some $\beta \in \mathrm{Lk}(p)$ such that $d(\alpha, \beta) \geq \pi$.*

Proof. The implication (1) $\implies$ (2) is clear from Lemmas 10.3 and 10.5. And (2) $\implies$ (3) is immediate from the fact that radial projection $T_p Y \to \mathrm{Lk}(p)$ is a bijective map on geodesics (see the proof of Proposition I.5.10(1) in [9]). Since Y is CAT(κ), each component of the link $\mathrm{Lk}(p)$ of p is CAT(1) and therefore has no geodesic circles of length $< \pi$; thus (3) $\implies$ (4).

Finally, we prove (4) $\implies$ (1). Let $r > 0$ be small enough that geodesics in Y of length $< 3r$ are uniquely determined by their endpoints. It suffices to show that for $p, q \in Y$ with $d(p,q) \leq r$, there is some $y_0 \in Y$ such that $d(p, y_0) = r$ and $b_q(y_0, p) = d(y_0, p)$ (recall $b_q(y_0, p) = d(q, y_0) - d(q, p)$). So let $f_p(y) = b_q(y, p)$. Let $\delta > 0$, and define $A_\delta = \left\{y \in Y \mid f_p(y) \geq (1-\delta)d(y,p)\right\}$. Since Y is proper and A_δ is closed, $A_\delta \cap \overline{B}(p,r)$ compact. Thus some $y = y_\delta \in A_\delta \cap \overline{B}(p,r)$ maximizes f_p on $A_\delta \cap \overline{B}(p,r)$. If $d(p,y) < r$ then by (4) and the density of Σ_p in $\mathrm{Lk}(p)$, some $z \in Y$ with $d(y,z) \leq r - d(p,y)$ satisfies $f_y(z) \geq (1-\delta)d(z,y) > 0$ (assume $\delta < 1$). Since $f_p(z) = f_y(z) + f_p(y)$ by the cocyle property of Busemann functions, we obtain $f_p(z) > f_p(y)$; furthermore, $z \in A_\delta$ by the triangle inequality. But this contradicts the maximality of y, hence we must have $d(p,y) = r$. Now take a sequence $\delta_n \to 0$ and let y_0 be a limit point of y_{δ_n}. Then $d(p, y_0) = r$ and $f_p(y_0) = d(y_0, p)$, as required. $\square$

CHAPTER 11

Cross-Ratios

Our proof of mixing of the geodesic flow on Bowen-Margulis measures is inspired by Babillot's treatment for the smooth manifold case ([3]), which involves the cross-ratio for endpoints of geodesics. So we will extend the theory of cross-ratios to CAT(0) spaces.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Assume that $|\partial X| > 2$, and that X admits a rank one axis.

First we need to describe the space where cross-ratios will be defined.

Definition 11.1. For $v^-, w^-, v^+, w^+ \in \partial X$, call (v^-, w^-, v^+, w^+) a *quadrilateral* if there exist rank one geodesics with endpoints (v^-, v^+), (w^-, w^+), (v^-, w^+), and (w^-, v^+). Denote the set of quadrilaterals by $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$.

Definition 11.2. Let $\mathcal{Q}_{\mathrm{SX}} = \{(v, w) \in SX \times SX \mid (v^-, w^-, v^+, w^+) \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}\}$.

By Lemma 4.7, $\mathcal{Q}_{\mathrm{SX}}$ is an open neighborhood of the diagonal in $\mathcal{R} \times \mathcal{R}$.

Definition 11.3. For a quadrilateral (v^-, w^-, v^+, w^+), define its *cross-ratio* by

$$\mathrm{B}(v^-, w^-, v^+, w^+) = \beta_p(v^-, v^+) + \beta_p(w^-, w^+) - \beta_p(v^-, w^+) - \beta_p(w^-, v^+),$$

for $p \in X$ arbitrary.

Note that we removed reference to $p \in X$ in writing B in Definition 11.3. This omission is justified by the following lemma.

Lemma 11.4. *The cross-ratio* $\mathrm{B}(\xi, \xi', \eta, \eta')$ *of a quadrilateral* (ξ, ξ', η, η') *does not depend on choice of* $p \in X$.

Proof. Let $v_0 \in \mathrm{E}^{-1}(\xi,\eta)$, $v_1 \in \mathrm{E}^{-1}(\xi,\eta')$, $v_2 \in \mathrm{E}^{-1}(\xi',\eta')$, and $v_3 \in \mathrm{E}^{-1}(\xi',\eta)$. By Lemma 6.2 and the definition of the cross-ratio,

$$\begin{aligned}\mathrm{B}(\xi,\xi',\eta,\eta') =& [b_\xi(v_0(0),p)+b_\eta(v_0(0),p)]+[b_{\xi'}(v_2(0),p)+b_{\eta'}(v_2(0),p)] \\ &-[b_\xi(v_1(0),p)+b_{\eta'}(v_1(0),p)]-[b_{\xi'}(v_3(0),p)+b_\eta(v_3(0),p)]\end{aligned}$$

for any $p \in X$. Using the cocycle property of Busemann functions, this gives us

$$\begin{aligned}\mathrm{B}(\xi,\xi',\eta,\eta') =& b_\xi(v_0(0),v_1(0))+b_\eta(v_0(0),v_3(0)) \\ &+b_{\eta'}(v_2(0),v_1(0))+b_{\xi'}(v_2(0),v_3(0)),\end{aligned}$$

which is independent of $p \in X$. □

Definition 11.5. Define the *cross-ratio* of a pair of geodesics $(v,w) \in \mathcal{Q}_{\mathrm{SX}}$ by

$$\mathrm{B}(v,w) = \mathrm{B}(v^-,w^-,v^+,w^+).$$

Remark. One word of caution: We have followed the convention in the literature in our ordering of terms in the cross-ratio of a quadrilateral. However, this means that the cross-ratio of a pair of geodesics is not given by $\mathrm{B}\circ(\mathrm{E}\times\mathrm{E})$, but rather by the map $\mathrm{B}\circ\tau\circ(\mathrm{E}\times\mathrm{E})$, where τ is the map the flips that second and third components of $(\partial X)^4$.

The following proposition summarizes some of the basic properties of the cross-ratio (cf. [23]). The proofs are straightforward.

Proposition 11.6. *The cross-ratio on $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ is continuous and satisfies all the following.*

1. B *is invariant under the diagonal action of* $\operatorname{Isom} X$ *on* $(\partial X)^4$*,*
2. $\mathrm{B}(\xi,\xi',\eta,\eta') = -\mathrm{B}(\xi,\xi',\eta',\eta)$*,*
3. $\mathrm{B}(\xi,\xi',\eta,\eta') = \mathrm{B}(\eta,\eta',\xi,\xi')$*,*
4. $\mathrm{B}(\xi,\xi',\eta,\eta')+\mathrm{B}(\xi,\xi',\eta',\eta'') = \mathrm{B}(\xi,\xi',\eta,\eta'')$*, and*
5. $\mathrm{B}(\xi,\xi',\eta,\eta')+\mathrm{B}(\xi',\eta,\xi,\eta')+\mathrm{B}(\eta,\xi,\xi',\eta') = 0.$

Property (4) above is called the *cocycle property* of the cross-ratio. We will not use the curious property (5) later in this work.

We will now show (Lemma 11.9 below) that the translation length of any hyperbolic isometry of X is given by some appropriately chosen cross-ratio, up to a factor of 2. For negatively curved manifolds, the result is known and due to Otal ([38]). The proof outline given by Dal'bo ([14]) for Fuchsian groups extends readily to CAT(0) spaces; we include the details of the proof for completeness.

Write $\ell(\gamma)$ for the *translation length* $\ell(\gamma) = \inf_{x \in X} d(x, \gamma x)$ of any $\gamma \in \operatorname{Isom} X$. If there is some $x \in X$ such that $d(x, \gamma x) = \ell(\gamma)$, we say γ is *hyperbolic*. Then $x = v(0)$ for some geodesic $v \in SX$ with $\gamma v = g^{\ell(\gamma)} v$. Such a geodesic $v \in SX$ is called an *axis* of γ. For any hyperbolic isometry $\gamma \in \operatorname{Isom} X$, write $\gamma^+ = v^+$ and $\gamma^- = v^-$ for some (any) axis v of γ.

Lemma 11.7. *Let γ be a hyperbolic isometry of X. Then for all $x \in X$,*

$$b_{\gamma^-}(x, \gamma^{-1} x) = b_{\gamma^-}(\gamma x, x) = b_{\gamma^+}(x, \gamma x) = b_{\gamma^+}(\gamma^{-1} x, x) = \ell(\gamma).$$

Proof. The statement holds for all x on an axis of γ. Since isometries fixing $\xi \in \partial X$ preserve the foliation of X by horospheres based at ξ, the statement must hold for all $x \in X$. □

Lemma 11.8. *Let γ be a hyperbolic isometry of X. Then*

$$\beta_x(\gamma\xi, \gamma\eta) = \beta_x(\xi, \eta) + (b_\xi + b_\eta)(x, \gamma^{-1} x)$$

for all $\xi, \eta \in \partial X$ and $x \in X$.

Proof. Let $v \in \mathrm{E}^{-1}(\xi, \eta)$. Using the definition of β_x and the cocycle property of Busemann functions,

$$\begin{aligned}
\beta_x(\gamma\xi, \gamma\eta) &= (b_{\gamma v^-} + b_{\gamma v^+})(\gamma v(0), x) \\
&= (b_{v^-} + b_{v^+})(v(0), \gamma^{-1} x) \\
&= (b_{v^-} + b_{v^+})(v(0), x) + (b_{v^-} + b_{v^+})(x, \gamma^{-1} x) \\
&= \beta_x(\xi, \eta) + (b_\xi + b_\eta)(x, \gamma^{-1} x).
\end{aligned}$$

□

Lemma 11.9. *Let γ be a hyperbolic isometry of X. Then*

$$\mathrm{B}(\gamma^-, \gamma^+, \gamma\xi, \xi) = 2\ell(\gamma)$$

for all $\xi \in \partial X$ that are Tits distance $> \pi$ from both γ^- and γ^+.

