Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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# A projection from filling currents to Teichmuller space

#### Jenya Sapir

#### Binghamton University

Joint work with Sebastian Hensel

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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Geodes	sic currents o	on a surface <i>S</i>	;		

Let S be a surface of genus  $g \ge 2$ , no boundary or punctures. Fix a hyperbolic metric  $X_0$ .

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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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• Geodesics on S

Objects •	Geodesic currents	Intersection number	Structure 000000	Idea of proof 00	Questions 0



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- ${\scriptstyle \bullet}$  Geodesics on S
- Measured laminations  $\mathcal{ML}(S)$

Objects •	Geodesic currents	Intersection number	Structure 000000	Idea of proof 00	Questions 0



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- Geodesics on S
- Measured laminations  $\mathcal{ML}(S)$
- Teichmuller space  $\mathcal{T}(S)$

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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- Geodesics on S
- Measured laminations  $\mathcal{ML}(S)$
- Teichmuller space  $\mathcal{T}(S)$

... and many more. Unified by space of **geodesic currents**.

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Geode	esic currents				



Identify 
$$\tilde{X}_0 = \mathbb{H}^2$$
.

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Geode	sic currents				



Identify  $\tilde{X}_0 = \mathbb{H}^2$ .

 $ilde{\mathcal{G}}$  - set of geodesics on  $\mathbb{H}^2$ 

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Identify  $\tilde{X}_0 = \mathbb{H}^2$ .

 $ilde{\mathcal{G}}$  - set of geodesics on  $\mathbb{H}^2$ 

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Have  $\pi_1(S)$  action on  $\tilde{\mathcal{G}}$ .

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Geode	sic currents				



Definition (Bonahon '86:) A geodesic current on S is a Borel,  $\pi_1(S)$ -invariant measure on  $\tilde{\mathcal{G}}$ 

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Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Geode	sic currents				



Definition (Bonahon '86:) A geodesic current on S is a Borel,  $\pi_1(S)$ -invariant measure on  $\tilde{\mathcal{G}}$ 

 $\mathcal{C}(S)$  - space of geodesic currents.

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Geode	sic currents				



Definition (Bonahon '86:) A geodesic current on S is a Borel,  $\pi_1(S)$ -invariant measure on  $\tilde{\mathcal{G}}$ 

 $\mathcal{C}(S)$  - space of geodesic currents. Independent of choice of  $X_0$ .

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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## Geodesic currents



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Example: Closed geodesics  $\mathcal{G}^c \subset \mathcal{C}(S)$ 

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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#### Geodesic currents



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Example: Closed geodesics  $\mathcal{G}^c \subset \mathcal{C}(S)$ 

- Take  $\gamma$  on  $X_0$
- $\bullet$  Lift to  $\tilde{\gamma}$  on  $\mathbb{H}^2$
- $\mu_{\gamma}$  Dirac measure on  $\tilde{\gamma} \subset \tilde{\mathcal{G}}$

Objects	Geodesic currents	Intersection number	Idea of proof	Questions
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Objects	Geodesic currents	Intersection number		Idea of proof	Questions
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• Consider geometric intersection number on  $\mathcal{G}^c$ 

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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- Consider geometric intersection number on  $\mathcal{G}^c$
- Extend bilinearly on weighted multicurves: If  $\alpha, \beta, \gamma \in \mathcal{G}^c$ , c > 0,

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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- Consider geometric intersection number on  $\mathcal{G}^c$
- Extend bilinearly on weighted multicurves: If  $\alpha, \beta, \gamma \in \mathcal{G}^c$ , c > 0,

$$i(c\alpha + \beta, \gamma) = ci(\alpha, \gamma) + i(\beta, \gamma)$$

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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• Bonahon: The set  $\mathbb{R}_+\mathcal{G}^c$  is dense in  $\mathcal{C}(S)$ !

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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- Bonahon: The set  $\mathbb{R}_+\mathcal{G}^c$  is dense in  $\mathcal{C}(S)$ !
- Bonahon: Intersection number extends *continuously* to all of  $\mathcal{C}(S)$

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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- Symmetric
- Bilinear
- Mapping class group invariant

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Let  $\gamma \in \mathcal{G}$ ,  $Y \in \mathcal{T}(S)$ .

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Let  $\gamma \in \mathcal{G}$ ,  $Y \in \mathcal{T}(S)$ . Currents:  $\mu_{\gamma}, \mu_{Y} \in \mathcal{C}(S)$ .

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Let  $\gamma \in \mathcal{G}$ ,  $Y \in \mathcal{T}(S)$ . Currents:  $\mu_{\gamma}, \mu_{Y} \in \mathcal{C}(S)$ . Then,

$$i(\mu_{\gamma},\mu_{Y}) = \ell_{Y}(\gamma)$$

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Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Intersection gives geodesic length!

