Math 401-02 **Homework 3** Fall 2021

(Due: **Wednesday, September 22**)

Instruction: Turn in All problems. Use Tex/Latex if possible.

Problem 1. Let G be a group. The center of G, denoted by $\mathbf{Z}(G)$, is the subset of elements in G that commute with every element of G, that is,

$$\mathbf{Z}(G) = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

(1) Show that the center $\mathbf{Z}(G)$ of G is an abelian subgroup of G.
(2) Compute $\mathbf{Z}(\mathrm{D}_8)$ and $\mathbf{Z}(\mathrm{S}_3)$.
(3) Let $a \in G$. The centralizer of a in G, denoted by $\mathbf{C}_G(a)$, is the set of all elements of G that commute with a, that is,

$$\mathbf{C}_G(a) = \{x \in G : xa = ax\}.$$

Show that $\mathbf{Z}(G) = \cap_{a \in G} \mathbf{C}_G(a)$.

Problem 2. Let G be a finite group with identity e. Let $a, b \in G$.

(1) Show that $o(a) = o(a^{-1})$.
(2) Show that $o(bab^{-1}) = o(a)$.
(3) Show that $o(ab) = o(ba)$.
(4) Assume that $a \neq e, o(b) = 2$ and $bab^{-1} = a^2$. Find $o(a)$.

Problem 3.

(1) Let $G = (\mathbb{Z}_{10}, +)$. List all the elements G; determine the order of each element and list all the cyclic subgroups of G.
(2) Do the same for the group $(\mathbb{Z}_{14}^*, \cdot)$.

Problem 4.

(1) Prove or disprove that the union of two proper subgroups of a group is a subgroup.
(2) Let H be a finite non-empty subset of a group G which is closed under multiplication, that is, $ab \in H$ for all $a, b \in H$. Prove that H is a subgroup of G.