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Pointwise Carleman Estimates, Global Uniqueness, Observability, and Stabilization for Schrödinger Equations on Riemannian Manifolds at the $H^1(\Omega)$-Level

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Abstract. We consider a general, non-conservative Schrödinger equation defined on an open, bounded, connected set Ω of a complete, n-dimensional, Riemannian manifold M with metric g. The boundary $\Gamma = \partial\Omega \equiv \overline{\Gamma_0 \cup \Gamma_1}$ is subject to Dirichlet and, as a main focus, to Neumann boundary conditions over the entire boundary Γ. Here, Γ_0 and Γ_1 are the unobserved (or uncontrolled) and observed (or controlled) parts of the boundary, respectively, both being relatively open in Γ. The Schrödinger equation includes energy level ($H^1(\Omega)$-level) terms, which accordingly may be viewed as unbounded perturbations. The present paper generalizes all the results of [**L-T-Z.2**] from the Euclidean setting to the Riemannian setting, along the same line of arguments, carried out, however, in the technically more demanding Riemannian context (via the Levi-Civita connection). Thus, the first goal of this paper is to provide *pointwise* Carleman-type inequalities at the H^1-level, which then yield integral-type, Carleman-type inequalities, which do *not* contain lower-order terms (*unlike* those in [**T-Y.1**], still in the Riemannian setting). This is a distinguishing feature of the present paper over most of the literature. From Carleman-type inequalities with no lower-order terms, one then obtains the sought-after benefits. These consist of deducing, in one shot, as part of the same flow of arguments, two important implications: (i) *global uniqueness* results for H^1-solutions satisfying over-determined boundary conditions, and—above all—(ii) continuous observability (or stabilization) inequalities with an *explicit* constant (hence exact controllability by duality). The more demanding purely Neumann boundary conditions requires the same geometrical conditions on the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ that arise in the corresponding problem for second-order hyperbolic equations in the Riemannian setting [**T-Y.1**, Appendix B], which in turn generalize sufficient conditions in the Euclidean setting [**L-T-Z.1**, Appendix A]. The most general result, with weakest geometrical conditions is, in fact, deferred to Section 9. Instead, Sections 1 through 8 provide the main body of our treatment with one vector field under a preliminary working geometrical condition, which is then removed in Section 9, by use of two suitable vector fields.

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1. Introduction. Problem Statement. Assumptions

Preliminary notation. Throughout this paper, M is a complete n-dimensional, Riemannian manifold of class C^3 with C^3-metric $g(\,\cdot\,,\,\cdot\,) = \langle\,\cdot\,,\,\cdot\,\rangle$ and squared norm $|X|^2 = g(X,X)$. We may, on occasion, append a subscript g : $\langle\,\cdot\,,\,\cdot\,\rangle_g$; $|\cdot|_g$. We shall denote it by (M,g). Let Ω be an *open, bounded, connected* set of M with smooth boundary (say, of class C^2) $\partial\Omega \equiv \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$. Here, Γ_0 is the uncontrolled or unobserved part of Γ and Γ_1 is the controlled or observed part of Γ, both relatively open in Γ. We let ν denote the outward unit normal field along the boundary Γ. Further, we denote by ∇ the gradient, by D the Levi-Civita connection, by D^2 the Hessian, by $\Delta = \operatorname{div}(\nabla)$ the Laplace (Laplace-Beltrami) operator [**Do.1**, p. 55, p. 83, p. 141], [**Le.1**, p. 28, pp. 43–44, p. 54, p. 68].

Model. In this paper, we consider the following Schrödinger equation in the (complex-valued) unknown $w(t,x)$ defined on Q,

$$\mathcal{P}w \equiv iw_t + \Delta w = F(w) + f \qquad \text{in } Q \equiv (0,T] \times \Omega. \tag{1.1}$$

In (1.1), we have set

$$F(w) = \langle P(t,x), Dw\rangle + q_0(t,x)w, \tag{1.2a}$$

where q_0 is a complex-valued function on $[0,T] \times \Omega$ and $P(t)$ is a complex-valued vector field on M for $t > 0$. A standing *assumption* on the energy level term F is that

$$|F(w)|^2 \le C_T\{|Dw|^2 + |w|^2\}, \qquad \forall\ (t,x) \in Q \text{ a.e.}, \tag{1.2b}$$

where $Dw = \nabla w$ for the scalar function w. Thus, $Dw \in \mathcal{X}(M)$ = the set of all C^2-complex-valued vector fields on M. Furthermore, we assume throughout that the forcing term f in (1.1) satisfies (at least)

$$f \in L_2(0,T;L_2(\Omega)) \equiv L_2(Q); \qquad \int_Q |f|^2 dQ < \infty, \tag{1.3}$$

where $dQ = d\Omega\, dt$ is the volume element [**Do.1**, p. 45], [**Le.1**, p. 29], of the manifold M in its Riemannian metric g.

Remark 1.1. Property (1.2b) is fulfilled if $q_0 \in L_\infty(Q)$, $P \in L_\infty(0,T;\mathcal{X}(M))$ [**He.1**]. The aforementioned regularity assumptions suffice for the first main result, Theorem 2.1.1. In effect, for this theorem, we could relax the standing assumption of the lower-order coefficient q_0 and just require $q_0 \in L_p(Q)$, $p = n+1$, by invoking a Sobolev embedding theorem. Beyond that, in particular, to obtain Theorem 2.1.3, stronger regularity assumptions are needed. □

Remark 1.2. Modulo a first-order additive operator $(Df)(w)$, we have that $\Delta_g w$ models the principal part of a second-order elliptic operator with variable coefficients $a_{ij}(x)$ in space, defined on an n-dimensional Euclidean bounded domain, see Section 10.1, Eqn. (10.1.7). In this case, the Riemannian manifold is $(\mathbb{R}^n, g)$, where the Riemannian metric g is derived from an inversion on the symmetric matrix $\{a_{ij}(x)\}$. See [**C-H.1**], [**Du.1**]. □

Main assumptions. In addition to the standing assumptions (1.2b), (1.3) on the first-order operator F and the forcing term f, the following hypothesis is postulated throughout Section 8 of this paper.

(A.1) Given the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, there exists a non-negative, real-valued function $d : \overline{\Omega} \Rightarrow \mathbb{R}^+$ of class C^3 that is strictly convex in the metric g. This means that the Hessian $D^2 d$ of d (a two-order tensor) satisfies $D^2 d(X, X) > 0$, $\forall\ x \in \overline{\Omega}$, $\forall\ X \in T_x M$ = the tangent space at x. By compactness of Ω, we can always achieve that: There exists a constant $\rho > 0$ such that

$$D^2 d(X, \overline{X}) \equiv \langle D_X(Dd), \overline{X}\rangle_g \geq \rho |X|_g^2, \quad \forall\ x \in \Omega,\ \forall\ X \in T_x M. \tag{1.4}$$

A working assumption throughout Section 8, to be later relaxed in Section 9, is that $d(x)$ has no critical point on $\overline{\Omega}$.

(A.2)

$$\inf_{x \in \Omega} |Dd| = p > 0. \tag{1.5}$$

Remark 1.3. In effect, assumption (A.2) = (1.5) is needed to hold true only for $x \in \Gamma_0$ (uncontrolled or unobserved part of the boundary Γ); that is, it can be replaced by: $\inf |Dd| = p > 0$, where the inf is taken just over Γ_0. Indeed, a critical point of $d(x)$ at a point (necessarily interior) of Ω, or at a point of Γ_1 (controlled or observed part of Γ) can always be eliminated, by smoothly redefining $d(x)$, first locally around the critical point, and then away from Γ_0, as to make the new critical point fall off of Ω, while preserving the positivity condition (1.4). $\square$

Remark 1.4. Assumption (A.2) will be dispensed with in Section 9, *which will provide the most general results of the present paper*; see Theorem 9.1.2. This can be achieved (as in the case of second-order hyperbolic equations on an Euclidean domain [**L-T-Z.1**, Section 10] or on a Riemannian manifold [**T-Y.1**, Section 10]; or as in the case of Schrödinger equations on a Euclidean domain [**L-T-Z.1**, Section 9]) by splitting Ω as $\Omega = \Omega_1 \cup \Omega_2$, for two suitably overlapping sets Ω_1 and Ω_2, and working with two strictly convex functions (in the Riemann metric) d_1 and d_2, where each d_i satisfies assumption (1.5), but only on Ω_i, $i = 1, 2$. Since the full statement of results without (A.2) requires a rather lengthy preparatory background, we have opted (as in our past aforementioned efforts) for the strategy of retaining assumption (A.2) throughout Section 8. $\square$

Pseudo-convex function $\varphi(t, x)$. Having chosen, on the strength of assumption (A.1), a strictly convex potential function $d(x) \geq 0$ satisfying condition (1.4), we next introduce the pseudo-convex function $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ of class C^3 by setting

$$\varphi(x, t) = d(x) - c\left(t - \frac{T}{2}\right)^2; \quad 0 \leq t \leq T,\ x \in \Omega, \tag{1.6a}$$

where $T > 0$ is arbitrary, and where $c = c_T$ is chosen large enough as to have

$$cT^2 > 4 \max_{\overline{\Omega}} d(x), \qquad \text{so that } cT^2 > 4 \max_{\overline{\Omega}} d(x) + 4\delta, \tag{1.6b}$$

for a suitably small $\delta > 0$, henceforth kept fixed. Unless otherwise explicitly noted, $\varphi(x, t)$ is selected as described above and kept fixed henceforth. Such function $\varphi(x, t)$ has the following properties:

(a) for constant $\delta > 0$, fixed in (1.6b), we have

$$\varphi(x, 0) = \varphi(x, T) = d(x) - c\frac{T^2}{4} \leq \max_{\overline{\Omega}} d(x) - c\frac{T^2}{4} \leq -\delta \text{ uniformly in } x \in \overline{\Omega}; \tag{1.7}$$

(b) there exist t_0 and t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that

$$\min_{x\in\overline{\Omega},t\in[t_0,t_1]} \varphi(x,t) \geq -\frac{\delta}{2}, \tag{1.8}$$

since $\varphi(x,\frac{T}{2}) = d(x) \geq 0$ for all $x \in \Omega$ (actually, we shall only need the weaker property $\min_{\overline{\Omega}\times[t_0,t_1]} \varphi(x,t) \geq \sigma > -\delta$ in place of (1.8)).

Throughout this paper, we set

$$E(t) \equiv \int_\Omega |Dw(t)|^2 d\Omega; \quad \mathbb{E}(t) = \int_\Omega [|Dw(t)|^2 + |w(t)|^2] d\Omega = \|w(t)\|^2_{H^1(\Omega)}. \tag{1.9}$$

Boundary conditions. In seeking continuous observability inequalities, in Sections 2.3 and 2.4, we shall supplement Eqn. (1.1) with either purely Dirichlet B.C. or purely Neumann B.C. In each case, a geometric condition related to the normal derivative $\langle Dd, \nu\rangle$ on Γ_0 is needed, with d introduced in assumption (A.1). They are:

Purely Dirichlet B.C. With $\Sigma = (0,T] \times \Gamma$,

$$w|_\Sigma \equiv 0, \text{ in which case we assume } \langle Dd, \nu\rangle \leq 0 \text{ on } \Gamma_0. \tag{1.10}$$

Purely Neumann B.C.

$$\langle Dw, \nu\rangle \equiv 0 \text{ on } \Sigma, \text{ in which case we assume } \langle Dd, \nu\rangle = 0 \text{ on } \Gamma_0. \tag{1.11}$$

The geometric requirement in (1.10) in the case of Dirichlet B.C. is standard on Γ_0 (but also the geometrical condition $\langle Dd, \nu\rangle \geq 0$ on Γ_1 was traditionally made, which, however, we dispense with here, as in [**L-T-Z.1**], [**L-T-Z.2**]. The stronger geometrical requirement (1.11) in the case of Neumann B.C. was introduced in [**Tr.1**, Section 5]. Reference [**L-T-Z.1**, Appendices A–C] provides, by different mathematical techniques, *several classes* of triples $\{\Omega, \Gamma_0, \Gamma_1\}$, $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$, in the Euclidean setting $\mathbb{R}^n$, $n \geq 2$, where assumptions (A.1) = (1.4), (A.2) = (1.5), as well as the geometrical requirement on (1.11) are satisfied for a suitably constructed strictly convex function d. Reference [**T-Y.1**, Appendix B] extends a general sufficient condition of [**L-T-Z.1**] to the Riemannian setting. These results are also reported in [**G-L-L-T.1**]. They will also be recalled and used in Section 10 dealing with examples.

The more demanding purely Neumann B.C. case (1.11) will be the primary focus of the present paper. Naturally, a combination of Dirichlet/Neumann B.C. can be readily derived as well from the present treatment.

Goals. Literature. The main goal of the present paper is to generalize essentially all results of [**L-T-Z.2**] for non-conservative Schrödinger equations from the Euclidean setting in [**L-T-Z.2**] to the Riemannian setting as in (1.1). In particular, this includes the generalization, on a bounded Euclidean domain, from the Euclidean Laplacian to a second-order elliptic differential operator with space variable coefficients, as in Section 10.1. This latter case alone justifies the treatment in the Riemannian setting.

First Goal. Thus, as in [**L-T-Z.2**], the first *goal* is therefore to establish Carleman-type inequalities at the $H^1(\Omega)$-level for w—the basic energy level—for these solutions *without lower-order terms.* The proofs of the present paper therefore follow closely those of [**L-T-Z.2**], albeit at the more technical Riemannian setting, based on the Levi-Civita connection. This is particularly more demanding in the first part of the present paper, to include Section 3, where the desired *pointwise* Carleman inequality is achieved via, at times, a more precise analysis than in [**L-T-Z.2**]. In essence, the present paper is a generalization from the Euclidean [**L-T-Z.2**] to the Riemannian setting for general, non-conservative Schrödinger equations, in the same way that [**T-Y.2**] is a generalization still from the Euclidean [**L-T-Z.1**] to the Riemannian setting for general, on-conservative, second-order, hyperbolic equations. In all these four works—[**L-T-Z.1**], [**L-T-Z.2**] in the Euclidean case, [**T-Y.2**] and the present paper in the Riemannian case—the emphasis is on *pointwise* Carleman estimates *without lower-order terms*, following the original inspiration of [**L-R-S.1**] for wave equations in the Euclidean setting. In contrast, Carleman inequalities for non-conservative Schrödinger equations, however, *with (interior) lower-order terms* were obtained in [**Tr.2**, Theorems 2.1.1 and 2.1.2, pp. 464–466] in the Euclidean setting and [**T-Y.1**, Theorems 3.3 and 3.4, pp. 640–641] in the Riemannian setting (or in the setting of Section 10.1). In the present work, as in the aforementioned references, the boundary terms (traces of w) of the solutions w to Eqn. (1.1) which appear in the Carleman estimate are *explicit.* Moreover, for the final observability results in Theorem 2.3.1 (Dirichlet case) and Theorem 2.4.1 (Neumann case), the *explicit* constant in estimates (2.3.2a) and (2.4.2) is Ce^{-CN^2}, where N is the norm in (2.3.2b) penalizing the coefficients q_0 and R_1.

As in past work [**L-T.4**], [**L-T-Z.1**], [**L-T-Z.2**], [**L-T-Y.1**], [**L-T-Y.2**], [**Tr.2**], [**T-Y.1**], the basic assumption of the present paper is: the existence of a strictly convex function $d(x)$, in the Riemannian metric g, in $\overline{\Omega}$. Many classes of examples for this assumption to hold true in $\{\mathbb{R}^n, g\}$ (the case of Section 10.1) were already given in [**L-T-Y.1**], [**L-T-Y.2**], [**Y.1**], [**Y.2**]. Additional classes in $\{M, g\}$ were provided in [**B-G-L.1**] and in [**G-L-L-T.1**]. In Section 10.2, we augment the collection of such examples.

Second goal. As a consequence of the Carleman estimates without lower-order terms and explicit boundary terms or traces (our first goal), we then achieve—precisely as in [**L-T-Z.2**]—our second goal: that is, we obtain global uniqueness results as well as continuous observability/uniform stabilization inequalities at the basic energy level $H^1(\Omega)$, in one shot; that is, as part of the same flow of arguments. Further elaboration on the advantages of Carleman estimates without lower-order terms are noted in [**L-T-Z.2**, Section 1].

Literature. A detailed analysis of the literature—from pure Schrödinger equations to more general Schrödinger equations, in this latter case, with or without lower-order terms—is given in [**L-T-Z.2**, Section 1]. It puts the results quoted above and others in an historical perspective. In short, regarding control-theoretic properties such as exact controllability and uniform stabilization, we quote: [**L-T.3**], [**Ma.1**], [**Leb.1**], for pure Schrödinger equations ($F \equiv 0$) in the Euclidean setting; [**Tr.2**] (Euclidean setting), [**T-Y.2**] (Riemannian setting), [**Ta.1**]-[**Ta.6**] (pseudo-differential setting). All these papers obtain exact controllability by duality on continuous observability estimates. Instead, [**H-L.1**], [**H-L.2**]—which deal with variable coefficient principal part and $F \equiv 0$—give exact controllability results *directly*, while reading off the boundary control as trace of the required solution, following the approach in [**Li-Ta.1**].

In the case of Schrödinger equations, the local theory of [**Ta.4**] excludes the case of Neumann B.C. (since it does not satisfy the strong Lopatinski Condition) and conjectures that the same unique continuation result continues to hold true also in this case [**Ta.4**, Remark 5.7, p. 406]. We also refer to [**Ta.5**], [**Ta.6**]. The global unique continuation in Section 2.4 below extend those of [**L-T-Z.2**] to the present Riemannian setting.

2. Main Results under Assumptions (A.1) and (A.2)

2.1. Carleman Estimates without Lower-Order Terms for $H^{2,2}(Q)$-Solutions of Eqn. (1.1) and No boundary Conditions under Assumptions (A.1) = (1.4), (A.2) = (1.5).

THEOREM 2.1.1. *(First version) Let $T > 0$ be arbitrary, and let $c = c_T$ be defined by (1.6b). Let $d(x) \in C^3(\overline{\Omega})$ be the non-negative, real, strictly convex function satisfying assumptions (A.1) = (1.4) and (A.2) = (1.5). Define accordingly $\varphi(x,t)$ by (1.6a). Let w be a solution of Eqn. (1.1) [with no boundary conditions imposed] in the following class:*

$$w \in H^{2,2}(Q) \equiv L_2(0,T;H^2(\Omega)) \cap H^2(0,T;L_2(\Omega)), \tag{2.1.1a}$$

so that [**L-M.1**] *with $\nu =$ the outward unit normal field along Γ:*

$$\langle Dw, \nu\rangle \in L_2(0,T;H^{\frac{1}{2}}(\Gamma));\ \ w_t \in L_2(0,T;H^1(\Omega));\ \ w_t|_\Gamma \in L_2(0,T;H^{\frac{1}{2}}(\Gamma)), \tag{2.1.1b}$$

where (1.1) is subject to the standing assumption (1.2b) for $F(w)$ and (1.3) for f. Then, for all τ sufficiently large, the following one-parameter family of estimates holds true:

$$B_\Sigma(w) + 4\int_0^T\int_\Omega e^{2\tau\varphi}|f|^2 d\Omega\, dt$$

$$\geq\ m_{\rho,p,\tau,C_T}\left\{\int_0^T\int_\Omega e^{2\tau\varphi}[|Dw|^2 + |w|^2]d\Omega\, dt\right\} - (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T) + \mathbb{E}(0)] \tag{2.1.2}$$

$$\geq\ m_{\rho,p,\tau,C_T} e^{-\delta\tau}\int_{t_0}^{t_1}\mathbb{E}(t)dt - (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T) + \mathbb{E}(0)]; \tag{2.1.3}$$

$$m_{\rho,p,\tau,C_T} \equiv \min\left\{\left[\delta_0\left(2\tau p - \frac{1}{2}\right) - 4C_T\right], \left[4\tau^3\rho p^2(1-\delta_0) + \mathcal{O}(\tau^2) - 4C_T\right]\right\} \nearrow \infty \text{ as } \tau \nearrow \infty, \tag{2.1.4}$$

where $\rho > 0$, $p > 0$, $\delta > 0$ are defined by (1.4), (1.5), (1.6b); $C_{d,T}$ is a positive constant depending on d and T, and δ_0 is a constant $0 < \delta_0 < 1$. Moreover, $\mathbb{E}(t)$ is defined by (1.9), while t_0, t_1 are as in (1.8). The boundary terms $B_\Sigma(w)$ are given *explicitly as follows, where $\xi = Re\ w$, $\eta = Im\ w$:*

(2.1.5)

$$\begin{aligned}
B_\Sigma(w) \quad = \quad & 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi} \left[2\tau^2 |Dd|^2 + \Phi\right] |w|^2 \langle Dd, \nu\rangle d\Gamma\, dt \\
& - 4c\tau \int_0^T \int_\Gamma e^{2\tau\varphi} \left(t - \frac{T}{2}\right) [\eta\langle D\xi, \nu\rangle - \xi\langle D\eta, \nu\rangle]\, d\Gamma\, dt \\
& - 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi} \left[\xi_t \eta - \xi\eta_t\right] \langle Dd, \nu\rangle d\Gamma\, dt \\
& + \int_0^T \int_\Gamma e^{2\tau\varphi} \left[2\tau^2 |Dd|^2 - \tau\Delta d\right] [\bar{w}\langle Dw, \nu\rangle + w\langle D\bar{w}, \nu\rangle]\, d\Gamma\, dt \\
& + 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi} \langle Dd, [D\bar{w}\langle Dw, \nu\rangle + Dw\langle D\bar{w}, \nu\rangle]\rangle\, d\Gamma\, dt \\
& - 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi} |Dw|^2 \langle Dd, \nu\rangle d\Gamma\, dt.
\end{aligned}$$

Here, c is the constant in (1.6b), while the function Φ occurring in (2.1.5) may be taken to satisfy either $\Phi \equiv 0$ or else $\Phi \equiv \tau\Delta d(x)$, see (4.3b) and (4.4). Thus, from (2.1.5), the following estimate holds true:

$$(2.1.6) \quad B_\Sigma(w) \leq C_{\phi,\tau} \int_0^T e^{2\tau\varphi} \left[\|w(t)\|^2_{H^1(\Gamma)} + \|\langle Dw, \nu\rangle\|^2_{L_2(\Gamma)} + \|w_t\|^2_{H^{-1}(\Gamma)}\right] dt.$$

Theorem 2.1.1 is proved in Sections 3 through 5 (for $H^{2,2}(Q)$-solutions). In particular, inequalities (2.1.2), (2.1.3) are established in inequalities (5.1), (5.2) below (for $\epsilon = 1$).

