A projection from filling currents to Teichmuller space

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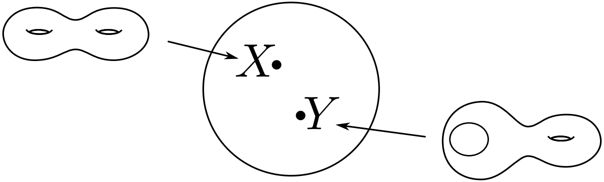
Joint work with Sebastian Hensel

Geodesic currents on a surface S

Let S be a surface of genus $g \geq 2$, no boundary or punctures. Fix a hyperbolic metric X_0.

Geodesic currents on a surface S



- Geodesics on S

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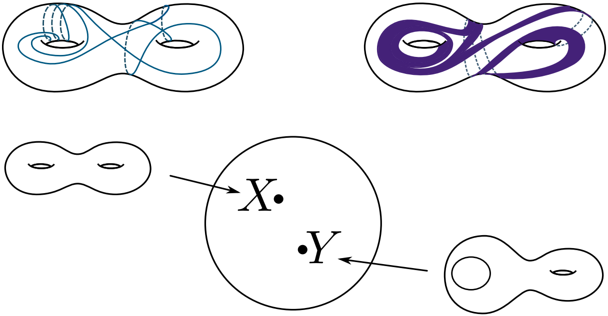
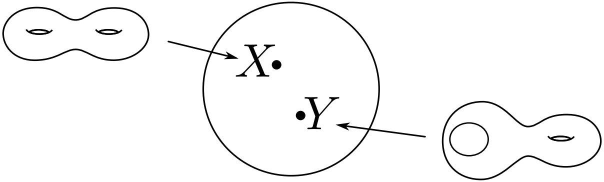


- Geodesics on S
- Measured laminations $\mathcal{ML}(S)$

Geodesic currents on a surface S



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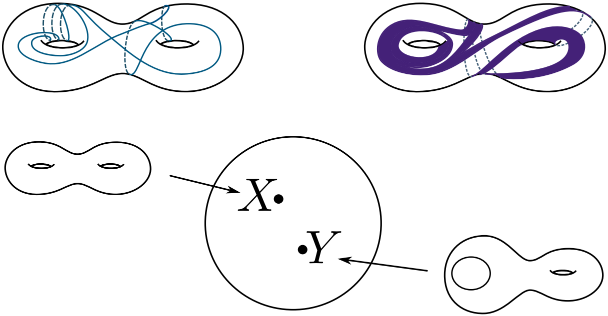

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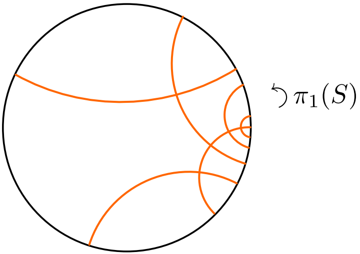
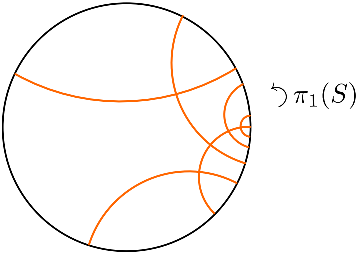
... and many more. Unified by space of **geodesic currents**.

Geodesic currents



Identify $\tilde{X}_0 = \mathbb{H}^2$.

Geodesic currents



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$\tilde{\mathcal{G}}$ - set of geodesics on $\mathbb{H}^2$

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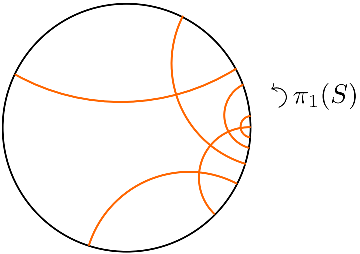

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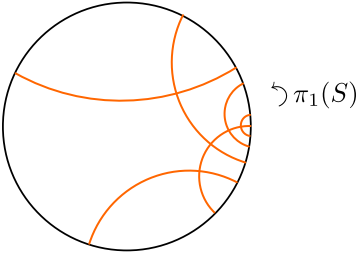
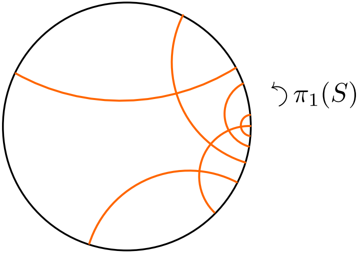
Have $\pi_1(S)$ action on $\tilde{\mathcal{G}}$.

