Applications of a combinatorial model for curves

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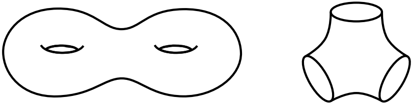
Notation and Background

- $\mathcal{S}$ - surface, $\mathcal{P}$ - pair of pants



$\mathcal{S}$ $\mathcal{P}$

Negatively curved (usually hyperbolic), geodesic boundary

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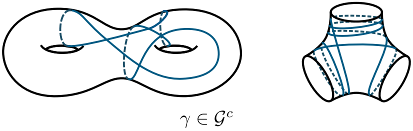
- $\mathcal{G}^c$ - closed geodesics



Non-simple, primitive

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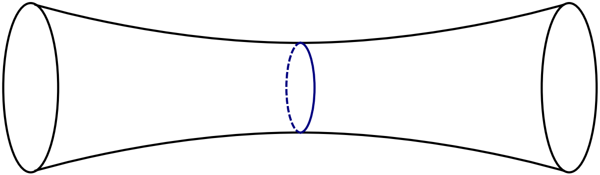

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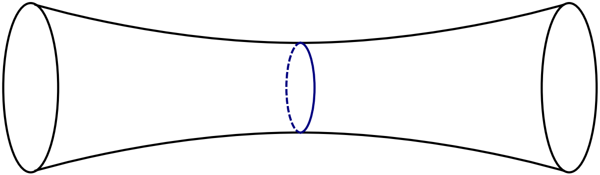

So, geodesics $\leftrightarrow$ free homotopy classes

Counting non-simple closed geodesics

Counting with respect to length

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Theorem (Margulis)

If $\mathcal{S}$ has finite volume, then

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NB: $\mathcal{S}$ hyperbolic $\implies \delta = 1$.

Counting with respect to length

Aside:
Lattice counting problem

Counting with respect to length and intersection number

$$\mathcal{G}^c(L, K) = \{\gamma \in \mathcal{G}^c \mid l(\gamma) \leq L, i(\gamma, \gamma) \leq K\}$$

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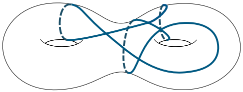
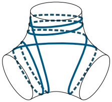


$\gamma \in \mathcal{G}^c(L, 4)$



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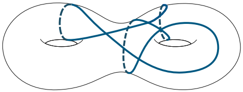

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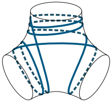

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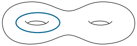
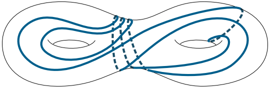
Question

If $K = f(L)$, what is the asymptotic growth of $\mathcal{G}^c(L, K)$ as $L \to \infty$?

$K = 0$

$\mathcal{G}^c(L, 0) -$ simple closed curves





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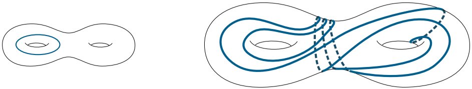

Theorem (Mirzakhani)

For an arbitrary hyperbolic surface $\mathcal{S}$,

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for $c(\mathcal{S})$ a constant depending only on the geometry of $\mathcal{S}$.

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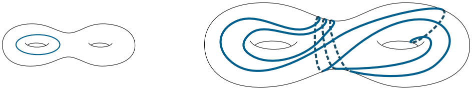

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$\mathcal{S}$ - genus g, n punctures

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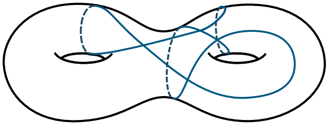

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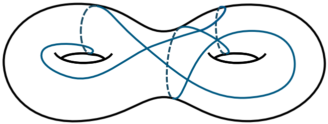

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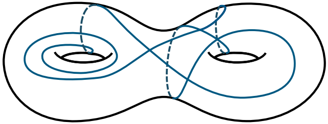

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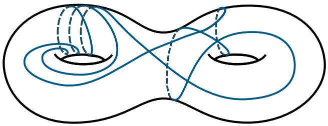

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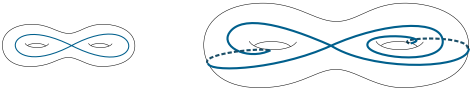
Therefore,

$$\#\mathcal{G}^c(L,0) \sim c_{\mathcal{S}} L^{6g-6+2n}$$

$K = 1, 2, 3 \ . \ . \ .$ and other fixed K?