Remark 2.1.1. The Carleman estimate of Theorem 2.1.1 is essentially the one in [**T-Y.1**, Theorem 3.3, p. 640], *except for the critical improvement* that the present version, unlike [**T-Y.1**], in Theorem 2.1.1 does *not* include an interior lower-order term. In turn, [**T-Y.1**, Theorem 3.3, p. 640] was the extension to the Riemannian case of the prior Carleman estimate in the Euclidean case of [**Tr.1**, Theorem 2.1.1, p. 464], which likewise did contain an interior lower-order term. The technique of proof of the present paper (leading to pointwise Carleman estimates) is vastly different from the technique of proof in [**T-Y.1**], which is based on Riemannian multipliers. □

Corollary 2.1.2. *(Global uniqueness) Assume the setting of Theorem 2.1.1. In particular, let w be a solution of Eqn. (1.1) in the class $H^{2,2}(Q)$ defined in (2.1.1).*

(i) Neumann case*: Assume further that such w satisfies, in addition, the following B.C.:*

$$(2.1.7) \qquad \langle Dw, \nu\rangle \equiv 0 \text{ on } \Sigma \text{ and } w|_{\Sigma_1} \equiv 0, \quad \text{where } \langle Dd, \nu\rangle = 0 \text{ on } \Gamma_0,$$

with $\Sigma \equiv (0, T] \times \Gamma$; $\Sigma_1 \equiv (0, T] \times \Gamma_1$. Then, in fact, such solution must vanish: $w \equiv 0$ in $(0, T] \times \Omega$.

(ii) Dirichlet case*: Assume further that such w satisfies, in addition, the following B.C.:*

$$w|_{\Sigma} \equiv 0 \text{ and } \langle Dw, \nu\rangle \equiv 0 \text{ on } \Sigma_1, \quad \text{where } \langle Dd, \nu\rangle \leq 0 \text{ on } \Gamma_0. \tag{2.1.8}$$

Then, in fact, such solution must vanish: $w \equiv 0$ in $(0,T] \times \Omega$.

Corollary 2.1.2 is established at the end of Section 5, after completion of the proof of Theorem 2.1.1.

Imposition of structural properties. Henceforth, we shall specialize to the case dim $M \geq 2$, and leave the case dim $M = 1$ to Appendix A below. Thus, let dim $M \geq 2$. In this case, as in [**Tr.1**], [**T-Y.1**], [**L-T-Z.1**], in order to refine Theorem 2.1.1, we need to specialize the first-order operator $F(w)$ by imposing a *structural* property. This is stated by the following assumption, whereby, eventually, without loss of generality, we can assume that the vector field $P(t,x)$ in (1.2a) be purely imaginary (this is the case of the magnetic potential [**R-S.1**, p. 173], even in the Euclidean case):

(A.3) We assume that $P(t,x) = -iR_1(t,x)$, where $R_1(t,x)$ is a *real-valued* vector field on $\mathbb{R}_t \times M$. Moreover, in this case, we make the following additional regularity assumption (over $P \in L_\infty(0,T;\mathcal{X}(M))$, $q_0 \in L_\infty(Q)$, behind assumption (1.2b)) on the vector field R_1 and the function q_0. Recall that the covariant differential (a $2-0$ tensor T_2^0) of $R_1 \in \mathcal{X}(M)$ determines a bilinear form on $TM \times TM$, for each $x \in M$, defined by $DR_1(X,Y) = \langle D_X R_1, Y\rangle_g$. Then, we require that

$$\begin{cases} |DR_1(X,Y)| = |\langle D_X R_1, Y\rangle| \leq C|X|\,|Y|, \quad 0 \leq t \leq T, \\ \text{or } DR_1 \in L_\infty(0,T;T_2^0). \end{cases} \tag{2.1.9a}$$

$$\text{either } q_0 \in \begin{cases} L_1(0,T;W^{1,2}(\Omega)), \\ L_1(0,T;W^{1,2+\epsilon}(\Omega)), \\ L_1(0,T;W^{1,\dim M}(\Omega)), \end{cases} \tag{2.1.9b}$$

$$\text{or else } (q_0)_t \in \begin{cases} L_1(0,T;L_1(\Omega)), & \dim M = 1, \\ L_1(0,T;L_{1+\epsilon}(\Omega)), & \dim M = 2, \\ L_1(0,T;L_{\frac{\dim M}{2}}(\Omega)), & \dim M \geq 3. \end{cases}$$

Sufficient conditions are $|Dq_0|$, $(q_0)_t \in L_\infty(Q)$, respectively. The proof of Lemma 6.1(v) will use the assumption on q_0 on the LHS of (2.1.9b), while Remark 6.1 will use $(q_0)_t$ on the RHS of (2.1.9b).

Consequence of (A.3). It is documented in Appendix A, on the basis of the counterpart of precisely the same argument as in [**L-T-Z.2**, Appendix A], that: under assumption (A.3), we can always assume, without loss of generality, that Eqn. (1.1) be of the form (magnetic potential [**R-S.1**, Theorem X.22, p. 173], at least in the Euclidean case):

$$iw_t + \Delta w = -i\langle R_1(t,x), Dw\rangle + q_0(t,x)w + f, \tag{2.1.10}$$

where q_0 is a real-valued scalar function on $\mathbb{R}_t \times M$, while $R_1(t,x)$ is a real-valued vector field on $\mathbb{R}_t \times M$, such that assumptions (2.1.9a–b) hold true, and—moreover—such that

$$\begin{cases} \text{either } |\langle R_1, \mu\rangle|^2 \le \langle R_1, \nu\rangle < 1 & \text{on } \Gamma, \qquad (2.1.11a) \\ \text{or else } \langle R_1, \nu\rangle \equiv 0 & \text{on } \Gamma_0, \qquad (2.1.11b) \end{cases}$$

where μ is any unit tangent vector on $\partial\Omega = \Gamma$, so that $\langle D\,\cdot\,, \mu\rangle$ is the tangential gradient. Moreover, in the case dim $\Omega = 1$, we can even take $R_1(t,x) \equiv 0$. □

The structural property $P = -iR_1$, R_1 real, is *critical* precisely as in [**Tr.1**], [**T-Y.1**], [**L-T-Z.2**]. We shall use the first "w.l.o.g."-assumption (2.1.11a) in the case of Neumann B.C. (1.11); and the second "w.l.o.g."-assumption (2.1.11b) in the case of Dirichlet B.C. (1.10).

The use of assumption (A.3)—which includes (2.1.9a–b) as well as its consequences (2.1.10), (2.1.11a–b)—will be seen in Lemma 6.1(ii), (iv), Eqns. (6.7), (6.9), (6.11) via (6.15). Such an assumption will permit us to obtain a second version, more refined, of the Carleman-type estimate (2.1.3) of Theorem 2.1.1.

THEOREM 2.1.3. *(Second version) Assume the setting, in particular (A.1) = (1.4), (A.2) = (1.5), and the notation of Theorem 2.1.1. In addition, we assume hypothesis (A.3) = (2.1.9a–b), so that the dynamics under consideration may be taken to be (2.1.10), and properties (2.1.11a–b) hold true. Assume that* $f \in L_2(0,T;H^1(\Omega))$*. Let w be a solution of Eqn. (1.1)—ultimately (2.1.10)—in the class* $H^{2,2}(Q)$ *defined in (2.1.1a).*

Then, for all $\tau > 0$ sufficiently large, the following one-parameter family of estimates holds true, where $k_{\varphi,\tau} > 0$*:*

$$\tilde{B}_\Sigma(w) + 4\int_0^T\int_\Omega e^{2\tau\varphi}|f|^2 d\Omega\, dt + C_{p,q,\rho,\tau}\|f\|^2_{L_2(0,T;H^1(\Omega))} \tag{2.1.12}$$

$$\ge \left\{ m_{\rho,p,\tau,C_T} e^{-\delta\tau}\frac{(t_1-t_0)}{2}e^{-C_T T} - (C_{d,T})\tau e^{-2\tau\delta}\right\}[\mathbb{E}(T) + \mathbb{E}(0)]$$

$$\ge k_{\varphi,\tau}[\mathbb{E}(T) + \mathbb{E}(0)], \tag{2.1.13}$$

with explicit m_{ρ,p,τ,C_T}*, which is noted in Eqn. (2.1.4) or Eqn. (5.3) below, while* $\mathbb{E}(t)$ *is defined in (1.9). Moreover, the boundary terms* $\tilde{B}_\Sigma(w)$ *(2.1.12) are given by*

$$\tilde{B}_\Sigma(w) = B_\Sigma(w) + C_{\varphi,p,q}\left\{\int_0^T\int_\Gamma |\langle Dw,\nu\rangle|\Big[|w_t| + |\langle Dw,\nu\rangle|\,|\langle R_1,\nu\rangle \right. \tag{2.1.14}$$

$$\left. + \,\mathcal{O}_c| + |q_0 w| + |w| + |f|\Big] d\Gamma\, dt\right\},$$

with constant $C_{\varphi,p,q}$*, which may be given explicitly via the LHS of (2.1.12) and* $G(T)$ *in (6.13). Moreover, we have that* $\mathcal{O}_c = 0$ *in the case of the Dirichlet B.C.* $w|_\Sigma \equiv 0$*; and* $\mathcal{O}_c = 1$ *otherwise.*

Theorem 2.1.3 is proved in Section 6 (for $H^{2,2}(Q)$-solutions).

2.2. Extension of Carleman Estimate (2.1.3) to Finite Energy Solutions.

THEOREM 2.2.1. *Assume the hypotheses of Theorem 2.1.3: (1.2b), (A.3) for* $F(w)$, $f \in L_2(0,T;H^1(\Omega))$, *in addition to (A.1) = (1.4) and (A.2) = (1.5). Then, estimate (2.1.3) of Theorem 2.1.1 and estimate (2.1.13) of Theorem 2.1.3 can be extended to finite energy solutions in the class*

$$w \in C([0,T];H^1(\Omega));\ \langle Dw,\nu\rangle,\ w_t \in L_2(0,T;L_2(\Gamma)), \tag{2.2.1}$$

provided that the (unbounded) term $F(w) = -i\langle R_1, Dw\rangle + q_0 w$ *in (2.1.10) satisfies the following additional hypothesis of convenience:*

(2.2.2)

(A.4) the vector field R_1 *and the function* q_0 *are time-independent.*

[See [**L-T-Z.2**, below (2.2.2)] for the time-dependent case.] The proof of Theorem 2.2.1 is given in Section 7.

2.3. Continuous Observability. Global Uniqueness. Dirichlet B.C. We first list our main results in our treatment of Sections 1 through 8.

Purely Dirichlet problem. Here, we consider the following problem:

$$\begin{cases} iw_t + \Delta w = F(w) + f \equiv -i\langle R_1, Dw\rangle + q_0 w + f & \text{in } (0,T]\times\Omega \equiv Q; & \text{(2.3.1a)} \\ w(0,\ \cdot\ ) = w_0 & \text{in } \Omega; & \text{(2.3.1b)} \\ w|_\Sigma = 0 & \text{in } (0,T]\times\Gamma \equiv \Sigma. & \text{(2.3.1c)} \end{cases}$$

In the case of the Dirichlet problem (2.1.3), the extension of Theorem 2.1.3 as given by Theorem 2.2.1 is *not* needed. Indeed, the extension of the final sought-after continuous observability inequality (2.3.2) below can be readily accomplished from $H^{2,2}(Q)$-solutions to $H^{1,1}(Q)$-solutions just by virtue of the regularity theorem ("direct" inequality obtained in [**L-T.2**] (which is valid also in the Riemannian setting by virtue of the techniques in [**T-Y.1**] which generalize to the Riemannian setting those of [**L-T.2**] originally performed in the Euclidean setting. Thus, the additional assumption (2.2.2) of Theorem 2.2.1 may be dispensed with. We obtain

THEOREM 2.3.1. *Let* w *be a solution of problem (2.3.1) with I.C.* $w_0 \in H_0^1(\Omega)$ *and with* $f \in L_2(0,T;H^1(\Omega))$, *under the standing assumption (1.2b) on* $F(w) = -\langle R_1, Dw\rangle + q_0 w$, *as dictated by assumption (A.3), with* R_1 *and* q_0 *satisfying (2.1.9a–b). Assume further (A.1) = (1.4) and (A.2) = (1.5) for* $d(x)$ *(see also Remark 1.4). Assume* Γ_0 *as in (1.10); that is,* $\langle Dd,\nu\rangle \le 0$ *on* Γ_0. *Let* $\Gamma_1 = \Gamma\setminus\Gamma_0$. *Let* $T > 0$ *be arbitrary. Then:*

(a) There exists a constant $C_T > 0$ *such that the following continuous observability inequality holds true:*

$$C_T\mathbb{E}(0) \le \int_0^T \int_{\Gamma_1} |\langle Dw,\nu\rangle|^2 d\Gamma_1\, dt + \|f\|^2_{L_2(0,T;H^1(\Omega))}. \tag{2.3.2a}$$

The constant C_T *in (2.3.2a) is explicit: it is of the order of* Ce^{-CN^2}, *where* N *is the appropriate norm of the coefficients* R_1 *and* q_0. *For instance, under (1.2b) and (2.1.9a–b) for* $n = \dim M \ge 3$, *we have*

$$N = |q_0|_{L_\infty(Q)} + |q_0|_{L_1(0,T;W^{1,n}(\Omega))} + |R_1|_{L_\infty(0,T;W^{1,\infty}((\Omega)^n))}, \tag{2.3.2b}$$

and analogously for $n = \dim M = 1, 2$.

(b) Let now $f \equiv 0$ in (2.3.1). Then the following global uniqueness result holds true: with $T > 0$, let w be a $H^{1,1}(Q)$-solution of Eqn. (2.3.1a) and over-determined B.C.'s:

(2.3.3)

$$w|_{\Sigma} \equiv 0 \ \textit{and} \ \langle Dw, \nu\rangle \equiv 0 \ \textit{on} \ \Sigma_1 = (0,T] \times \Gamma_1, \ \textit{where} \ \langle Dd, \nu\rangle \leq 0 \ \textit{on} \ \Gamma_0 = \Gamma \setminus \Gamma_1.$$

Then, in fact, $w \equiv 0$ in Q, indeed in $\mathbb{R}^+_t \times \Omega$.

Theorem 2.3.1(a) is established as Theorem 8.2(c) in Section 8. See also [**Ta.6**, Section 7].

2.4. Continuous Observability. Global Uniqueness. Neumann B.C.. We first list our main continuous observability inequalities and related global uniqueness results in our treatment of Sections 1 through 8.

Purely Neumann problem. Here we consider the following problem:

$$\begin{cases} iw_t + \Delta w = F(w) + f \equiv -i\langle R_1, Dw\rangle + q_0 w + f & \text{in } (0,T] \times \Omega \equiv Q; & (2.4.1a) \\ w(0, \ \cdot\ ) = w_0 & \text{in } \Omega; & (2.4.1b) \\ \langle Dw, \nu\rangle \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma. & (2.4.1c) \end{cases}$$

As explained below in Section 7, it is for the Neumann problem (2.4.1) that the extension Theorem 2.2.1 is called for. Accordingly, we inherit its additional (non-critical) assumption (2.2.2).

THEOREM 2.4.1. *Let w be the solution of problem (2.4.1) with I.C. $w_0 \in H^1(\Omega)$ and with $f \in L_2(0,T; H^1(\Omega))$, under the standing assumption (1.2b) on $F(w)$. Assume, further, hypotheses (A.1) = (1.4), (A.2) = (1.5), (A.3) [that is, $P = -iR_1$, with R_1 and q_0 satisfying (2.1.9a–b)], as well as (A.4). Assume further Γ_0 as in (1.11); that is, $\langle Dd, \nu\rangle \equiv 0$ on Γ_0. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$. Let $T > 0$ be arbitrary. Then:*

(a) there exists a constant $C_T > 0$ such that the following continuous observability inequality holds true where $\mathbb{E}(\ \cdot\ )$ is defined in (1.9):

$$C_T \mathbb{E}(0) \leq \int_0^T \int_{\Gamma_1} |w|^2 + |w_t|^2 d\Gamma_1 dt + \|f\|^2_{L_2(0,T;H^1(\Omega))}. \tag{2.4.2}$$

The constant C_T in (2.4.2) is explicit. It is given as in Theorem 2.3.1.

(b) Let $f \equiv 0$ in (2.4.1). Then the following global uniqueness result holds true: with $T > 0$, let w be a $H^{1,1}(Q)$-solution of Eqn. (2.4.1) and over-determined B.C.'s:

$$\langle Dw, \nu\rangle \equiv 0 \ \textit{on} \ \Sigma \ \textit{and} \ w|_{\Sigma_1} \equiv 0, \ \textit{where} \ \langle Dd, \nu\rangle = 0 \ \textit{on} \ \Gamma_0 = \Gamma \setminus \Gamma_1, \tag{2.4.3}$$

$\Sigma_1 = (0,T] \times \Gamma_1$. Then, in fact, $w \equiv 0$ in Q, indeed in $\mathbb{R}^+_t \times \Omega$.

(c) Let $f \equiv 0$. Then, inequality (2.4.2) can be improved to:

$$C_T \mathbb{E}(0) \leq \int_0^T \int_{\Gamma_1} |w_t|^2 d\Gamma \, dt. \tag{2.4.4}$$

The above results are the perfect counterpart of the Euclidean case [**L-T-Z.2**, Sections 2.3, 2.4]. See also [**Ta.6**, Section 7].

In effect, we shall first prove the global uniqueness statement of part (b) of Theorem 2.4.1 in Section 8, as a direct consequence of the Carleman estimate without lower-order terms of Theorem 2.1.3, first for smooth $H^{2,2}(Q)$-solutions and next extended to $H^{1,1}(Q)$-solutions in Section 7. Next, part (b) will be used to establish part (a) in Theorem 8.4 of Section 8 by virtue also of the trace result of Theorem 8.3. The above global uniqueness result appears to be new over (mostly local) results of the literature [**Ho.1**], [**Is.1**], [**Is.2**], [**Ta.3**]–[**Ta.6**] also in view of the passage from local to global results given in [**Lit.2**].

2.5. Exact Controllability. By the usual duality between exact controllability and continuous observability [**L-T.3**], [**Ma.1**], [**G-L-L-T.1**, Section 0, pp. 76–78], we obtain exact controllability results.

Purely Dirichlet problem. We consider the following problem:

$$\begin{cases} iy_t + \Delta y = F(y) = -i\langle R_1(t,x), Dy\rangle + q_0(t,x)y, & \text{in } (0,T] \times \Omega \equiv Q; \quad (2.5.1a)\\ y(0,\ \cdot\ ) = y_0 & \text{in } \Omega; \quad (2.5.1b)\\ y|_{\Sigma_0} \equiv 0,\ y|_{\Sigma_1} = u, & \text{in } \Sigma_i. \quad (2.5.1c) \end{cases}$$

By duality on the COI of Theorem 2.3.1 with $f \equiv 0$ (as in [**L-T.5**, Lemma 3.1, p. 529]), we obtain the following topologically *optimal* result. See also [**Ta.6**, Section 7] and [**T-Y.1**]. In the Euclidean setting we recover [**L-T-Z.2**], [**Tr.2**], and for $F = 0$ we obtain [**L-T.3**], [**Ma.1**]; in this latter case [**Leb.1**] obtains the result under the geometric optics condition, but requires analytic boundary Γ.

