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Periodic solutions for a class of nonautonomous Hamiltonian systems

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1. Introduction

In this paper, we study the existence of periodic solutions for a Hamiltonian system

$$-J\dot{z}-B(t)z=\nabla H(t,z),\quad z\in \mathbf{R}^{2N},\ \ t\in \mathbf{R}, \tag{1}$$

where $B(t)$ is a given T-periodic and symmetric $2N\times 2N$-matrix function of C^1 class in t, $H\in C^1(\mathbf{R}\times\mathbf{R}^{2N},\mathbf{R})$ is T-periodic in t, $\nabla H:=\nabla_z H\in C(\mathbf{R}\times\mathbf{R}^{2N},\mathbf{R}^{2N})$ and

$$J=\begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

is the standard symplectic matrix. The main results of this paper are the following:

Theorem 1.1. *For $T>0$, suppose that H satisfies the following conditions*:
(H1) $H\in C^1(S_T\times\mathbf{R}^{2N},\mathbf{R})$, $S_T=\mathbf{R}/(T\mathbf{Z})$.

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(H2) *There are constants* $\mu > 2$ *and* $r > 0$ *such that*

$$0 < \mu H(t,z) \le z \nabla H(t,z), \quad \forall |z| \ge r.$$

(H3) $H(t,z) = o(|z|^2)$, uniformly in t *as* $z \to 0$.
(H4) *There exists a constant* $\bar{a}$ *such that*

$$\varliminf_{|z| \to \infty} \frac{H_t(t,z)}{H(t,z)} \ge \bar{a} > -\frac{2}{T}, \quad uniformly\ in\ t.$$

Then Eq. (1) *has a nontrivial* T*-periodic solution in each of the following two cases*:
(i) *The boundary value problem*

$$-J\dot{z} = B(t)z, \qquad z(0) = z(T), \tag{2}$$

has only the trivial solution.
(ii) *There is a constant* $\rho > 0$ *such that* $H(t,z) > 0$ (*or* $H(t,z) < 0$) *for all* z *satisfying* $0 < |z| < \rho$.

Theorem 1.2. *Suppose that* H *satisfies* (H1)–(H3) *and the following* (H5).
There are constants $c, d > 0$, *such that* $|\nabla H(t,z)| \le c(\nabla H(t,z), z) + d$, $\forall z \in \mathbf{R}^{2N}$.
Then Eq. (1) *has a nontrivial* T*-periodic solution in each of case* (i) *and* (ii) *in Theorem* 1.1.

For the autonomous case, i.e. H is independent of t, in his pioneering work [9] Rabinowitz first proved the existence of at least one periodic solution for Eq. (1). Many works have been done on this problem. For example, in [1, 2, 4–6, 9–12] some existence results of Eq. (1) are proved. We refer to [1, 12] for further references. These results have further restrictions on $\nabla_z H(t,z)$ in addition to (H1)–(H3). In this paper, we prove the existence of periodic solutions for Eq. (1) under a different and new condition (H4), which measures the difference of Eq. (1) from the autonomous systems. Define $H(t,z) = f(t)\mathrm{e}^{|z|^\alpha}$ for large $|z|$, with $\alpha > 0$ and $f \in C^1(S_T, \mathbf{R})$ satisfying $f'(t)/f(t) > -2/T$ for all t. Such kinds of functions as above satisfy the conditions of our Theorem 1.1, but are not contained in the above mentioned papers. Our Theorem 1.2 generalizes Theorem 2.1 of [2], where [2] requires $|\nabla H(t,z)|^p \le c\nabla H(t,z)z + d$, for all $z \in \mathbf{R}^{2N}$, where $p > 1$. One may also compare our theorems with Theorem 1.4 of [10].

