**MATH 304-01**
**Fall 2019**
**Test 1**
**September 25, 2019**
**Time Limit: 60 Minutes**

**Name (Print):** ____________________

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This test contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page. Question 1 is a **True/False** question. Clearly PRINT your correct answer. Question 2 is multiple choice.

You are required to show your work on Questions 3 – 5 on this exam. Question 6 is a bonus question. Only do this question when you finish other questions. The following rules apply:

- **Turn off and put away your cell phone.**
- **No note nor book is allowed.**
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.**

Do not write in the table to the right.

| Question | Points | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 15 | |
| 3 | 20 | |
| 4 | 15 | |
| 5 | 20 | |
| 6 | 10 | |
| Total: | 110 | |

1. (30 points) For each statement below, indicate whether it is **True** or **False**:

(a) T If the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not a linear combination of the columns of $A$.

(b) F The columns of a matrix $A$ are linearly independent if and only if the equation $A\mathbf{x} = \mathbb{O}$ has the trivial solution.

(c) T If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ are linearly independent vectors and if $v_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

(d) T If $A$ is a $4 \times 6$ matrix, then the columns of $A$ are linearly dependent.

(e) F If $\mathbf{x}$ is a nontrivial solution of the homogeneous equation $A\mathbf{x} = \mathbb{O}$, then every entry in $\mathbf{x}$ is nonzero.

(f) F Let $A$ and $B$ be matrices. If $AB$ is defined, then $(AB)^T = A^T B^T$.

(g) F If $A$ is a square $n \times n$ matrix, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an onto map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(h) F Let $A$ be a square $n \times n$ matrix. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some vector $\mathbf{b} \in \mathbb{R}^n$, then $A$ is invertible.

(i) T If the columns of a square $n \times n$ matrix $A$ are linearly independent, then $A^T$ is invertible.

(j) T If $A$ and $B$ are square $n \times n$ matrices and $AB = I_n$, then both $A$ and $B$ are invertible.

2. (15 points) Answer the following multiple choice questions.

   (*i*) Consider a system of linear equations with **augmented** matrix

$$C = \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 2 & 0 & -3 & 1 \\ 0 & 0 & -1 & 4 & -3 \end{array}\right).$$

   Circle ALL the **correct** answers.

   (*a*) This system has 4 variables and 3 equations.
   (*b*) This system has two free variables.
   (*c*) The system has a unique solution since $C$ has a pivot in every row.
   (*d*) The system has infinitely many solutions since it has a free variable.
   (*e*) This system is inconsistent since it has a nonzero row.

   (*ii*) Let $T$ be a linear transformation with standard matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 3 & 4 \end{pmatrix}.$$

   Circle **ALL the correct** answers.

   (*a*) The domain of $T$ is $\mathbb{R}^3$.
   (*b*) The codomain of $T$ is $\mathbb{R}^3$.
   (*c*) $T$ is one to one.
   (*d*) $T$ is onto.
   (*e*) $T$ is invertible since $A$ has a pivot position in every row.

   (*iii*) Let $A$ be a square $4 \times 4$ matrix. Assume that the equation $A\mathbf{x} = \mathbf{b}$ is **inconsistent** for some vector $\mathbf{b} \in \mathbb{R}^4$. Which of the following statements are **correct**?

   (*a*) The equation $A\mathbf{x} = \mathbb{O}$ has only trivial solution.
   (*b*) The equation $A\mathbf{x} = \mathbb{O}$ has infinitely many solutions.
   (*c*) The columns of $A$ do not span $\mathbb{R}^4$.
   (*d*) The columns of $A$ span $\mathbb{R}^4$.
   (*e*) $A$ is not invertible.

3. (20 points) Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 3 & 3 \\ 1 & -1 & 1 & -4 \end{pmatrix}.$$

(a) Find the reduced row-echolon form (RREF) of $A$. List all the elementary row operations performed.

(b) Which columns of $A$ are pivot columns?

(c) Find all vectors $\mathbf{x} \in \mathbb{R}^4$ such that $T(\mathbf{x}) = \mathbb{O}$. Write the answers in parametric vector form.