Geodesic currents



Definition (Bonahon '86:)
A geodesic current on S is a Borel, $\pi_1(S)$-invariant measure on $\tilde{\mathcal{G}}$

Geodesic currents



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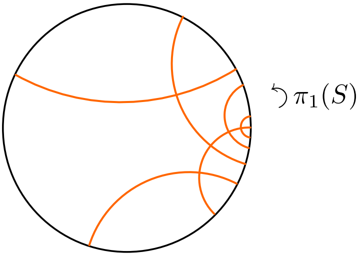

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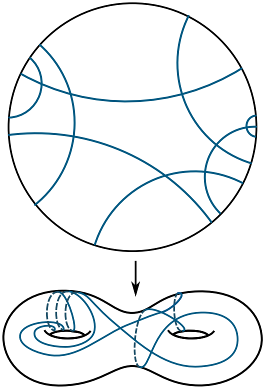
Geodesic currents



Example: Closed geodesics $\mathcal{G}^c \subset \mathcal{C}(S)$

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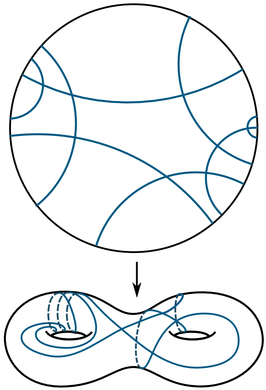
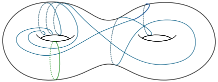


Example: Closed geodesics $\mathcal{G}^c \subset \mathcal{C}(S)$

- Take γ on X_0
- Lift to $\tilde{\gamma}$ on $\mathbb{H}^2$
- μ_γ - Dirac measure on $\tilde{\gamma} \subset \tilde{\mathcal{G}}$

Intersection number for geodesic currents



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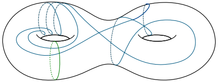

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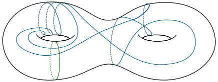

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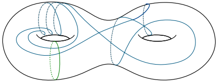

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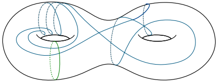

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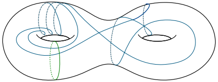

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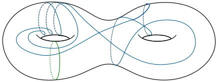

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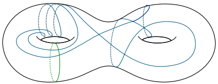

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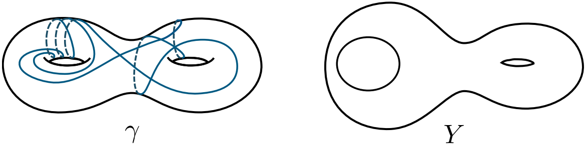
  - Symmetric
  - Bilinear
  - Mapping class group invariant

Intersection number unifies structures



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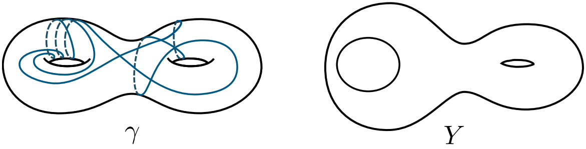

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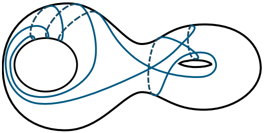

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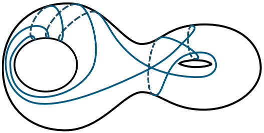

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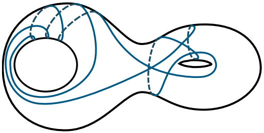

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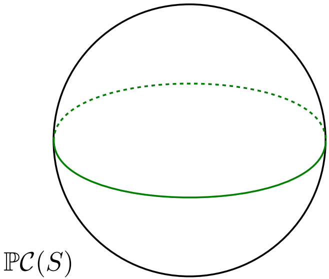
Same holds when Y is any metric on S.