2. Proofs of main results

In this section, we consider the Hamiltonian system

$$-J\dot{z} - B(t)z = \nabla H(t,z), \quad z \in \mathbf{R}^{2N}, \ t \in \mathbf{R}$$

with $B(t)$ being a given continuous T-periodic and symmetric matrix function and H being T-periodic in t. Let $X := W^{1/2,2}(S_T, \mathbf{R}^{2N})$ be the Sobolev space of T-periodic

$\mathbf{R}^{2N}$-valued functions with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. Define two self-adjont operators $A, B \in \mathscr{L}(X)$ by extending the bilinear forms

$$(Ax, y) = \int_0^T (-J\dot{x}, y)\,\mathrm{d}t, \qquad (Bx, y) = \int_0^T (B(t)x, y)\,\mathrm{d}t, \quad \forall x, y \in X.$$

By [7] and standard spectral theory, B is compact on X. Denote the eigenvalues of $A-B$ on X by

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 (= \lambda_0) < \lambda_1 \leq \lambda_2 \leq \cdots,$$

where when $\dim\ker(A-B) = 0$, $\lambda_0 \notin \sigma(A-B)$. Let $\{e_{\pm j}\}$ be the eigenvectors of $A-B$ corresponding to $\{\lambda_{\pm j}\}$, respectively. Define $X_+ = \mathrm{span}\{e_1, e_2, \ldots\}$, $X_- = \mathrm{span}\{e_{-1}, e_{-2}, \ldots\}$, $X_0 = \ker(A-B)$. Hence there exists a decomposition $X = X_+ \oplus X_0 \oplus X_-$ with $\dim X_0 < \infty$, $\dim X_+ = \dim X_- = \infty$ and an equivalent inner product in X, denoted by $\langle\cdot,\cdot\rangle$, for $u = u^+ + u^0 + u^-$ and $v = v^+ + v^0 + v^- \in X = X_+ \oplus X_0 \oplus X_-$, define

$$\langle u, v\rangle = ((A-B)u^+, v^+)_X - ((A-B)u^-, v^-)_X + (u^0, v^0)_X.$$

Hence, we have

$$\int_0^T (-J\dot{u} - B(t)u)u\,\mathrm{d}t = ((A-B)u, u)_X = \|u^+\|^2 - \|u^-\|^2.$$

Note that $\dim X_0 > 0$ if and only if the boundary value problem

$$-J\dot{z} = B(t)z, \qquad z(0) = z(T)$$

has at least a nontrivial solution.

Set $\alpha_0 = \min_{|z|=r_0, t\in S_T} H(t,z)$, $\beta_0 = \max_{|z|\leq r_0, t\in S_T} |H(t,z)|$. Conditions (H1) and (H2) imply that for some $\beta_3 \geq 0$

$$\alpha_0|z|^\mu \leq H(t,z), \quad \forall |z| \geq r_0,$$
$$\alpha_0|z|^\mu \leq H(t,z) + \beta_0 \leq \frac{1}{\mu}(\nabla H(t,z)z + \beta_3), \quad \forall z \in \mathbf{R}^{2N}.$$

Modifying [5] (cf. appendix of [5]), choose $\sigma \in (0,1)$, such that $\mu\sigma > 2$, we truncate H as in the following proposition:

Proposition 2.1. *Assume conditions* (H1) *and* (H2), *then there exist two sequences* $\{K_n\}$ *and* $\{K_n'\}$ *in* $\mathbf{R}$ *and a sequence of functions* $\{H_n\}$ *such that*

(i) $0 < K_0 < K_n < K_{n+1}$, $\forall n \in \mathbf{N}$, *and* $K_n \to \infty$, *as* $n \to \infty$, *where* $K_0 = \max\{1, r, \beta_0/\alpha_0(1-\sigma)\}$; *and* $K_n < K_n'$, $\forall n \in \mathbf{R}$.

(ii) $H_n t\, C(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *and for any given* $t \in S_T$, $H_n(t,\cdot) \in C^1(\mathbf{R}^{2N}, \mathbf{R})$, *for every* $n \in \mathbf{N}$.

(iii) $H_n(t,z) = H(t,z)$, $\forall |z| \leq K_n$, *for every* $n \in \mathbf{N}$; *and* $H_n(t,z) = (\tau_n + 1)|z|^{\mu\lambda}$, $\forall |z| \geq K_n'$, *for every* $n \in \mathbf{N}$.

(iv) $H_n(t,z) \le H_{n+1}(t,z) \le H(t,z)$, $\forall (t,z) \in S_T \times \mathbf{R}^{2N}$.
(v) $0 < \mu\sigma H_n(t,z) \le \nabla H_n(t,z)z$, $\forall |z| \ge r_0$, *for every* $n \in \mathbf{N}$.

Note that in [5] the truncating functions are constructed for autonomous Hamiltonian functions. But the proof also works for time-dependent $H(t,z)$.