Proof. By Lemma 11.8,

$$\beta_x(\gamma^-, \gamma\xi) - \beta_x(\gamma^-, \xi) = \beta_x(\gamma(\gamma^-), \gamma(\xi)) - \beta_x(\gamma^-, \xi) = (b_{\gamma^-} + b_\xi)(x, \gamma^{-1} x)$$

and

$$\beta_x(\gamma^+,\xi)-\beta_x(\gamma^+,\gamma\xi) = -(b_{\gamma^+}+b_\xi)(x,\gamma^{-1}x).$$

So

$$\begin{aligned}
\mathrm{B}(\gamma^-,\gamma^+,\gamma\xi,\xi) &= (b_{\gamma^-}+b_\xi)(x,\gamma^{-1}x)-(b_{\gamma^+}+b_\xi)(x,\gamma^{-1}x)\\
&= b_{\gamma^-}(x,\gamma^{-1}x)-b_{\gamma^+}(x,\gamma^{-1}x)\\
&= 2\ell(\gamma)
\end{aligned}$$

by Lemma 11.7. □

In the case that X is a tree, Lemma 11.9 implies that $\mathrm{B}(\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}})$ contains all the translation lengths of hyperbolic elements of $\operatorname{Isom} X$. The following lemma implies, in particular, the slightly stronger statement that if X is a tree (with no vertices of valence 2), then $\mathrm{B}(\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}})$ contains all the edge lengths of X. We will use this fact in the proof of Lemma 12.6.

Lemma 11.10. *Suppose the link of $p,q \in X$ each has ≥ 3 components. Then there is some $(\xi,\xi',\eta,\eta') \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ such that $\mathrm{B}(\xi,\xi',\eta,\eta') = 2d(p,q)$.*

Proof. Let $r = d(p,q)$, and let $\rho_p\colon \partial X \to \mathrm{Lk}(p)$ and $\rho_q\colon \partial X \to \mathrm{Lk}(q)$ be radial projection onto the links of p and q. Find geodesics $v,w \in SX$ such that

1. $v(0) = w(r) = p$ and $v(r) = w(0) = q$,
2. $\rho_p(v^-),\rho_p(w^+),\rho_p(v^+)$ lie in distinct components of $\mathrm{Lk}(p)$, and
3. $\rho_q(v^+),\rho_q(w^-),\rho_q(w^+)$ lie in distinct components of $\mathrm{Lk}(q)$.

One easily computes $\mathrm{B}(v^-,w^-,v^+,w^+) = 2r$. □

By Lemma 11.9, we can calculate the translation length of any hyperbolic isometry of X in terms of cross-ratios. The next lemma shows that we can calculate any cross-ratio in $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ in terms of translation lengths of hyperbolic isometries of X. For negatively curved manifolds, the result is due to Kim ([28]) and Otal ([38]). Our proof follows the one given by Dal'bo ([14]) for Fuchsian groups.

Lemma 11.11. *Let $g_1,g_2 \in \Gamma$ be rank one hyperbolic isometries with g_1^-, g_1^+, g_2^-, and g_2^+ all distinct. Then*

$$\mathrm{B}(g_1^-,g_2^-,g_1^+,g_2^+) = \lim_{n\to\infty}\left[\ell(g_1^n)+\ell(g_2^n)-\ell(g_1^n g_2^n)\right].$$

Proof. By Lemma 4.9, $g_1^n g_2^n$ is hyperbolic for all sufficiently large n. Let $\xi_n = (g_1^n g_2^n)^+$. Then for all $x \in X$,

$$\ell(g_1^n) + \ell(g_2^n) - \ell(g_1^n g_2^n) = b_{g_1^-}(x, g_1^{-n} x) + b_{g_2^-}(x, g_2^{-n} x) + b_{(g_1^n g_2^n)^-}(x, (g_1^n g_2^n)^{-1} x)$$

by Lemma 11.7. But this equals

$$b_{g_1^-}(x, g_1^{-n} x) + b_{g_2^-}(x, g_2^{-n} x) + b_{g_2^n \xi_n}(x, g_1^{-n} x) + b_{\xi_n}(x, g_2^{-n} x)$$

by the cocycle property of Busemann functions. So this equals

$$\left[\beta_x(g_1^n g_1^-, g_1^n g_2^n \xi_n) - \beta_x(g_1^-, g_2^n \xi_n)\right] + \left[\beta_x(g_2^n g_2^-, g_2^n \xi_n) - \beta_x(g_2^-, \xi_n)\right],$$

by Lemma 11.8. This equals

$$\beta_x(g_1^-, \xi_n) + \beta_x(g_2^-, g_2^n \xi_n) - \beta_x(g_2^-, g_2^n \xi_n) - \beta_x(g_2^-, \xi_n),$$

which equals $\mathrm{B}(g_1^-, g_2^-, \xi_n, g_2^n \xi_n)$ by definition.

We now show $\xi_n \to g_1^+$ and $g_2^n \xi_n \to g_2^+$. Let $U, V, U', V' \subset \overline{X}$ be pairwise-disjoint neighborhoods of $g_1^+, g_2^+, g_1^-, g_2^-$ (respectively), and let $x \in X$. By Lemma 4.9, for all sufficiently large n we have $g_1^n(U \cup V) \subset U$, $g_2^n(U \cup V) \subset V$, and $g_2^n x \in V$. Hence $(g_1^n g_2^n)^k x \in U$ for all $k > 0$, and therefore $\xi_n = (g_1^n g_2^n)^+ \in U$ for all sufficiently large n. Then $g_2^n \xi_n \in V$ for all sufficiently large n, too. But this holds for arbitrarily small neighborhoods U, V of g_1^+, g_2^+ (respectively), so $\xi_n \to g_1^+$ and $g_2^n \xi_n \to g_2^+$. Thus

$$\lim_{n\to\infty} \left[\ell(g_1^n) + \ell(g_2^n) - \ell(g_1^n g_2^n)\right] = \lim_{n\to\infty} \mathrm{B}(g_1^-, g_2^-, \xi_n, g_2^n \xi_n) = \mathrm{B}(g_1^-, g_2^-, g_1^+, g_2^+)$$

by continuity of the cross-ratio, which proves the lemma. □

The next lemma describes how the cross-ratio detects, to some extent, the non-integrability of the stable and unstable horospherical foliations.

Lemma 11.12. *Suppose $(\xi, \xi', \eta, \eta') \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$. Let $v_0 \in \mathrm{E}^{-1}(\xi, \eta)$, and recursively choose $v_1 \in H^u(v_0)$ with $v_1^+ = \eta'$, $v_2 \in H^s(v_1)$ with $v_2^- = \xi'$, $v_3 \in H^u(v_2)$ with $v_3^+ = \eta$, and $v_4 \in H^s(v_3)$ with $v_4^- = \xi$. Then $v_4 \sim g^{t_0} v_0$, for $t_0 = \mathrm{B}(\xi, \xi', \eta, \eta')$.*

Proof. As in the proof of Lemma 11.4, we know

$$\begin{aligned}\mathrm{B}(\xi, \xi', \eta, \eta') = &b_\xi(v_0(0), v_1(0)) + b_\eta(v_0(0), v_3(0)) \\ &+ b_{\eta'}(v_2(0), v_1(0)) + b_{\xi'}(v_2(0), v_3(0)).\end{aligned}$$

But $b_\xi(v_0(0), v_1(0)) = b_{\eta'}(v_1(0), v_2(0)) = b_{\xi'}(v_2(0), v_3(0)) = b_\eta(v_3(0), v_4(0)) = 0$ by choice of $v_1, \ldots, v_4$, so

$$\begin{aligned} \mathrm{B}(\xi, \xi', \eta, \eta') &= b_\eta(v_0(0), v_3(0)) + b_\eta(v_3(0), v_4(0)) \\ &= b_\eta(v_0(0), v_4(0)) \end{aligned}$$

by the cocycle property of Busemann functions. On the other hand, $v_4 \parallel v_0$ by construction, and so by Proposition 6.7, $v_4 \sim g^t v_0$ for the value $t \in \mathbb{R}$ such that $b_\eta(v_0(t), v_4(0)) = 0$. But

$$b_\eta(v_0(t), v_4(0)) = -t + b_\eta(v_0(0), v_4(0)) = -t + \mathrm{B}(\xi, \xi', \eta, \eta')$$

for all t, which shows $v_4 \sim g^{t_0} v_0$ for $t_0 = \mathrm{B}(\xi, \xi', \eta, \eta')$. $\square$

Lemma 11.13. *The cross-ratio* B *is the continuous extension of the function*

$$\mathcal{B}(p, p', q, q') = d(p, q) + d(p', q') - d(p, q') - d(p', q)$$

on X^4 *to* $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$.

Proof. Fix $p \in X$. For all $z \in \overline{X}$, let $h_z(y) = b_z(y, x)$ for $y \in X$. Define, for $Q = (p, p', q, q') \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}} \cup X^4$,

$$\begin{aligned} d_1(Q) &= \inf_{y \in X}(h_p(y) + h_q(y)) \\ d_2(Q) &= \inf_{y \in X}(h_{p'}(y) + h_{q'}(y)) \\ d_3(Q) &= \inf_{y \in X}(h_p(y) + h_{q'}(y)) \\ d_4(Q) &= \inf_{y \in X}(h_{p'}(y) + h_q(y)). \end{aligned}$$

Now $y \in X$ minimizes $h_p(y) + h_q(y)$ if and only if y lies on a geodesic joining p to q in X. So if $Q \in X^4$, let y_0 be an arbitrary point on the geodesic joining p to q in X, and we have

$$\begin{aligned} \inf_{y \in X}(h_p(y) + h_q(y)) &= (d(p, y_0) - d(p, x)) + (d(q, y_0) - d(q, x)) \\ &= d(p, q) - d(p, x) - d(q, x). \end{aligned}$$

Similarly,

$$
\begin{aligned}
d_2(Q) &= d(p',q') - d(p',x) - d(q',x) \\
d_3(Q) &= d(p,q') - d(p,x) - d(q',x) \\
d_4(Q) &= d(p',q) - d(p',x) - d(q,x).
\end{aligned}
$$

Thus $\mathcal{B}(Q) = d_1(Q) + d_2(Q) - d_3(Q) - d_4(Q)$ for $Q \in X^4$. Hence we may define $\mathcal{B}$ on $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ by $\mathcal{B}(Q) = d_1(Q) + d_2(Q) - d_3(Q) - d_4(Q)$ for all $Q \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$.

