MATH 304-01
Fall 2019
Exam 2
October 30, 2019
Time Limit: 90 Minutes

Name (Print): ________________________

This exam contains 6 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page. Question 1 is a **True/False** question. Clearly CIRCLE your correct answer. You are required to show your work on Questions 2 to 7 on this exam.

Instruction:

- **All solutions must be written on the blue book.**
- **At the end of the exam, please turn in both this exam and the blue book.**
- **Turn off and put away your cell phone.**
- **Notes, the textbooks, and digital devices are not permitted.**
- **Discussion or collaboration is not allowed.**
- **Justify your answers, and write clearly.**
- **Mysterious or unsupported answers will not receive full credit.**

Do not write in the table to the right.

Question	Points	Score
1	10	
2	15	
3	10	
4	15	
5	20	
6	25	
7	10	
Total:	105	

1. (10 points) In each question circle either True or False. No justification is needed.

(a) **True** Let A be a square matrix. If the columns of A are linearly dependent, then $\det(A) = 0$.

(b) **True** If A is a 5×5 matrix and $\text{Col}A = \mathbb{R}^5$, then for each $\mathbf{b} \in \mathbb{R}^5$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(c) **False** Let A be a 5×8 matrix. If the null space of A has a basis consisting of 2 vectors. Then $\text{rank}(A) = 3$.

(d) **False** If A and B are 3×3 matrices, then $\det(A + B) = \det(A) + \det(B)$.

(e) **True** The column space of A is the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

(f) **False** Let A be an $m \times n$ matrix. Then $\dim \text{Col}A + \dim \text{Row}A = n$.

(g) **True** If two matrices A and B are row equivalent, then their row spaces are the same and if B is in row echelon form, then the nonzero rows of B form a basis for the row space of A.

(h) **False** If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

(i) **True** Let V be a vector space. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is a set of vectors in V and S spans V, then some subset of S is a basis for V.

(j) **True** Let V be a finite-dimensional vector space. If every set of p elements in V fails to span V, then $\dim V > p$.

2. (15 points) Which of the following sets is a subspace of the given vector space? If it is not a subspace, find a specific example (two vectors or a vector and a scalar) to show that it is not a subspace.

(a) The set $H = \left\{ \begin{bmatrix} 2a+b \\ a+b \\ a-3b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ in the vector space $\mathbb{R}^3$.

(b) The set $H = \{ \begin{bmatrix} a \\ b \end{bmatrix} : a^2 + 4b^2 \le 1, a, b \text{ in } \mathbb{R}\}$ in the vector space $\mathbb{R}^2$.

(c) The set H of all polynomials $p(t)$ in $\mathbb{P}_2$ satisfying $p(0) = 1$, where $\mathbb{P}_2$ is the vector space of all polynomials of degree at most 2.

Solutions.

(a) We have $H = \text{Span}\{v_1, v_2\}$, where $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. So H is a subspace of $\mathbb{R}^3$.

(b) Let $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $v \in H$ as $1^2 + 4 \cdot 0^2 = 1 \le 1$. Let $c = 2$. Then $cv = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. However $cv \notin H$ since $2^2 + 4 \cdot 0^2 = 4 > 1$. We can also take $u = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$. Then $u, v \in H$ but $u + v \notin H$. Hence H is not a subspace of $\mathbb{R}^2$.

(c) H is not a subspace of $\mathbb{P}_2$ since the zero polynomial is not in H.

3. (10 points) Compute the following determinants.

(a) $\begin{vmatrix} 1 & -2 & -2 \\ 2 & 1 & 2 \\ -1 & 3 & 3 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 1 & 0 & 2 & -6 \\ 4 & 10 & 30 & 20 & 19 \\ 0 & 2 & -1 & -2 & -9 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -6 & 0 \end{vmatrix}$

Solutions.