$$\mathcal{G}^c(L,1):$$



The growth rate

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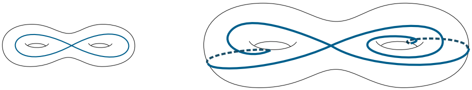

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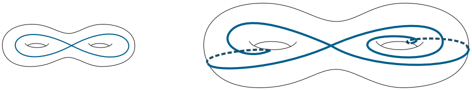

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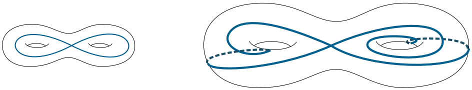

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– Previously, asymptotics for some K or some S by Rivin, Erlandsson-Souto

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K fixed

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Problem

Interpolate between these extremes with $K = f(L)$.

What is $c_{\mathcal{S}}(K)$?

Again, Mirzakhani's approach: cut $\mathcal{G}^c(L, K)$ into $\mathrm{Mod}_{\mathcal{S}}$ orbits.

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  This is a finite set!

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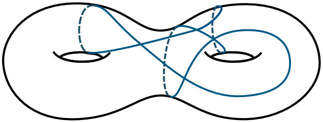
Moral

Asymptotic growth of $\#\mathcal{G}^c(L, K)$ $\leftarrow$ Asymptotic growth of $\#\mathcal{O}(\cdot, K)$

Counting Mod_S orbits of closed geodesics

Bounds on number of orbits



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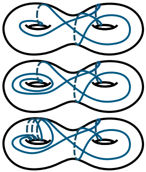

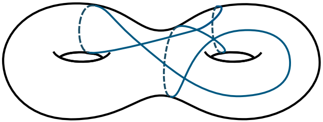

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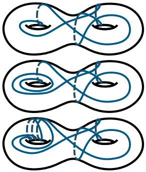

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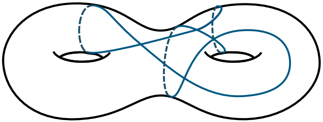

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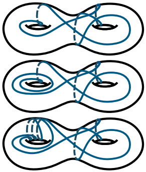

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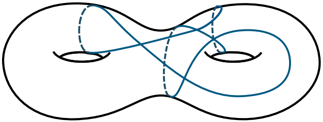

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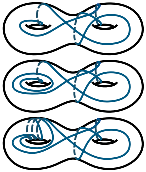

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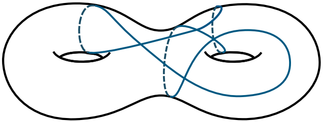

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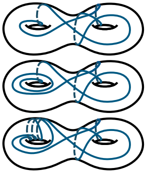

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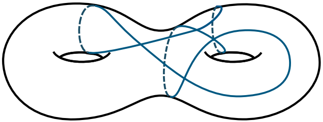
- Allows us to estimate $c_{\mathcal{S}}(K)$
- Bounds rather far apart: we dig deeper!

Keeping track of length in orbits



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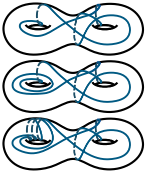

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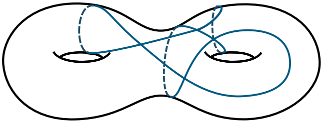

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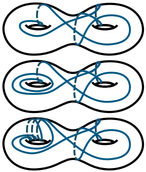

We cut $\mathcal{G}^c(L, K)$ into $\mathrm{Mod}_{\mathcal{S}}$ orbits.

Definition

$$\mathcal{O}(L, K) = \{\mathrm{Mod}_{\mathcal{S}} \cdot \gamma \mid \mathrm{Mod}_{\mathcal{S}} \cdot \gamma \cap \mathcal{G}^c(L, K) \neq \emptyset\}$$

So, orbits that actually contain length L curves!

Not all orbits have curves of length L!

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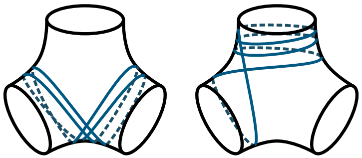
Theorem (Basmajian, Gaster, Aougab-Gaster-Patel-S.)

Suppose γ shortest in $Mod_{\mathcal{S}} \cdot \gamma$ and $i(\gamma, \gamma) = K$, then

$$c_1\sqrt{K} \leq l(\gamma) \leq c_2 K$$

where c_1, c_2 depend only on geometry of $\mathcal{S}$. These bounds are tight!