Now integrating (v) yields

$$H_n(t,z) \ge a|z|^{\mu\sigma} - b, \quad \forall z \in \mathbf{R}^{2N},$$

for some n-independent constants a and b. Let $\Psi_n(u) = \int_0^T H_n(t,u)\,\mathrm{d}t$. Define a functional $I_n : X \to \mathbf{R}$ by

$$\begin{aligned} I_n(u) &= \frac{1}{2}\int_0^T (-Ju' - B(t)u)u\,\mathrm{d}t - \int_0^T H_n(t,u)\,\mathrm{d}t \\ &= \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi_n(u). \end{aligned}$$

It is well known that $I_n \in C^1(X,\mathbf{R})$, and

$$\begin{aligned} \langle I_n'(u), v\rangle &= \int_0^T (-Ju' - B(t)u)v\,\mathrm{d}t - \int_0^T \nabla H_n(t,u)v\,\mathrm{d}t \\ &= \left\langle u^+ - u^-, v\right\rangle - \langle \Psi_n'(u), v\rangle \end{aligned}$$

and Ψ_n' is compact as in [12]. So finding T-periodic solutions of Eq. (1) with H replaced by H_n is equivalent to finding critical points of I_n in X.

We will use Theorem 1.3 of [2] to prove that I_n has a critical point u_n which is different from 0. Similarly to the proof of [2], it is easy to show that the functional I_n satisfies (I2), (I3) and (I4) in Theorem 1.3 of [2] without using (H4) or (H5). Different from [2], we also prove (I1) without using (H4) or (H5) as the following.

Lemma 2.1. I_n *satisfies* $(PS)^*$.

Proof. Suppose $\{u_k\}$ is a sequence in X such that

$$u_k \in X_k, \quad I_n(u_k) \le C < \infty \quad \text{and} \quad P_k I_n'(u_k) \to 0 \quad \text{as } k \to \infty.$$

Then for large n and $v = u_k$,

$$\begin{aligned} C + \|u_k\|_X &\ge I_n(u_k) - \tfrac{1}{2}\langle P_k I_n'(u_k), u_k\rangle \\ &= \int_0^T (\tfrac{1}{2}\nabla H_n(t,u_k)u_k - H_n(t,u_k))\,\mathrm{d}t \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu\sigma}\right)\int_0^T \nabla H_n(t,u_k)u_k\,\mathrm{d}t - c_1 \end{aligned}$$

$$\geq \left(\frac{\mu\sigma}{2} - 1\right)\int_0^T H_n(t,u_k)\,\mathrm{d}t - c_2$$

$$\geq c_3\|u_k\|_{L^{\mu\sigma}}^{\mu\sigma} - c_4 \tag{3}$$

via (H2) and the growth of H_n at infinity. Writing

$$u_k = u_k^+ + u_k^- + u_k^0 \in X_+ \oplus X_- \oplus X_0.$$

Because X_0 is a finite-dimensional space, it follows from Eq. (3) that

$$\|u_k^0\|_X \leq c_5(1 + \|u_k\|_X^{1/\mu\sigma}).$$

Taking $v = u_k^+$ in the inequality $|\langle P_k I_n'(u_k), v\rangle| \leq \|v\|$ (which holds for large n), we have

$$\|u_k^+\|_X^2 - \left|\int_0^T \nabla H_n(t,u_k)u_k^+\,\mathrm{d}t\right| \leq \|u_k\|_X.$$

Using the Hölder inequality and $\|u\|_{L^{\mu\sigma}} \leq C_{\mu\sigma}\|u\|_X$, by Eq. (3) we have

$$\|u_k^+\|_X^2 \leq \left\{\int_0^T |\nabla H_n(t,u_k)|^{\mu\sigma/(\mu\sigma-1)}\,\mathrm{d}t\right\}^{(\mu\sigma-1)/\mu\sigma} \|u_k^+\|_{L^{\mu\sigma}} + \|u_k^+\|_X$$

$$= \left\{\int_{|u_k|\leq K_n'+1} + \int_{|u_k|>K_n'+1} |\nabla H_n(t,u_k)|^{\mu\sigma/(\mu\sigma-1)}\,\mathrm{d}t\right\}^{(\mu\sigma-1)/\mu\sigma}$$