(d) Is $T$ one to one ? onto? Justify your answer.

a)

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 3 & 3 \\ 1 & -1 & 1 & -4 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & 0 & -3 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[R_1-R_3]{R_2-R_3} \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

b) The first three columns of $A$ are pivot columns.

c) $T(x) = \mathbb{O} \Leftrightarrow Ax = \mathbb{O}$.

we have

$$x_1 - 3x_4 = 0 \qquad x_1 = 3x_4$$
$$x_2 + 3x_4 = 0 \qquad x_2 = -3x_4$$
$$x_3 + 2x_4 = 0 \qquad x_3 = -2x_4$$

$x_1, x_2, x_3$: basic variables,

$x_4$: free variable.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ -3x_4 \\ -2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 3 \\ -3 \\ -2 \\ 1 \end{bmatrix}.$$

d) $T$ is not one to one since $T(x) = \mathbb{O}$ has infinitely many solutions.

$T$ is onto since $A$ has a pivot position in every row. So $T(x) = b$ or $Ax = b$ has a solution for each $b \in \mathbb{R}^3$.

4. (15 points) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 4 \\ 1 & -1 & 4 \end{pmatrix}$.

1. Determine whether $A$ is invertible or not. Find $A^{-1}$ if it exists.
2. Is $A^T$ invertible? Find $(A^T)^{-1}$ if it exists.

1. $$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & -1 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow[R_1 + R_3]{R_2 - 4R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 4 & 5 & -4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -4 & -5 & 4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right).$$

So $A$ is invertible and

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -4 & -5 & 4 \\ -1 & -1 & 1 \end{pmatrix}.$$

2. Since $A$ is invertible, $A^T$ is invertible by the Invertible Matrix Theorem and

$$(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} 0 & -4 & -1 \\ -1 & -5 & -1 \\ 1 & 4 & 1 \end{pmatrix}$$

5. (20 points) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ 2x_2 + x_3 \\ x_1 + 5x_2 + 4x_3 \end{bmatrix}.$$

(a) Determine the standard matrix $A$ for $T$.

(b) Show that $T$ is not onto. Describe the set of all $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$ for which $\mathbf{y}$ is in the range of $T$.

a) The standard matrix $A$ of $T$ is

$$A = [T(e_1)\ T(e_2)\ T(e_3)] = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix}$$ where

$e_1, e_2, e_3$ are columns of the 3x3 identity matrix.

b)

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & 3 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $A$ does not have a pivot position in every row, $T$ is not onto. Now $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is in the range of $T$ if and only if the equation $T(x) = y$ or $Ax = y$ has at least one solution.

$$(A|y) = \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 2 & 1 & y_2 \\ 1 & 5 & 4 & y_3 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 2 & 1 & y_2 \\ 0 & 6 & 3 & y_3 - y_1 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 2 & 1 & y_2 \\ 0 & 0 & 0 & y_3 - y_1 - 3y_2 \end{array}\right).$$

This system is consistent if and only if $y_3 - y_1 - 2y_2 = 0$.

Thus $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is in the range of $T$ if and only if

$y_3 - y_1 - 2y_2 = 0$.

6. (10 points) (Bonus) Let $A$ be an $m \times n$ matrix with $1 \leq m < n$. Is it possible to find an $n \times m$ matrix $B$ such that $BA = I_n$? Explain.

No, you cannot find an $n \times m$ matrix $B$ such that $B \cdot A = I_n$.

Suppose by contradiction that such an $n \times m$ matrix $B$ exists. Then $B \cdot A = I_n$.

Since $m < n$, the system $AX = \mathbb{0}$ has at least one free variable and so $AX = \mathbb{0}$ has a nontrivial solution, say $v \in \mathbb{R}^n$. We have $Av = \mathbb{0}$ and $v \neq \mathbb{0}$. Now $v = I_n v = (BA)v = B(Av) = B \cdot \mathbb{0} = \mathbb{0}$, which is a contradiction. Thus $B$ does not exist.