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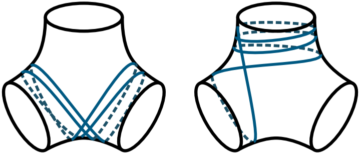

Thus: $\mathcal{O}(L, K) = \mathcal{O}(\cdot, K)$ only when $L \geq c_2 K$.

Get tighter bounds on $\#\mathcal{O}(L, K)$:

Theorem (S-)

On any $\mathcal{S}$,

$$\#\mathcal{O}(L, K) \leq \min\left\{ e^{d_{\mathcal{S}}\sqrt{K}\log\left(c_{\mathcal{S}}\frac{L}{\sqrt{K}}+c_{\mathcal{S}}\right)}, e^{d_{\mathcal{S}}\sqrt{K}\log d_{\mathcal{S}}\sqrt{K}} \right\}$$

where $c_{\mathcal{S}}$ depends on metric, $d_{\mathcal{S}}$ only on topology of $\mathcal{S}$.

What is the typical shortest curve?

Theorem (Lalley)

Let $\mathcal{S}$ be a closed surface. Choosing $\gamma_L \in \mathcal{G}^c(L)$ at random for each L,

$$i(\gamma_L, \gamma_L) \sim \kappa^2 L^2 \text{ almost surely}$$

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Conjecture

This is evidence for:

$$\#\mathcal{O}(\frac{1}{\kappa}\sqrt{K}, K) \sim \#\mathcal{O}(\cdot, K)$$

Back to counting curves

Our conjecture implies

$$c_1 e^{c_1\sqrt{K}} \leq \#\mathcal{O}(\cdot, K) \leq c_2 e^{c_2\sqrt{K}}$$

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$$c_1' e^{c_1\sqrt{K}} L^{6g-6+2n} \leq \#\mathcal{G}^c(L, K) \leq c_2' e^{c_2\sqrt{K}} L^{6g-6+2n}$$

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We should understand shortest curves in $\mathrm{Mod}_{\mathcal{S}}$ orbits better!

Curve lengths in Teichmüller space: New work

Two questions

Let $\gamma \in \mathcal{G}^c$. If $\phi \in \mathrm{Mod}_{\mathcal{S}}$, note

$$l_X(\gamma) = l_{\phi \cdot X}(\phi \cdot \gamma)$$

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Question (Minimize in thick part)

Find a metric Y so that γ is as short as possible.

Question (Minimize everywhere)

Fix a metric X. If $\gamma' \in \mathrm{Mod}_{\mathcal{S}} \cdot \gamma$ is shortest, what is $l_X(\gamma')$?

Theorem (Aougab, Gaster, Patel, S-)

Given γ with $i(\gamma,\gamma) = K$, we construct metric Y on $\mathcal{S}$ so that

$$l_Y(\gamma) \le c_{\mathcal{S}}\sqrt{K}$$

and $inj(Y) \ge \frac{1}{\sqrt{K}}$.

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Using Lenzhen-Rafi-Tao, this implies:

Corollary

If γ is shortest curve in $Mod_{\mathcal{S}} \cdot \gamma$ for metric any X, then

$$l_X(\gamma) \leq c_X K$$

The combinatorial model

Further applications model:

- Can construct many families $\{\gamma_K\}$ where

$$l_X(\gamma_K) = O(\sqrt{K})$$

  (generic curves)

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  (generic curves)
- and where

$$l_X(\gamma_K) = O(K)$$

  (worst case scenario)
- Given any metric X, curve γ, can bound $l_X(\gamma)$ from below.

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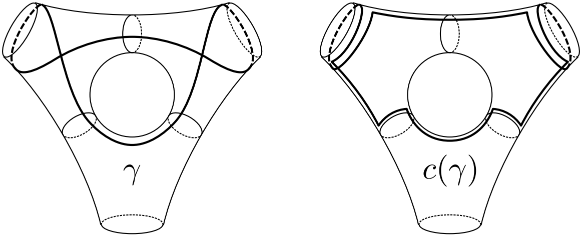

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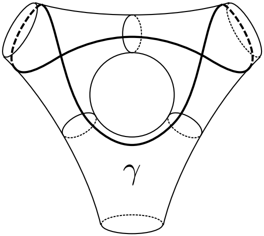

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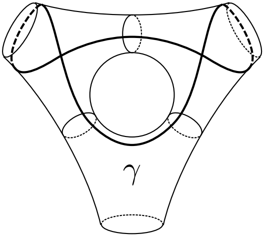