THEOREM 2.5.1. *As in Theorem 2.3.1, assume the hypotheses (1.2b) and (A.3) (to include (2.1.9a–b)) on $F(y) = -\langle R_1, Dy\rangle + q_0 y$; and the hypotheses (A.1) = (1.4) and (A.2) = (1.5) on the strictly convex function $d(x)$. Let Γ_0 be defined as in (1.10): i.e., $\langle Dd, \nu\rangle \le 0$ on Γ_0. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$. Let $T > 0$ be arbitrary. Given $y_T \in H^{-1}(\Omega)$ [respectively, given $y_0 \in H^{-1}(\Omega)$], there exists a boundary control $u \in L_2(0,T;L_2(\Gamma_1))$, such that the corresponding solution of problem (2.5.1a-b-c) with $y_0 = 0$ [respectively, of problem (2.5.1a-b-c) for such given $y_0 \in H^{-1}(\Omega)$] satisfies $y(T) = y_T$ [respectively, satisfies $y(T) = 0$].*

Purely Neumann problem. Here we consider the following problem:

$$\begin{cases} iy_t + \Delta y = F(y) = -i\langle R_1(t,x), Dy\rangle + q_0(t,x)y, & \text{in } (0,T] \times \Omega \equiv Q; \quad (2.5.2a)\\ y(0,\ \cdot\ ) = y_0 & \text{in } \Omega; \quad (2.5.2b)\\ \langle Dy, \nu\rangle \equiv 0 \text{ on } \Sigma_0,\ \langle Dy, \nu\rangle|_{\Sigma_1} = u, & \text{in } \Sigma_i. \quad (2.5.2c) \end{cases}$$

By duality on the COI of Theorem 2.4.1(c) with $f \equiv 0$, Eqn. (2.4.4), we obtain:

THEOREM 2.5.2. *As in Theorem 2.4.1, assume the hypotheses (1.2b), (A.3) (to include (2.1.9a–b)), (A.4) = (2.2.2) on $F(y)$, as well as (A.1) = (1.4), (A.2) = (1.5) on the strictly convex function $d(x)$. Let Γ_0 be defined as in (1.11): $\langle Dy, \nu\rangle \equiv 0$ on Γ_0; or else $\langle Dy, \nu\rangle \le 0$ on Γ_0 if $y|_{\Sigma_0} \equiv 0$. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$. Let $T > 0$ be arbitrary. Given $y_T \in H^1(\Omega)$ [respectively, given $y_0 \in H^1(\Omega)$], there is a boundary control $u \in L_2(0,T;L_2(\Gamma_1))$, such that the corresponding solution of problem (2.5.2a-b-c) with $y_0 = 0$ [respectively, of problem (2.5.2a-b-c) for such given $y_0 \in H^1(\Omega)$] satisfies $y(T) = y_T$ [respectively, satisfies $y(T) = 0$].*

Remark 2.5.1. Theorem 2.5.2 is not optimal: with the class of $L_2(0,T;L_2(\Gamma_1))$-controls in the Neumann B.C. (2.5.2c), the corresponding problem (2.5.2a-b-c) is exactly controllable on the space $L_2(\Omega)$ [rather than $H^1(\Omega)$], at least in the Euclidean case, see [**L-T.6**, Theorem 1.4], essentially already contained in [**L-T-Z.2**, Section 11]. For the Riemannian case, see [**Tr.3**].

This latter result requires $L_2(\Omega)$-energy estimates (rather than $H^1(\Omega)$-energy estimates, see [**L-T-Z.2**, Section 10]), which no doubt extend also to the Riemannian setting. [**L-T-Z.2**, Remark 2.6.2]. $\square$

2.6. Uniform Stabilization with Neumann Dissipation. In this section, we briefly treat also a uniform stabilization result, as it follows readily from prior estimates. We consider the following problem:

$$\begin{cases} w_t = i\Delta w, & \text{in } (0,T] \times \Omega \equiv Q; & (2.6.1a) \\ w(0,\ \cdot\ ) = w_0 & \text{in } \Omega; & (2.6.1b) \\ \langle Dw, \nu \rangle + w_t \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma. & (2.6.1c) \end{cases}$$

For the Euclidean version, see [**L-T-Z.2**][Section 7 and Appendix C.2, Eqns. (C.18a-b-c)]. For a related problem still in the Euclidean case, see [**Ma.1**]. As in [**L-T-Z.2**, Eqn. (C.20)], we define the operator

(2.6.2)
$$Aw = i\Delta w, \mathcal{D}(A) = \{w \in H^2(\Omega) : \Delta w|_\Gamma \in L_2(\Gamma) \text{ and } [\langle Dw, \nu\rangle + i\Delta w]_\Gamma = 0\}$$

on the space

$$\mathcal{H} \equiv H^1(\Omega)/\text{const}; \ (f,g)_{\mathcal{H}} = \int_\Omega \langle Df, D\bar{g}\rangle d\Omega. \tag{2.6.3}$$

Theorem 2.6.1. *(a) The operator A in (2.6.2) is dissipative*

$$\begin{aligned} Re\{(Aw,w)_{\mathcal{H}}\} &= Re\left\{ i \int_\Omega \langle D(\Delta w), D\bar{w}\rangle d\Omega \right\} \\ &= -\int_\Gamma |\langle Dw, \nu\rangle|^2 d\Gamma \leq 0, \quad \forall\ w \in \mathcal{D}(A). \end{aligned} \tag{2.6.4}$$

Indeed, A is maximal dissipative on H and hence it is the generator of a s.c. contraction semigroup e^{At} on $\mathcal{H}$.

(b) With reference to problem (2.6.1), the following estimate holds true:

$$\int_0^T \int_{\Gamma_1} |w_t|^2 d\Gamma_1 dt \geq C_T[\mathbb{E}(T) + \mathbb{E}(0)], \tag{2.6.5}$$

recalling (1.9) for $\mathbb{E}(\ \cdot\ )$. Accordingly, there is $T > 0$ such that

$$\|e^{AT}\|_{\mathcal{L}(\mathcal{H})} < 1 \quad \textit{and so} \quad \mathbb{E}(t) \leq Me^{-\delta t}\mathbb{E}(0), \tag{2.6.6}$$

for some $M \geq 1$ and $\delta > 0$.

Remark 2.6.1. In Theorem 2.6.1, we could modify the B.C. (2.6.1c) in the usual way. That is, we could require the dissipative condition in (2.6.1c) only on Σ_1, while on Σ_0 taking either the Dirichlet homogeneous condition $w|_{\Sigma_0} \equiv 0$ under the geometric assumption $\langle Dd, \nu\rangle \leq 0$ on Γ_0 as in (1.10); or else the Neumann homogeneous condition $\langle Dw, \nu\rangle \equiv 0$ on Σ_0 under the geometric assumption $\langle Dd, \nu\rangle = 0$ on Γ_0 as in (1.11). $\square$

Remark 2.6.2. (Counterpart of Remark 2.5.1) A far more attractive uniform stabilization result—with a Neumann boundary dissipation involving only w, and not w_t; and the mathematically more desirable space $L_2(\Omega)$, rather than $H^1(\Omega)$—is given in [**L-T-Z.2**, Section 11] in the linear case and in [**L-T.6**] in the nonlinear case. As is the case in Remark 2.5.1, these more attractive linear and nonlinear uniform decay results require $L_2(\Omega)$-energy estimates (rather than $H^1(\Omega)$-energy estimates, see [**L-T-Z.2**, Section 10]), which no doubt extend to the Riemannian setting as well [**L-T-Z.2**, Remark 2.6.2]. □

The proof of Theorem 2.6.1 is given at the end of Section 8.

2.7. The Euler-Bernoulli Equation. In this section, we transfer the estimates of the Schrödinger problem, given in the previous section, into estimates for the Euler-Bernoulli (E-B) equation with 'hinged' B.C. Thus, we consider the following problem:

$$\begin{cases} w_{tt} + \Delta^2 w = f & \text{in } (0,T] \times \Omega \equiv Q; & \text{(2.7.1a)} \\ w(0, \cdot\,) = w_0,\ w_t(0, \cdot\,) = w_1 & \text{in } \Omega; & \text{(2.7.1b)} \\ w|_\Sigma \equiv 0,\ \Delta w|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma = \Sigma, & \text{(2.7.1c)} \end{cases}$$

where now $w(t,x)$ is real-valued. Writing the E-B equation as an iteration of two Schrödinger equations, in the usual way

$$w_{tt} + \Delta^2 w = (\Delta + i\partial_t)(\Delta - i\partial_t)w = f \text{ in } Q, \tag{2.7.2}$$

and setting

$$\begin{cases} v = iw_t - \Delta w; & \text{(2.7.3a)} \\ |v|^2 = w_t^2 + |\Delta w|^2;\ |\nabla v|^2 = |\nabla w_t|^2 + |\nabla \Delta w|^2, & \text{(2.7.3b)} \end{cases}$$

we rewrite problem (2.7.1) via (2.7.3) as

$$\begin{cases} iv_t + \Delta v = f & \text{in } Q; & \text{(2.7.4a)} \\ v(0) = iw_1 - \Delta w_0 & \text{in } \Omega; & \text{(2.7.4b)} \\ v|_\Sigma = 0 & \text{in } \Sigma. & \text{(2.7.4c)} \end{cases}$$

Setting in this section

$$E_w(t) \equiv \int_\Omega [|\nabla \Delta w(t)|^2 + |\nabla w_t(t)|^2] d\Omega \tag{2.7.5a}$$

$$= E_v(t) = \int_\Omega |\nabla v(t)|^2 d\Omega, \tag{2.7.5b}$$

via (2.7.3b), we can apply Theorem 2.3.1 (Dirichlet case) to the v-problem (2.7.4) and obtain

Theorem 2.7.1. *Let w be a solution of problem (2.7.1) with $\{w_0, w_1\} \in H^3(\Omega) \times H^1(\Omega)$. Let $f \in L_2(0,T; H^1(\Omega))$. Let $d(x)$ be the function satisfying properties (1.4), (1.5) (see also Remark 1.3 and 1.4). Let, as in Theorem 2.3.1, Γ_0 be defined by (1.10), i.e., $\langle Dd, \nu \rangle \leq 0$ on Γ_0. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$, and let $T > 0$ be arbitrary. Then, there exists a constant $C_T > 0$, such that the following continuous observability inequality holds true:*

$$C_T E_w(0) \leq \int_0^T \int_{\Gamma_1} \left[|\langle D\Delta w, \nu \rangle|^2 + |\langle Dw_t, \nu \rangle|^2 \right] d\Gamma_1\, dt + \|f\|^2_{L_2(0,T;H^1(\Omega))}. \tag{2.7.6}$$

The constant C_T in (2.7.6) is explicit and is given by the expression in Theorem 2.3.1: Ce^{-CN^2}, with N defined by (2.3.2b).

Proof. Apply Theorem 2.3.1 with $v(0) \in H^1(\Omega)$, $|\nabla v(0)|^2 = |\nabla w_1|^2 + |\nabla \Delta w_0|^2$. $\square$

Remark 2.7.1. Classical results, even in the Euclidean case, yielded a result such as Theorem 2.7.1, however, with a *non-explicit* constant C_T, resulting from a compactness/uniqueness argument by contradiction to absorb lower-order terms. In the Euclidean case, the improvement of estimate (2.7.6) with an *explicit* constant C_T is given in [**L-T-Z.2**, Theorem 2.5.1], of which the present result is a generalization to the Riemannian case. In the Riemannian case, Carleman estimates for plate-like equations with general energy level terms are given in [**L-T-Y.2**]; they contain, however, l.o.t. $\square$

3. A Fundamental Lemma

The following lemma is the key starting point of our analysis. It gives a pointwise estimate which is the counterpart, in the present Riemannian metric, of [**L-T-Z.2**, Lemma 3.1]. Its proof is mostly a parallel development, however, in the Riemannian metric, of that given in [**L-T-Z.2**] in the Euclidean metric.

Lemma 3.1. *Let*

$$w(t,x) \in C^2(\mathbf{R}_t \times M; \mathbf{C}),\ \ell(t,x) \in C^3(\mathbf{R}_t \times M; \mathbf{R}),\quad \Phi(t,x),\ \Psi(t,x) \in C^1(\mathbf{R}_t \times M; \mathbf{R}), \tag{3.1}$$

be given functions, all real-valued, except for $w(t,x)$ which is, instead, complex-valued. Let

$$\nabla(\ell_t) \equiv 0; \quad \theta(t,x) = e^{\ell(t,x)}; \quad v(t,x) = \theta(t,x) w(t,x), \tag{3.2}$$

where the condition on ℓ is an assumption, while the other two relations in (3.2) are definitions. Let $\epsilon > 0$ be arbitrary. Then, the following pointwise inequality holds true:

$$\begin{aligned} &\left(1 + \frac{1}{\epsilon}\right)\theta^2 |iw_t + \Delta w|^2 - \frac{\partial \mathcal{M}}{\partial t} + \mathrm{div} V \\ \geq\ & \Big\{ -2(\Psi + \Delta\ell)\Delta\ell + 4D^2\ell(D\ell, D\ell) + 2\langle D\ell, D(\Phi - \Delta\ell)\rangle \\ &+ 2\langle D\ell, D(\Psi + \Delta\ell)\rangle - \epsilon|\Psi + \Delta\ell|^2 \\ &- \frac{1}{\epsilon}|D(\Delta\ell)|^2 - [\Psi^2 + \Phi^2 - 2\Phi\Delta\ell] + \ell_{tt} \Big\} |v|^2 \\ &+ 2\Big\{ D^2\ell(Dv, D\bar{v}) + D^2\ell(D\bar{v}, Dv) - (\Psi + \Delta\ell)|Dv|^2 \Big\} - \epsilon|Dv|^2. \end{aligned} \tag{3.3}$$

The left side of (3.3) is defined as follows. Call $\xi \equiv \mathrm{Re}\ w$ and $\eta \equiv \mathrm{Im}\ w$. Then, we set:

$$\mathcal{M} = \mathcal{M}(w) \equiv \theta^2[2\langle D\ell, D\xi\rangle\eta - \xi\langle D\ell, D\eta\rangle - \ell_t|w|^2]; \tag{3.4}$$

$$V = V(w) \equiv 2\theta^2\Big\{\Big[2|D\ell|^2 - (\Delta\ell + \Psi) + \Phi\Big]\,|w|^2 D\ell + \ell_t(\eta D\xi - \xi D\eta) - (\xi_t\eta - \xi\eta_t)D\ell + \frac{1}{2}(2|D\ell|^2 + \Delta\ell)D(|w|^2) + \langle D\ell, D\bar{w}\rangle Dw + \langle D\ell, Dw\rangle D\bar{w} - |Dw|^2 D\ell\Big\}. \tag{3.5}$$

Remark 3.1. Inequality (3.3) is a streamlined variation of [**L-T-Z.2**, Eqn. (3.3)]; both expressions produce the sought-after consequences. □

Proof. Step 1. From $v = \theta w$, we obtain by direct computation:

$$\begin{cases} \theta_t = \theta\ell_t;\ v_t = \theta_t w + \theta w_t = \ell_t v + \theta w_t; & (3.6a) \\ \theta\Delta w = \Delta v - 2\langle D\ell, Dv\rangle + (|D\ell|^2 - \Delta\ell)v. & (3.6b) \end{cases}$$

Then by (3.6a–b), we compute

$$|\theta(iw_t + \Delta w)|^2 = \Big|i(v_t - \ell_t v) + \Delta v - 2\langle D\ell, Dv\rangle + (|D\ell|^2 - \Delta\ell)v\Big|^2 \tag{3.7}$$

$$= |I_1 - I_2 + I_3|^2 \tag{3.8}$$

$$= |I_1|^2 + |I_2|^2 + |I_3|^2 - (I_1\bar{I}_2 + I_2\bar{I}_1) - (I_2\bar{I}_3 + I_3\bar{I}_2) + (I_1\bar{I}_3 + I_3\bar{I}_1), \tag{3.9}$$

where, after adding and subtracting Φv and Ψv in (3.7), we have

$$\begin{cases} I_1 \equiv iv_t + \Delta v + (|Dl|^2 - \Delta\ell)v - \Psi v, & (3.10a) \\ I_2 = 2\langle D\ell, Dv\rangle + \Phi v + i\ell_t v; \quad I_3 = (\Psi + \Phi)v. & (3.10b) \end{cases}$$

Thus, dropping $|I_1|^2 + |I_2|^2$ in (3.9), we obtain

$$|\theta(iw_t + \Delta w)|^2 \geq |I_3|^2 - (I_1\bar{I}_2 + I_2\bar{I}_1) - (I_2\bar{I}_3 + I_3\bar{I}_2) + (I_1\bar{I}_3 + I_3\bar{I}_1). \tag{3.11}$$

Step 2: With reference to (3.10b), in this Step 2 we shall establish that

$$I_2\bar{I}_3 + I_3\bar{I}_2 = 2 \text{ div}\left[(\Psi + \Phi)|v|^2 D\ell\right] + 2(\Psi + \Phi)(\Phi - \Delta\ell)|v|^2 - 2\langle D(\Psi + \Phi), D\ell\rangle|v|^2. \tag{3.12}$$

Proof of (3.12). By invoking the definitions in (3.10b), we compute

$$I_2\bar{I}_3 + I_3\bar{I}_2 = [2\langle D\ell, Dv\rangle + \Phi v + i\ell_t v](\Psi + \Phi)\bar{v} + (\Psi + \Phi)v[2\langle D\ell, D\bar{v}\rangle + \Phi\bar{v} + i\ell_t\bar{v}]$$

$$= 2(\Psi + \Phi)[\langle D\ell, Dv\rangle\bar{v} + \langle D\ell, D\bar{v}\rangle v] + 2\Phi(\Phi + \Psi)|v|^2 \tag{3.13}$$

$$= 2(\Psi + \Phi)\langle D\ell, D(|v|^2)\rangle + 2\Phi(\Psi + \Phi)|v|^2, \tag{3.14}$$

after a cancellation of $[i\ell_t(\Phi + \Psi)|v|^2]$ to arrive at (3.13). We next observe that

$$2 \text{ div}[(\Psi+\Phi)|v|^2 D\ell] = 2\langle D(\Psi+\Phi), D\ell\rangle|v|^2 + 2(\Psi+\Phi)\Delta\ell|v|^2 + 2(\Psi+\Phi)\langle D\ell, D(|v|^2)\rangle. \tag{3.15}$$

Extracting $2(\Psi + \Phi)\langle D\ell, D(|v|^2)\rangle$ from (3.15) and substituting into the right side of (3.14) yields (3.12), as desired.

Step 3: With reference to (3.10), in this step we shall establish that

$$
\begin{aligned}
(3.16)\quad I_1\bar{I}_3 + I_3\bar{I}_1 &= \Phi(I_1\bar{v} + \bar{I}_1 v) + i\Psi(v_t\bar{v} - \bar{v}_t v) + \operatorname{div}(\Psi D(|v|^2)) \\
&\quad - \langle D\Psi, D(|v|^2)\rangle - 2\Psi|Dv|^2 + 2\Psi(|D\ell|^2 - \Delta\ell - \Psi)|v|^2.
\end{aligned}
$$

Proof of (3.16). By invoking the definitions in (3.10), we compute

$$
\begin{aligned}
(3.17)\quad I_1\bar{I}_3 + I_3\bar{I}_1 &= I_1(\Psi + \Phi)\bar{v} + (\Psi + \Phi)v\bar{I}_1 = \Phi(I_1\bar{v} + \bar{I}_1 v) + I_1\Psi\bar{v} + \Psi v\bar{I}_1 \\
&= \Phi(I_1\bar{v} + \bar{I}_1 v) + [iv_t + \Delta v + (|D\ell|^2 - \Delta\ell)v - \Psi v]\Psi\bar{v} \\
&\quad + \Psi v[-i\bar{v}_t + \Delta\bar{v} + (|D\ell|^2 - \Delta\ell)\bar{v} - \Psi\bar{v}] \\
&= \Phi(I_1\bar{v} + \bar{I}_1 v) + i\Psi(v_t\bar{v} - \bar{v}_t v) + \Psi[\Delta v\bar{v} + \Delta\bar{v}v] \\
&\quad + 2\Psi[|D\ell|^2 - \Delta\ell - \Psi]|v|^2.
\end{aligned}
$$

On the other hand, using $\Delta(|v|^2) = \Delta(v \cdot \bar{v}) = \Delta v\bar{v} + \Delta\bar{v}v + 2|Dv|^2$, we have

$$
(3.18)\qquad \operatorname{div}(\Psi D(|v|^2)) = \Psi[\Delta v\bar{v} + \Delta\bar{v}v] + 2\Psi|Dv|^2 + \langle D\Psi, [\bar{v}Dv + vD\bar{v}]\rangle.
$$

Extracting $\Psi[\Delta v\bar{v} + \Delta\bar{v}v]$ from (3.18) and substituting into the right side of (3.17) yields (3.16), as desired.