Structure of $\mathbb{P}\mathcal{C}(S)$



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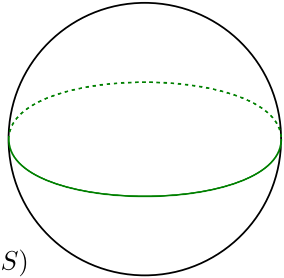

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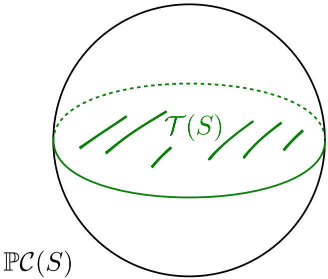

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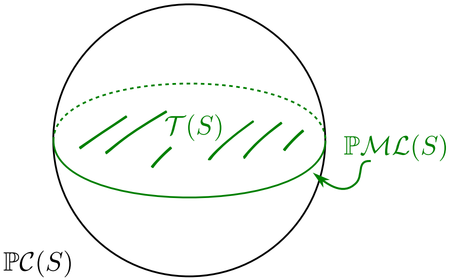

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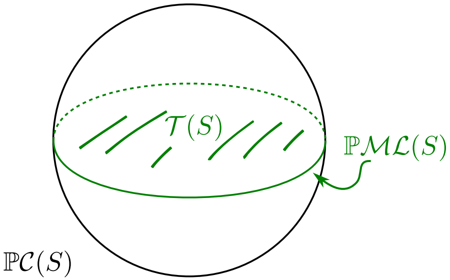

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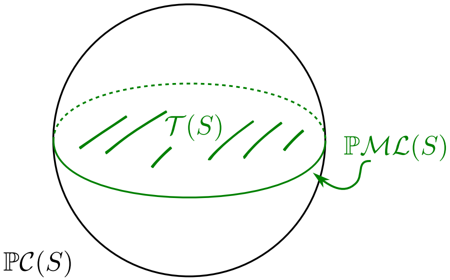

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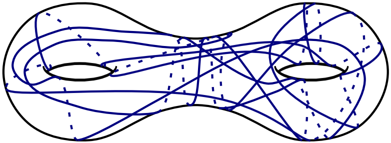
What about the rest of $\mathbb{P}\mathcal{C}(S)$?

Structure of $\mathbb{P}\mathcal{C}(S)$



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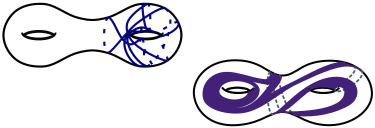

Non-filling

A current μ is filling if $i(\mu, \nu) > 0$ for all $\nu \in \mathcal{C}(S)$.

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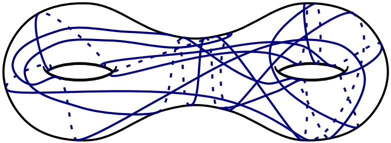

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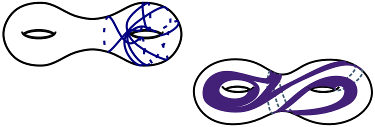

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Examples: filling curves,

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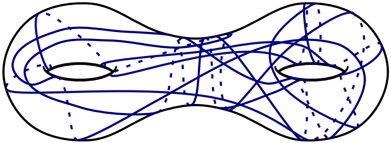

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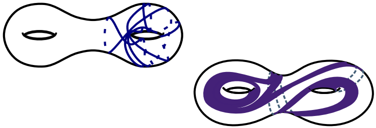

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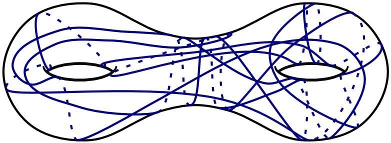

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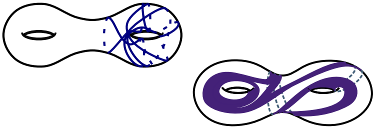