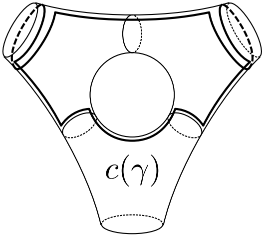

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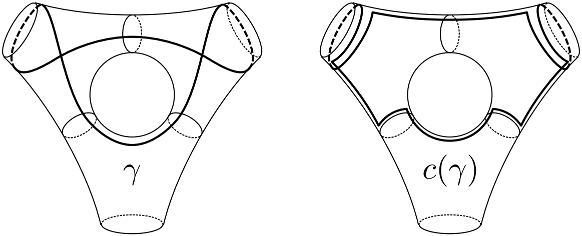

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- Geodesic γ
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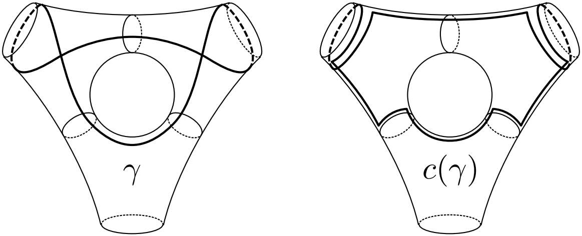

Ingredients:

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- Cut pairs of pants along matching seams

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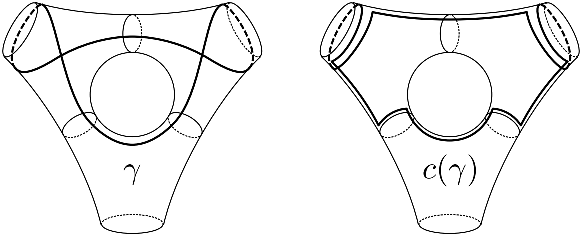

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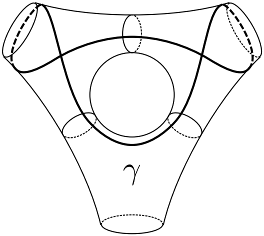

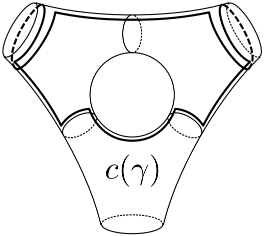

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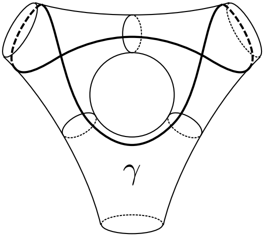

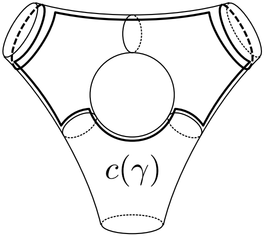

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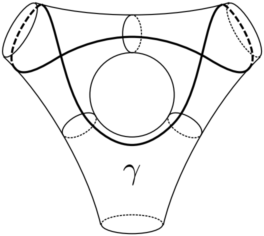

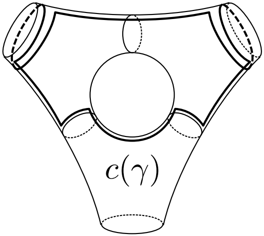

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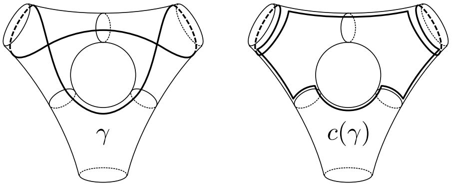
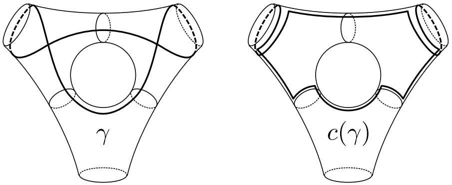
- Arcs along seams

Goal: relate $l(\gamma)$ and $i(\gamma, \gamma)$ to properties of $c(\gamma)$.



Length



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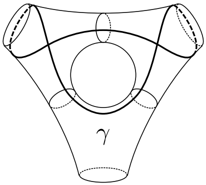
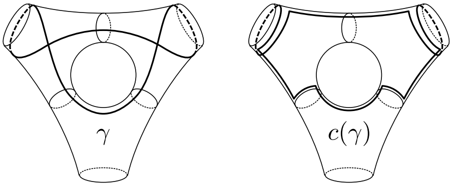


Choose curves lengths for Π, get metric X.