(a) $\begin{vmatrix} 1 & -2 & -2 \\ 2 & 1 & 2 \\ -1 & 3 & 3 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & 1 \\ -1 & 3 \end{matrix} = 1{\cdot}1{\cdot}3+(-2){\cdot}2{\cdot}(-1)+(-2){\cdot}2{\cdot}3-(-1){\cdot}1{\cdot}(-2)-3{\cdot}2{\cdot}1-3{\cdot}2{\cdot}(-2)$
$= 3 + 4 - 12 - 2 - 6 + 12 = -1.$

(b) $\begin{vmatrix} 0 & 1 & 0 & 2 & -6 \\ 4 & 10 & 30 & 20 & 19 \\ 0 & 2 & -1 & -2 & -9 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -6 & 0 \end{vmatrix} = -4 \begin{vmatrix} 1 & 0 & 2 & -6 \\ 2 & -1 & -2 & -9 \\ 1 & 0 & -2 & 0 \\ 2 & 0 & -6 & 0 \end{vmatrix} = (-4)(-1) \begin{vmatrix} 1 & 2 & -6 \\ 1 & -2 & 0 \\ 2 & -6 & 0 \end{vmatrix} = (-4)(-1) \begin{vmatrix} 1 & 2 & -6 \\ 1 & -2 & 0 \\ 2 & -6 & 0 \end{vmatrix} =$
$= 4(-6) \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} = (-24)(1(-6) - 2(-2)) = 48.$

4. (15 points) Let A, B and C be 5×5 matrices. Assume that $\det(A) = 2, \det(B) = -3$ and $\det(C) = -4$. Compute

(a) $\det(ABC^2)$.

(b) $\det(A^{-2}B^2C^T)$.

(c) $\det(2A)$.

(d) Let P be an invertible 5×5 matrix. What is $\det(P^TAP^{-1})$?

(e) Assume that Q is a 5×5 matrix such that $Q^TAQ = A$. Show that $\det Q = \pm 1$.

Solutions.

(a) $\det(ABC^2) = \det(a)\det(B)\det(C^2) = \det(A)\det(B)\det(C)^2 = 2(-3)(-4)^2 = -96.$

(b) $\det(A^{-2}B^2C^T) = \det(A^{-2})\det(B^2)\det(C^T) = \frac{1}{\det(A)^2}\det(B)^2\det(C) = \frac{1}{2^2} \cdot (-3)^2 \cdot (-4) = -9.$

(c) $\det(2A) = 2^5\det(A) = 31 \cdot 2 = 64.$

(d) $\det(P^TAP^{-1}) = \det(P^T)\det(A)\det(P^{-1}) = \det(P)\det(A)\det(P)^{-1} = \det(A).$

(e) Since $Q^TAQ = A$, we have $\det(A) = \det(Q^T)\det(A)\det(Q) = \det(Q)^2\det(A)$. Since $\det(A) = 2$, we see that $\det(Q)^2 = 1$. Thus $\det Q = \pm 1$.

5. (20 points) Assume that the matrix A is row equivalent to the matrix B, where

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 3 & 6 & -3 & 3 & 6 \\ 2 & 4 & -1 & 5 & 3 \\ 1 & 2 & 0 & 4 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find Rank$A$, dim Nul A and dim Row$A$.
(b) Find a basis for Col$A$.
(c) Find a basis for Row$A$.
(d) Find a basis for Nul$A$.

Solutions.

(a)
- Rank$A = 2$ since A has two pivot columns.
- $\dim \text{Row}A = 2$ since B is in row echelon form, B has two non-zero rows and B is row equivalent to A.
- $\dim \text{Nul}A = 5 - 2 = 3$ by applying the Rank Theorem, where A has five columns.

(b) $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for Col$A$ as these are the pivot columns of A.

(c) $\{(1, 2, -1, 1, 2), (0, 0, 1, 3, -1)\}$ is a basis for Row$A$ as these are the non-zero rows of an echelon form of A (which are the rows of B).

(d) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul$A$.