Step 4. With reference to (3.10), in this Step 4 we shall establish that

$$
\begin{aligned}
(3.19\text{a})\qquad I_1\bar{I}_2 + I_2\bar{I}_1 &= \Phi(I_1\bar{v} + \bar{I}_1 v) + i\Big\{2[\langle D\ell, D\bar{v}\rangle v]_t - 2\operatorname{div}[D\ell(\bar{v}_t v)] \\
&\quad + 2\Delta\ell(v\bar{v}_t) + \operatorname{div}[\ell_t(vD\bar{v} - \bar{v}Dv)]\Big\} \\
&\quad + (\ell_t|v|^2)_t - \ell_{tt}|v|^2 + (C);
\end{aligned}
$$

where

$$
\begin{aligned}
(3.19\text{b})\qquad (C) &= (B) - 4D^2\ell(Dv, D\bar{v}) + 2\Delta\ell|Dv|^2 \\
&\quad - 2\left[|D\ell|^2 - \Delta\ell - \Psi\right]\Delta\ell|v|^2 \\
&\quad - 2\Big\{2D^2\ell(D\ell, D\ell) - \langle D(\Delta\ell), D\ell\rangle - \langle D\Psi, D\ell\rangle\Big\}|v|^2; \\
(3.19\text{c})\qquad (B) &= 2\operatorname{div}\Big\{\langle D\ell, D\bar{v}\rangle Dv + \langle D\ell, Dv\rangle D\bar{v} - |Dv|^2 D\ell \\
&\quad + \left[|D\ell|^2 - \Delta\ell - \Psi\right]|v|^2 D\ell\Big\}.
\end{aligned}
$$

Proof of (3.19). We shall accomplish this in a few steps.
(i) First, we show that

$$
\begin{aligned}
I_1\bar{I}_2 + I_2\bar{I}_1 &= \Phi(I_1\bar{v} + \bar{I}_1 v) + 2i\langle D\ell, v_tD\bar{v} - \bar{v}_tDv\rangle + 2\langle D\ell, \Delta vD\bar{v} + \Delta\bar{v}Dv\rangle \\
(3.20)\qquad &\quad + \ell_t(|v|^2)_t + 2\langle D\ell, D(|v|^2)\rangle[|D\ell|^2 - \Delta\ell - \Psi] + i\ell_t[v\Delta\bar{v} - \bar{v}\Delta v].
\end{aligned}
$$

In fact, to prove (3.20), we recall (3.10) and write

$$
\begin{aligned}
(3.21)\quad I_1\bar{I}_2 + I_2\bar{I}_1 &= I_1\left[2\langle D\ell, D\bar{v}\rangle + \Phi\bar{v} - i\ell_t\bar{v}\right] + \left[2\langle D\ell, Dv\rangle + \Phi v + i\ell_t v\right]\bar{I}_1 \\
&= \Phi(I_1\bar{v} + \bar{I}_1 v) \\
&\quad + \left[iv_t + \Delta v + (|D\ell|^2 - \Delta\ell)v - \Psi v\right]\left[2\langle D\ell, D\bar{v}\rangle - i\ell_t\bar{v}\right] \\
&\quad + \left[-i\bar{v}_t + \Delta\bar{v} + (|D\ell|^2 - \Delta\ell)\bar{v} - \Psi\bar{v}\right]\left[2\langle D\ell, Dv\rangle + i\ell_t v\right] \\
&= \Phi(I_1\bar{v} + \bar{I}_1 v) + 2iv_t\langle D\ell, D\bar{v}\rangle + iv_t(-i\ell_t\bar{v}) \\
&\quad - 2iv_t\langle D\ell, Dv\rangle - i\bar{v}_t(i\ell_t v) \\
&\quad + \left[\Delta v + (|D\ell|^2 - \Delta\ell)v - \Psi v\right]2\langle D\ell, D\bar{v}\rangle \\
&\quad + \left[\Delta\bar{v} + (|D\ell|^2 - \Delta\ell)\bar{v} - \Psi\bar{v}\right]2\langle D\ell, Dv\rangle \\
&\quad - i\ell_t\bar{v}\left[\Delta v + (|D\ell|^2 - \Delta\ell)v - \Psi v\right] \\
&\quad + i\ell_t v\left[\Delta\bar{v} + (|D\ell|^2 - \Delta\ell)\bar{v} - \Psi\bar{v}\right] \\
(3.22)\qquad\qquad &= \Phi(I_1\bar{v} + \bar{I}_1 v) + 2i\langle D\ell, v_t D\bar{v} - \bar{v}_t Dv\rangle \\
&\quad + 2\langle D\ell, \Delta v D\bar{v} + \Delta\bar{v} Dv\rangle + \ell_t(|v|^2)_t \\
&\quad + 2\langle D\ell, vD\bar{v} + \bar{v}Dv\rangle\left[|D\ell|^2 - \Delta\ell - \Psi\right] + i\ell_t\left[v\Delta\bar{v} - \bar{v}\Delta v\right].
\end{aligned}
$$

after a cancellation of terms, $i\ell_t(|D\ell|^2 - \Delta\ell)|v|^2$ and $i\ell_t\Psi|v|^2$. Then, (3.22) readily yields (3.20).

Remark 3.2. So far, the proof has followed closely—in a Riemannian setting using intrinsic, rather than coordinate-based quantities—the Euclidean proof in [**L-T-Z.2**, Lemma 3.1]. Henceforth, some tactical modifications will take place over [**L-T-Z.2**], with the common goal of establishing that the RHS of Eqn. (3.20) does coincide with the RHS of (3.19a); that is, we next seek to establish that

$$\text{RHS of (3.20)} = \text{RHS of (3.19a)}. \qquad \square$$

(ii) Next, we shall simplify the RHS of (3.19a) to match (3.20). First, just using repeatedly the identity: $\operatorname{div}(fX) = f \operatorname{div} X + X(f)$, $X(f) = \langle\nabla_g f, X\rangle_g$, $\nabla_g f = Df$, as well as $|Dv|^2 = Dv\cdot D\bar{v}$, etc., on the expression of (B) in (3.19c), we readily obtain

$$
\begin{aligned}
(3.23)\qquad (B) \;=\; 2\Big\{&\langle D\langle D\ell, D\bar{v}\rangle, Dv\rangle + \langle D\ell, D\bar{v}\rangle\Delta v + \langle D\langle D\ell, Dv\rangle, D\bar{v}\rangle \\
&+ \langle D\ell, Dv\rangle\Delta\bar{v} - \Delta\ell|Dv|^2 - \langle D\ell, D\langle Dv, D\bar{v}\rangle\rangle \\
&+ \langle D\langle D\ell, D\ell\rangle - D(\Delta\ell) - D\Psi, D\ell\rangle|v|^2 \\
&+ \left[\langle D\ell, D\ell\rangle - \Delta\ell - \Psi\right]\Delta\ell|v|^2 \\
&+ \left[\langle D\ell, D\ell\rangle - \Delta\ell - \Psi\right]\langle D\ell, vD\bar{v} + \bar{v}Dv\rangle\Big\},
\end{aligned}
$$

where $vD\bar{v}+\bar{v}Dv = D(|v|^2)$. Next, we substitute the expression (3.23) for (B) into the definition (3.19b) for (C) and obtain

$$
\begin{aligned}
(3.24)\quad (C) \quad = \quad & 2\langle D\langle D\ell, D\bar{v}\rangle, Dv\rangle + 2\langle D\ell, D\bar{v}\rangle\Delta v \\
& + 2\langle D\langle D\ell, Dv\rangle, D\bar{v}\rangle + 2\langle D\ell, Dv\rangle\Delta\bar{v} \\
& - \cancel{2\Delta\ell|Dv}|^2 - 2\langle D\ell, D\langle Dv, D\bar{v}\rangle\rangle \\
& + 2\langle D\langle D\ell, D\ell\rangle, D\ell\rangle|v|^2 - 2\cancel{\langle D(\Delta\ell), D\ell\rangle}|v|^2 - 2\cancel{\langle D\Psi, D\ell\rangle}|v|^2 \\
& + 2\left[\langle D\ell, D\ell\rangle \cancel{- \Delta\ell} - \Psi\right]\Delta\ell|v|^2 \\
& + 2\left[\langle D\ell, D\ell\rangle - \Delta\ell - \Psi\right]\langle D\ell, vD\bar{v}+\bar{v}Dv\rangle \\
& - 4D^2\ell(Dv, D\bar{v}) + \cancel{2\Delta\ell|Dv}|^2 - 2\left[|D\ell|^2 \cancel{- \Delta\ell} - \Psi\right]\Delta\ell|v|^2 \\
& - 4D^2\ell(D\ell, D\ell)|v|^2 + 2\cancel{\langle D(\Delta\ell, D\ell\rangle}|v|^2 + 2\cancel{\langle D\Psi, D\ell\rangle}|v|^2.
\end{aligned}
$$

After the cancellations noted above in (3.24), and after combining the 6^{th} term with the 11^{th} term; and the 7^{th} term with the 16^{th} term, we obtain

$$
\begin{aligned}
(3.25)\quad (C) \quad = \quad & 2\langle D\langle D\ell, D\bar{v}\rangle, Dv\rangle + 2\langle D\ell, D\bar{v}\rangle\Delta v + 2\langle D\langle D\ell, Dv\rangle, D\bar{v}\rangle \\
& + 2\langle D\ell, Dv\rangle\Delta\bar{v} - 2\langle D\ell, D(|Dv|^2)\rangle \\
& + 2\left[|D\ell|^2 - \Delta\ell - \Psi\right]\langle D\ell, vD\bar{v}+\bar{v}Dv\rangle + (D) - 4D^2\ell(Dv, D\bar{v}),
\end{aligned}
$$

where the term (D) is defined by

$$
(3.26)\qquad (D) \equiv 2\langle D\langle D\ell, D\ell\rangle, D\ell\rangle|v|^2 - 4D^2\ell(D\ell, D\ell)|v|^2 \equiv 0.
$$

We shall show below that $(D)=0$, after which we then obtain via (3.25), (3.26):

$$
\begin{aligned}
(3.27)\quad (C) \quad = \quad 2\Big\{ & \langle D\langle D\ell, D\bar{v}\rangle, Dv\rangle + \langle D\ell, D\bar{v}\rangle\Delta v + \langle D\langle D\ell, Dv\rangle, D\bar{v}\rangle \\
& + \langle D\ell, Dv\rangle\Delta\bar{v} - \langle D\ell, D\langle Dv, D\bar{v}\rangle\rangle \\
& + \left[\langle D\ell, D\ell\rangle - \Delta\ell - \Psi\right]\langle D\ell, vD\bar{v}+\bar{v}Dv\rangle \\
& - D^2\ell(Dv, D\bar{v}) - D^2\ell(D\bar{v}, Dv)\Big\}.
\end{aligned}
$$

We shall next establish several identities, one of which will imply $(D)=0$, as claimed in (3.26).

Claim. Let $X, Y, Z \in \mathcal{X}(M)$. Then:

(3.28a) (a) $\langle D\langle X, Y\rangle, Z\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle$;

(3.28b) (b) $\langle D\langle D\ell, D\ell\rangle, D\ell\rangle = 2D^2\ell(D\ell, D\ell)$;

(3.28c) (c) $\langle D\langle D\ell, D\bar{v}\rangle, Dv\rangle = D^2\ell(D\bar{v}, Dv) + D^2\bar{v}(D\ell, Dv)$;

(3.28d) (d) $\langle D\langle D\ell, Dv\rangle, D\bar{v}\rangle = D^2\ell(Dv, D\bar{v}) + D^2v(D\ell, D\bar{v})$;

(3.28e) (e) $\langle D\ell, D\langle Dv, D\bar{v}\rangle\rangle = D^2\bar{v}(D\ell, Dv) + D^2v(D\ell, D\bar{v})$.

Proof of Claim. (a) In fact, let $f = \langle X, Y\rangle$ so that $\langle Df, Z\rangle = Z(f)$ and

$$\langle D\langle X, Y\rangle, Z\rangle = \langle Df, Z\rangle = Z(f) = Z\langle X, Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle,$$

recalling, in the last step, the metric compatibility of the Levi-Civita connection [**Do.1**, p. 54]. Thus, (3.28a) is established.

(b) We take $X = Y = Z = D\ell$. Then, (3.28a) yields (3.28b) by the very definition of $D^2\ell(\,\cdot\,,\,\cdot\,)$, see (1.4).

(c) We take $X = D\ell$, $Y = D\bar{v}$, $Z = Dv$. Then, (3.28a)

$$\langle D\langle D\ell, D\bar{v}\rangle, Dv\rangle = \langle D_{Dv}(D\ell), D\bar{v}\rangle + \langle D\ell, D_{Dv}(D\bar{v})\rangle$$

yields (3.28c) by definition of the Hessian.

(d) We take $X = D\ell$, $Y = Dv$, $Z = D\bar{v}$.

(e) We take $X = Dv$, $Y = D\bar{v}$, $Z = D\ell$. Then (3.28a),

$$\langle D\langle Dv, D\bar{v}\rangle, D\ell\rangle = \langle D_{D\ell}(Dv), D\bar{v}\rangle + \langle Dv, D_{D\ell}(D\bar{v})\rangle$$

yields (3.28e).

Finally, we obtain at once that (3.28b) implies $(D) = 0$ in (3.26), as desired.
Substituting (3.28c-d-e) into (3.27) yields

$$\begin{aligned}(3.29)\qquad (C) \;=\; 2\Big\{ & \big[D^2\ell\cancel{(D\bar{v}, Dv)} + D^2\bar{v}\cancel{(D\ell, Dv)}\big] + \langle D\ell, D\bar{v}\rangle\Delta v \\ + \; & \big[D^2\ell\cancel{(Dv, D\bar{v})} + D^2v\cancel{(D\ell, D\bar{v})}\big] + \langle D\ell, Dv\rangle\Delta\bar{v} \\ - \; & \big[D^2\bar{v}\cancel{(D\ell, Dv)} + D^2v\cancel{(D\ell, D\bar{v})}\big] \\ + \; & \left[\langle D\ell, D\ell\rangle - \Delta\ell - \Psi\right]\langle D\ell, vD\bar{v} + \bar{v}Dv\rangle \\ - \; & D^2\ell\cancel{(Dv, D\bar{v})} - D^2\ell\cancel{(D\bar{v}, Dv)}\Big\}.\end{aligned}$$

After the cancellations noted above in (3.29), we finally obtain

$$\begin{aligned}(3.30)\qquad (C) \;=\; 2\Big\{ & \langle D\ell, D\bar{v}\rangle\Delta v + \langle D\ell, Dv\rangle\Delta\bar{v} \\ + \; & \left[\langle D\ell, D\ell\rangle - \Delta\ell - \Psi\right]\langle D\ell, vD\bar{v} + \bar{v}Dv\rangle\Big\}.\end{aligned}$$

Orientation. Our goal, as stated in Remark 3.1 and at the outset of Step (ii), is to establish that the RHS of Eqn. (3.20) does coincide with the RHS of (3.19a). Thus, if we compare the RHS of (3.20), already proved in Step (i), with the RHS of (3.19a), we see that the term $\Phi(I_1\bar{v} + \overline{I}_1 v)$ is in common. Having simplified, in (3.29) above, the expression of (C) defined on the RHS of (3.19a), we now address, in Step (iii) below, the term $\{\quad\}$ on the RHS of (3.19a) and rewrite it in a more suitable way to obtain the desired match with the RHS of (3.20), as stated in Remark 3.1. □

(iii) We consider three terms in the expression within $\{\quad\}$ on the RHS of (3.19a), and simplify them as follows; since $D(\ell_t) \equiv 0$ by assumption (3.2):

(iii1)

$$(3.31a)\qquad 2[\langle D\ell, D\bar{v}\rangle v]_t = 2\langle D\ell, D\bar{v}_t\rangle v + 2\langle D\ell, D\bar{v}\rangle v_t;$$

(iii2)

$$-2\ \mathrm{div}[D\ell(\bar{v}_t v)] = -2\Delta\ell(\bar{v}_t v) - 2\langle D\ell, D(\bar{v}_t v)\rangle = -2\Delta\ell(\bar{v}_t v) - 2\langle D\ell, D\bar{v}_t\rangle v - 2\langle D\ell, Dv\rangle v_t; \tag{3.31b}$$

(iii3)

$$\mathrm{div}[\ell_t v D\bar{v}] = \Delta\bar{v}(\ell_t v) + \langle D\bar{v}, Dv\rangle \ell_t; \tag{3.31c}$$

(iii4)

$$-\mathrm{div}[\ell_t \bar{v} Dv] = -\Delta v(\ell_t \bar{v}) + \langle Dv, D\bar{v}\rangle \ell_t. \tag{3.31d}$$

Then, summing up the terms in (3.31a-b-c-d) along with $2\Delta\ell(v\bar{v}_t)$, we obtain—after two cancellations—that the expression $\{\quad\}$ on the RHS of (3.19a) is simplified as follows:

$$\begin{aligned}&\Big\{2\left[\langle D\ell, D\bar{v}\rangle v\right]_t - 2\ \mathrm{div}[D\ell(v_t v)] + 2\Delta\ell(v\bar{v}_t) + \mathrm{div}\left[\ell_t(vD\bar{v} - \bar{v}Dv)\right]\Big\}\\ &\qquad = 2\langle D\ell, D\bar{v}\rangle v_t - 2\langle D\ell, Dv\rangle \bar{v}_t + \Delta\bar{v}(\ell_t v) - \Delta v(\ell_t v).\end{aligned} \tag{3.32}$$

Moreover, with reference to the two additional terms after $\{\quad\}$ in the RHS of (3.19a), we have

$$(\ell_t|v|^2)_t - \ell_{tt}|v|^2 = \ell_t(|v|^2)_t. \tag{3.33}$$

Finally, combining (3.32) in $\{\quad\}$, (3.33) and (3.30) in (C), we obtain returning to (3.19a),

$$\begin{aligned}&[\text{RHS of (3.19a)}] - \Phi(I_1\bar{v} + \overline{I}_1 v)\\
= \;& i\Big\{2\left[\langle D\ell, D\bar{v}\rangle v\right]_t - 2\ \mathrm{div}[D\ell(\bar{v}_t v)]\\
&+\ 2\Delta\ell(v\bar{v}_t) + \mathrm{div}\left[\ell_t(vD\bar{v} - \bar{v}Dv)\right]\}\\
&+ \left[(\ell_t|v|^2)_t - \ell_{tt}|v|^2\right] + (C)\\
= \;& i\Big\{2\langle D\ell, D\bar{v}\rangle v_t - 2\langle D\ell, Dv\rangle \bar{v}_t + \Delta\bar{v}(\ell_t v) - \Delta v(\ell_t \bar{v})\Big\}\\
&+ \ell_t(|v|^2)_t + 2\Big\{\langle D\ell, D\bar{v}\rangle \Delta v + \langle D\ell, Dv\rangle \Delta\bar{v}\\
&+\ \left[|D\ell|^2 - \Delta\ell - \Psi\right]\langle D\ell, D(|v|^2)\rangle\Big\}\\
= \;& [\text{RHS of (3.20)}] - \Phi(I_1\bar{v} + \overline{I}_1 v),\end{aligned} \tag{3.34}$$

and the sought-after identity in Remark 3.2 is established. Thus, identity (3.19a) in Step 4 has been verified.

Step 5. In this step, we shall establish that

$$\begin{aligned}|\theta(iw_t + \Delta w)|^2 \;&\geq\; |I_3|^2 - (I_1\bar{I}_2 + I_2\bar{I}_1) - (I_2\bar{I}_3 + I_3\bar{I}_2) + (I_1\bar{I}_3 + I_3\bar{I}_1)\\ &=\; X_1 + X_2 + X_3 + X_4.\end{aligned} \tag{3.35}$$

with

$$\begin{aligned}
(3.36a)\qquad X_1 &= \Big\{2(\Psi+\Delta\ell)(|D\ell|^2-\Delta\ell)+4D^2\ell(D\ell,D\ell)+2\langle D\Phi,D\ell\rangle \\
&\quad -2\langle D\ell,D(\Delta\ell)\rangle-\Psi^2-\Phi^2+2\Phi\Delta\ell+\ell_{tt}\Big\}|v|^2 \\
&\quad +2\Big\{D^2\ell(Dv,D\bar v)+D^2\ell(D\bar v,Dv)-(\Psi+\Delta\ell)|Dv|^2\Big\}
\end{aligned}$$

$$\begin{aligned}
(3.36b)\qquad X_2 &= -[\ell_t|v|^2]_t-2\ \mathrm{div}\Big[(|D\ell|^2-\Delta\ell+\Phi)|v|^2D\ell-\frac{\Psi}{2}D(|v|^2) \\
&\quad +\langle D\ell,D\bar v\rangle Dv+\langle D\ell,Dv\rangle D\bar v-D\ell|Dv|^2\Big]
\end{aligned}$$

$$(3.36c)\qquad X_3 = -\langle D\Psi,D(|v|^2)\rangle+i(\Psi+\Delta\ell)(v_t\bar v-\bar v_tv)$$

$$\begin{aligned}
(3.36d)\qquad X_4 &= -i\Big\{[2\langle D\ell,D\bar v\rangle v+\Delta\ell|v|^2]_t \\
&\quad +\ \ \mathrm{div}[\ell_t(vD\bar v-\bar vDv)-2D\ell(\bar v_tv)]\Big\},
\end{aligned}$$

where X_1, X_2, X_3, X_4 are all real valued.