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Let  $\gamma \in \mathcal{G}$ ,  $Y \in \mathcal{T}(S)$ . Currents:  $\mu_{\gamma}, \mu_{Y} \in \mathcal{C}(S)$ . Then,

$$i(\mu_{\gamma},\mu_{Y}) = \ell_{Y}(\gamma)$$

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Intersection gives geodesic length! Same holds when Y is any metric on S.

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Struct	ure of $\mathbb{P}\mathcal{C}(S)$	)			

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Let  $\mathbb{P}C(S) = C(S)/\mathbb{R}_+$ .

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Struct	ure of $\mathbb{P}C(S)$				

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Let  $\mathbb{P}C(S) = C(S)/\mathbb{R}_+$ . Bonahon showed:

•  $\mathbb{P}\mathcal{C}(S)$  is compact

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Structu	ure of $\mathbb{P}C(S)$				

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Let  $\mathbb{P}C(S) = C(S)/\mathbb{R}_+$ . Bonahon showed:

- $\mathbb{P}\mathcal{C}(S)$  is compact
- $\mathcal{T}(S) \hookrightarrow \mathbb{P}\mathcal{C}(S)$

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Struct	ure of $\mathbb{P}\mathcal{C}(S)$	)			



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Let  $\mathbb{P}C(S) = C(S)/\mathbb{R}_+$ . Bonahon showed:

- $\mathbb{P}C(S)$  is compact
- $\mathcal{T}(S) \hookrightarrow \mathbb{P}\mathcal{C}(S)$
- $\partial \mathcal{T}(S) = \mathbb{P}\mathcal{ML}(S)$

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Struct	ure of $\mathbb{D}\mathcal{C}(S)$				



Let  $\mathbb{P}C(S) = C(S)/\mathbb{R}_+$ . Bonahon showed:

- $\mathbb{P}C(S)$  is compact
- $\mathcal{T}(S) \hookrightarrow \mathbb{P}\mathcal{C}(S)$
- $\partial \mathcal{T}(S) = \mathbb{P}\mathcal{ML}(S) \leftarrow$  Thurston compactification

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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Struct	ure of $\mathbb{D}\mathcal{C}(S)$				



Let  $\mathbb{P}C(S) = C(S)/\mathbb{R}_+$ . Bonahon showed:

•  $\mathbb{P}C(S)$  is compact

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What about the rest of  $\mathbb{PC}(S)$ ?

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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## Structure of $\mathbb{P}\overline{\mathcal{C}(S)}$



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A current  $\mu$  is filling if  $i(\mu, \nu) > 0$  for all  $\nu \in \mathcal{C}(S)$ .

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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A current  $\mu$  is filling if  $i(\mu, \nu) > 0$  for all  $\nu \in C(S)$ . Examples: filling curves,

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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A current  $\mu$  is filling if  $i(\mu, \nu) > 0$  for all  $\nu \in C(S)$ . Examples: filling curves, all metrics

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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A current  $\mu$  is filling if  $i(\mu, \nu) > 0$  for all  $\nu \in C(S)$ . Examples: filling curves, all metrics

 $\mathcal{C}_{fill}(S)$  - filling currents.

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Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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A current  $\mu$  is filling if  $i(\mu, \nu) > 0$  for all  $\nu \in C(S)$ . Examples: filling curves, all metrics

> $C_{fill}(S)$  - filling currents.  $\mathbb{P}C_{fill}(S)$  - filling projective currents.

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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Burger-lozzi-Parreau-Pozzetti '19: non-filling projective currents are closed in  $\mathbb{PC}(S)$ .

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Structuro	of $\mathbb{D}\mathcal{C}(S)$		



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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Projec	tions to $\mathcal{T}(S)$	5)			

#### A point $X \in \mathcal{T}(S)$ is the length minimizer of $\mu \in \mathcal{C}(S)$ iff

$$i(X,\mu) \leq i(Y,\mu)$$

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for all  $Y \in \mathcal{T}(S)$ .

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Projec	tions to $\mathcal{T}(S)$	5)			

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for all  $Y \in \mathcal{T}(S)$ .