It now suffices, by symmetry, to show that d_1 is continuous on $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}} \cup X^4$. So let $Q = (p, p', q, q') \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}} \cup X^4$; this gives us four geodesics with respective endpoints (p,q), (p',q'), (p,q'), and (p',q). Note the footpoints of all four geodesics (or the whole geodesic segment, if finite) lie in some ball $B(x,r)$ about $x \in X$, so by Lemma 4.7 we may find neighborhoods U, U', V, V' of p, p', q, q' and $R > 0$ such that every pair $(\xi, \eta) \in (U \cup U') \times (V \cup V')$ is connected by a geodesic, and every such geodesic enters $B(x,R)$. By properness of X, the compact set $K = \overline{B}(x,R)$ is compact. Now take any sequence $Q_n \to Q$; we may assume each $Q_n \in U \times U' \times V \times V'$. If $Q_n = (p_n, p'_n, q_n, q'_n)$ then we must have $(h_{p_n}, h_{p'_n}, h_{q_n}, h_{q'_n}) \to (h_p, h'_p, h_q, h'_q)$ uniformly on K. Hence $\inf_K(h_{p_n} + h_{q_n}) \to \inf_K(h_p + h_q)$. But $\inf_K(h_p + h_q) = \inf_X(h_p + h_q) = d_1(Q)$ and $\inf_K(h_{p_n} + h_{q_n}) = \inf_X(h_{p_n} + h_{q_n}) = d_1(Q_n)$ by choice of R since the infimum is obtained at every point on the corresponding geodesic. Thus $d_1(Q_n) \to d_1(Q)$, and $\mathcal{B}$ is continuous on $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}} \cup X^4$. □

CHAPTER 12

Mixing

We now establish mixing.

Standing Hypothesis. In this chapter, let Γ be a group acting properly, cocompactly, by isometries on a proper, geodesically complete CAT(0) space X. Assume that $|\partial X| > 2$, and that X admits a rank one axis.

The following lemma, which we will not prove, comes from the results in the first chapter of Babillot's paper ([3]). For context, recall that for a locally compact group G acting measurably on a space Z, a finite G-invariant measure ν on Z is *mixing* under the action of G if, for every pair of measurable sets $A, B \subseteq Z$, and every sequence $g_n \to \infty$ in G, we have $\nu(A \cap g_n B) \to \frac{\nu(A)\,\nu(B)}{\nu(Z)}$.

Lemma 12.1. *Let $(Y, \mathcal{B}, \nu, (T_t)_{t \in A})$ be a measure-preserving dynamical system, where $(Y, \mathcal{B})$ is a standard Borel space, ν a Borel measure on $(Y, \mathcal{B})$ and $(T_t)_{t \in A}$ an action of a locally compact, second countable, Abelian group A on Y by measure-preserving transformations. Let $\varphi \in L^2(\nu)$ be a real-valued function on Y such that $\int \varphi \, d\nu = 0$ if ν is finite.*

If there exists a sequence (t_n) going to infinity in A such that $\varphi \circ T_{t_n}$ does not converge weakly to 0, then there exists a sequence (s_n) going to infinity in A and a nonconstant function ψ in $L^2(\nu)$ such that $\varphi \circ T_{s_n} \to \psi$ and $\varphi \circ T_{-s_n} \to \psi$ weakly in $L^2(\nu)$. Furthermore, both the forward and backward Cesaro averages

$$A^+_{N^2} = \frac{1}{N^2} \sum_{n=1}^{N^2} \varphi \circ T_{s_n} \qquad \text{and} \qquad A^-_{N^2} = \frac{1}{N^2} \sum_{n=1}^{N^2} \varphi \circ T_{-s_n}$$

converge to ψ a.e.

Lemma 12.2. *Let $\psi \colon SX \to \mathbb{R}$ be a measurable function. Suppose $\Omega \subseteq \mathcal{R}^E$ is a set of full μ-measure such that ψ is constant on both $H^s(v) \cap \mathrm{E}^{-1}(\Omega)$ and $H^u(v) \cap \mathrm{E}^{-1}(\Omega)$ for every $v \in \mathrm{E}^{-1}(\Omega)$. Further suppose the map $t \mapsto \psi(g^t v)$ is continuous for every $v \in \mathrm{E}^{-1}(\Omega)$. Then* *there is a set $\Omega' \subseteq \Omega$ of full μ-measure such that for every $v \in \mathrm{E}^{-1}(\Omega')$, every nonzero $\left|\mathrm{B}(v^-, w^-, v^+, w^+)\right|$ is a period of $t \mapsto \psi(g^t v)$, for $(v^-, w^-, v^+, w^+) \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$.*

Proof. Let $U \times U' \times V \times V' \subseteq \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ be an arbitrary nonempty product neighborhood. Since $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ is second countable, it suffices to show that the conclusion of the lemma holds for a.e. $(\xi, \xi', \eta, \eta') \in U \times U' \times V \times V'$. So let

$$\begin{aligned}
\Omega_- &= \{\xi \in U \mid (\xi, \eta') \in \Omega \text{ for a.e. } \eta' \in V'\} \\
\Omega'_- &= \{\xi' \in U' \mid (\xi', \eta') \in \Omega \text{ for a.e. } \eta' \in V'\} \\
\Omega_+ &= \{\eta \in V \mid (\xi', \eta) \in \Omega \text{ and } \xi' \in \Omega'_- \text{ for a.e. } \xi' \in U'\}.
\end{aligned}$$

By Fubini's theorem, Ω_- has full measure in U and Ω'_- has full measure in U'. Since $\Omega \cap (\Omega'_- \times V)$ has full measure in $U' \times V$, Ω_+ has full measure in V. Thus $\Omega_- \times \Omega_+$ has full measure in $U \times V$.

Let $(\xi, \eta) \in \Omega_- \times \Omega_+$. Because $\eta \in \Omega_+$, a.e. $\xi' \in U'$ has $(\xi', \eta) \in \Omega$ and $\xi' \in \Omega'_-$, so let ξ' be such a point in U'. Because $\xi \in \Omega_-$ and $\xi' \in \Omega'_-$, a.e. $\eta' \in V'$ has both $(\xi, \eta'), (\xi', \eta') \in \Omega$, so let η' be such a point in V'. Thus all four pairs $(\xi, \eta), (\xi', \eta), (\xi, \eta'), (\xi', \eta')$ lie in Ω.

Let $v \in \mathrm{E}^{-1}(\xi, \eta)$. Follow the procedure in the statement of Lemma 11.12 to choose $v_1, \ldots, v_4 \in \mathcal{R}$. Since all our geodesics lie in $\mathrm{E}^{-1}(\Omega)$ by construction, $\psi(v) = \psi(v_1) = \psi(v_2) = \psi(v_3) = \psi(v_4)$ by hypothesis. But $v_4 \sim g^{t_0} v$ by Lemma 11.12, where $t_0 = \mathrm{B}(\xi, \xi', \eta, \eta')$, so $\psi(g^{t_0} v) = \psi(v)$. Thus $\mathrm{B}(\xi, \xi', \eta, \eta')$ is a period of $t \mapsto \psi(g^t v)$ for every $v \in \mathrm{E}^{-1}(\xi, \eta)$.

Since μ has full support, there is a sequence (ξ'_n, η'_n) in $\mathcal{R}^E$ converging to (ξ', η') such that all four pairs $(\xi, \eta), (\xi'_n, \eta), (\xi, \eta'_n), (\xi'_n, \eta'_n)$ lie in Ω. By continuity of B, either $\mathrm{B}(\xi, \xi'_n, \eta, \eta'_n)$ is eventually constant at $\mathrm{B}(\xi, \xi', \eta, \eta')$, or the subgroup generated by $\{\mathrm{B}(\xi, \xi'_n, \eta, \eta'_n)\}$ is all of $\mathbb{R}$. This concludes the proof of the lemma. $\square$

Lemma 12.3. *Either m_Γ is mixing under the geodesic flow g^t_Γ on $\Gamma \backslash SX$, or there is some $c \in \mathbb{R}$ such that every cross-ratio $\mathrm{B}(v^-, w^-, v^+, w^+)$ of $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ lies in $c\mathbb{Z}$.*

Proof. Suppose m_Γ is not mixing. Then there is a continuous function $\bar{\varphi}$ on $\Gamma \backslash SX$ such that $\bar{\varphi} \circ g^t_\Gamma$ does not converge weakly to a constant function. By Lemma 12.1, there is a nonconstant function $\bar{\psi}_0$ on $\Gamma \backslash SX$ which is the a.e.-limit of Cesaro averages of $\bar{\varphi}$ for both positive and negative times.

Let $\varphi \colon SX \to \mathbb{R}$ be the lift of $\bar{\varphi}$ and let $\psi_0 \colon SX \to \mathbb{R}$ be the lift of $\bar{\psi}_0$. Note φ and ψ_0 are Γ-invariant, and there is a sequence $t_n \to +\infty$ such that

$$\psi_0 = \lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N^2} \varphi \circ g^{t_n} = \lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N^2} \varphi \circ g^{-t_n}$$

on a set $\Omega_0 \subseteq SX$ of full measure. We may assume $\Omega_0 \subseteq \mathcal{S}$.