Proof of (3.35): It is enough to substitute (3.10b) (for I_3), (3.19a), (3.12), (3.16) in the first line of (3.35): after cancelling some terms, and using identity (3.33), as well as $\Delta\ell(|v|^2)_t=(\Delta\ell|v|^2)_t$, due to assumption (3.2): $D(\ell_t)=0$, we obtain (3.35), as desired.

Step 6. Orientation. So far, all terms in estimate (3.35) are expressed in terms of the variable $v=\theta w$, rather than the original variable w. Beginning with this step, we return from the variable v to the variable w in terms under the time-derivative $\frac{\partial}{\partial t}$ and the space divergence div. This is the same strategy that was used in [**L-T-Z.1**], [**L-T-Z.2**]. More precisely, in our present Schrödinger case, we shall proceed as follows: (i) we shall leave X_1 unchanged until the very end of the proof; (ii) we shall pass from the variable v to the variable w in the terms X_2 and X_4; these will still yield—though now in the variable w—time-derivative terms $\frac{\partial}{\partial t}$ and space divergence div terms. This is accomplished in (3.57) below for X_2 and in (3.38) below for X_4. The time-derivative $\frac{\partial}{\partial t}$ terms will define $\frac{\partial}{\partial t}M$, see (3.4); while the div terms will contribute to the definition of div V, see (3.5); (iii) from X_3—or better from the lower bound on X_3—we shall extract one more term in div to complete the definition of div V in (3.5). This is accomplished in (   ) below. In conclusion: the full terms X_2 and X_4 as well as one component term of X_3 are those that contribute to $[-\frac{\partial M}{\partial t}+\mathrm{div}\ V]$ on the left side of (3.3). □

To return from v to w in selected terms, as identified in the above Orientation, we shall use the following identities:

$$(3.37)\quad \theta_t=\theta\ell_t;\ v_t=\theta_tw+\theta w_t=\theta(\ell_tw+w_t);\ D\theta=\theta D\ell,\ Dv=\theta[Dw+wD\ell].$$

Step 7. Here we shall show that X_4 expressed in terms of v in (3.36d), can be rewritten in terms of $w=\xi+i\eta$, $\xi=\mathrm{Re}\ w$, $\eta=\mathrm{Im}\ w$ as follows:

$$X_4 = \frac{\partial}{\partial t}\left\{2\theta^2\langle D\ell, \eta D\xi - \xi D\eta\rangle\right\} + \operatorname{div}\left\{2\theta^2[\ell_t(\eta D\xi - \xi D\eta) - D\ell(\xi_t\eta - \xi\eta_t)]\right\}. \tag{3.38}$$

Proof of (3.38). We recall the definition (3.36d) and by direct use of (3.37), we get

$$\begin{aligned} X_4 &= -i\left\{[2\langle D\ell, D\bar{v}\rangle v + \Delta\ell|v|^2]_t + \operatorname{div}[\ell_t(vD\bar{v} - \bar{v}Dv) - 2D\ell(\bar{v}_t v)]\right\} \\ &= -i\left\{[2\langle D\ell, D\bar{v}\rangle v]_t - 2\operatorname{div}[D\ell(\bar{v}_t v)]\right. \\ &\quad \left. + [\Delta\ell|v|^2]_t + \operatorname{div}[\ell_t(vD\bar{v} - \bar{v}Dv)]\right\} \\ &= -i\left\{1 + 2 + 3\right\}, \end{aligned} \tag{3.39}$$

after interchanging the order of the terms. Performing the indicated time-differentiations $[\quad]_t$ using $D(\ell_t) \equiv 0$, $\Delta\ell_t \equiv 0$ from assumption (3.2), we obtain after a cancellation of $2\langle \ell, D\bar{v}_t\rangle v$:

$$X_4 = -i\left\{2\langle D\ell, v_t D\bar{v} - \bar{v}_t Dv\rangle + \Delta\ell(v_t\bar{v} - v\bar{v}_t) + \operatorname{div}[\ell_t(vD\bar{v} - \bar{v}Dv)]\right\}. \tag{3.40}$$

Now we transform each of the three terms in (3.40) from v to w using (3.37):

$$\begin{aligned} 1 &\equiv 2\langle D\ell, v_t D\bar{v} - \bar{v}_t Dv\rangle \\ &= 2\theta^2\langle D\ell, (w_t D\bar{w} - \bar{w}_t Dw) + D\ell(\bar{w}w_t - w\bar{w}_t) + \ell_t(wD\bar{w} - \bar{w}Dw)\rangle, \end{aligned} \tag{3.41}$$

after regrouping and a cancellation of $D\ell(\ell_t|w|^2)$;

$$2 \equiv \Delta\ell(v_t\bar{v} - v\bar{v}_t) = \theta^2\Delta\ell(w_t\bar{w} - \bar{w}_t w), \tag{3.42}$$

after regrouping and a cancellation of $w\bar{w}\ell_t$;

$$3 \equiv \operatorname{div}[\ell_t(vD\bar{v} - \bar{v}Dv)] = \operatorname{div}[\theta^2\ell_t(wD\bar{w} - \bar{w}Dw)], \tag{3.43}$$

after a cancellation of $|w|^2 D\ell$.

Our next step is to use $w = \xi + i\eta$ in (3.41) for 1; (3.42) for 2; (3.43) for 3. As to term 1, after a cancellation of numerous terms ($\xi_t D\xi; \eta_t D\eta; D\ell[\xi\xi_t + \eta\eta_t]; \ell_t[\xi D\xi + \eta D\eta]$), and regrouping, we obtain

$$(-i)1 = 2\theta^2\langle D\ell, 2(\eta_t D\xi - \xi_t D\eta) + 2D\ell(\xi\eta_t - \eta\xi_t) + 2\ell_t(\eta D\xi - \xi D\eta)\rangle. \tag{3.44}$$

As to the term 2, after a cancellation of $\Delta\ell[\xi\xi_t + \eta\eta_t]$ and regrouping, we obtain

$$(-i)2 = 2\theta^2\Delta\ell(\xi\eta_t - \eta\xi_t). \tag{3.45}$$

As to term 3, after a cancellation of $[\xi D\xi + \eta D\eta]$ and regrouping, we obtain

$$(-i)3 = 2\operatorname{div}[\theta^2\ell_t(\eta D\xi - \xi D\eta)]. \tag{3.46}$$

Summing up (3.44), (3.45), (3.46) on the RHS of (3.39), then yields

$$
\begin{aligned}
X_4 &= 2\theta^2\langle D\ell, 2(\eta_t D\xi - \xi_t D\eta) + 2\ell_t(\eta D\xi - \xi D\eta) + 2D\ell(\xi\eta_t - \eta\xi_t)\rangle \\
&\quad + 2\theta^2\Delta\ell[(\eta_t\xi - \xi_t\eta)] + 2\,\mathrm{div}[\theta^2\ell_t(\eta D\xi - \xi D\eta)] \qquad (3.47)\\
&= 2\theta^2\left[2\langle D\ell, \eta_t D\xi - \xi_t D\eta\rangle + 2\ell_t\langle D\ell, \eta D\xi - \xi D\eta\rangle\right. \qquad (3.48)\\
&\quad \left. + \; (2|D\ell|^2 + \Delta\ell)(\xi\eta_t - \eta\xi_t))\right] + 2\,\mathrm{div}[\theta^2\ell_t(\eta D\xi - \xi D\eta)].
\end{aligned}
$$

We now compute the right side of (3.38). Recalling from (3.2) that $D(\ell_t) = 0$ and invoking (3.37), we obtain

(3.49) RHS of (3.38)

$$
\begin{aligned}
&= \frac{\partial}{\partial t}\left\{2\theta^2\langle D\ell, \eta D\xi - \xi D\eta\rangle\right\} \\
&\quad + \mathrm{div}\left\{2\theta^2[\ell_t(\eta D\xi - \xi D\eta) - D\ell(\xi_t\eta - \xi\eta_t)]\right\} \\
&= 4\theta^2\ell_t\langle D\ell, \eta D\xi - \xi D\eta\rangle + 2\theta^2\langle D\ell, \eta_t D\xi + \eta D\xi_t - \xi_t D\eta - \xi D\eta_t\rangle \\
&\quad - 4\theta^2|D\ell|^2(\xi_t\eta - \xi\eta_t) - 2\theta^2\Delta\ell(\xi_t\eta - \xi\eta_t) \\
&\quad - 2\theta^2\langle D\ell, \eta D\xi_t + \xi_t D\eta - \eta_t D\xi - \xi D\eta_t\rangle \\
&\quad + \mathrm{div}[2\theta^2\ell_t(\eta D\xi - \xi D\eta)] \\
&= 4\theta^2[\ell_t\langle D\ell, \eta D\xi - \xi D\eta\rangle + \langle D\ell, \eta_t D\xi - \xi_t D\eta\rangle] \\
&\quad -2\theta^2[2|D\ell|^2 + \Delta\ell](\xi_t\eta - \xi\eta_t) + \;\mathrm{div}[2\theta^2\ell_t(\eta D\xi - \xi D\eta)] \\
&= \text{RHS of (3.48)} = X_4.
\end{aligned}
$$

Thus, (3.49) proves (3.38), as desired.

Step 8. We next consider X_3 in (3.36c) and rewrite it in terms of w, recalling $v = \theta w$ as well as (3.37):

$$
\begin{aligned}
X_3 &= -\langle D\Psi, D(|v|^2)\rangle + i(\Psi + \Delta l)(v_t\bar{v} - \bar{v}_t v) \qquad (3.50)\\
&= i(\Psi + \Delta\ell)\theta^2[(w_t + \ell_t w)\bar{w} - (\bar{w}_t + \ell_t\bar{w})w] - \langle D\Psi, D(|v|^2)\rangle \\
&= (\Psi + \Delta\ell)\theta^2[(iw_t)\bar{w} + \overline{(iw_t)}w] - \langle D\Psi, D(|v|^2)\rangle \\
&= \theta^2(\Psi + \Delta\ell)[(\mathcal{P}w)\bar{w} + w(\overline{\mathcal{P}w}) - \Delta w\bar{w} - \Delta\bar{w}w] - \langle D\Psi, D(|v|^2)\rangle,
\end{aligned}
$$

where in the last step in (3.50) we have invoked the Schrödinger operator $\mathcal{P}w = iw + \Delta w$, from (1.1). Next, we shall use the identity $\Delta(w\bar{w}) = \Delta w\bar{w} + \Delta\bar{w}w + 2|Dw|^2$, to rewrite X_3 in (3.50) as follows:

$$
\begin{aligned}
X_3 = \theta^2(\Psi + \Delta\ell)[(\mathcal{P}w)\bar{w} + w(\overline{\mathcal{P}w})] + \theta^2(\Psi + \Delta\ell)[2|Dw|^2 \\
- \Delta(|w|^2) - \langle D\Psi, D(|v|^2)\rangle.
\end{aligned}
\tag{3.51}
$$

Next, on the terms $|Dw|^2$ and $\Delta(|w|^2)$ in (3.51), we use the following identities:

$$
\theta^2|Dw|^2 = |Dv - vD\ell|^2 = |Dv|^2 + |D\ell|^2|v|^2 - \langle D\ell, D(|v|^2)\rangle; \tag{3.52}
$$

$$\begin{aligned}(3.53)\qquad \operatorname{div}[\theta^2(\Psi+\Delta\ell)D(|w|^2)] &= \theta^2(\Psi+\Delta\ell)\Delta(|w|^2)\\ &+ \langle D(|\theta|^2)(\Psi+\Delta\ell), D(|w|^2)\rangle\\ &+ \langle \theta^2 D(\Psi+\Delta\ell), D(|w|^2)\rangle,\end{aligned}$$

$D(|\theta|^2)=2\theta^2 D\ell$ by (3.2), whereby then (3.51) is rewritten as

$$\begin{aligned}(3.54)\qquad X_3 &= \theta^2(\Psi+\Delta\ell)[(\mathcal{P}w)\bar{w}+w(\overline{\mathcal{P}w})]\\ &+ 2(\Psi+\Delta\ell)\Big[|Dv|^2+|D\ell|^2|v|^2-\langle D\ell, D(|v|^2)\rangle\Big]\\ &- \operatorname{div}[\theta^2(\Psi+\Delta\ell)D(|w|^2)]+2\theta^2(\Psi+\Delta\ell)\langle D\ell, D(|w|^2)\rangle\\ &+ \theta^2\langle D\Psi+D(\Delta\ell), D(|w|^2)\rangle-\langle D\Psi, D(|v|^2)\rangle.\end{aligned}$$

Next, in (3.54), we use twice, in the 4^{th} and 5^{th} term, the identity $\theta^2 D(|w|^2)=D(|v|^2)-2|v|^2D\ell$; and obtain

$$\begin{aligned}(3.55)\qquad X_3 &= \theta^2(\Psi+\Delta\ell)[(\mathcal{P}w)\bar{w}+w(\overline{\mathcal{P}w})]-\operatorname{div}[\theta^2(\Psi+\Delta\ell)D(|w|^2)]\\ &+ 2(\Psi+\Delta\ell)[|Dv|^2+|D\ell|^2|v|^2-\langle D\ell, D(|v|^2)\rangle\\ &+ \langle D\ell, D(|v|^2)\rangle-2|v|^2D\ell)]\\ &+ (D\Psi+D(\Delta\ell), D(|v|^2)-2|v|^2D\ell-(D\Psi, D(|v|^2)\rangle\\ (3.56)\qquad &= \theta^2(\Psi+\Delta\ell)[(\mathcal{P}w)\bar{w}+w(\overline{\mathcal{P}w})]-\operatorname{div}[\theta^2(\Psi+\Delta\ell)D(|w|^2)]\\ &+ 2(\Psi+\Delta\ell)[|Dv|^2-|D\ell|^2-|D\ell|^2|v|^2]\\ &- 2(D\Psi+D(\Delta\ell), D\ell)|v|^2+(D(\Delta\ell), D(|v|^2)).\end{aligned}$$

To pass from (3.55) to (3.56) we have cancelled two terms $\langle D\ell, D(|v|^2)\rangle$ and $\langle D\Psi, D(|v|^2)\rangle$.

Step 9. In this step, we return to the term X_2 in (3.36b), use here $Dv=\theta[Dw+wD\ell]$ from (3.37), as well as $D(\theta^2)=2\theta^2 D\ell$ from (3.2), and obtain:

$$\begin{aligned}(3.57)\qquad X_2 &= -[\ell_t|v|^2]_t-2\operatorname{div}\Bigg[(|D\ell|^2-\Delta\ell+\Phi)D\ell|v|^2-\frac{\Psi}{2}D(|v|^2)\\ &+ \langle D\ell, D\bar{v}\rangle Dv+\langle D\ell, Dv\rangle D\bar{v}-D\ell|Dv|^2\Bigg]\\ &= -[\ell_t\theta^2|w|^2]_t-2\operatorname{div}\Bigg[\theta^2(|D\ell|^2-\Delta\ell+\Phi)D\ell|w|^2\\ &- \theta^2\frac{\Psi}{2}[D(|w|^2)+2|w|^2D\ell]\\ &+ \theta^2\Big\{\langle D\ell, D\bar{w}+\bar{w}D\ell\rangle(Dw+wD\ell)+\langle D\ell, Dw+wD\ell\rangle(D\bar{w}+\bar{w}D\ell)\\ &- D\ell\langle Dw+wD\ell, D\bar{w}+\bar{w}D\ell\rangle\Big\}\Bigg]\\ &= -[\ell_t\theta^2|w|^2]_t-2\operatorname{div}\Bigg[\theta^2(|D\ell|^2-\Delta\ell+\Phi-\Psi)D\ell|w|^2-\theta^2\frac{\Psi}{2}D(|w|^2)\\ &+ \theta^2\Big\{\langle D\ell, D\bar{w}\rangle Dw+\langle D\ell, Dw\rangle D\bar{w}+|D\ell|^2D(|w|^2)\end{aligned}$$

$$
\begin{aligned}
&+\ |D\ell|^2|w|^2 D\ell - |Dw|^2 D\ell\Big\}\Big]\\
&=\ [-\ell_t\theta^2|w|^2]_t - 2\ \mathrm{div}\Big[\theta^2(2|D\ell|^2 - \Delta\ell + \Phi - \Psi)D\ell|w|^2\\
&+\ \theta^2\left(|D\ell|^2 - \frac{\Psi}{2}\right)D(|w|^2)\\
&+\ \theta^2\Big\{\langle D\ell, D\bar{w}\rangle Dw + \langle D\ell, Dw\rangle D\bar{w} - |Dw|^2 D\ell\Big\}\Big].
\end{aligned}
$$

Step 10. We return to inequality (3.35), rewritten here for convenience

$$|\theta(iw_t + \Delta w)|^2 \geq X_1 + X_2 + X_3 + X_4. \tag{3.58}$$

Now we leave X_1 untouched; and substitute (3.38) for X_4, (3.56) for X_3, (3.57) for X_2, in (3.58). We obtain:

(3.59)

$$
\begin{aligned}
&|\theta(iw_t + \Delta w)|^2\\
\geq\ & X_1 + \Bigg[\Bigg[[-\ell_t\theta^2|w|^2]_t - 2\ \mathrm{div}\Big[\theta^2(2|D\ell|^2 - \Delta\ell + \Phi - \Psi)D\ell|w|^2\\
+\ & \theta^2\left(|D\ell|^2 - \frac{\Psi}{2}\right)D(|w|^2) + \theta^2\Big\{\langle D\ell, D\bar{w}\rangle Dw + \langle D\ell, Dw\rangle D\bar{w} - D\ell|Dw|^2\Big\}\Big]\Bigg]\Bigg]\\
+\ & \Bigg\{\Bigg\{\theta^2(\Psi + \Delta\ell)[(\mathcal{P}w)\bar{w} + w(\overline{\mathcal{P}w})] - \mathrm{div}\Big[\theta^2(\Psi + \Delta\ell)D(|w|^2)]\\
+\ & 2(\Psi + \Delta\ell)[|Dv|^2 - |D\ell|^2|v|^2\Big] - 2\langle D\Psi + D(\Delta\ell), D\ell\rangle|v|^2 + \langle D(\Delta\ell), D(|v|^2)\rangle\Bigg\}\Bigg\}\\
+\ & \Bigg(\Bigg(\frac{\partial}{\partial t}\{2\theta^2(D\ell, \eta D\xi - \xi D\eta)\} + \mathrm{div}\{2\theta^2[\ell_t(\eta D\xi - \xi D\eta) - D\ell(\xi_t\eta - \xi\eta_t)]\}\Bigg)\Bigg),
\end{aligned}
$$

where we have used the following notation: $X_2 = [[\quad]]$; $X_3 = \{\{\quad\}\}$; $X_4 = ((\quad))$. Thus, we see that the term $\mathrm{div}[\theta^2\Psi D(|w|^2)]$ in X_2 cancels out with the term of opposite sign in X_3. We obtain from (3.58),

$$
\begin{aligned}
&|\theta(iw_t + \Delta w)|^2 \qquad (3.60)\\
&\geq\ \text{RHS of (3.58)}\\
&=\ X_1 + \theta^2(\Psi + \Delta\ell)[(\mathcal{P}w)\bar{w} + w(\overline{\mathcal{P}w})] + \langle D(\Delta\ell), D(|v|^2)\rangle \qquad (3.61)\\
&\quad + 2(\Psi + \Delta\ell)[|Dv|^2 - |D\ell|^2|v|^2] - 2\langle D\Psi + D(\Delta\ell), D\ell\rangle|v|^2\\
&\quad + \frac{\partial}{\partial t}\Big\{2\theta^2\langle D\ell, \eta D\xi - \xi D\eta\rangle - \ell_t\theta^2|w|^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \operatorname{div} \Big\{ 2\theta^2 \Big[(2|D\ell|^2 - \Delta\ell + \Phi - \Psi) D\ell |w|^2 + \left(|D\ell|^2 + \frac{1}{2}\Delta\ell \right) D(|w|^2) \\
&+ \langle D\ell, D\bar{w} \rangle Dw + \langle D\ell, Dw \rangle D\bar{w} - D\ell |Dw|^2 \\
&+ \ \ell_t(\eta D\xi - \xi D\eta) - D\ell(\xi_t \eta - \xi \eta_t) \Big] \Big\} \\
= \quad & X_1 + \theta^2 (\Psi + \Delta\ell)[(\mathcal{P}w)\bar{w} + w(\overline{\mathcal{P}w})] + \langle D(\Delta\ell), D(|v|^2) \rangle \\
&+ 2(\Psi + \Delta\ell)[|Dv|^2 - |D\ell|^2 |v|^2] - 2\langle D\Psi + D(\Delta\ell), D\ell \rangle |v|^2 \\
&+ \frac{\partial}{\partial t} M - \operatorname{div} V,
\end{aligned}
$$

recalling (3.4) for M and (3.5) for V.