**Remark**: X only depends on the projective class of  $\mu$ !

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Projec	tions to $\mathcal{T}(S)$	5)			

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Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Drojoc	tions to $\tau(c)$	2)			



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Kerckhoff: If  $\mu, \nu \in \mathcal{ML}(S)$ ,  $\mu + \nu \in \mathcal{C}_{fill}(S)$ ,

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Droio	stions to $\tau(c)$	2)			
Projec	ctions to 7 (S				

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Kerckhoff: If  $\mu, \nu \in \mathcal{ML}(S)$ ,  $\mu + \nu \in \mathcal{C}_{fill}(S)$ , then for all  $t \in (0, 1)$ ,  $t\mu + (1-t)\nu$ 

Objects O	Geodesic currents 000	Intersection number 00	Structure 000●00	ldea of proof 00	Questions 0
Droio	stions to $\tau(c)$	2)			
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 $(1-t)\nu$ 

 $\nu$ 

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 $\mathbb{P}\mathcal{C}(S)$ 

Kerckhoff: If  $\mu, \nu \in \mathcal{ML}(S)$ ,  $\mu + \nu \in \mathcal{C}_{fill}(S)$ , then for all  $t \in (0, 1),$ 

$$t\mu + (1-t)\nu$$

has a unique length minimizer  $X_t$ .

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Proiec	tions to $\mathcal{T}(S)$	5)			



Kerckhoff: If  $\mu, \nu \in \mathcal{ML}(S)$ ,  $\mu + \nu \in C_{fill}(S)$ , then for all  $t \in (0, 1)$ ,

$$t\mu + (1-t)\nu$$

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has a unique length minimizer  $X_t$ . Line of minima  $t \to X_t$  is a continuous, proper map.

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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## Projections to $\mathcal{T}(S)$



#### Theorem (Hensel-S)

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Projec	tions to $\mathcal{T}(S)$	5)			



#### Theorem (Hensel-S)

There is a map  $\pi : \mathbb{P}C_{fill}(S) \to \mathcal{T}(S)$  sending  $[\mu]$  to its length minimizer that is

• Continuous

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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## Projections to $\mathcal{T}(S)$



#### Theorem (Hensel-S)

- Continuous
- Proper

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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## Projections to $\mathcal{T}(S)$



#### Theorem (Hensel-S)

- Continuous
- Proper
- Identity on  $\mathcal{T}(S)$

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## Projections to $\mathcal{T}(S)$ .



#### Theorem (Hensel-S)

- Continuous
- Proper
- Identity on  $\mathcal{T}(S)$
- Mapping class group invariant

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Projec	tions to $\mathcal{T}(\mathbf{S})$	5)			



#### Theorem (Hensel-S)

There is a continuous, proper, Mod(S)-invariant map  $\pi : \mathbb{P}C_{fill}(S) \to \mathcal{T}(S)$  sending  $[\mu]$  to its length minimizer.

Objects	Geodesic currents	Intersection number	Structure	Idea of proof	Questions
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Projec	tions to $\mathcal{T}(\mathbf{S})$	5)			



#### Theorem (Hensel-S)

There is a continuous, proper, Mod(S)-invariant map  $\pi : \mathbb{P}C_{fill}(S) \to \mathcal{T}(S)$  sending  $[\mu]$  to its length minimizer. On the other hand, if  $[\mu]$  is non-filling, it has no length minimizer in X.

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Idea o	f proof: Prop	perness			

#### Extend inequalities for closed curves:

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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ldea o	of proof: Prop	perness			

#### Extend inequalities for closed curves:

• Collar lemma: For  $\mu \in \mathcal{C}(S)$ ,  $\alpha$  - simple,  $X \in \mathcal{T}(S)$ ,

 $i(\mu, \alpha) \mathsf{Col}_X(\alpha) \leq i(\mu, X)$ 

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Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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ldea o	f proof: Prop	perness			

#### Extend inequalities for closed curves:

• Collar lemma: For  $\mu \in \mathcal{C}(S)$ ,  $\alpha$  - simple,  $X \in \mathcal{T}(S)$ ,

 $i(\mu, \alpha) \mathsf{Col}_X(\alpha) \leq i(\mu, X)$ 

• Minimal length bound:

$$i(\mu, \pi(\mu)) \leq c\sqrt{i(\mu, \mu)}$$

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Take  $c = 4\sqrt{2\chi(S)}$  by Aougab-Souto.

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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ldea o	f proof				

**Existence:** Wolpert: Given  $\mu \in C(S)$ , the function

$$\ell_{\mu}:\mathcal{T}(\mathcal{S})
ightarrow\mathbb{R} \ X\mapsto i(\mu,X)$$

is strictly convex with respect to Weil-Petersson metric on  $\mathcal{T}(S)$ .

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**Identity on**  $\mathcal{T}(S)$ **:** Bonahon:

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for all  $Y \in \mathcal{T}(S)$ .

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Mapping class group invariant:  $i(\cdot, \cdot)$  is invariant

Objects	Geodesic currents	Intersection number	Structure	ldea of proof	Questions
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Given  $\pi : \mathbb{P}\mathcal{C}_{fill}(S) \to \mathcal{T}(S)$ , have  $\pi^{-1}(X)$  is compact.

Question: What can we say about points in  $\pi^{-1}(X)$ ?

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 and  $i(\pi(\mu), \cdot)$ 

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