Math 402 **Practice 3** Spring 2021

(Due: Tuesday, March. 2)

Problem 1. Let R be a commutative ring with identity. Assume that R has a prime characteristic p and $a \in R$ is nilpotent, that is, $a^n = 0$ for some positive integer n. Show that there exists a positive integer k such that $(1+a)^k = 1$.

(Hint. You can use the Binomial Theorem and the fact that $p \mid \binom{p}{k}$ for $1 \le k < p$.)

Problem 2. Let R be a commutative ring with identity, and let I, J be ideals of R. Show that IJ and $I : J$ are ideals of R.

Problem 3. Let R be a commutative ring with identity 1.

(1) Show that a is an idempotent, i.e. $a^2 = a$ if and only if there exists $b \in R$ such that $ab = 0$ and $a + b = 1$.

(2) Let E be the set of all idempotents of R. Assume that the characteristic of R is 2. Is E a subring of R?

Problem 4. Let $R = \mathbb{Z}[\sqrt{-5}]$.

(1) Find all units of R.

(2) The integer 21 can be factored as $21 = 3 \times 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$ in R. Show that $3, 7, 1 \pm 2\sqrt{-5}$ are irreducible in R. Furthermore, 3 is not associate to $1 \pm 2\sqrt{-5}$ and 7 is not associate to $1 \pm 2\sqrt{-5}$. This shows that R is not a UFD. Can you find an irreducible element of R that is not a prime?