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$$\mathcal{C}_{fill}(S) \text{ - filling currents.}$$

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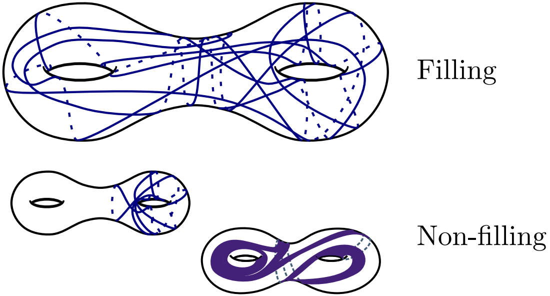

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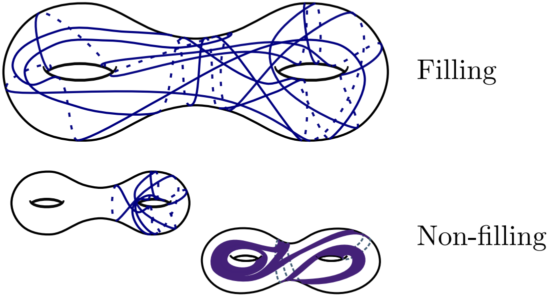

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Burger-Iozzi-Parreau-Pozzetti '19: non-filling projective currents are closed in $\mathbb{P}\mathcal{C}(S)$.

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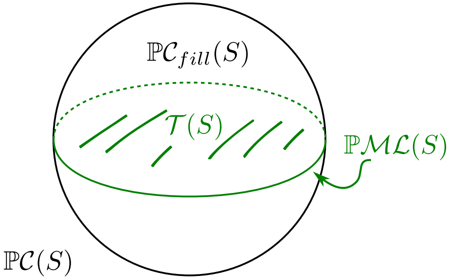

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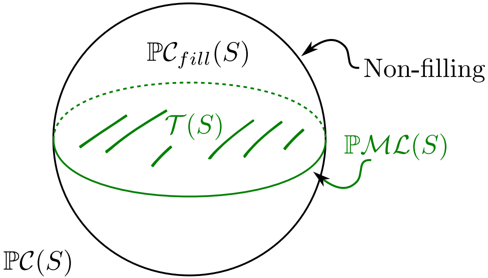

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A point $X \in \mathcal{T}(S)$ is the length minimizer of $\mu \in \mathcal{C}(S)$ iff

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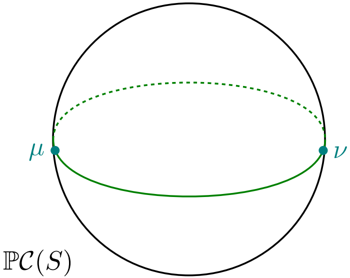

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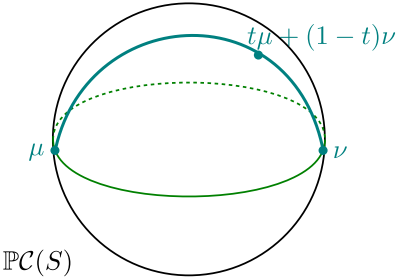

Kerckhoff: If $\mu, \nu \in \mathcal{ML}(S)$, $\mu + \nu \in \mathcal{C}_{fill}(S)$, then for all $t \in (0,1)$,

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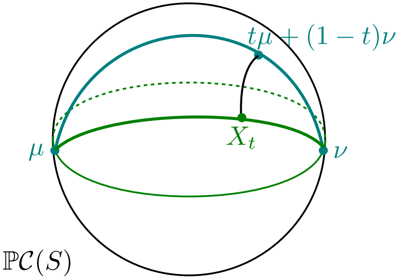
Projections to $\mathcal{T}(S)$



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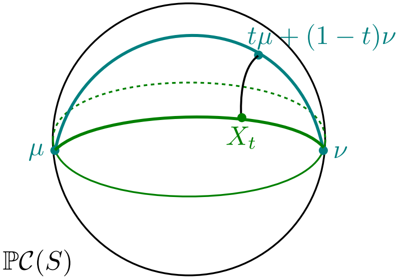

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Line of minima $t \to X_t$ is a continuous, proper map.