Now for each $\alpha > 0$, let ψ_α be the smoothing $\psi_\alpha(v) = \frac{1}{\alpha}\int_0^\alpha \psi_0(g^s v)\, ds$ of ψ_0. By Fubini's theorem, there is a set $\Omega_0' \subseteq \mathcal{R}^E$ of full measure such that for every $v \in \mathrm{E}^{-1}(\Omega_0')$, we have $g^t v \in \Omega_0$ for a.e. $t \in \mathbb{R}$, and $\psi_\alpha(v)$ well-defined for all $\alpha > 0$. Write $\Omega = \mathrm{E}^{-1}(\Omega_0')$. Note that for every $v \in \Omega$ and every $\alpha > 0$, the map $t \mapsto \psi_\alpha(g^t v)$ is not only well-defined but absolutely continuous on all $\mathbb{R}$.

We claim ψ_{α_0} is not constant a.e., for some $\alpha_0 > 0$. Otherwise, every $\alpha > 0$ must have ψ_α constant a.e. But then for every $\alpha > 0$, by Fubini's theorem we have a set $\Omega_\alpha' \subseteq \Omega_0'$ of full measure such that for every $v \in \mathrm{E}^{-1}(\Omega_\alpha')$, the map $t \mapsto \psi_\alpha(g^t v)$ is constant a.e. on $\mathbb{R}$. By continuity, this map must be constant on all $\mathbb{R}$. Note we may assume $\Omega_0' \subseteq \Omega_\alpha'$ for every rational $\alpha > 0$. So let $v \in \Omega$. For every rational $\alpha > 0$, we have

$$\psi_\alpha(v) = \frac{1}{2}\left(\psi_{\frac{1}{2}\alpha}(v) + \psi_{\frac{1}{2}\alpha}(g^{\frac{1}{2}\alpha} v)\right) = \psi_{\frac{1}{2}\alpha}(v).$$

Thus $\psi_{2^{-k}}(v)$ does not depend on $k \in \mathbb{Z}$. But by Lebesgue density, $\psi_0(g^t v) = \lim_{k\to\infty} \psi_{2^{-k}}(g^t v)$ for a.e. $t \in \mathbb{R}$. Hence $\psi_0(g^t v) = \psi_1(g^t v)$ for a.e. $t \in \mathbb{R}$. But $v \in \Omega$ was arbitrary, so $\psi_0 = \psi_1$ a.e. by Fubini's theorem. Thus ψ_0 is constant a.e., which contradicts our choice of $\bar{\psi}_0$. Therefore, there is some $\alpha_0 > 0$ such that ψ_{α_0} is not constant a.e., as claimed. Write $\psi = \psi_{\alpha_0}$.

Let f be the map taking $v \in \Omega$ to the closed subgroup of $\mathbb{R}$ generated by the periods of the map $t \mapsto \psi(g^t v)$. Clearly f is both Γ- and g^t-invariant. By Theorem B.5, f is measurable; hence f is constant a.e. by Corollary 9.17. Replacing Ω by a smaller g^t-invariant set if necessary, we may therefore assume that f is constant everywhere on Ω. Now suppose $f(v) = \mathbb{R}$ for every $v \in \Omega$. Then for every $v \in \Omega$, the map $t \mapsto \psi(g^t v)$ must be constant by continuity. Hence ψ must be g^t-invariant a.e. By Corollary 9.17, ψ must be constant a.e., contradicting our choice of ψ. Thus there must be some $c \geq 0$ such that $f(v) = c\mathbb{Z}$ for every $v \in \Omega$.

Let

$$\varphi_N^+ = \frac{1}{N^2}\sum_{n=1}^{N^2} \varphi \circ g^{t_n} \qquad \text{and} \qquad \varphi_N^- = \frac{1}{N^2}\sum_{n=1}^{N^2} \varphi \circ g^{-t_n}.$$

Now ψ is the smoothed a.e.-limit of φ_n^+, that is,

$$\psi(v) = \int_0^{\alpha_0} \psi_0(g^s v)\, ds = \int_0^{\alpha_0} \lim_{n\to\infty} \varphi_n^+(g^s v)\, ds,$$

for all $v \in \Omega$. Since φ is bounded by compactness of $\Gamma \backslash SX$, $\{\varphi_n^+\}$ is uniformly bounded. Thus

$$\psi(v) = \lim_{n\to\infty} \int_0^{\alpha_0} \varphi_n^+(g^s v)\, ds \qquad \text{for all } v \in \Omega.$$

Similarly, ψ is the smoothed a.e.-limit of φ_n^-, so

$$\psi(v) = \lim_{n\to\infty} \int_0^{\alpha_0} \varphi_n^-(g^s v)\,ds \qquad \text{for all } v \in \Omega.$$

Set

$$\widetilde{\varphi}_n^+(v) = \int_0^{\alpha_0} \varphi_n^+(g^s v)\,ds \qquad \text{and} \qquad \widetilde{\varphi}_n^-(v) = \int_0^{\alpha_0} \varphi_n^-(g^s v)\,ds,$$

and let $\psi^+ = \limsup \widetilde{\varphi}_n^+$ and $\psi^- = \limsup \widetilde{\varphi}_n^-$. Since $\psi = \lim \widetilde{\varphi}_n^+ = \lim \widetilde{\varphi}_n^-$ on Ω, we have $\psi = \psi^+ = \psi^-$ on Ω.

By uniform continuity of φ and strong recurrence of $\mathcal{S}$, every $v \in \mathcal{S}$ has ψ^+ constant along $H^s(v)$ and ψ^- constant along $H^u(v)$. Thus $\psi|_\Omega$ is constant along every $H^s(v)$ and $H^u(v)$. Since $f = c\mathbb{Z}$ on all Ω, applying Lemma 12.2 we see that, for a.e. $(\xi,\eta) \in \mathcal{R}^E$, every cross-ratio $\mathrm{B}(\xi,\xi',\eta,\eta')$ must lie in $c\mathbb{Z}$. □

Lemma 12.4. *Suppose $p \in X$ and $\xi,\eta \in \partial X$. Then $\beta_p(\xi,\eta) = 0$ if and only if $\angle_p(\xi,\eta) = \pi$.*

Proof. Both statements are equivalent to the existence of a geodesic in X that joins ξ and η and passes through the point p. □

Lemma 12.5. *Suppose all cross-ratios in $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ take values in a fixed discrete subgroup of the reals. Then $\mathcal{R} = SX$.*

Proof. Suppose, by way of contradiction, that some $v \in SX$ has $d_T(v^-,v^+) = \pi$. Find $\xi,\eta \in \partial X$ isolated in the Tits metric such that ξ,η,v^-,v^+ are distinct. Since the set $\mathcal{S}$ is dense in SX, there is a sequence $v_k \to v$ such that v_k^- and v_k^+ both are isolated in the Tits metric for every k. We may assume $v_k^-, v_k^+ \in \partial X \setminus \{\xi,\eta,v^-,v^+\}$, hence $(v_k^-,\xi,v_k^+,\eta),(v_k^-,\xi,v^-,\eta) \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$. Then

$$\mathrm{B}(v_k^-,\xi,v_k^+,\eta) = \mathrm{B}(v_k^-,\xi,v^-,\eta)$$

for all k by discreteness and continuity of cross-ratios. Thus

$$\beta_p(v_k^-,v_k^+) - \beta_p(\xi,v_k^+) = \beta_p(v_k^-,v^-) - \beta_p(\xi,v^-), \tag{$*$}$$

where $p \in X$ is arbitrary.

Recall from Lemma 6.3 that $\beta_p \colon \partial X \times \partial X \to [-\infty,\infty)$ is upper semicontinuous on $\partial X \times \partial X$. By Lemma 6.2, $\beta_p(\xi,v^-)$ is finite but $\beta_p(v^-,v^-) = -\infty$, so we have

$$\lim_{k\to\infty} (\beta_p(v_k^-,v^-) - \beta_p(\xi,v^-)) = \beta_p(v^-,v^-) - \beta_p(\xi,v^-) = -\infty$$

by upper semicontinuity since $v_k^- \to v^-$. Thus $\beta_p(v_k^-,v^-) \to -\infty$. But $\left\{\beta_p(\xi,v_k^+)\right\}$ is bounded because $\beta_p(\xi, v_k^+) \to \beta_p(\xi, v^+)$ and β_p is continuous on $\mathcal{R}^E$. Therefore, $\beta_p(v_k^-, v^+) \to -\infty$ by $(*)$ and upper semicontinuity.

On the other hand, $\beta_p \circ \mathrm{E}$ is continuous on SX. For if $v_k \to v$ in SX, then $(v_k^-, v_k^+) \to (v^-, v^+)$ in ∂X, so $(b_{v_k^-} + b_{v_k^+}) \to (b_{v^-} + b_{v^+})$ uniformly on compact subsets. Also, $v_k(0) \to v(0)$ in X, so $(b_{v_k^-} + b_{v_k^+})(v_k(0), p) \to (b_{v^-} + b_{v^+})(v(0), p)$. Thus $\beta_p(v_k^-, v_k^+)$ converges to $\beta_p(v^-, v^+)$.

Hence $\beta_p(v_k^-, v_k^+)$ must converge to $\beta_p(v^-, v^+)$, which is finite by Lemma 6.2; but this contradicts $\beta_p(v_k^-, v^+) \to -\infty$. Therefore, $\mathcal{R} = SX$. $\square$

We use Lemma 12.5 implicitly in the proof of the next lemma to guarantee $(\xi, \xi', \eta, \eta') \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ whenever $\xi, \xi', \eta, \eta' \in \partial X$ are distinct.

Lemma 12.6. *Suppose all cross-ratios in $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ take values in a fixed discrete subgroup of the reals. Then there is some $c > 0$ such that X is isometric to a tree with all edge lengths in $c\mathbb{Z}$.*

Proof. Suppose all cross-ratios of X lie in $a\mathbb{Z} \subset \mathbb{R}$, for some $a > 0$. We will prove that the link $\mathrm{Lk}(p)$ of p is discrete at every point $p \in X$. So fix $p \in X$, and let $\rho \colon \partial X \to \mathrm{Lk}(p)$ be radial projection.