Length



Choose curves lengths for Π, get metric X. Then,

$$l(\gamma) \asymp c_X l(c(\gamma))$$

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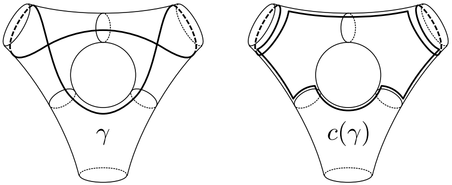

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Relationship depends on X.
NB. $l(c(\gamma))$ can be estimated from its combinatorics.

Combinatorics of $c(\gamma)$

Intermediate step: cut $c(\gamma)$ into pieces:

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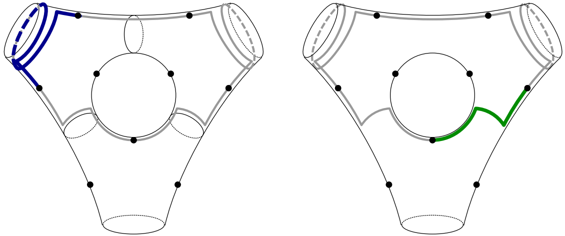

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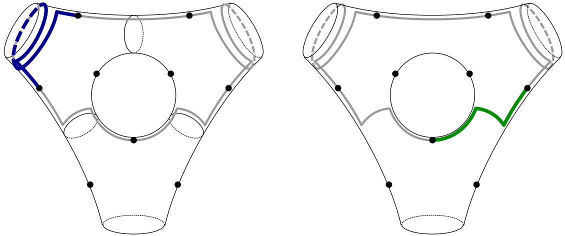

- Choose seam points on seam edges

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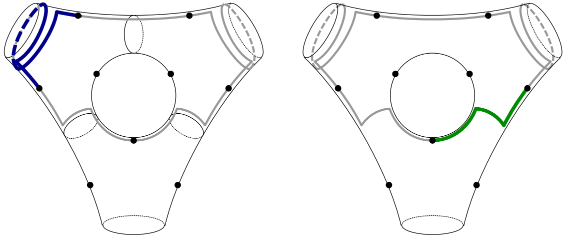

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- Cut $c(\gamma)$ into τ-arcs and β-arcs

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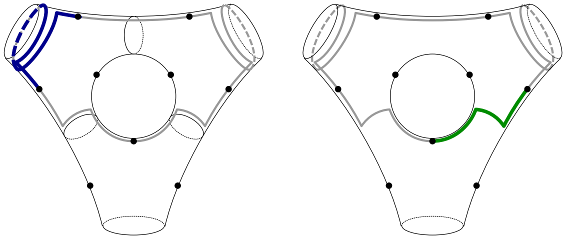

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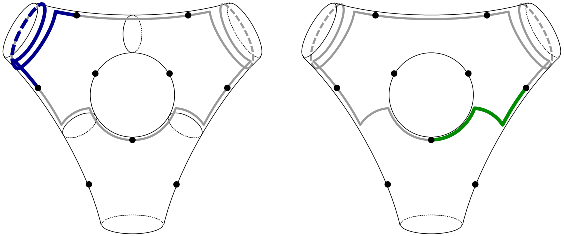

- Choose seam points on seam edges
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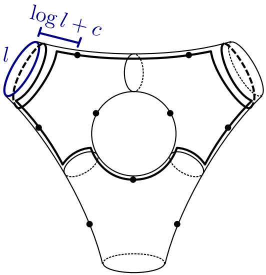
Revised goal: Relate $l(\gamma)$ and $i(\gamma, \gamma)$ to twisting numbers!

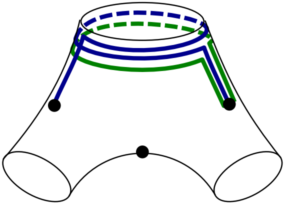
Twisting numbers, length and intersection

- Given lengths of pants curves, estimate $l(c(\gamma))$ by twisting numbers.



Twisting numbers, length and intersection

- If τ_i, τ_j have twisting numbers t_i, t_j, then they contribute roughly $\min\{t_i, t_j\}$ to intersection.