Remark 3.3. So far, in handling X_1, X_2, X_3, X_4, we have managed to keep "identities" all along. Thus, (3.61) above is a bit sharper than its counterpart in the Euclidean case [**L-T-Z.2**, Eqn. (3.61)], where, instead, the term X_3 was estimated by an inequality. $\square$

Step 11. For any $\epsilon > 0$, we note the following two inequalities:

$$
(\Psi + \Delta\ell)[(\mathcal{P}w)\bar{w} + w(\overline{\mathcal{P}w})] \ \geq \ -\epsilon |\Psi + \Delta\ell|^2 |w|^2 - \frac{1}{\epsilon} |\mathcal{P}w|^2 \tag{3.62a}
$$

$$
\langle D(\Delta\ell), D(|v|^2) \rangle \ \geq \ -\epsilon |Dv|^2 - \frac{1}{\epsilon} |D(\Delta\ell)|^2 |v|^2 \tag{3.62b}
$$

Next, substituting inequalities (3.62a–b) on the RHS of (3.60), and recalling here the definition of X_1 in (3.36a), we obtain

$$
\begin{aligned}
&\theta^2 |iw_t + \Delta w|^2 \\
\geq \quad & \langle\!\langle \big\{ 2(\Psi + \Delta\ell)(\cancel{|D\ell|^2 -} \Delta\ell) + 4D^2\ell(D\ell, D\ell) + 2(D\Phi, D\ell) \\
&- \ \ 2(D\ell, D(\Delta\ell)) - \Psi^2 - \Phi^2 + 2\Phi\Delta\ell + l_{tt} \big\} |v|^2 \\
&+ \ \ 2\big\{ D^2\ell(Dv, D\bar{v}) + D^2\ell(D\bar{v}, Dv) - (\Psi + \cancel{\Delta\ell)|Dv|^2} \big\} \rangle\!\rangle \\
&+ 2(\Psi + \cancel{\Delta\ell)[|Dv|^2} - \cancel{|D\ell|^2|v|^2}] - 2(D\Psi + D(\Delta\ell), D\ell)|v|^2 \\
&- \epsilon |\Psi + \Delta\ell|^2 |v|^2 - \epsilon |Dv|^2 - \frac{1}{\epsilon} |D(\Delta\ell)|^2 |v|^2 \\
&- \frac{\theta^2}{\epsilon} |\mathcal{P}w|^2 + \frac{\partial}{\partial t} M - \operatorname{div} V,
\end{aligned} \tag{3.63}
$$

where we have used the notation $X_1 = \langle\!\langle \quad \rangle\!\rangle$, and we have noted two cancellations. We regroup the remaining terms, we recall $\mathcal{P}w = iw_t + \Delta w$ from (1.1), we use the identity

$$
\begin{aligned}
&2\langle D\Phi, D\ell \rangle - 2\langle D\Psi, D\ell \rangle - 4\langle D\ell, D(\Delta\ell) \rangle \\
&\quad = \quad 2\langle D\ell, D(\Phi - \Delta\ell) \rangle + 2\langle D\ell, D(\Psi + \Delta\ell) \rangle,
\end{aligned} \tag{3.64}
$$

and we thus obtain

$$\begin{aligned}(3.65)\quad &\left(1+\frac{1}{\epsilon}\right)\theta^2|iw_t+\Delta w|^2-\frac{\partial M}{\partial t}+\operatorname{div} V\\ &\geq \Big\{-2(\Psi+\Delta\ell)\Delta\ell+4D^2\ell(D\ell,D\ell)\\ &\quad+2\langle D\ell,D(\Phi-\Delta\ell)\rangle+2\langle D\ell,D(\Psi+\Delta\ell)\rangle-\epsilon|\Psi+\Delta\ell|^2\\ &\quad-\frac{1}{\epsilon}|D(\Delta\ell)|^2-\Psi^2-\Phi^2+2\Phi\Delta\ell+\ell_{tt}\Big\}|v|^2\\ &\quad+2\left\{D^2\ell(Dv,D\bar{v})+D^2\ell(D\bar{v},Dv)\right\}-\epsilon|Dv|^2,\end{aligned}$$

and (3.65) establishes (3.3), as desired. The proof of Lemma 3.1 is complete. □

4. A Basic Pointwise Inequality

We now make suitable choices of the functions $\ell(t,x)$, $\Psi(t,x)$, and $\Phi(t,x)$ occurring in Lemma 3.1, precisely as in [**L-T-Z.2**, Theorem 4.1].

THEOREM 4.1. *Let*

$$w(t,x)\in \mathbf{C}^2(\mathbb{R}_t\times\overline{\Omega};\mathbb{C});\ \ d(x)\in C^3(\overline{\Omega};\mathbb{R}) \tag{4.1}$$

be two given functions, w being complex-valued, while d being real-valued. [At this stage, w and d need not be the solution of Eqn. (1.1) and the function provided by assumptions (A.1), (A.2), respectively.] If τ is a positive parameter, we select the function $\ell(t,x)$, $\Psi(t,x)$ and $\Phi(t,x)$ in Lemma 3.1 as follows:

$$\ell(t,x)=\tau\left[d(x)-C\left(t-\frac{T}{2}\right)^2\right]\equiv\tau\varphi(t,x)\in C^3(\mathbb{R}_+\times\overline{\Omega};\mathbb{R}); \tag{4.2}$$

$$\Psi(t,x)\ \equiv\ -\Delta\ell(t,x)\in C^1(\mathbb{R}_+\times\overline{\Omega};\mathbb{R}), \tag{4.3a}$$

$$\textit{either}\ \Phi(t,x)\ \equiv\ \Delta\ell(t,x);\ \ \textit{or else}\ \Phi(t,x)\equiv 0, \tag{4.3b}$$

so that the required assumption $D(\ell_t)\equiv 0$ in (3.2) is satisfied. The function $\varphi(t,x)$ in (4.2) is defined consistently with (1.6a). In particular, $T>0$ is arbitrary, while the constant c appearing in (4.2) is selected as in (1.6b).

(a) Then, with the above choices, Lemma 3.1 specializes as follows:

$$\begin{aligned}&D\ell=\tau Dd;\ |D\ell|^2=\tau^2|Dd|^2;\ D(\ell_t)\equiv 0;\ \Delta\ell=\tau\Delta d;\\ &\ell_t=-2c\tau\left(t-\frac{T}{2}\right);\ \ell_{tt}=-2c\tau;\end{aligned} \tag{4.4}$$

$$4D^2\ell(D\ell,D\ell)=4\tau^3D^2d(Dd,Dd), \tag{4.5}$$

where $D^2\ell(\,\cdot\,,\,\cdot\,)$ and $D^2d(\,\cdot\,,\,\cdot\,)$ are, respectively, the Hessian of ℓ and d (a two-order tensor);

$$-(\Psi^2+\Phi^2)+2\Phi\Delta\ell=-(\Phi-\Delta\ell)^2\quad \textit{for}\ \Psi=-\Delta\ell. \tag{4.6}$$

Moreover, still with $\Psi = -\Delta\ell$, *we obtain*

$$2\langle D[\Phi - \Delta\ell], D\ell\rangle - (\Psi^2 + \Phi^2) + 2\Phi\Delta\ell$$
$$= 2\langle D[\Phi - \Delta\ell], D\ell\rangle - (\Phi - \Delta\ell)^2, \ \ \Psi = -\Delta\ell$$

$$\begin{aligned} &(4.7a) \\ &(4.7b) \end{aligned} \qquad = \begin{cases} 0, & \textit{if } \Phi = \Delta\ell \\ -\tau^2[2\langle D(\Delta d), Dd\rangle + (\Delta d)^2], & \textit{if } \Phi \equiv 0, \end{cases}$$

according to the two choices in (4.3b).

(b) As a consequence of (4.2)–(4.7), the pointwise estimate (3.3) of Lemma 3.1 becomes as follows:

(b$_1$) *Under the choice* $\Psi = -\Delta\ell$, *we obtain from (3.3):*

$$(4.8) \qquad \left(1 + \frac{1}{\epsilon}\right)\theta^2[iw_t + \Delta w]^2 - \frac{\partial\mathcal{M}}{\partial t} + \textit{div}\ V$$
$$\begin{aligned} \geq\ & 2\left\{D^2\ell(Dv, D\bar{v}) + D^2\ell(D\bar{v}, Dv)\right\} - \epsilon|Dv|^2 \\ +\ & \Big\{4D^2\ell(D\ell, D\ell) + 2\langle D[\Phi - \Delta\ell], D\ell\rangle \\ -\ & (\Phi - \Delta\ell)^2 + \ell_{tt} - \frac{1}{2}|D(\Delta\ell)|^2\Big\}\,|v|^2 \end{aligned}$$
$$\begin{aligned} (4.9) \qquad =\ & 2\tau\left\{D^2d(Dv, D\bar{v}) + D^2d(D\bar{v}, Dv)\right\} - \epsilon|Dv|^2 \\ +\ & \Big\{4\tau^3D^2d(Dd, Dd) + 2\langle D[\Phi - \Delta\ell], D\ell\rangle \\ -\ & (\Phi - \Delta\ell)^2 - 2c\tau - \frac{\tau^2}{\epsilon}|D(\Delta d)|^2\Big\}\,|v|^2. \end{aligned}$$

(b$_2$) *Either with the further choice* $\Phi = 0$ *made in (4.9) (whereby* $(\Phi - \Delta\ell)^2 = \tau^2(\Delta d)^2$*), or else with the further choice* $\Phi = \Delta\ell$ *made in (4.9), we likewise obtain in both cases from (4.9) the final pointwise estimate:*

$$(4.10) \qquad \left(1 + \frac{1}{\epsilon}\right)\theta^2|iw_t + \Delta w|^2 - \frac{\partial\mathcal{M}}{\partial t} + \textit{div}\ V$$
$$\begin{aligned} \geq\ & 2\tau\left\{D^2d(Dv, D\bar{v}) + D^2d(D\bar{v}, Dv)\right\} - \epsilon|Dv|^2 \\ +\ & \left\{4\tau^3D^2d(Dd, Dd) + \mathcal{O}(\tau^2)\right\}\theta^2|w|^2, \quad \forall\ x, t \in Q, \end{aligned}$$

$v = \theta w$, *which is valid for the choices* $\Psi = -\Delta\ell$, *and either* $\Phi \equiv 0$ *or else* $\Phi = \Delta\ell$, *where the constant in* $\mathcal{O}$ *depends on* d, c, ϵ.

(c) Moreover, only under the choice in (4.2) for ℓ *and in (4.3a) for* $\Psi = -\Delta\ell$, *and leaving* Φ *uncommitted, the vector field* V *in (3.5) specializes to the following*

expression:

$$
\begin{aligned}
(4.11)\quad V = V(w) \quad = \quad & 2\theta^2 \Big\{ [2\tau^2|Dd|^2 + \Phi]\tau Dd|w|^2 + (\ell_t\eta)D\xi - (\ell_t\xi)D\eta \\
& - \tau Dd(\xi_t\eta - \xi\eta_t) + [\tau^2|Dd|^2 + \frac{\tau}{2}\Delta d]D(|w|^2) \\
& + \tau\langle Dd, D\bar{w}\rangle Dw + \tau\langle Dd, Dw\rangle D\bar{w} - \tau Dd|Dw|^2 \Big\}.
\end{aligned}
$$

Proof. The proof is a direct verification. The choice $\Psi + \Delta\ell \equiv 0$ causes the vanishing of three terms in (3.3) for the coefficient of $|v|^2$; the vanishing of one term in (3.3) for the coefficient of $|Dv|^2$; and the vanishing of one term in the coefficient of $|w|^2$ in (3.5) for V. The remaining terms in (3.3) then yield (4.8) and next (4.9), also by virtue of the obvious identities (4.4)–(4.7). Note that $\mathcal{O}(\tau^2)$ comes from the term $|D(\Delta\ell)|^2$, regardless of the choice of Φ. The verification of (4.11) is immediate. □

The pointwise estimate of interest, given in Corollary 4.2 below, is then obtained for the choice of the function $d(x) \in C^3(\overline{\Omega})$ coming from assumptions (A.1) = (1.4) and (A.2) = (1.5).

COROLLARY 4.2. *Let $d(x) \in C^3(\overline{\Omega})$ satisfy assumption (A.1) = (1.4) and (A.2) = (1.7). Define then, ℓ, Ψ, and Φ as in (4.2), (4.3), with constant $c > 0$ in (4.2) selected as in (1.6b) for $T > 0$ arbitrary. Let $w \in C^2(\mathbb{R}_t \times \Omega; \mathbb{C})$ as in (4.1).*

(i) Then, with these choices, inequality (4.10) specializes as follows: for any $\epsilon > 0$ small, we have the following estimate for sufficiently large τ:

$$
\begin{aligned}
(4.12)\quad & \left(1 + \frac{1}{\epsilon}\right)\theta^2|iw_t + \Delta w|^2 - \frac{\partial \mathcal{M}}{\partial t} + div\ V \\
& \geq \ [4\tau\rho - \epsilon]|Dv|^2 + [4\tau^3\rho|Dd|^2 + \mathcal{O}(\tau^2)]\theta^2|w|^2 \\
(4.13)\quad & \geq \ \delta_0\left[2\tau\rho - \frac{\epsilon}{2}\right]\theta^2|Dw|^2 + [2\tau^3\rho p^2(1-\delta_0) + \mathcal{O}(\tau^2)]\theta^2|w|^2,
\end{aligned}
$$

for some $1 > \delta_0 > 0$, where the constant in $\mathcal{O}$ depends on d, c, ϵ.

(ii) Moreover, for future use below, we note here that on the boundary $\Gamma = \partial\Omega$ with outward unit normal field $\nu(x)$ along Γ, the vector field V in (3.5) yields the following identity via the divergence theorem [**Le.1**, p. 43]*, where $D\ell = \tau Dd$, and where Φ is left uncommitted:*

$$
\begin{aligned}
(4.14)\qquad \int_\Omega div\ V\, d\Omega \ &= \ \int_\Gamma \langle V, \nu\rangle d\Gamma \\
\textit{(by (3.5))}\qquad &= \ 2\int_\Gamma \theta^2\left[2\tau^2|Dd|^2 + \Phi\right]\tau|w|^2\langle Dd,\nu\rangle d\Gamma \\
& \quad + 2\int_\Gamma \theta^2\ell_t[\eta\langle D\xi,\nu\rangle - \xi\langle D\eta,\nu\rangle]d\Gamma \\
& \quad - 2\int_\Gamma \theta^2[\xi_t\eta - \xi\eta_t]\tau\langle Dd,\nu\rangle d\Gamma
\end{aligned}
$$

$$+ \int_\Gamma \theta^2 \Big[2\tau^2|Dd|^2 - \tau\Delta d\Big]\Big[\bar{w}\langle Dw,\nu\rangle + w\langle D\bar{w},\nu\rangle\Big]\,d\Gamma$$

$$+ 2\int_\Gamma \theta^2\tau\,\langle Dd, [D\bar{w}\langle Dw,\nu\rangle + Dw\langle D\bar{w},\nu\rangle]\rangle\,d\Gamma$$

$$- 2\int_\Gamma \theta^2|Dw|^2\tau\langle Dd,\nu\rangle d\Gamma.$$

(iii) Finally, as for the $\mathcal{M}$-term in (3.4), we likewise note for future reference that

$$\int_\Omega\int_0^T \frac{\partial\mathcal{M}}{\partial t}dt\,d\Omega = \left[\int_\Omega \mathcal{M}d\Omega\right]_0^T \le \tau C_{d,T}\left[\int_\Omega e^{2\tau\varphi}[|Dw|^2+|w|^2]d\Omega\right]_0^T \tag{4.15}$$

$$\le (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)], \tag{4.16}$$

where $\mathbb{E}(t) = \|w(t)\|^2_{H^1(\Omega)}$ as defined in (1.9).

Proof. (i) Inequality (4.12) follows at once from estimate (4.10) by direct use of assumptions (A.1) = (1.4) and (A.2) = (1.5). Next, by (3.2): $Dv = \theta Dw + wD\theta$, $D\theta = \theta D\ell = \theta\tau Dd$, and hence

$$2|Dv|^2 \ge \theta^2|Dw|^2 - 2\tau^2|Dd|^2\theta^2|w|^2. \tag{4.17}$$

Then, with reference to the RHS of (4.12), we estimate with $0 < \delta_0 < 1$:

$$\begin{aligned} [4\tau\rho-\epsilon]|Dv|^2 &> [4\tau\rho-\epsilon]\delta_0|Dv|^2 \\ \text{(by (4.17))} &\ge [4\tau\rho-\epsilon]\delta_0\left[\frac{\theta^2}{2}|Dw|^2 - \tau^2|Dd|^2\theta^2|w|^2\right] \\ &= \left[2\tau\rho - \frac{\epsilon}{2}\right]\delta_0\theta^2|Dw|^2 - 4\tau^3\rho\delta_0|Dd|^2\theta^2|w|^2 + \epsilon\delta_0\tau^2|Dd|^2\theta^2|w|^2. \end{aligned} \tag{4.18}$$

Using estimate (4.18) on the RHS of (4.12), we obtain

$$\text{RHS of (4.12)} \ge \left[2\tau\rho - \frac{\epsilon}{2}\right]\delta_0\theta^2|Dw|^2 + [4\tau^3\rho|Dd|^2(1-\delta_0)+\mathcal{O}(\tau^2)]\theta^2|w|^2. \tag{4.19}$$

Finally, recalling $|Dd|^2 \ge p^2 > 0$ from assumption (A.2) = (1.5), we see that (4.19) yields (4.13), as claimed.

(iii) Recalling (3.4) with $D\ell = \tau Dd$, and ℓ_t in (4.4), we obtain

$$\left|\int_\Omega\int_0^T \frac{\partial\mathcal{M}}{\partial t}\,dt\,d\Omega\right| = \left|\left[\int_\Omega \mathcal{M}d\Omega\right]_0^T\right| \tag{4.20}$$

$$\text{(by (3.4))} \quad = \left|\left[\int_\Omega \theta^2[2\tau\langle Dd, D\xi\rangle\eta - \xi\tau\langle Dd, D\eta\rangle + 2c\tau\left(t-\frac{T}{2}\right)|w|^2]d\Omega\right]_0^T\right|$$

$$\le \tau\tilde{c}_{d,T}\left|\left[\int_\Omega \theta^2[|Dw|\,|w| + |w|^2]d\Omega\right]_0^T\right| \tag{4.21}$$

$$\le \tau c_{d,T}\left|\left[\int_\Omega \theta^2[|Dw|^2+|w|^2]d\Omega\right]_0^T\right|, \tag{4.22}$$

where in going from (4.20) to (4.21) we have recalled $\xi = \text{Re } w$, $\eta = \text{Im } w$. Then, inequality (4.22) establishes inequality (4.15) via $\theta = e^{\ell} = e^{\tau\varphi}$ by (3.2), (4.2). Finally, (4.15) yields (4.16) by recalling property (1.7) for $\varphi|_{t=0}$ and $\varphi|_{t=T}$, as well as the definition (1.9) for $\mathbb{E}(t)$. $\square$

5. Proof of Theorem 2.1.1 and Corollary 2.1.2. Carleman Estimates for $H^{2,2}(Q)$-Solutions of Eqn. (1.1). First Version

The next key result yields a Carleman estimate, as in [**L-T-Z.2**]. This is Theorem 2.1.1 rewritten here for convenience; however, at this stage, it is obtained only for $H^{2,2}(Q)$-solutions of Eqn. (1.1). Achievement of its extension Theorem 2.2.1 in its full strength, i.e., as extended to solutions of Eqn. (1.1) in the class (2.2.1) will take place in Section 7.