For $\eta \in \partial X$, let $A_p(\eta) = \left\{\xi \in \partial X \mid \angle_p(\xi, \eta) = \pi\right\}$. Clearly $\rho(A_p(\eta))$ is closed in $\mathrm{Lk}(p)$. We claim every $\rho(A_p(\eta))$ is also open. For if $\rho(A_p(\eta_0))$ is not open for some $\eta_0 \in \partial X$, there is a point $\xi_0 \in A_p(\eta_0)$ and a sequence (ξ_k) in ∂X such that $\angle_p(\xi_0, \xi_k) \to 0$ but each $\angle_p(\xi_k, \eta_0) < \pi$. For each ξ_k, choose $\eta_k \in A_p(\xi_k)$. Passing to a subsequence, $(\xi_k, \eta_k) \to (\xi_0', \eta_0') \in \partial X \times \partial X$. By continuity of $\angle_p$, we have $\angle_p(\xi_0', \eta_0') = \lim_{k\to\infty} \angle_p(\xi_k, \eta_k) = \pi$ and $\angle_p(\xi_0, \xi_0') = 0$. Hence

$$\angle_p(\xi_0, \eta_0') = \angle_p(\xi_0', \eta_0') = \pi = \angle_p(\xi_0, \eta_0) = \angle_p(\xi_0', \eta_0),$$

with the left- and right-most equalities coming from the triangle inequality. Thus

$$\mathrm{B}(\xi_0, \xi_0', \eta_0, \eta_0') = \beta_p(\xi_0, \eta_0) + \beta_p(\xi_0', \eta_0') - \beta_p(\xi_0, \eta_0') - \beta_p(\xi_0', \eta_0)$$

equals zero by Lemma 12.4. By discreteness and continuity of cross-ratios, we have a neighborhood $U \times V$ of (ξ_0', η_0') in $\partial X \times \partial X$ such that $\mathrm{B}(\xi_0, \xi, \eta_0, \eta) = 0$ for all $(\xi, \eta) \in U \times V$. Thus for large k, since $(\xi_k, \eta_k) \in U \times V$, we have $\mathrm{B}(\xi_0, \xi_k, \eta_0, \eta_k) = 0$. But we know $0 = \beta_p(\xi_0, \eta_0) = \beta_p(\xi_k, \eta_k)$, hence

$$0 = \mathrm{B}(\xi_0, \xi_k, \eta_0, \eta_k) = -\beta_p(\xi_0, \eta_k) - \beta_p(\xi_k, \eta_0).$$

Both terms on the right being nonnegative, they must both equal zero. Hence we have $\angle_p(\xi_k, \eta_0) = \pi$, contradicting our assumption on ξ_k. Thus every $\rho(A_p(\eta))$ must be both open and closed in $\mathrm{Lk}(p)$.

It follows from the previous paragraph that no component of $\mathrm{Lk}(p)$ can contain points distance $\geq \pi$ apart. But $\mathrm{Lk}(p)$ is geodesically complete by Proposition 10.6, and no closed geodesic in $\mathrm{Lk}(p)$ can have length less than π because $\mathrm{Lk}(p)$ is CAT(1). Thus $\mathrm{Lk}(p)$ must be discrete. Therefore X, being proper and geodesically complete, must be a metric simplicial tree. So $2a\mathbb{Z}$ includes all edge lengths of X by Lemma 11.10. □

Lemmas 12.3 and 12.6 give us Theorem 4.

Theorem 12.7 (Theorem 4)**.** *Let X and Γ satisfy the assumptions of Theorem 1. The following are equivalent:*

1. *The Bowen-Margulis measure m_Γ is not mixing under the geodesic flow on $\Gamma \backslash SX$.*
2. *The length spectrum is arithmetic—that is, the set of all translation lengths of hyperbolic isometries in Γ must lie in some discrete subgroup $c\mathbb{Z}$ of $\mathbb{R}$.*
3. *There is some $c \in \mathbb{R}$ such that every cross-ratio of $\mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$ lies in $c\mathbb{Z}$.*
4. *There is some $c > 0$ such that X is isometric to a tree with all edge lengths in $c\mathbb{Z}$.*

Proof. The case $X = \mathbb{R}$ is clear, so we may assume $|\partial X| > 2$. Lemma 12.3 shows (1) $\Longrightarrow$ (3), and Lemma 12.6 shows (3) $\Longrightarrow$ (4). If X is a tree with all edge lengths in $c\mathbb{Z}$, then the geodesic flow factors continuously over the circle, so m_Γ is not even weak mixing; this proves (4) $\Longrightarrow$ (1). Now $\mathrm{supp}(\mu_x) = \partial X$, so by Theorem 9.1, $\mathcal{R}^E$ is dense in $\partial X \times \partial X$. Since the rank one axes are weakly dense in $\mathcal{R}$ by Proposition 7.5, every point $(\xi, \xi', \eta, \eta') \in (\partial X)^4$ is a limit of points $(v^-, w^-, v^+, w^+) \in \mathcal{Q}_{\mathcal{R}^{\mathrm{E}}}$, where v and w are rank one axes. Thus Lemma 11.11 shows (2) $\Longrightarrow$ (3); meanwhile, (3) $\Longrightarrow$ (2) is immediate from Lemma 11.9. □

Remark. Suppose Γ is a group acting properly discontinuously and by isometries (but not necessarily cocompactly) on a proper, geodesically complete CAT(-1) space. In this case, Roblin ([41]) has constructed Bowen-Margulis measures on SX and $\Gamma \backslash SX$; he has also shown that m_Γ is ergodic. If m_Γ is finite and $\mathrm{supp}(\mu_x) = \partial X$, the proofs from Lemma 12.3 and Lemma 12.6 apply verbatim, with the exception that in the proof of Lemma 12.3, one simply requires $\bar{\varphi}$ to have compact support, and then φ is bounded. Thus we have characterized mixing in this case also.

APPENDIX A

Fundamental Domains and Quotient Measures

In this appendix, we describe a simple way to push forward a measure modulo a group action. We use fundamental domains (see Definition A.1) for the group action; these sets are allowed to have large (but still finite) point stabilizers on sets of large measure. We will first consider general measure spaces, and then restrict to Borel measures on topological spaces.

Consider, for example, the natural $\mathbb{Z}$-action on $\mathbb{R}$ by translations. If we push forward the Lebesgue measure λ on $\mathbb{R}$ directly, we get a measure λ_* that only takes values 0 or ∞. However, in this case we have a fundamental domain for the action—an open set $U = (0,1) \subset \mathbb{R}$ such that every $t \in \mathbb{R}$ either has $n+t \in U$ for exactly one $n \in \mathbb{Z}$, or $n+t \in \partial U = \{0,1\}$ for some (two) $n \in Z$, and $\lambda(\partial U) = 0$. Thus we may restrict λ to U and then push it forward.

However, requiring that fundamental domains only intersect each orbit at a single point with trivial stabilizer makes finding fundamental domains difficult, or even impossible. For instance, the natural action on $\mathbb{Z} \subset \mathbb{R}$ of the standard $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ subgroup in $\mathrm{Isom}(\mathbb{R})$ has nontrivial stabilizers at every point.

A.1 General Measure Spaces

Let G be a countable group acting measurably (that is, the map $z \mapsto gz$ is measurable for every $g \in G$) on a measurable space $(Z, \mathfrak{M})$, and let ν be a G-invariant measure on Z. Let $\mathrm{pr}\colon Z \to G\backslash Z$ be the canonical projection. Let $G\backslash\mathfrak{M}$ be the σ-algebra on $G\backslash Z$ given by $G\backslash\mathfrak{M} = \{\mathrm{pr}(A) \mid A \in \mathfrak{M}\}$. Notice that G permutes the σ-algebra $\mathfrak{M}$, with $g \in G$ sending $A \in \mathfrak{M}$ to $gA = \{gz \in Z \mid z \in A\}$. Hence $G\backslash\mathfrak{M}$ is naturally in bijective correspondence with the σ-algebra $\mathfrak{M}^G = \{A \in \mathfrak{M} \mid gA = A \text{ for all } g \in G\}$ of G-invariant subsets of Z. These sets are all of the form GA, with $A \in \mathfrak{M}$.

Our goal is to show (Proposition A.6) that, under fairly weak hypotheses, one can construct a measure $\widehat{\nu}$ on $G\backslash Z$ satisfying both the following properties:

(†) Let $A \subseteq Z$ be measurable and $h\colon Z \to [0,\infty]$ be a G-invariant measurable function. Define $f_A\colon Z \to [0,\infty]$ by $f_A(z) = |\{g \in G \mid gz \in A\}|$. Both h, f_A descend to measurable functions $\bar{h}, \bar{f}_A\colon G\backslash Z \to [0,\infty]$, and we have

$$\int_A h\,d\nu = \int_{G\backslash Z} (\bar{h}\cdot \bar{f}_A)\,d\widehat{\nu}.$$

(‡) For any ν-preserving map $\phi\colon Z \to Z$ such that $\phi \circ g = g \circ \phi$ for all $g \in G$, the factor map $\phi_G\colon G\backslash Z \to G\backslash Z$ defined by $\phi_G \circ \mathrm{pr} = \mathrm{pr} \circ \phi$ preserves $\widehat{\nu}$.

Observe that condition (†) allows us to reconstruct ν from $\widehat{\nu}$, and allows us to transfer information about ν to $\widehat{\nu}$, and vice versa. Condition (‡) ensures that certain actions on Z (such as the geodesic flow g^t on $\mathcal{G}^E \times \mathbb{R}$) will descend well to the quotient.

We begin by defining fundamental domains. Our definition allows more flexibility than is typical in the literature; in particular, we do not require our fundamental domains to be open (or closed), connected, or to project one-to-one onto $G\backslash Z$ almost everywhere. However, we do need them to project onto $G\backslash Z$ almost everywhere finite-to-one, but not uniformly so (that is, we do not require any uniform bound on the size of the fibers).

Definition A.1. Call a set $F \subseteq Z$ a *fundamental domain* for the action if it satisfies both the following conditions:

1. $\nu(Z \setminus GF) = 0$.
2. For every $z \in F$, there are only finitely many $g \in G$ such that $gz \in F$.

