Math 401-02 **Practice 2** Fall 2021

(Due: **Tuesday, Sep. 14**)

Instruction: Work in small groups

Problem 1. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 8 & 6 & 4 & 5 & 3 & 7 & 10 & 9 \end{pmatrix} \in \mathbf{S}_{10}$.

(1) Write σ as a product of disjoint cycles.
(2) Let $\tau = (1\,3\,5\,7\,8)$. Find the smallest positive integer n such that $\tau^n = e$.
(3) Compute $\sigma^{-1}, (\sigma\rho)^{-1}$ and $\sigma^{-1}\rho^{-1}$, where $\rho = (126)$.
(4) (*) Let $f = (a_1\, a_2 \cdots\, a_r)$ and $g = (b_1\, b_2 \cdots\, b_s)$ be two disjoint cycles in S_n for some integer $n > 1$. Show that $fg = gf$.

Problem 2. Let a, b, c be integers with $a \neq 0$. Show that

(1) If $a \mid b$ and $a \mid c$, then $a \mid bu + cv$ for all integers u, v.
(2) Assume that $d = ax + by$ is a positive integer for some integers x, y. Does it imply that $d = \gcd(a, b)$?
(3) Show that $\gcd(a, b) = 1$ if and only if $1 = ax + by$ for some integers x, y.
(4) Find the multiplicative inverse of $[43]$ in $\mathbb{Z}_{102}$ if it exists.

Problem 3. Construct the Cayley tables of $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_8^*, \cdot)$. Use the tables to find the inverse of each element in the group.

Problem 4. Let G be a group with identity e. Show that G is abelian if one of the following conditions holds.

(1) $a^2 = e$ for all $a \in G$.
(2) $(ab)^2 = a^2b^2$ for all $a, b \in G$.
(3) (*) $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$, for all $a, b \in G$.