$$\times \|u_k^+\|_{L^{\mu\sigma}} + \|u_k^+\|_X$$

$$\leq \{C_0(n) + (\mu\sigma R)^{\mu\sigma/(\mu\sigma-1)}\|u_k\|_{L^{\mu\sigma}}^{\mu\sigma}\}^{(\mu\sigma-1)/\mu\sigma}\|u_k^+\|_{L^{\mu\sigma}} + \|u_k^+\|_X$$

$$\leq C_1(n)(1 + \|u_k\|_{L^{\mu\sigma}}^{\mu\sigma-1})\|u_k^+\|_X,$$

i.e.,

$$\|u_k^+\|_X \leq C_1(n)(1 + \|u_k\|_{L^{\mu\sigma}}^{\mu\sigma-1}) \leq C_2(n)(1 + \|u_k\|_X^{(\mu\sigma-1)/\mu\sigma}),$$

where $C_i(n)$'s are constants depending on n. Similarly, for $v = u_k^-$ we have

$$\|u_k^-\|_X \leq C_3(n)(1 + \|u_k\|_X^{(\mu\sigma-1)/\mu\sigma}).$$

Hence,

$$\|u_k\|_X \leq C_4(n)(1 + \|u_k\|_X^{(\mu\sigma-1)/\mu\sigma})$$

i.e., $\{u_k\}$ is bounded on X. Since

$$u_k^+ - u_k^- - P_k\Psi_n'(u_k) = P_k I_n'(u_k) \to 0 \quad \text{as } k \to \infty,$$

Ψ_n' is a compact operator, and $\{u_k^0\} \subset X^0$ is bounded, $\{u_k\}$ has a convergence subsequence, i.e., $(PS)^*$ holds. □

Proof of Theorem 1.1. By our above discussions, I_n satisfies the hypotheses of Theorem 1.3 of [2]. So I_n possesses a nontrivial critical point u_n. We shall prove $\|u_n\|_C \le K_n$ for large n.

We first prove that there is a constant $M>0$ such that $I_n(u_n) \le M$, for every $n \in \mathbf{N}$. If every one of $\{u_n\}$ is gained in the first case in the proof of Theorem 1.3 of [2] (p. 228), $I_n(u_n)<0$ holds for every $n \in \mathbf{N}$. Otherwise, there exists an n_0 such that u_{n_0} is gained in the second case. Note that $I_n \le I_{n_0}$ for $n > n_0$ (since $H_n \ge H_{n_0}$ for $n > n_0$), we replace I_{n_0} by I_n only in the proof of the Theorem 1.3 of [2] (pp. 228–230), and use the same $\Phi, \Gamma, \mathscr{B}$, $\mathscr{H}_m, Q_m, G$ as gained for I_{n_0} and B_1^m for I_n. Then we can gain a critical point u_n of I_n such that $\alpha_n \le I_n(u_n) \le I_{n_0}(u_{n_0})$, i.e., $0 < \alpha_n \le c_n \le c_{n_0}$. Thus, we have constant $M>0$ such that there exists a critical point u_n of I_n such that $I_n(u_n) \le M$.

Now we show that $\|u_n\|_C \le K_n$ for large n. Since $I_n'(u_n)=0$, similarly to Eq. (3) we have

$$\int_0^T \nabla H_n(t,u_n)u_n \, \mathrm{d}t \le M_1, \qquad \int_0^T H_n(t,u_n)\, \mathrm{d}t \le M_2 \tag{4}$$

for some constants M_1 and M_2 independent of n.

Denote by $\tilde{H}_n(t,z) = \frac{1}{2}\langle B(t)z,z\rangle + H_n(t,z)$. Then (H1), (H2) and (H4) also hold for $\tilde{H}_n$ with some $\tilde{\mu}$, $\tilde{r}$ independent of n and the same $\bar{a}$. Thus we can omit $\langle B(t)z,z\rangle$ in the following proof.

Denote

$$A_n = \{t \in S_T \mid |u_n(t)| < K_n\}.$$

By Eq. (4) we have

$$M_2 \ge \int_0^T H_n(t,u_n)\,\mathrm{d}t \ge \alpha_0 \|u_n\|_{L^{\mu\sigma}}^{\mu\sigma} + b$$

for some n-independent constant b. Thus we know for large n, $A_n \ne \emptyset$ and $measure(A_n) > T/2$. Since $u_n \in C^1$, A_n is open. Let $A_n = \bigcup_{j=1}^{\infty} (a_{n,j}, b_{n,j})$. It suffices to prove $A_n = S_T$.