Let $\mathfrak{F}$ be the collection of finite subsets of G, and let $\mathfrak{F}_1$ the subcollection of finite subsets of G containing the identity. For $A \subseteq Z$ measurable and $B \in \mathfrak{F}$, define

$$Z_B^A = \{z \in Z \mid gz \in A \text{ if and only if } g \in B\}.$$

Since G is countable and

$$Z_B^A = \Big(\bigcap_{g\in B} g^{-1}A\Big) \cap \Big(\bigcap_{h \in G\setminus B} (Z \setminus h^{-1}A)\Big),$$

each Z_B^A is measurable. Let $Z_\infty^A = \{z \in Z \mid gz \in A \text{ for infinitely many } g \in G\}$; clearly $Z_\infty^A = Z \setminus \bigcup_{B\in\mathfrak{F}} Z_B^A$, so Z_∞^A is also measurable.

Note that if F is a fundamental domain, by condition (2) the collection $\left\{Z_B^F\right\}_{B\in\mathfrak{F}_1}$ forms a countable partition of F, and $\left\{Z_B^F\right\}_{B\in\mathfrak{F}}$ forms a countable partition of GF. Fundamental domains allow us to transfer all the information from ν to a smaller measure ν' ($= \nu_F$ in the following lemma), and reconstruct ν from pushing ν' around by G.

Lemma A.2. *Suppose $F \subseteq Z$ is a fundamental domain. Then there is a measure ν_F on Z with $\nu_F(Z \setminus F) = 0$ such that $\nu = \sum_{g\in G} g_*\nu_F$. Moreover, if $\nu(F) < \infty$ then ν_F is finite.*

Proof. Define ν_F by

$$\nu_F(A) = \sum_{B\in\mathfrak{F}_1} \frac{1}{|B|}\nu(A \cap Z_B^F)$$

for all measurable $A \subseteq Z$. Then for any measurable $A \subseteq Z$,

$$\sum_{g\in G} g_*\nu_F(A) = \sum_{g\in G} \nu_F(g^{-1}A) = \sum_{\substack{g\in G\\ B\in\mathfrak{F}_1}} \frac{1}{|B|}\nu(g^{-1}A \cap Z_B^F)$$

by definition. Because ν is G-invariant, we may rewrite this expression as

$$\sum_{\substack{g\in G\\ B\in\mathfrak{F}_1}} \frac{1}{|B|}\nu(A \cap gZ_B^F) = \sum_{\substack{g\in G\\ B\in\mathfrak{F}_1}} \frac{1}{|B|}\nu(A \cap Z_{Bg^{-1}}) = \sum_{\substack{g\in G\\ B\in\mathfrak{F}_1 g^{-1}}} \frac{1}{|B|}\nu(A \cap Z_B^F).$$

But for each $B \in \mathfrak{F}$, we have $B \in \mathfrak{F}_1 g^{-1}$ if and only if $g \in B$. Hence we may again rewrite this expression as

$$\sum_{B\in\mathfrak{F}} \frac{1}{|B|} \sum_{g\in B} \nu(A \cap Z_B^F) = \sum_{B\in\mathfrak{F}} \nu(A \cap Z_B^F) = \nu(A \cap GF)$$

because $\left\{Z_B^F\right\}_{B\in\mathfrak{F}}$ is a countable partition of GF. Therefore, $\sum_{g\in G} g_*\nu_F(A) = \nu(A)$ by condition (1) of Definition A.1.

Since $\left\{Z_B^F\right\}_{B\in\mathfrak{F}_1}$ is a countable partition of F, it is clear from the definition of ν_F that $\nu_F(Z \setminus F) = 0$. Moreover, $\sum_{B\in\mathfrak{F}_1} \nu(F \cap Z_B^F) = \nu(F)$. So $\nu_F(Z) = \nu_F(F) \le \nu(F)$, hence if $\nu(F)$ is finite then so is $\nu_F(Z)$. □

We can now show a version of condition (†) for ν' such that $\nu = \sum_{g\in G} g_*\nu'$.

Lemma A.3. *Suppose ν' is a measure on Z such that $\nu = \sum_{g\in G} g_*\nu'$. Let $A \subseteq Z$ be measurable, and let $f_A\colon Z \to [0,\infty]$ be $f_A(z) = |\{g \in G \mid gz \in A\}|$. Then for any G-invariant measurable function $h\colon Z \to [0,\infty]$,*

$$\int_A h\,d\nu = \int_Z (h \cdot f_A)\,d\nu'.$$

Proof. Note that $f_A = \infty \cdot \chi_{Z^A_\infty} + \sum_{B\in\mathfrak{F}} |B| \chi_{Z^A_B}$, using the usual measure-theoretic convention that $0 \cdot \infty = 0$. In particular, f_A is measurable. The proof splits into two cases, depending on whether or not $\nu(Z^A_\infty) = 0$.

Suppose first that $\nu(Z^A_\infty) = 0$. Then $\left\{Z^A_B\right\}_{B\in\mathfrak{F}_1}$ partitions A, so

$$\int_A h\,d\nu = \sum_{B\in\mathfrak{F}_1} \int_{Z^A_B} h\,d\nu = \sum_{B\in\mathfrak{F}_1}\sum_{g\in G}\int_{Z^A_B} h\,dg_*\nu'$$

by hypothesis on ν'. By G-invariance of h, this expression equals

$$\sum_{B\in\mathfrak{F}_1}\sum_{g\in G}\int_{g^{-1}Z^A_B} h\,d\nu' = \sum_{B\in\mathfrak{F}_1}\sum_{g\in G}\int_{Z^A_{Bg}} h\,d\nu' = \sum_{g\in G}\sum_{B\in\mathfrak{F}_1 g}\int_{Z^A_B} h\,d\nu'.$$

But $B \in \mathfrak{F}_1 g$ if and only if $g \in B$, so

$$\int_A h\,d\nu = \sum_{B\in\mathfrak{F}}\sum_{g\in B}\int_{Z^A_B} h\,d\nu' = \sum_{B\in\mathfrak{F}} |B| \int_{Z^A_B} h\,d\nu' = \int_Z (h\cdot f_A)\,d\nu'.$$

Suppose now that $\nu(Z^A_\infty) > 0$. If $\left\{z \in Z^A_\infty \mid h(z) > 0\right\}$ has zero ν-measure, then we may set $h' = h \cdot \chi_{Z \setminus Z^A_\infty}$, and we have

$$\int_A h\,d\nu = \int_A h'\,d\nu = \int_Z (h'\cdot f_A)\,d\nu' = \int_Z (h\cdot f_A)\,d\nu'$$

by the previous paragraph. Otherwise, $\nu(\left\{z \in A \cap Z^A_\infty \mid h(z) > 0\right\}) > 0$, so there exist some $\delta, \epsilon > 0$ such that the set $U_\delta = \left\{z \in A \cap Z^A_\infty \mid h(z) \geq \delta\right\}$ has $\nu(U_\delta) \geq \epsilon$. Thus

$$\int_{U_\delta} h\,d\nu = \sum_{g\in G}\int_{U_\delta} h\,dg_*\nu' = \sum_{g\in G}\int_{g^{-1}U_\delta} h\,d\nu'$$

using our hypothesis on ν'. But $\int_{U_\delta} h\,d\nu > 0$ by construction, so $\int_{g_0^{-1}U_\delta} h\,d\nu' > 0$ for some $g_0 \in G$. Define the sets A_g (for $g \in G$) by $A_g = g_0^{-1}U_\delta \cap g^{-1}U_\delta$. Now $g_0^{-1}U_\delta \subseteq Z^A_\infty$, while h and Z^A_∞ are G-invariant, so every $z \in g_0^{-1}U_\delta$ is in A_g for infinitely many $g \in G$. Equivalently, $\bigcup_{g\in G\setminus B} A_g = g_0^{-1}U_\delta$ for all $B \in \mathfrak{F}$.

We claim $\nu(U_\delta) = \infty$. For if not, then

$$\sum_{g\in G}\nu'(A_g) = \sum_{g\in G}\nu'(g_0^{-1}U_\delta \cap g^{-1}U_\delta) \leq \sum_{g\in G}\nu'(g^{-1}U_\delta) = \sum_{g\in G} g_*\nu'(U_\delta) = \nu(U_\delta) < \infty.$$

Hence there is some $B \in \mathfrak{F}$ such that $\sum_{g\in G\setminus B}\nu'(A_g) < \epsilon$. But this means $\nu'(\bigcup_{g\in G\setminus B} A_g) <$

$\nu'(U_\delta)$, which contradicts the fact that $\bigcup_{g\in G\smallsetminus B} A_g = U_\delta$ for all $B \in \mathfrak{F}$. Therefore, we must have $\nu(U_\delta) = \infty$. Thus $\int_A h\, d\nu \geq \int_{U_\delta} \epsilon\, d\nu = \infty$, and

$$\int_Z (h \cdot f_A)\, d\nu' \geq \int_{g_0^{-1}U_\delta} (h \cdot f_A)\, d\nu' \geq (\epsilon \cdot \infty)\, \nu'(g_0^{-1}U_\delta) = \infty,$$

which proves $\int_A h\, d\nu = \int_Z (h \cdot f_A)\, d\nu'$. □

It now follows that for G-invariant subsets A of Z, $\nu_F(A)$ does not depend on F.

Corollary A.4. *Suppose E and F are two fundamental domains for the action, and let ν_E and ν_F be the measures given by Lemma A.2. Then for all G-invariant subsets A of Z, $\nu_E(A) = \nu_F(A)$.*

Proof. Let $A \subseteq Z$ be G-invariant. By our definition of ν_E in Lemma A.2,

$$\nu_E(A) = \int_E \frac{1}{f_E} \chi_A\, d\nu$$

because conditions (1) and (2) of Definition A.1 force $\nu(Z^E_\varnothing) = 0$ and $\nu(Z^E_\infty) = 0$, respectively. So

$$\nu_E(A) = \int_E \frac{1}{f_E} \chi_A\, d\nu = \int_Z \frac{1}{f_E} \chi_A \cdot f_E\, d\nu_F = \nu_F(A)$$

by Lemma A.3, since A and f_E are G-invariant. □

We now prove a version of condition (‡) for ν_F coming from a fundamental domain.