Twisting numbers, length and intersection

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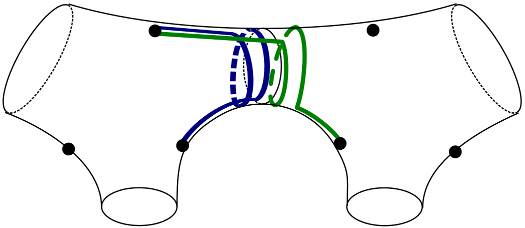
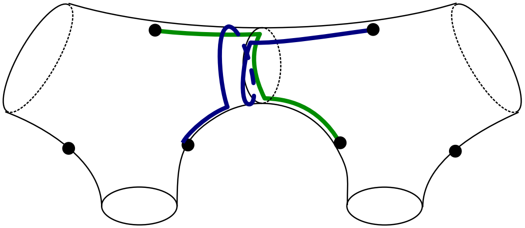


Twisting numbers, length and intersection

- If β_i, β_j have twisting numbers b_i, b_j, then they contribute roughly $|b_i - b_j|$ to intersection.



Optimal metric

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Optimal metric

To build a metric X on $\mathcal{S}$ where $l(\gamma) \leq c_X\sqrt{K}$:

- Choose a good pants decomposition
- Choose lengths of pants curves
- Use twisting numbers to relate length and intersection number

Counting orbits

To bound $\#\mathcal{O}(L,K)$, take one pants decomposition from each $\mathrm{Mod}_{\mathcal{S}}$ orbit.

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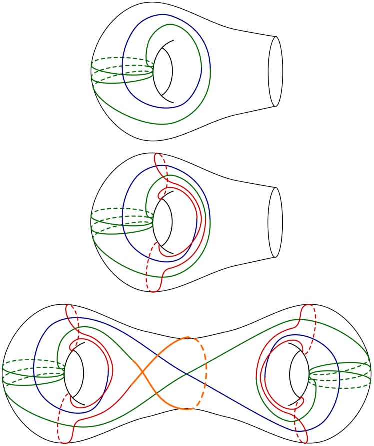
- Each $c(\gamma)$ determined by its combinatorics
- Bound possible twist numbers using L, K
- Any set of twist numbers $\{t_1, \ldots, t_n\}$ and $\{b_1, \ldots, b_m\} \leftrightarrow$ finite number of $c(\gamma)$.

Examples of curves

Can construct γ whose

- length is minimized in thick part of Teichmüller space
- length is minimized in thin part of Teichmüller space

Length minimized in thick part



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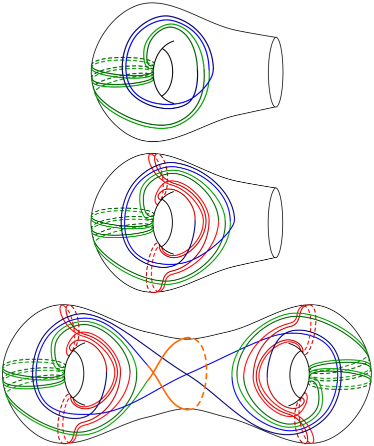

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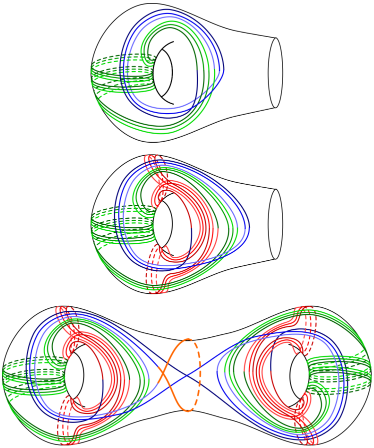

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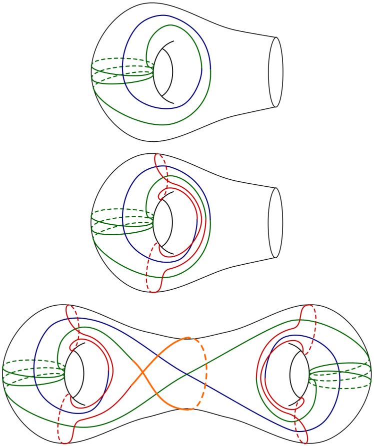

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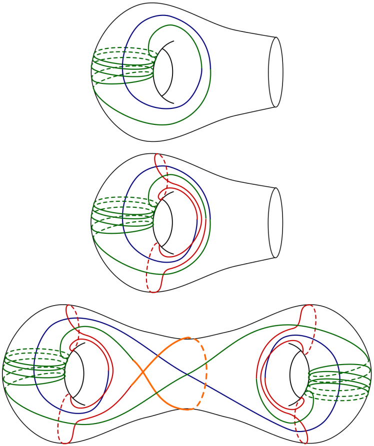

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