We need to solve the system with augmented matrix $(B|\mathbf{0})$.

$$(B|\mathbf{0}) = \begin{pmatrix} 1 & 2 & -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 2 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\begin{cases} x_1 + 2x_2 + 4x_4 + x_5 = 0 \\ \qquad\quad x_3 + 3x_4 - x_5 = 0 \end{cases}$$

or

$$\begin{cases} x_1 = -2x_2 - 4x_4 - x_5 \\ x_3 = -3x_4 \quad + \quad x_5 \end{cases}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 - x_5 \\ x_2 \\ -3x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

6. (25 points) Let $p_1(t) = t + 1$, $p_2(t) = t^2 + t$ and $p_3(t) = t^2 + t - 2$ be polynomials in $\mathbb{P}_2$, the vector space of all polynomials in the variable t of degree at most 2. Let $\mathcal{B} = \{1, t, t^2\}$ be a basis for $\mathbb{P}_2$.

   (a) Show that the set $\{p_1(t), p_2(t), p_3(t)\}$ is linearly independent (Hint. Use $\mathcal{B}$-coordinate vectors or definition of linear independence).

   (b) Without further calculation explain why $\mathcal{C} = \{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\mathbb{P}_2$?

   (c) Find the change-of-coordinates matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

   (d) Let $p(t) = t^2 + 4t + 5$. Find the coordinate vector $[p(t)]_{\mathcal{C}}$.

   (e) Suppose $q(t) \in \mathbb{P}_2$ and $[q(t)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Find the polynomial $q(t)$.

Solutions.

(a) We use the $\mathcal{B}$-coordinates. We have

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Consider the matrix $A = [[p_1(t)]_{\mathcal{B}}\ [p_2(t)]_{\mathcal{B}}\ [p_3(t)]_{\mathcal{B}}] = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Row reduce A to an echelon form:

$$\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since every columns of A is a pivot columns, the set $\{[p_1(t)]_{\mathcal{B}}\ [p_2(t)]_{\mathcal{B}}\ [p_3(t)]_{\mathcal{B}}\}$ is linearly independent. Since the coordinate mapping is an isomorphism, we deduce that the set $\{p_1(t), p_2(t), p_3(t)\}$ is linearly independent.

(b) As $\dim \mathbb{P}_2 = 3$ and $\mathcal{C}$ has exactly three vectors and is a linearly independent set by part (a), the Basis Theorem implies that $\mathcal{C}$ is a basis for $\mathbb{P}_2$.

(c) The change-of-cooridnates matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [[p_1(t)]_{\mathcal{B}}\ [p_2(t)]_{\mathcal{B}}\ [p_3(t)]_{\mathcal{B}}] = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(d) We have $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[p(t)]_{\mathcal{C}} = [p(t)]_{\mathcal{B}}$. Thus to find $[p(t)]_{\mathcal{C}}$, we need to solve the system with augmented matrix $(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\ [p(t)]_{\mathcal{B}})$, where $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$. We have

$$\left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\ [p(t)]_{\mathcal{B}}\right) = \begin{pmatrix} 1 & 0 & -2 & 5 \\ 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_3/(-2)} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_2-3R_3} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1+2R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Therefore $[p(t)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

(e) We have $[q(t)]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\ [q(t)]_{\mathcal{C}} = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

Thus $q(t) = -3 \cdot 1 + 2 \cdot t + 1 \cdot t^2 = t^2 + 2t - 3$.

7. (10 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for a vector space V. Show that for each $\mathbf{x} \in V$, there exists a unique set of scalars $c_1, c_2, \cdots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.

Solutions. This is the Unique Representation Theorem (Theorem 7 in Section 4.4 of the textbook.)

Since $\mathcal{B}$ spans V and $\mathbf{x} \in V$, there exists scalars $c_1, c_2, \cdots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$. Assume that $\mathbf{x}$ also has a representation $\mathbf{x} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \cdots + d_n\mathbf{b}_n$, for scalars $d_1, d_2, \cdots, d_n$. By subtracting, we have

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + (c_2 - d_2)\mathbf{b}_2 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Since $\mathcal{B}$ is linearly independent, all the weights in the previous equation must be 0, that is, $c_1 - d_1 = 0, c_2 - d_2 = 0, \cdots, c_n - d_n = 0$. Thus $c_i = d_i$ for all $1 \leq i \leq n$.