Lemma A.5. *Suppose F is a fundamental domains for the action, and let ν_F be the measure given by Lemma A.2. Further suppose that $\phi\colon Z \to Z$ is a ν-preserving map such that $\phi \circ g = g \circ \phi$ for all $g \in G$. Then for all G-invariant subsets A of Z, $\phi_*(\nu_F)(A) = \nu_F(A)$.*

Proof. Let $A \subseteq Z$ be G-invariant, and adopt the notation from the proof of Lemma A.2. We show first that $\phi^{-1}F$ is a fundamental domain. Condition (1) is clear from the hypotheses on ϕ. But $\left\{Z^F_B\right\}_{B\in\mathfrak{F}_1}$ is a partition of F, so $\left\{\phi^{-1}Z^F_B\right\}_{B\in\mathfrak{F}_1}$ is a partition of $\phi^{-1}F$. Since ϕ commutes with the action of G, we have $\phi^{-1}Z^F_B = Z^{\phi^{-1}F}_B$, and condition (2) follows. Thus $\phi^{-1}F$ is a fundamental domain.

Since $\phi^{-1}F$ is a fundamental domain, by definition of $\nu_{\phi^{-1}F}$ we have

$$\nu_{\phi^{-1}F}(\phi^{-1}A) = \sum_{B\in\mathfrak{F}_1} \frac{1}{|B|} \nu(\phi^{-1}A \cap Z^{\phi^{-1}F}_B) = \sum_{B\in\mathfrak{F}_1} \frac{1}{|B|} \nu(\phi^{-1}A \cap \phi^{-1}Z^F_B).$$

Now on the left side, $\nu_{\phi^{-1}F}(\phi^{-1}A) = \nu_F(\phi^{-1}A) = \phi_*(\nu_F)(A)$ by Corollary A.4, and on the right,

$$\sum_{B\in\mathfrak{F}_1} \frac{1}{|B|}\nu(\phi^{-1}A\cap\phi^{-1}Z_B^F) = \sum_{B\in\mathfrak{F}_1} \frac{1}{|B|}\nu(A\cap Z_B^F)$$

because ϕ preserves ν. This last expression is the very definition of $\nu_F(A)$, and thus we have shown that $\phi_*(\nu_F)(A) = \nu_F(A)$. □

Finally, we collect our results into a proposition that will give us a good quotient measure in general.

Proposition A.6. *Let G be a countable group acting measurably on a measurable space $(Z,\mathfrak{M})$, and let ν be a G-invariant measure on Z. Suppose the action admits a fundamental domain. Then there is a unique measure ν_G on the quotient space $(G\backslash Z, G\backslash\mathfrak{M})$ such that the following property holds:*

- (†) *Let $A\subseteq Z$ be measurable and $h\colon Z\to[0,\infty]$ be a G-invariant measurable function. Define $f_A\colon Z\to[0,\infty]$ by $f_A(z) = |\{g\in G \mid gz\in A\}|$. Both h, f_A descend to measurable functions $\bar{h},\bar{f}_A\colon G\backslash Z\to[0,\infty]$, and we have*

$$\int_A h\,d\nu = \int_{G\backslash Z}(\bar{h}\cdot\bar{f}_A)\,d\nu_G.$$

Moreover, ν_G satisfies the following property:

- (‡) *For any ν-preserving map $\phi\colon Z\to Z$ such that $\phi\circ g = g\circ\phi$ for all $g\in G$, the factor map $\phi_G\colon G\backslash Z\to G\backslash Z$ defined by $\phi_G\circ\mathrm{pr} = \mathrm{pr}\circ\phi$ preserves ν_G.*

Proof. Let $F\subseteq Z$ be a fundamental domain, and let ν_F be the finite measure on Z constructed in Lemma A.2. Let $\mathrm{pr}\colon Z\to G\backslash Z$ be the canonical projection, and push ν_F forward by pr to obtain a measure $\nu_G = \mathrm{pr}_*(\nu_F)$ on $G\backslash Z$. Then ν_G satisfies condition (†) by Lemma A.3. To prove uniqueness, suppose ν_G is a measure on $G\backslash Z$ that satisfies (†). For any measurable $A\subseteq Z$, let $h = \frac{1}{f_{F\cap GA}}\chi_{GA}$; then $\nu_G(\mathrm{pr}(A)) = \int_{F\cap GA} h\,d\nu$, which shows that $\nu_G(\mathrm{pr}(A))$ is determined by (†), hence ν_G is unique. Since $\nu = \sum_{g\in G} g_*\nu_F$ by Lemma A.2, (‡) holds by Lemma A.5. □

Notice that putting $h = 1$ in (†) gives us the following corollary.

Corollary A.7. *If $A\subseteq Z$ is a measurable set satisfying both $\nu(A)<\infty$ and $\nu(Z\smallsetminus GA) = 0$, then ν_G is finite.*

Similarly, putting $h = \chi_{GA}$ in (†) gives us the next corollary.

Corollary A.8. *For all measurable $A \subseteq Z$, the following are equivalent:*

1. $\nu_G(\mathrm{pr}(A)) = 0$.
2. $\nu(GA) = 0$.
3. $\nu(A) = 0$.

A.2 Topological Spaces

Let Z be a topological space, and $\mathfrak{B}(Z)$ its Borel σ-algebra. From Corollary A.8 we see that if ν is a Borel measure, then $z \in \mathrm{supp}(\nu)$ if and only if $\nu_G(\mathrm{pr}(U)) > 0$ for every neighborhood U of z in Z. However, we cannot quite say that $z \in \mathrm{supp}(\nu)$ if and only if $\mathrm{pr}(z) \in \mathrm{supp}(\nu_G)$. This is because the Borel σ-algebra $\mathfrak{B}(G\backslash Z)$ on $G\backslash Z$ may not equal the quotient σ-algebra $G\backslash\mathfrak{B}(Z)$ on $G\backslash Z$.

For example, consider the natural action of the group $G = \mathbb{Q}$ on the space $Z = \mathbb{R}$ by translations, with Borel σ-algebra $\mathfrak{B}(\mathbb{R})$. Then $\mathbb{Q}\backslash\mathfrak{B}(\mathbb{R})$ contains each $\mathbb{Q}$-orbit of $\mathbb{R}$, but the Borel σ-algebra $\mathfrak{B}(\mathbb{Q}\backslash\mathbb{R})$ on $\mathbb{Q}\backslash\mathbb{R}$ is trivial, comprising only $\varnothing$ and $\mathbb{Q}\backslash\mathbb{R}$. Furthermore, if ν is the counting measure on the subspace $\mathbb{Q} \subset \mathbb{R}$, then $\{0\} \subset \mathbb{R}$ is a fundamental domain; thus even having a fundamental domain does not solve the problem.

However, the quotient σ-algebra $G\backslash\mathfrak{B}(Z)$ always contains the Borel σ-algebra $\mathfrak{B}(G\backslash Z)$ on $G\backslash Z$. Hence any measure constructed of $G\backslash\mathfrak{B}(Z)$ will define a measure on the Borel σ-algebra of $G\backslash Z$ by restriction. (This measure is just the pushforward by the canonical projection $\mathrm{pr}\colon Z \to G\backslash Z$, which is continuous and therefore Borel).

Nevertheless, it is often convenient, when possible, to know that the Borel σ-algebra $\mathfrak{B}(G\backslash Z)$ on $G\backslash Z$ coincides with the quotient σ-algebra $G\backslash\mathfrak{B}(Z)$. This is the case under certain hypotheses on the topologies of Z and $G\backslash Z$. The next theorem follows from Theorem 2.1.14 and Theorem A.7 in [45] (due to Glimm ([21]), Effros ([18]), and Kallman ([27])).

Theorem A.9. *Let G be a locally compact, second countable group acting continuously on a complete, separable metric space Z. If the G-orbit of every point in Z is locally closed in Z, there is a Borel section $G\backslash Z \to Z$ of the canonical projection* $\mathrm{pr}\colon Z \to G\backslash Z$.

Remark. In the setting of Theorem A.9, the condition that every G-orbit is locally closed in Z is equivalent to the condition that $G\backslash Z$ is T_0. It is also equivalent to the action being *smooth*—that is, that $G\backslash Z$ is countably separated.

Theorem A.9 gives us the following.

Lemma A.10. *Let G be a locally compact, second countable group acting continuously on a complete, separable metric space Z. Let $\mathfrak{B}(Z)$ and $\mathfrak{B}(G\backslash Z)$ be the Borel σ-algebras on Z and $G\backslash Z$, respectively. If every G-orbit is locally closed, then $G\backslash\mathfrak{B}(Z) = \mathfrak{B}(G\backslash Z)$.*

Proof. The inclusion $\mathfrak{B}(G\backslash Z) \subseteq G\backslash\mathfrak{B}(Z)$ is clear, since the images under pr of G-invariant open sets of Z generate $\mathfrak{B}(G\backslash Z)$ but are also elements of $G\backslash\mathfrak{B}(Z)$. On the other hand, suppose $A \in \mathfrak{B}(Z)$ is G-invariant. Then $A = \mathrm{pr}^{-1}(\mathrm{pr}(A))$, although $\mathrm{pr}(A)$ is not necessarily Borel. Let $\iota\colon G\backslash Z \to Z$ be the Borel section given by Theorem A.9. Since ι is Borel, $\iota^{-1}(A) = \iota^{-1}(\mathrm{pr}^{-1}(\mathrm{pr}(A)))$ is Borel. But $\iota^{-1}(\mathrm{pr}^{-1}(\mathrm{pr}(A))) = \mathrm{pr}(A)$ because ι is a section. Thus $\mathrm{pr}(A)$ is Borel. □

Proposition A.11. *Let G be a countable group acting properly discontinuously and by homeomorphisms on a proper metric space Z, preserving a Borel measure ν on Z. Then there is a unique Borel measure ν_G on $G\backslash Z$ such that the following property holds:*