THEOREM 5.1. *Let $T > 0$ and let c be defined accordingly by (1.6b). Let $d(x) \in C^3(\overline{\Omega})$ be a (real) function satisfying (A.1) = (1.4) and (A.2) = (1.5). Define $\varphi(t,x)$ as in (1.6a) = (4.2). Let $w \in C^2(\mathbb{R}_t \times M; \mathbb{C})$ be a solution of Eqn. (1.1) [and no B.C.] under the standing assumption (1.2) for F with constant C_T and (1.3) for f. Then,*

(i) the following one-parameter family of estimates hold true for all $\tau > 0$ sufficiently large and all $\epsilon > 0$:

$$B_\Sigma(w) + 2\left(1 + \frac{1}{\epsilon}\right)\int_0^T \int_\Omega e^{2\tau\varphi}|f|^2 d\Omega\, dt \tag{5.1}$$

$$\geq m_{\rho,p,\tau,C_T,\epsilon}\left\{\int_0^T \int_\Omega e^{2\tau\varphi}[|Dw|^2 + |w|^2] d\Omega\, dt\right\}$$

$$- C_{d,T}\ \tau e^{-2\tau\delta}[\mathbb{E}(T) + \mathbb{E}(0)]$$

$$\geq m_{\rho,p,\tau,C_T,\epsilon} e^{-\delta\tau}\int_{t_0}^{t_1} \mathbb{E}(t) dt - C_{d,T}\ \tau e^{-2\tau\delta}[\mathbb{E}(T) + \mathbb{E}(0)]; \tag{5.2}$$

$$m_{\rho,p,\tau,C_T,\epsilon} \equiv min\left\{\left[\delta_0\left(2\tau p - \frac{\epsilon}{2}\right) - 2C_T\left(1 + \frac{1}{\epsilon}\right)\right], \left[4\tau^3\rho p^2(1-\delta_0)\right.\right. \tag{5.3}$$

$$\left.\left. + \mathcal{O}(\tau^2) - 2C_T\left(1 + \frac{1}{\epsilon}\right)\right]\right\} \nearrow \infty, \quad as\ \tau \nearrow \infty.$$

On the LHS of (5.1), the boundary terms $B_\Sigma(w)$ are defined by

$$
\begin{aligned}
\text{(5.4a)} \qquad B_\Sigma(w) &\equiv \int_0^T \int_\Omega div\, V \, d\Omega \, dt = \int_0^T \int_\Gamma \langle V, \nu \rangle d\Gamma \, dt \\
\text{(5.4b)} \qquad &= 2\int_0^T \int_\Gamma e^{2\tau\varphi}[2\tau^2|Dd|^2 + \Phi]\tau|w|^2\langle Dd, \nu\rangle d\Gamma \, dt \\
&- 2\int_0^T \int_\Gamma e^{2\tau\varphi} 2c\tau\left(t - \frac{T}{2}\right)[\eta\langle D\xi, \nu\rangle - \xi\langle D\eta, \nu\rangle] \, d\Gamma \, dt \\
&- 2\int_0^T \int_\Gamma e^{2\tau\varphi}[\xi_t\eta - \xi\eta_t]\tau\langle Dd, \nu\rangle d\Gamma \, dt \\
&+ \int_0^T \int_\Gamma e^{2\tau\varphi}[2\tau^2|Dd|^2 - \tau\Delta d][\bar{w}\langle Dw, \nu\rangle + w\langle D\bar{w}, \nu\rangle] d\Gamma \, dt \\
&+ 2\int_0^T \int_\Gamma e^{2\tau\varphi} \; \tau\langle Dd, D\bar{w}\langle Dw, \nu\rangle + Dw\langle D\bar{w}, \nu\rangle\rangle d\Gamma \, dt \\
&- 2\int_0^T \int_\Gamma e^{2\tau\varphi}|Dw|^2\tau\langle Dd, \nu\rangle d\Gamma \, dt
\end{aligned}
$$

(ii) The above inequality (5.2) may then be extended by density to all $w \in H^{2,2}(Q)$.

Proof. We return to estimate (4.13) and integrate it over $Q = (0, T] \times \Omega$. On the LHS, we invoke Eqn. (1.1) as well as identity (4.14) for div V and inequality (4.16) for $\frac{\partial M}{\partial t}$. We thus obtain, recalling θ and ℓ_t from (3.2), (4.2), (4.4), and $B_\Sigma(w)$ from (5.4a):

$$
\begin{aligned}
\text{(5.5)} \qquad &\left(1 + \frac{1}{\epsilon}\right) \int_0^T \int_\Omega \theta^2 |F(w) + f|^2 d\Omega \, dt + (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T) + \mathbb{E}(0)] + B_\Sigma(w) \\
&\geq \left(1 + \frac{1}{\epsilon}\right) \int_0^T \int_\Omega \theta^2 |iw_t + \Delta w|^2 d\Omega \, dt \\
&\quad - \left[\int_\Omega \mathcal{M} \, d\Omega\right]_0^T + \int_0^T \int_\Omega \text{div } V \, d\Omega \, dt \\
&\geq \delta_0 \left[2\tau\rho - \frac{\epsilon}{2}\right] \int_0^T \int_\Omega \theta^2 |Dw|^2 d\Omega \, dt \\
&\quad + \left[4\rho(1 - \delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)\right] \int_0^T \int_\Omega \theta^2 |w|^2 d\Omega \, dt.
\end{aligned}
$$

Next, invoking estimate (1.2b) for $F(w)$ on the LHS of (5.5), we finally obtain

$$
\begin{aligned}
(5.6)\quad & 2\left(1+\frac{1}{\epsilon}\right)\int_0^T\int_\Omega e^{2\tau\varphi}|f|^2 d\Omega\, dt + B_\Sigma(w) \\
& \geq \left[\delta_0\left(2\tau\rho - \frac{\epsilon}{2}\right) - 2C_T\left(1+\frac{1}{\epsilon}\right)\right]\int_0^T\int_\Omega e^{2\tau\varphi}|Dw|^2 d\Omega\, dt \\
& + \left[4\tau^3\rho p^2(1-\delta_0) + \mathcal{O}(\tau^2) - 2C_T\left(1+\frac{1}{\epsilon}\right)\right]\int_0^T\int_\Omega e^{2\tau\varphi}|w|^2 d\Omega\, dt \\
& - (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)],
\end{aligned}
$$

with C_T the constant in (1.2b). Thus (5.6) establishes (5.1) by using (5.3) with τ sufficiently large. Finally, using property (1.8) on φ and (1.9) on $\mathbb{E}(t)$ in the integral term in the RHS of (5.1), we obtain estimate (5.2). $\square$

Theorem 5.1 coincides with Theorem 2.1.1 by taking $\epsilon = 1$.

Proof of Corollary 2.1.2. Step 1. We return to the definition (2.5) or (5.4b) of the boundary terms $B_\Sigma(w)$ and verify directly that

$$
\begin{aligned}
(5.7a)\quad & \begin{cases} \text{condition (2.1.7)} \ \Rightarrow B_\Sigma(w) \ \equiv \ 0 \\ \text{condition (2.1.8)} \ \Rightarrow B_\Sigma(w) \ = \ 2\tau\displaystyle\int_0^T\int_{\Gamma_0} e^{2\tau\varphi}|\langle Dw,\nu\rangle|^2\langle Dd,\nu\rangle d\Sigma_0 \leq 0. \end{cases} \\
(5.7b)\quad &
\end{aligned}
$$

Indeed, in the case, say, of condition (2.1.7), the individual terms in (2.1.5) (or (5.4b)) vanish either because $\langle Dw,\nu\rangle \equiv 0$ on all of Γ; or else because of the combination $\langle Dd,\nu\rangle \equiv 0$ on Γ_0 and $w \equiv 0$ on Γ_1, whereby then, with $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, we have

$$|Dw|^2 = |\langle Dw,\nu\rangle|^2 + |\langle Dw,\mu\rangle|^2 \equiv 0 \text{ on } \Sigma_1.$$

Here μ = tangential vector field along Γ_1. A similar analysis shows (5.7b).

Step 2. With $B_\Sigma(w) \leq 0$ as obtained in either case (5.7a) and (5.7b) above, then estimate (2.1.3) (or (5.2)) with $f \equiv 0$ yields

$$0 \geq m_{\rho,p,\tau,C_T} e^{-\delta\tau}\int_{t_0}^{t_1}\mathbb{E}(t)dt - (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)]; \tag{5.8}$$

$$\frac{(C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)]}{m_{\rho,p,\tau,C_T}} \geq \int_{t_0}^{t_1}\mathbb{E}(t)dt. \tag{5.9}$$

Letting $\tau \nearrow \infty$, whereby $m_{\rho,p,\tau,C_T} \nearrow \infty$ at the rate of τ, see (2.1.4) or (5.3), we conclude that

$$0 = \int_{t_0}^{t_1}\mathbb{E}(t)dt, \text{ hence } w \equiv 0 \text{ on } (t_0,t_1)\times\Omega, \tag{5.10}$$

recalling $\mathbb{E}(t)$ in (1.9), where we recall from (1.8) that $0 < t_0 < \frac{T}{2} < t_1 < T$. With $T > 0$ given and fixed for which (2.7) or (2.8) hold true, we may repeat the above argument for all intervals smaller than T and get, accordingly, at least, $w \equiv 0$ in $(0,\frac{T}{2})\times\Omega$, actually $(0,t_1)\times\Omega$, the original t_1 in (5.10). Finally, we recall from the line below (1.8) that actually we may take as interval (t_0,t_1) any interval where $\varphi(t,x) \geq \sigma > -\delta$ uniformly in Ω, with σ any number arbitrarily close to $-\delta$. We then conclude that the preceding argument actually yields $w \equiv 0$ on $(0,T]\times\Omega$, hence $w \equiv 0$ on $[0,T]\times\Omega$, since $w \in (C[0,T]; L_2(\Omega))$, *a-fortiori* from $w \in H^{2,2}(Q)$. $\square$

6. Proof of Theorem 2.1.3. Carleman Estimate for $H^{2,2}(Q)$-Solutions of Eqn. (1.1), or Eqn. (2.1.10) under (A.3). Second Version

Our starting point is Theorem 5.1, Eqn. (5.2), which coincides with Theorem 2.1.1, Eqn. (2.1.3), for $H^{2,2}(Q)$-solutions. This forms the basis upon which the sought-after results are based. From this present Section 6 on, our subsequent development merges with the one carried out in [**Tr.1**, Section 5] in the Euclidean setting, later adapted in [**T-Y.1**, Section 5] in the Riemannian setting $\{M,g\}$. In both these references a very different, multiplier-based approach was used yielding the same version of the Carleman estimate (2.1.3) of Theorem 2.1.1, *however polluted by interior lower-order terms*, as noted in Remark 2.1.1. This same development was taken up subsequently once more in [**L-T-Z.2**, Section 6], still in the Euclidean setting, of which the present section is the Riemannian faithful counterpart.

According to Section 2.1, under assumption (A.3), which includes (2.1.9a–b), we may w.l.o.g. consider our basic dynamics as being (2.1.10), that is

$$iw_t + \Delta w = F(w) + f = -i\langle R_1(t,x), Dw\rangle + q_0(t,x)w + f, \tag{6.1}$$

subject to the additional properties (2.1.11a) or (2.1.11b) on the real-valued vector field $R_1(t,x)$ for dim $\Omega \geq 2$. For dim $\Omega = 1$, we may further take $R_1(t,x) \equiv 0$. As mentioned at the outset of the present section, the proof is patterned after that of [**L-T-Z.2**, Section 6] in the Euclidean case (based in turn on [**Tr.1**, Section 2.3] (Euclidean case) and [**T-Y.1**, Section 5] (Riemannian case (M,g)). In this section, we shall need the Green's first identity [**Do.1**, p.], [**Le.1**, p. 44]

$$\int_\Omega (\Delta w)z\, d\Omega = \int_\Omega \langle Dw, Dz\rangle d\Omega - \int_\Gamma z\langle Dw,\nu\rangle d\Gamma, \tag{6.2}$$

as well as the divergence theorem with $X \in \mathcal{X}(M)$, where $X(f) = \langle Df, X\rangle$:

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle Df, X\rangle; \quad \int_\Omega \operatorname{div} X d\Omega = \int_\Gamma \langle X,\nu\rangle d\Gamma. \tag{6.3}$$

Combining the identities in (6.3), we finally obtain

$$\int_\Omega \langle Df, X\rangle d\Omega = \int_\Gamma f\langle X,\nu\rangle d\Gamma - \int_\Omega f \operatorname{div} X\, d\Omega, \tag{6.4}$$

to be invoked in the proof below.

Proof of Theorem 2.1.3. Step 1. Lemma 6.1(i). *Let w be a $H^{2,2}(Q)$-solution of the Schrödinger equation (1.1), with $f \in L_2(Q)$, with $F(w) = \langle P(t,x), Dw\rangle + q_0(t,x)w$ satisfying (1.2b). Then:*

(i_1) With $\mathbb{E}(t)$ defined by (1.9), we have for all t, s in $[0, T]$:

$$\begin{aligned}
\mathbb{E}(t) - \mathbb{E}(s) \quad &= \quad 2 \; Re\left\{\int_s^t \int_\Gamma \langle D\bar{w}, \nu\rangle w_t d\Gamma \, d\sigma\right\} \\
&+ 2 \; Re\left\{i\int_s^t \int_\Gamma \langle Dw, \nu\rangle \bar{w} d\Gamma \, d\sigma\right\} \\
&+ 2 \; Re\left\{\int_s^t \int_\Omega [F(w) + f][i\Delta\bar{w} - i\bar{w}] d\Omega \, d\sigma\right\}.
\end{aligned} \tag{6.5}$$

(i_2) Let q_0 satisfy assumptions (2.1.9b) (Left-Hand Side), then:

$$\begin{aligned}
Re\left\{i\int_\Omega \Delta\bar{w} q_0 w \, d\Omega\right\} \; &= \; Re\left\{i\int_\Gamma \langle D\bar{w}, \nu\rangle q_0 w \, d\Gamma\right\} \\
&- \; Re\left\{i\int_\Omega q_0 |Dw|^2 \, d\Omega\right\} - Re\left\{i\int_\Omega w\langle D\bar{w}, Dq_0\rangle d\Omega\right\}.
\end{aligned} \tag{6.6}$$

(ii) Assume further hypothesis (A.3) = (2.1.9), so that w.l.o.g., we can assume that the Schrödinger equation is in the form given by (6.1), i.e., (1.1) with $P(t, x) = -iR_1(t, x)$ and q_0 satisfying hypotheses (2.1.9a) and (2.1.9b) and R_1 fulfilling further w.l.o.g. properties (2.1.11a) or (2.1.11b). Then, for a unit tangent vector μ, we have:

(ii_1)

$$\begin{aligned}
Re\left\{i\int_\Omega \Delta\bar{w}\langle P(t,x), Dw\rangle d\Omega\right\} &= Re\left\{\int_\Omega \Delta\bar{w}\langle R_1(t,x), Dw\rangle d\Omega\right\} \\
= \quad & \frac{1}{2}\int_\Gamma |\langle Dw, \nu\rangle|^2 \langle R_1(t,x), \nu\rangle d\Gamma \\
+ \quad & Re\left\{\int_\Gamma \langle D\bar{w}, \nu\rangle \; \langle Dw, \mu\rangle \; \langle R_1(t,x), \mu\rangle d\Gamma\right\} \\
- \quad & \frac{1}{2}\int_\Gamma |\langle Dw, \mu\rangle|^2 \langle R_1(t,x), \nu\rangle d\Gamma \\
- \quad & \int_\Omega Re\left\{\langle D_{D\bar{w}} R_1, Dw\rangle\right\} d\Omega - \frac{1}{2}\int_\Omega |Dw|^2 div \; R_1 d\Omega;
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
Re\left\{i\int_\Omega \Delta\bar{w}\langle P(t,x), Dw\rangle d\Omega\right\} &= Re\left\{\int_\Omega \Delta\bar{w}\langle R_1(t,x), Dw\rangle d\Omega\right\} \\
\leq \quad & \int_\Gamma |\langle Dw, \nu\rangle|^2 d\Gamma \\
& - \frac{1}{2}\int_\Gamma |\langle Dw, \mu\rangle|^2 \Big[\langle R_1(t,x), \nu\rangle - |\langle R_1(t,x), \mu\rangle|^2\Big] d\Gamma \\
& - \int_\Omega Re\left\{\langle D_{D\bar{w}} R_1, Dw\rangle\right\} d\Omega - \frac{1}{2}\int_\Omega |Dw|^2 div \; R_1 d\Omega,
\end{aligned} \tag{6.8}$$

where $D_{D\bar{w}}R_1$ is the covariant derivative of the vector field R_1 with respect to $D\bar{w}$.

(ii_2)

$$
\begin{aligned}
(6.9)\quad & Re\left\{i\int_\Omega \Delta\bar{w}F(w)d\Omega\right\} = Re\left\{i\int_\Omega \Delta\bar{w}\,[\langle P(x,t),Dw\rangle + q_0 w]\,d\Omega\right\} \\
= \;& \frac{1}{2}\int_\Gamma |\langle Dw,\nu\rangle|^2\langle R_1(t,x),\nu\rangle d\Gamma \\
& + Re\left\{\int_\Gamma \langle D\bar{w},\nu\rangle\,\langle Dw,\mu\rangle\,\langle R_1(t,x),\mu\rangle d\Gamma\right\} \\
& - \frac{1}{2}\int_\Gamma |\langle Dw,\mu\rangle|^2\langle R_1(t,x),\nu\rangle d\Gamma + Re\left\{i\int_\Gamma \langle D\bar{w},\nu\rangle q_0 w\,d\Gamma\right\} \\
& - Re\left\{\int_\Omega \langle D_{D\bar{w}}R_1, Dw\rangle d\Omega\right\} - \frac{1}{2}\int_\Omega |Dw|^2\,div\;R_1 d\Omega \\
& - Re\left\{i\int_\Omega q_0|Dw|^2 d\Omega\right\} - Re\left\{i\int_\Omega w\langle D\bar{w},Dq_0\rangle d\Omega\right\}.
\end{aligned}
$$

(iii) Assume further that $f\in L_2(0,T;H^1(\Omega))$. Then, in the L_1-sense in t, on any subinterval of $[0,T]$:

(iii_1)

$$
(6.10)\quad Re\left\{i\int_\Omega \Delta\bar{w}f\,d\Omega\right\} = -Re\left\{i\int_\Gamma \langle D\bar{w},\nu\rangle f\,d\Gamma\right\} + Re\left\{i\int_\Omega \langle D\bar{w},Df\rangle d\Omega\right\}.
$$

(iv) Under the assumptions of parts (i) through (iii), we then have:

$$
\begin{aligned}
(6.11a)\quad \mathbb{E}(t)-\mathbb{E}(s) \;=\;& 2\;Re\left\{\int_s^t\int_\Gamma \langle D\bar{w},\nu\rangle[w_t + iq_0 w - if]\,d\Gamma\,d\sigma\right\} \\
+\;& \int_s^t\int_\Gamma |\langle Dw,\nu\rangle|^2\langle R_1(t,x),\nu\rangle d\Gamma\,d\sigma \\
+\;& 2\;Re\left\{i\int_s^t\int_\Gamma \langle Dw,\nu\rangle\bar{w}d\Gamma\,d\sigma\right\} \\
+\;& 2\;Re\left\{\int_s^t\int_\Gamma \langle D\bar{w},\nu\rangle\,\langle Dw,\mu\rangle\,\langle R_1(t,x),\mu\rangle d\Gamma\,d\sigma\right\} \\
-\;& \int_s^t\int_\Gamma |\langle Dw,\mu\rangle|^2\langle R_1(t,x),\nu\rangle d\Gamma\,d\sigma \\
-\;& 2\;Re\Bigg\{\int_s^t\int_\Omega \Big[\langle D_{D\bar{w}}R_1, Dw\rangle + iq_0|Dw|^2 \\
+\;& iw\langle D\bar{w},Dq_0\rangle - i\langle D\bar{w},Df\rangle + i[F(w)+f]\bar{w}\Big]d\Omega\,d\sigma\Bigg\} \\
-\;& \int_s^t\int_\Omega |Dw|^2\,div\;R_1 d\Omega\,d\sigma,
\end{aligned}
$$

where, for purposes other than the Dirichlet B.C., we estimate

$$\text{(6.11b)} \qquad 2\ Re\Big\{ \int_s^t \int_\Gamma \langle D\bar{w}, \nu\rangle\ \langle Dw, \mu\rangle\ \langle R_1, \mu\rangle d\Gamma\, d\sigma$$

$$- \int_s^t \int_\Gamma |\langle Dw, \mu\rangle|^2 \langle R_1(t,x), \nu\rangle d\Gamma\, d\sigma \Big\}$$

$$\leq \int_s^t \int_\Gamma |\langle Dw, \nu\rangle|^2 d\Gamma\, d\sigma$$

$$+ \int_s^t \int_\Gamma |\langle Dw, \mu\rangle|^2 \Big[|\langle R_1(t,x), \mu\rangle|^2 - \langle R_1(t,x), \nu\rangle \Big] d\Gamma\, d\sigma$$

$$\leq \int_s^t \int_\Gamma |\langle Dw, \nu\rangle|^2 d\Gamma\, d\sigma,$$

while in the Dirichlet case $w|_\Sigma \equiv 0$, the LHS of (6.11b) vanishes.