We prove this indirectly by assuming that this claim fails in a subsequence of $\{A_n\}$. Without loss generality, we still denote this subsequence by $\{A_n\}$. By Eq. (4), $H_n(t,u_n)|_{A_n} = H(t,u_n)$ and $K_n > r$, we have

$$M_2 \ge \int_0^T H_n(t,u_n)\,\mathrm{d}t \ge \int_{A_n} H_n(t,u_n)\,\mathrm{d}t = \sum_{j=1}^{\infty} \int_{a_{n,j}}^{b_{n,j}} H(t,u_n)\,\mathrm{d}t.$$

For every $(a_{n,j}, b_{n,j})$, let

$$B_j^n = \{t \in (a_{n,j}, b_{n,j}) \mid H(t,u_n(t)) < H(a_{n,j}, u_n(a_{n,j}))\} = \bigcup_{l=1}^{\infty} (c_l^j, d_l^j).$$

We have $H(a_{n,j},u_n(a_{n,j}))=H(c_l^j,u_n(c_l^j))=H(d_l^j,u_n(d_l^j))$, $\forall l\in\mathbf{N}$. Thus,

$$\begin{aligned}
&\int_{a_{n,j}}^{b_{n,j}} H(t,u_n)\,\mathrm{d}t\\
&\quad\geq (b_{n,j}-a_{n,j})H(a_{n,j},u_n(a_{n,j}))+\int_{B_j^n}[H(t,u_n(t))-H(a_{n,j},u_n(a_{n,j}))]\,\mathrm{d}t\\
&\quad=(b_{n,j}-a_{n,j})H(a_{n,j},u_n(a_{n,j}))+\sum_{l=1}^{\infty}\int_{c_l^j}^{d_l^j}\int_{c_l^j}^{t}H_s(s,u_n(s))\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

the last equality holds since $\dot{u}_n=J\nabla H(t,u_n)$. By (H4) there exists $N>r$ independent of n such that

$$\frac{H_t(t,z)}{H(t,z)}>-\frac{1}{T}+\frac{\bar{a}}{2},\quad \forall |z|>N.$$

When $|u_n(s)|\geq N$ and $H_s(s,u_n(s))<0$ for $s\in B_j^n$, we have

$$\frac{H_s(s,u_n(s))}{H(a_{n,j},u_n(a_{n,j}))}\geq\frac{H_s(s,u_n(s))}{H(s,u_n(s))}\geq-\frac{1}{T}+\frac{\bar{a}}{2}.$$

Let $\beta=\min_{s\in S_T,|z|\leq N}\{H_s(s,z),0\}$, then β is finite and independent of n. Hence we have

$$\begin{aligned}
\int_{a_{n,j}}^{b_{n,j}} H(t,u_n)\,\mathrm{d}t\geq H(a_{n,j},u_n(a_{n,j}))\left\{(b_{n,j}-a_{n,j})+\iint_{Q_1}\frac{H_s(s,u_n(s))}{H(a_{n,j},u_n(a_{n,j}))}\,\mathrm{d}s\,\mathrm{d}t\right\}\\
+\left(\iint_{Q_2}+\iint_{Q_3}\right)H_s(s,u_n(s))\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
Q_1&=\{s\in B_j^n\mid |u_n(s)|>N, H_s(s,u_n(s))<0\},\\
Q_2&=\{s\in B_j^n\mid |u_n(s)|>N, H_s(s,u_n(s))\geq 0\},\\
Q_3&=\{s\in B_j^n\mid |u_n(s)|\leq N\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_{a_{n,j}}^{b_{n,j}} H(t,u_n)\,\mathrm{d}t\\
&\quad\geq H(a_{n,j},u_n(a_{n,j}))\left[(b_{n,j}-a_{n,j})-\sum_{l=1}^{\infty}\int_{c_l^j}^{d_l^j}\int_{c_l^j}^{t}\left(-\frac{1}{T}+\frac{\bar{a}}{2}\right)\right]+\sum_{l=1}^{\infty}\int_{c_l^j}^{d_l^j}\int_{c_l^j}^{t}\beta\,\mathrm{d}t\\
&\quad\geq\left[(b_{n,j}-a_{n,j})-\frac{1}{4}(b_{n,j}-a_{n,j})^2\left(\frac{2}{T}-\bar{a}\right)\right]H(a_{n,j},u_n(a_{n,j}))+\frac{(b_{n,j}-a_{n,j})^2\beta}{2}
\end{aligned}$$