(†) *Let $A \subseteq Z$ be Borel and $h\colon Z \to [0,\infty]$ be a G-invariant Borel function. Define $f_A\colon Z \to [0,\infty]$ by $f_A(z) = |\{g \in G \mid gz \in A\}|$. Both h, f_A descend to Borel functions $\bar{h}, \bar{f}_A\colon G\backslash Z \to [0,\infty]$, and we have*

$$\int_A h\,d\nu = \int_{G\backslash Z} (\bar{h}\cdot\bar{f}_A)\,d\nu_G.$$

Moreover, ν_G satisfies the following property:

(‡) *For any ν-preserving map $\phi\colon Z \to Z$ such that $\phi\circ g = g\circ\phi$ for all $g \in G$, the factor map $\phi_G\colon G\backslash Z \to G\backslash Z$ defined by $\phi_G\circ\mathrm{pr} = \mathrm{pr}\circ\phi$ preserves ν_G.*

Proof. Recall that since Z is proper, requiring the G-action to be properly discontinuous is equivalent to requiring that every $z \in Z$ has a neighborhood $U \subseteq X$ such that $U \cap gU$ is nonempty for only finitely many $g \in G$ (see Remark I.8.3(1) of [9]). Hence every G-orbit is locally closed, and the stabilizer of ν-almost every point is finite. Furthermore, any proper metric space is complete and separable. So the image of the Borel section from Theorem A.9 is a fundamental domain. The rest follows from Proposition A.6 and Lemma A.10. □

Remark. From the proof, it is clear that Proposition A.11 holds under the weaker assumptions that G is a countable group acting by homeomorphisms on a complete, separable metric space Z, preserving a Borel measure ν on Z, that every G-orbit is locally closed, and that the stabilizer of ν-almost every point is finite.

Corollaries A.7 and A.8 give us the following.

Corollary A.12. *The measure ν_G from Proposition A.11 has the following properties:*

1. *If some Borel set $F \subseteq Z$ satisfies $\nu(F) < \infty$ and $\nu(Z \setminus GF) = 0$, then ν_G is finite.*
2. *For all Borel sets $A \subseteq Z$, we have $\nu_G(\mathrm{pr}(A)) = 0$ if and only if $\nu(A) = 0$ if and only if $\nu(GA) = 0$.*
3. *In particular, $z \in \mathrm{supp}(\nu)$ if and only if $\mathrm{pr}(z) \in \mathrm{supp}(\nu_G)$.*

APPENDIX B

Measurability of the Period Map

Let Ω be a topological space admitting a continuous $\mathbb{R}$-action. Our goal in this appendix is to prove the following theorem.

Theorem (Theorem B.5)**.** *Suppose $\psi\colon \Omega \to \mathbb{R}$ is a measurable function such that the map $t \mapsto \psi(t \cdot w)$ is continuous for every $w \in \Omega$. Let F be the map taking each $w \in \Omega$ to the closed subgroup of $\mathbb{R}$ generated by the periods of the map $t \mapsto \psi(g^t w)$. Then F is measurable.*

Let $C(\mathbb{R})$ denote the space of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, endowed with the topology of uniform convergence on compact subsets. Recall that $C(\mathbb{R})$ has a basis of open sets of the form

$$V(f, K, \epsilon) = \{g \in C(\mathbb{R}) \mid |f(x) - g(x)| < \epsilon \text{ for all } x \in K\},$$

where $f \in C(\mathbb{R})$, $K \subset \mathbb{R}$ is compact, and $\epsilon > 0$.

We want to use the following proposition, with $X = C(\mathbb{R})$ and $G = \mathbb{R}$. Here Σ is the space of closed subgroups of G, under the Fell topology.

Proposition B.1 (Proposition H.23 of [44])**.** *Suppose (G, X) is a topological transformation group with G locally compact, second countable, and X Hausdorff. Then the stabilizer map $\sigma\colon X \to \Sigma$ is a Borel map.*

But first we need to establish continuity of the $\mathbb{R}$-action we are considering on $C(\mathbb{R})$. Note that the following lemma fails to hold on the space of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with the topology of pointwise convergence, or with the topology of uniform convergence. However, it does hold for $C(\mathbb{R})$—that is, with respect to the topology of uniform convergence on compact subsets.

Lemma B.2. *The action of $\mathbb{R}$ on $C(\mathbb{R})$ given by $(t \cdot f)(x) = f(x + t)$ is continuous.*

Proof. Let $f \in C(\mathbb{R})$ and $t \in \mathbb{R}$, and suppose $V = V(t \cdot f, K, \epsilon)$ for some $\epsilon > 0$ and compact subset K of $\mathbb{R}$. We want to find a neighborhood W of (t, f) in $\mathbb{R} \times C(\mathbb{R})$ such that $s \cdot g \in V$

for all $(s,g) \in W$. So let $K' = \{x \in \mathbb{R} \mid d(x,K) \le 1\}$ and $K' = \{x \in \mathbb{R} \mid x+t \in K'\}$. Now f is uniformly continuous on K', so there is some $\delta \in (0,1)$ such that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in K'$ such that $|x-y| < \delta$. Thus $|(t \cdot f)(x) - (s \cdot f)(x)| < \epsilon/2$ whenever $x \in K$ and $|s-t| < \delta$. Let $U = V(f, K', \epsilon/2)$, so $|(s \cdot f)(x) - (s \cdot g)(x)| < \epsilon/2$ whenever $g \in U$ and $|s-t| < \delta$. Finally, let $W = (t-\delta, t+\delta) \times U$. Then for all $(s,g) \in W$ and $x \in K$, we have

$$\begin{aligned} |(t \cdot f)(x) - (s \cdot g)(x)| &\le |(t \cdot f)(x) - (s \cdot f)(x)| + |(s \cdot f)(x) - (s \cdot g)(x)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

so $s \cdot g \in V$ for all $(s,g) \in W$, as required. □

Hence the stabilizer map $\sigma \colon C(\mathbb{R}) \to \Sigma$ is a Borel map by Proposition B.1.

Now let $\psi_* \colon \Omega \to C(\mathbb{R})$ be given by $(\psi_*(\xi,\eta,s))(t) = \psi(\xi,\eta,s+t)$. Thus $(\psi_*(w))(t) = \psi(t \cdot w) = \psi \circ g^t$. On the other hand, $\mathbb{R}$ acts on $C(\mathbb{R})$ by $(t \cdot f)(x) = f(x+t)$. Therefore, $F = \sigma \circ \psi_*$, where $\sigma \colon C(\mathbb{R}) \to \Sigma$ is the stabilizer map.

Thus it suffices to show (and we will in Lemma B.4) that ψ_* is measurable. To prove this result we will use the following lemma.

Lemma B.3. *There is a basis for $C(\mathbb{R})$ consisting of countably many basic open sets of the form $V(f,K,\epsilon)$.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be the collection of polynomials $p_n \colon \mathbb{R} \to \mathbb{R}$ with rational coefficients. Suppose $f \in C(\mathbb{R})$, $K \subset \mathbb{R}$ is compact, and $\epsilon > 0$. Let $V = V(f,K,\epsilon)$. Find some positive integer m such that $1/m < \epsilon/2$ and $K \subseteq [-m,m]$. By the Weierstrass polynomial approximation theorem, there is some positive integer n such that $|p_n(x) - f(x)| < 1/m$ for all $x \in [-m,m]$. Then the open set $U = V(p_n, [-m,m], 1/m)$ contains f, and $U \subset V$. Thus the open sets of the form $V(p_n, [-m,m], 1/m)$, where m and n are positive integers, form a countable basis for $C(\mathbb{R})$. □

Lemma B.4. *The map $\psi_* \colon \Omega \to C(\mathbb{R})$ is Borel measurable.*

Proof. By Lemma B.3, it suffices to show that $\psi_*^{-1}(V)$ is measurable for every open set V of the form $V = V(f,K,\epsilon)$.

First suppose that $K = \{x_0\}$. Note in this case that

$$V = \{g \in C(\mathbb{R}) \mid |g(x_0) - y_0| < \epsilon\},$$

where $y_0 = f(x_0)$. Thus, using only the definitions, we have

$$\begin{aligned}
\psi_*^{-1}(V) &= \{w \in \Omega \mid \psi_*(w) \in V\} \\
&= \{w \in \Omega \mid |(\psi_*(w))(x_0) - y_0| < \epsilon\} \\
&= \left\{w \in \Omega \mid \left|\psi(g^{x_0} w) - y_0\right| < \epsilon\right\} \\
&= \left\{g^{-x_0} w \in \Omega \mid |\psi(w) - y_0| < \epsilon\right\} \\
&= g^{-x_0}(\{w \in \Omega \mid |\psi(w) - y_0| < \epsilon\}) \\
&= g^{-x_0}(\psi^{-1}(B(y_0, \epsilon))).
\end{aligned}$$

Since ψ is measurable, and g^{x_0} is continuous and therefore measurable, $\psi_*^{-1}(V)$ is therefore measurable.

Now let $K \subset \mathbb{R}$ be an arbitrary compact set. Since $\mathbb{R}$ is second countable, so is K; hence K admits a countable dense subset A. On the other hand, continuous functions are determined by their values on any countable dense subset, so $V(f, K, \epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{x \in A} V(f, \{x\}, \epsilon - 1/n)$. Thus $\psi_*^{-1}(V)$ is the countable union of countable intersections of sets of the form $\psi_*^{-1}(V(f, \{x\}, \epsilon - 1/n))$, which we showed were measurable in the previous paragraph. Therefore, $\psi_*^{-1}(V)$ is measurable. □

This completes the proof of our theorem.

Theorem B.5. *Suppose $\psi \colon \Omega \to \mathbb{R}$ is a measurable function such that the map $t \mapsto \psi(t \cdot w)$ is continuous for every $w \in \Omega$. Let F be the map taking each $w \in \Omega$ to the closed subgroup of $\mathbb{R}$ generated by the periods of the map $t \mapsto \psi(g^t w)$. Then F is measurable.*

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