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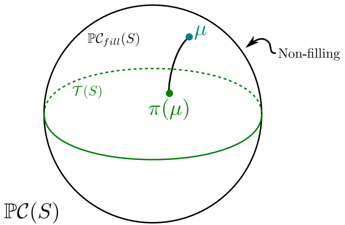

Theorem (Hensel-S)

There is a map $\pi : \mathbb{P}\mathcal{C}_{fill}(S) \to \mathcal{T}(S)$ *sending* $[\mu]$ *to its length minimizer that is*

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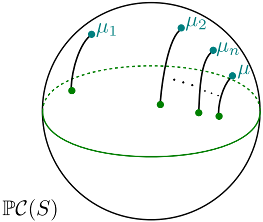

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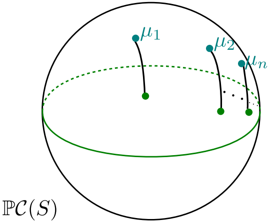

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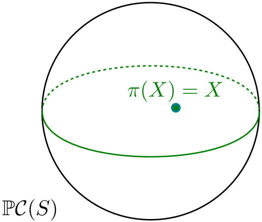

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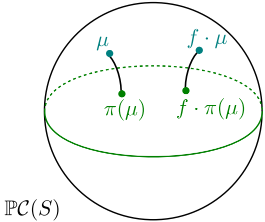

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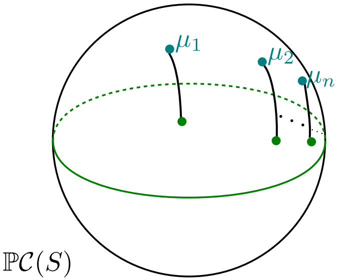
- *Mapping class group invariant*

Projections to $\mathcal{T}(S)$



Theorem (Hensel-S)

There is a continuous, proper, $Mod(S)$-invariant map $\pi : \mathbb{P}\mathcal{C}_{fill}(S) \to \mathcal{T}(S)$ sending $[\mu]$ to its length minimizer.

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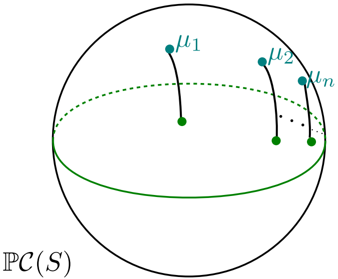

Theorem (Hensel-S)

There is a continuous, proper, Mod(S)-invariant map $\pi : \mathbb{P}\mathcal{C}_{fill}(S) \to \mathcal{T}(S)$ sending $[\mu]$ to its length minimizer.
On the other hand, if $[\mu]$ is non-filling, it has no length minimizer in X.

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Extend inequalities for closed curves:

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- Minimal length bound:

$$i(\mu, \pi(\mu)) \leq c\sqrt{i(\mu, \mu)}$$

Take $c = 4\sqrt{2\chi(S)}$ by Aougab-Souto.

Idea of proof

Existence: Wolpert: Given $\mu \in \mathcal{C}(S)$, the function

$$\begin{aligned} \ell_\mu : \mathcal{T}(S) &\to \mathbb{R} \\ X &\mapsto i(\mu, X) \end{aligned}$$

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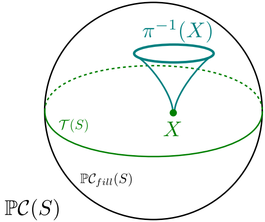
Mapping class group invariant: $i(\cdot, \cdot)$ is invariant

Questions



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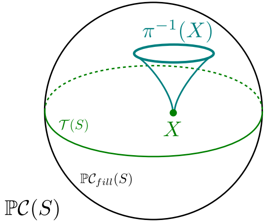

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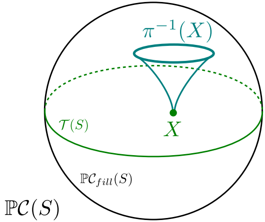

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relate? (For suitable normalization of $\mu \in [\mu]$.)