$$\geq \left[(b_{n,j}-a_{n,j})-\frac{1}{4}(b_{n,j}-a_{n,j})^2\left(\frac{2}{T}-\bar{a}\right)\right](\alpha_0|u_n(a_{n,j})|^{\mu}-b)$$

$$+\frac{(b_{n,j}-a_{n,j})^2\beta}{2}$$

$$\geq (b_{n,j}-a_{n,j})\left[\frac{2+T\bar{a}}{4}(\alpha_0K_n^{\mu}-b)+\frac{T\beta}{2}\right].$$

Thus, we have

$$M_2 \geq \sum_{j=1}^{\infty}\int_{a_{n,j}}^{b_{n,j}} H(t,u_n)\,\mathrm{d}t$$

$$\geq \sum_{j=1}^{\infty}(b_{n,j}-a_{n,j})\left[\frac{2+T\bar{a}}{4}(\alpha_0K_n^{\mu}-b)+\frac{T\beta}{2}\right]$$

$$\geq \frac{T}{2}\left[\frac{2+T\bar{a}}{4}(\alpha_0K_n^{\mu}-b)+\frac{T\beta}{2}\right].$$

Since $\mu>2$, $2/T+\bar{a}>0$ and $K_n\to\infty$ as $n\to\infty$, we have a contradiction. Hence $\|u_n\|_C\leq K_n$ for large n. Since $H_n(t,u_n)=H(t,u_n)$ for $\|u_n\|_C\leq K_n$, we have that u_n is a nontrivial solution of Eq. (1) for large n. Hence Theorem 1.1 is proved. □

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we have Eq. (4) and $A_n=\bigcup_{j=1}^{\infty}(a_{n,j},b_{n,j})$ for large n. By passing a subsequence, assume $A_n\neq S_T$, for $n\in\mathbf{N}$. Otherwise, we have the conclusion. From Eq. (4) and (H5)

$$M_1\geq\int_0^T\nabla H_n(t,u_n)u_n\,\mathrm{d}t\geq\int_{A_n}\nabla H(t,u_n)u_n\,\mathrm{d}t\geq\frac{1}{c}\int_{A_n}(|\nabla H(t,u_n)|-d)\,\mathrm{d}t.$$

Thus, we have

$$\sum_{j=1}^{\infty}\int_{a_{n,j}}^{b_{n,j}}|\dot{u}_n(t)|\,\mathrm{d}t=\int_{A_n}|\nabla H(t,u_n)|\,\mathrm{d}t\leq cM_1+\mathrm{d}T.$$

For $t\in(a_{n,j},b_{n,j})$, we have

$$|u_n(t)|-|u_n(a_{n,j})|\geq-\int_{a_{n,j}}^{t}|\dot{u}_n(s)|\,\mathrm{d}s\geq-(cM_1+\mathrm{d}T),$$

i.e., $|u_n(t)|\geq K_n-(cM_1+\mathrm{d}T)$. By Eq. (4) we have

$$M_2\geq\int_0^T H_n(t,u_n)\,\mathrm{d}t$$

$$\geq\int_0^T(\alpha_0|u_n|^{\mu\sigma}-b)\,\mathrm{d}t$$

$$\geq \int_0^T [\alpha_0(K_n - cM_1 - \mathrm{d}T)^{\mu\sigma} - b]\,\mathrm{d}t$$
$$= T[\alpha_0(K_n - cM_1 - \mathrm{d}T)^{\mu\sigma} - b].$$

Since $\alpha_0, b, c, d, \mu, M_1$ are independent of n and $K_n \to \infty$ as $n \to \infty$, we have a contradiction. Hence $A_n = S_T$ for large n, i.e., $\|u_n\|_C \leq K_n$. Since $H_n(t, u_n) = H(t, u_n)$ for $\|u_n\|_C \leq K_n$, we have that u_n is a nontrivial solution of Eq. (1) for large n. Hence Theorem 1.2 is proved. □

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