(v) Finally, under the assumptions of parts (i) through (iii), we then have the following estimate for all $0 \leq s \leq t \leq T$:

$$\text{(6.12a)} \qquad \mathbb{E}(t) - \mathbb{E}(s) = 2\ Re\Big\{ \int_s^t \int_\Gamma \langle D\bar{w}, \nu\rangle [w_t + iq_0 w - if]\, d\Gamma\, d\sigma \Big\}$$

$$+ \int_s^t \int_\Gamma |\langle Dw, \nu\rangle|^2 \langle R_1(t,x), \nu\rangle d\Gamma\, d\sigma$$

$$+ 2\ Re\Big\{ i \int_s^t \int_\Gamma \langle Dw, \nu\rangle \bar{w} d\Gamma\, d\sigma \Big\}$$

$$+ \mathcal{O}_c \left(\int_s^t \int_\Gamma |\langle Dw, \nu\rangle|^2 d\Gamma\, d\sigma \right)$$

$$+ \mathcal{O}\left(\int_s^t \mathbb{E}(\sigma) d\sigma + \|f\|^2_{L_2(s,t;H^1(\Omega))} \right).$$

$\mathcal{O}_c = 0$ in the Dirichlet case $w|_\Sigma \equiv 0$, and $\mathcal{O}_c = 1$ otherwise.

$$\text{(6.12b)} \qquad |\mathbb{E}(t) - \mathbb{E}(s)| \leq G(T) + c_T \int_s^t \mathbb{E}(\sigma) d\sigma;$$

$$\text{(6.13)} \qquad G(T) = 2 \int_0^T \int_\Gamma |\langle Dw, \nu\rangle| \Big[|w_t| + \frac{1}{2} |\langle Dw, \nu\rangle| |\langle R_1(t,x), \nu\rangle + \mathcal{O}_c|$$

$$+ \ |q_0|\,|w| + |w| + |f| \Big] d\Gamma\, dt + \int_0^T \int_\Omega [|Df|^2 + |f|^2] d\Omega\, dt,$$

where $c_T = c_{q_0,q_1}$ is a constant depending on the constant C_T in (1.2b), the norm $|div\ R_1|_{L_\infty(0,T)}$, $DR_1 \in L_\infty(0,T; T_2^0 M)$.

Proof. (i) We multiply Eqn. (1.1) by the multiplier $|i\Delta\bar{w} - i\bar{w}|$, take real parts—whereby 2 Re$\{\bar{w}w_t\} = \frac{\partial}{\partial t}|w|^2$ while the term $i|\Delta w|^2$ drops off—and integrate by parts, using similarly 2 Re$\{\langle D\bar{w}, Dw_t\rangle = \frac{\partial}{\partial t}|Dw|^2$. We readily obtain (6.5). Identity (6.6) is obtained by application of Green's first identity (6.2).

(ii) Identity (ii_1) is where the key structural hypothesis (A.3) = (2.1.9) is used, thus we provide details following [**Tr.1**], [**T-Y.1**], [**L-T-Z.2**]. We shall use the identity

$$\langle D\bar{w}, D\langle P, Dw\rangle\rangle = \langle D_{D\bar{w}}P, Dw\rangle + \langle P, D_{D\bar{w}}Dw\rangle, \tag{6.14}$$

which is obtained from (3.28a) with $X = P$, $Y = Dw$, $Z = D\bar{w}$. Next, we multiply (6.14) by "i," invoke the structural assumption (A.3) = (2.1.9a): $iP(t,x) = R_1(t,x)$ = real-valued vector field, take the real part, and obtain

$$\begin{aligned} &\text{(6.15a)} && \mathrm{Re}\,\{i\langle D\bar{w}, D\langle P, Dw\rangle\} = \mathrm{Re}\,\{\langle D_{D\bar{w}}R_1, Dw\rangle\} + \mathrm{Re}\,\{\langle R_1, D_{D\bar{w}}Dw\rangle\} \\ &\text{(6.15b)} && \qquad = \mathrm{Re}\,\{\langle D_{D\bar{w}}R_1, Dw\rangle\} + \left\langle \frac{R_1}{2}, D(|Dw|^2)\right\rangle, \end{aligned}$$

where in going from (6.15a) to (6.15b) we have invoked the identity

$$\begin{aligned} 2\mathrm{Re}\,\{\langle R_1, D_{D\bar{w}}Dw\rangle\} &= R_1(|Dw|^2) \\ &= \langle \nabla_g |Dw|^2, R_1\rangle = \langle D(|Dw|)^2), R_1\rangle, \end{aligned} \tag{6.15c}$$

whose first key part is established in [**T-Y.1**, Appendix A, identity (A.0), p. 368]. Thus, we compute the critical term in (6.7), at first by Green's first identity (6.2):

$$\begin{aligned} &\text{(6.16)} && \mathrm{Re}\left\{i\int_\Omega \Delta\bar{w}\langle P, Dw\rangle d\Omega\right\} = \mathrm{Re}\left\{i\int_\Gamma \langle D\bar{w}, \nu\rangle\,\langle P, Dw\rangle d\Gamma\right\} \\ &&& \qquad - \int_\Omega \mathrm{Re}\,\{i\langle D\bar{w}, D\langle P, Dw\rangle\}\; d\Omega \\ &\text{(6.17)} && \text{(by (6.15b))} \quad = \quad \mathrm{Re}\left\{i\int_\Gamma \langle D\bar{w}, \nu\rangle\,\langle P, Dw\rangle d\Gamma\right\} \\ &&& \qquad - \int_\Omega \mathrm{Re}\,\{\langle D_{D\bar{w}}R_1, Dw\rangle\}\; d\Omega - \frac{1}{2}\int_\Omega \langle R_1, D(|Dw|^2)\rangle d\Omega, \end{aligned}$$

after using identity (6.15b). We next evaluate the last term on the RHS of (6.17) by the divergence theorem (6.4), with $f = |Dw|^2$ and $X = R_1$, to obtain

$$\int_\Omega \langle R_1, D(|Dw|^2)\rangle d\Omega = \int_\Gamma |Dw|^2\langle R_1(t,x), \nu\rangle\, d\Gamma - \int_\Omega |Dw|^2 \mathrm{div}\ R_1 d\Omega. \tag{6.18}$$

Finally, regarding the boundary term on the RHS of (6.17), we again recall assumption (A.3): $iP = R_1$ (real-valued vector field) and compute with μ being a unit tangent vector field on Γ:

$$\begin{aligned} &\text{(6.19)} \quad \text{on } \Gamma : i\langle D\bar{w}, \nu\rangle\,\langle P, Dw\rangle \\ &\quad = \langle D\bar{w}, \nu\rangle\Big[\langle R_1, \langle Dw, \nu\rangle\nu + \langle Dw, \mu\rangle\mu\rangle\Big] \\ &\quad = \langle R_1, \nu\rangle|\langle Dw, \nu\rangle|^2 + \langle R_1, \mu\rangle\,\langle D\bar{w}, \nu\rangle\,\langle Dw, \mu\rangle. \end{aligned}$$

Finally, substituting (6.18) and (6.19) into (6.17) yields identity (6.7), using also

$$|Dw|^2 = |\langle Dw, \nu\rangle|^2 + |\langle Dw, \mu\rangle|^2,$$

by orthogonality of ν and μ. Then, inequality (6.8) follows readily from (6.7): in (6.8), we have evidenced the term $|\langle R_1, \mu\rangle|^2 - \langle R_1, \nu\rangle \leq 0$, via (2.1.11a) without loss of generality as in [**L-T-Z.2**, below Eqn. (6.10), in connection with Eqn. (C.7), Appendix C]. Identities (6.9) and (6.10) follow readily, from (6.6) and (6.7), respectively; the second via Green's first identity (6.2).

(iv) We rewrite (6.5) in detail:

$$\begin{aligned}
(6.20)\quad & \mathbb{E}(t) - \mathbb{E}(s) \\
= \;& 2\,\mathrm{Re}\left\{\int_s^t \int_\Gamma \langle D\bar{w}, w_t\rangle \, d\Gamma\, d\sigma\right\} + 2\,\mathrm{Re}\left\{i\int_s^t \int_\Gamma \langle Dw, \nu\rangle \bar{w}\, d\Gamma\, d\sigma\right\} \\
+ \;& 2\,\mathrm{Re}\left\{\int_s^t \int_\Omega i\Delta\bar{w}\, F(w) d\Omega\, d\sigma\right\} - 2\,\mathrm{Re}\left\{\int_s^t \int_\Omega i\bar{w}\, F(w) d\Omega\, d\sigma\right\} \\
+ \;& 2\,\mathrm{Re}\left\{\int_s^t \int_\Omega i\Delta\bar{w}\, f\, d\Omega\, d\sigma\right\} - 2\,\mathrm{Re}\left\{\int_s^t \int_\Omega i\bar{w}\, f\, d\Omega\, d\sigma\right\}.
\end{aligned}$$

Next, we substitute (6.9) for the third integral term and (6.10) for the fifth term, and obtain (6.11). Details are the direct counterpart of the Euclidean case [**L-T-Z.2**, Appendix B].

(v) Part (v) is the direct consequence of part (iv). The critical term in (6.11a) to analyze is

$$\begin{aligned}
(6.21)\qquad & 2\left|\mathrm{Re}\left\{i\int_s^t \int_\Omega w\langle D\bar{w}, Dq_0\rangle \, d\Omega\, d\sigma\right\}\right| \\
& \leq \int_s^t \int_\Omega |Dw|^2 d\Omega\, d\sigma + \int_s^t \int_\Omega |w|^2 |Dq_0|^2 d\Omega\, d\sigma.
\end{aligned}$$

The final step is therefore to prove that

$$(6.22)\qquad \text{RHS of (6.21)} \leq C\int_s^t \int_\Omega |Dw|^2 d\Omega\, d\sigma,$$

as desired, provided that

$$(6.23)\qquad q_0 \in L_1(0,T; W^{1,2}(\Omega)) \text{ for } n = 1;$$

$q_0 \in L_1(0,T; W^{1,2+\epsilon}(\Omega))$ for $n = 2$, $\epsilon > 0$; $q_0 \in L_1(0,T; W^{1,n}(\Omega))$ for $n \geq 3$, $n = \dim M$, as assumed in (2.1.9b) (LHS). Indeed, focusing first on the space variable, by embedding $H^1(\Omega) \subset L_\rho(\Omega)$, we obtain:

$$\begin{aligned}
(6.24)\qquad & w \in H^1(\Omega) \Rightarrow w \in L_\rho(\Omega) \text{ for } \rho \leq \infty,\ n = 1; \\
& \rho < \infty,\ n = 2;\ \rho \leq \frac{2n}{n-2},\ n > 2.
\end{aligned}$$

Thus, as a consequence, we obtain

$$\begin{aligned}
(6.25)\qquad & w \in H^1(\Omega) \Rightarrow |w|^2 \in L_\infty(\Omega),\ n = 1; \\
& w|^2 \in L_\rho(\Omega),\ n = 2,\ \rho < \infty;\ |w|^2 \in L_{\frac{n}{n-2}}(\Omega),\ n > 2.
\end{aligned}$$

According to (6.25), the second integral term in (6.21) yields (6.22) provided that

$$(6.26)\quad |Dq_0|^2 \in L_1(\Omega),\ n = 1;\ |Dq_0|^2 \in L_{1+\epsilon}(\Omega),\ n = 2;\ |Dq_0| \in L_{\frac{n}{2}}(\Omega),\ n = 3,$$

since $\rho' = \frac{n}{2}$ is the conjugate index of $\rho = \frac{n}{n-2}$, $\rho' = \frac{\rho}{\rho-1}$. Then (6.26) yields

$$(6.27)\qquad q_0 \in W^{1,2}(\Omega),\ n = 1;\ q_0 \in W^{1,2+\epsilon}(\Omega),\ n = 2;\ q_0 \in W^{1,n}(\Omega),\ n > 2.$$

Finally, starting from (6.27) and incorporating the analysis also the time derivative, we see that (6.21) yields (6.22), provided that (6.23) holds true.

Then, (6.22) used in (6.11a), yields $\mathcal{O}(\int_s^t \mathbb{E}(\sigma)d\sigma)$, as stated in (6.12a), provided that, in addition, $q_0 \in L_\infty(Q)$ and $DR_1 \in L_\infty(0,T;T_2^0)$, as assumed in (2.1.9a). $\square$

Remark 6.1. In the proof of Lemma 6.1, instead of multiplying Eqn. (1.1) by $[i\Delta\bar{w} - i\bar{w}]$, one could alternatively multiply Eqn. (1.1) by $\bar{w}_t (= -i\Delta\bar{w} + i\bar{F}(w) + \bar{f})$ as done in [**Tr.1**, p. 481], [**T-Y.1**], [**L-T-Z.2**, Remark 6.1]. In this case, one would integrate in time rather than integrate in space. This way, one would have to handle a term like

$$(6.28) \qquad \int_s^t q_0\, w\, \bar{w}_t\, d\Omega\, d\sigma, \quad \text{hence} \quad \int_s^t \int_\Omega (q_0)_t |w|^2 d\Omega\, d\sigma$$

[**Tr.1**, Eqn. (2.3.8), p. 482] rather than the term $\int_s^t \int_\Omega |Dq_0|^2 |w|^2 d\Omega\, d\sigma$ as in (6.21). Arguing as above, we see that this present method would likewise yield

$$(6.29) \qquad \int_s^t (q_0)_t |w|^2 d\Omega\, d\sigma \leq C \int_s^t \int_\Omega |Dw|^2 d\Omega\, d\sigma,$$

as desired, provided that

$$(6.30) \quad (q_0)_t \in L_1(0,T;L_1(\Omega)),\ n=1;\ (q_0)_t \in L_1(0,T;L_{1+\epsilon}(\Omega)),\ n=2,\ \epsilon>0;$$
$$(q_0)_t \in L_1(0,T;L_{\frac{n}{2}}(\Omega)),\ n>2,$$

the counterpart (in the space variable) of (6.26). $\square$

Step 2. Lemma 6.2. *Let w be a solution of Eqn. (1.1) in the class (2.1.1). Let $f \in L_2(0,T;H^1(\Omega))$. Assume further hypothesis (A.3). Then*

(ii) for any $0 \leq s \leq t \leq T$:

$$(6.31) \qquad \mathbb{E}(t) \leq [\mathbb{E}(s) + G(T)]e^{c_T(t-s)}; \quad \mathbb{E}(s) \leq [\mathbb{E}(t) + G(T)]e^{c_T(t-s)};$$

(iii)

$$(6.32) \qquad \mathbb{E}(t) \geq \frac{\mathbb{E}(0)+\mathbb{E}(T)}{2}\, e^{-c_T T} - G(T);\ 0 \leq t \leq T.$$

Proof. As usual, we apply the Gronwall inequality to the two inequalities that result from (6.12) and obtain (6.31), from which (6.32) follows. $\square$

Remark 6.2. The above Lemmas 6.1 and 6.2 are the perfect counterpart for the $H^1(\Omega)$-energy $\mathbb{E}(t)$ of the Euclidean case as in [**L-T-Z.2**, Section 6]. They are also the counterpart of [**Tr.1**, Lemma 2.3.1, p. 480], [**T-Y.1**, Lemma 5.1 and Proposition 5.2, p. 650], which were instead stated for the 'gradient energy' $E(t)$ in (1.9). This will be clear in Step 3 below. In [**Tr.1**], [**T-Y.1**] where the Carleman estimates (hence the observability/stabilization inequalities) contained interior lower-order terms, it was more expedient to work with $E(t)$ rather than $\mathbb{E}(t)$. $\square$

Step 3. We invoke inequality (6.32) for $\mathbb{E}(t)$ in Eqn. (2.1.3) = (5.2) and obtain

$$(6.33) \quad B_\Sigma(w) + 4\int_0^T \int_\Omega e^{2\tau\varphi}|f|^2 d\Omega\, dt + m_{\rho,p,\tau,C_T} e^{-\delta\tau} G(T)(t_1 - t_0)$$
$$\geq \left\{ m_{\rho,p,\tau,C_T} e^{-\delta\tau} \frac{e^{-c_T T}}{2}(t_1 - t_0) - (C_{d,T})\tau e^{-2\tau\delta} \right\} [\mathbb{E}(T) + \mathbb{E}(0)]$$
$$(6.34) \qquad = k_{\varphi,\tau,C_T}[\mathbb{E}(T) + \mathbb{E}(0)],$$

for all τ sufficiently large, so that $k_{\varphi,\tau,C_T} > 0$ [recall (2.1.4) = (5.3)]. Next, on the LHS of 6. the definition of $G(T)$, we then see that (6.34) yields (2.1.12) with $\tilde{B}_\Sigma(w)$ defined by (2.1.14). The proof of Theorem 2.1.3 is complete. □

7. Extension of Estimates to Finite Energy Solutions

So far, our estimates have been stated and proved only for $C^2(\mathbb{R}_t \times M; \mathbb{C})$-solutions, hence $H^{2,2}(Q)$-solutions, to Eqn. (1.1), with $f \in L_2(Q)$ (Theorem 5.1(ii)) or $f \in L_2(0,T; H^1(\Omega))$ (Section 6). In the present section, we extend all our previous estimates from $H^{2,2}(Q)$-solutions to finite energy solutions on the class (2.2.1), rewritten here for convenience

$$w \in C([0,T]; H^1(\Omega)); \quad w_t, \quad \langle Dw, \nu \rangle \in L_2(0,T; L_2(\Gamma)). \tag{7.1}$$

In order to achieve this goal, it suffices to extend the validity of estimate (5.2) of Theorem 5.1 from $H^{2,2}(Q)$-solutions to finite energy solutions defined by the above class (7.1) = (2.2.1). Here, the main difficulty is the fact that finite energy solutions subject to Neumann B.C. do not produce H^1-traces on the boundary (in the Euclidean case as well). By contrast, in the case of Dirichlet B.C., the regularity results of [**L-T.3**], [**L-T.4**, Chapter 10, Section 9]—that admit a perfect counterpart from the Euclidean to the Riemannian setting, by virtue of the techniques of [**T-Y.1**]—allows one to obtain the normal derivative $\langle Dw, \nu \rangle \in L_2(0,T; L_2(\Gamma)) \equiv L_2(\Sigma)$ for $w_0 \in H_0^1(\Omega)$. To overcome the aforementioned difficulty in the Neumann case, it is necessary to employ the regularization procedure of [**L-T-Z.2**, Section 7], as translated from the Euclidean setting there to the Riemannian setting in this section. Such regularization procedure was previously employed in the case of second-order hyperbolic equations in [**L-T-Z.1**, Section 8], still under Neumann B.C., and for the same reasons, with original inspiration coming from [**La-Ta.1**]. Plainly, the Schrödinger case is technically more demanding than the second-order hyperbolic case. This is so, since the term $F(w)$ in (1.2a) of the Schrödinger Eqn. (1.1) is an *unbounded* perturbation of the basic semigroup generator on the state space $H^1(\Omega)$, as established in the Euclidean case in [**L-T-Z.2**, Appendix C], while in the case of second-order hyperbolic equations in [**L-T-Z.1**, Section 8] (Euclidean case) and in [**T-Y.1**, Section 7] (Riemannian case) the corresponding first-order energy level term $\langle R_1(t,x), Dw \rangle + q_1(t,x) w_t$ is a *bounded* perturbation of the corresponding basic semigroup generator on $H^1(\Omega \times L_2(\Omega))$. The same proofs of [**L-T-Z.2**, Section 7] work also in the present Riemannian setting *mutatis mutandis*, using the Riemannian techniques of the present paper (or of [**T-Y.1**]). Thus, we shall confine ourselves to the relevant statements.

THEOREM 7.1. *Assume the following hypotheses on the term $F(w)$ of (1.1): (1.2b), (A.3) (which includes $P = -iR_1$ and (2.1.9a–b)) and (A.4) = (2.2.2). Let $f \in L_2(0,T; H^1(\Omega))$. Then, estimate (5.2) of Theorem 5.1 holds true also for solutions of Eqn. (1.1) in the class (7.1).*

Remark 7.1. Even in the Euclidean setting, the proof of Theorem 7.1 is rather lengthy and delicate. It goes, among other things, through the establishment of sharp regularity estimates of the following *non-homogeneous boundary dissipative* *(hence regularizing) problem:*

$$
\begin{cases}
iw_t + \Delta w = F(w) + f & \text{in } (0,T] \times \Omega \equiv Q; & (7.2a)\\
w(0,\,\cdot\,) = w_0 & \text{in } \Omega; & (7.2b)\\
\langle Dw, \nu\rangle + w_t = g & \text{in } (0,T] \times \Gamma \equiv \Sigma, & (7.2c)
\end{cases}
$$

for which the following interior and boundary estimates hold true. □

THEOREM 7.2. *(a) With reference to problem (7.2), assume the hypotheses of Theorem 7.1: (1.2b), (A.3), (A.4). Let*

$$w_0 \in H^1(\Omega): \ f \in L_2(0,T;H^1(\Omega)); \ g \in L_2(0,T;L_2(\Gamma)). \tag{7.3}$$

Then, there exists a unique solution of problem (7.2) which, moreover, satisfies the following estimates: there exists $C_T > 0$ such that, for all $0 < t \le T$:

$$\int_0^T\int_\Gamma |w_t|^2 d\Gamma\, d\sigma + \mathbb{E}(t) \le C_T \left\{\mathbb{E}(0) + \|f\|^2_{L_2(0,T;H^1(\Omega))} + \|g\|^2_{L_2(0,T;L_2(\Gamma))}|\right\}. \tag{7.4}$$

More specifically, the following regularity properties hold true for the solution of problem (7.2), under present assumptions, in particular (7.3) on the data:

$$\{w, \Delta w\} \in C([0,T];H^1(\Omega) \times H^{-1}(\Omega)), \ F(w) \in C([0,T];L_2(\Omega)), \quad w_t \in C([0,T];H^{-1}(\Omega)); \tag{7.5}$$

$$\langle Dw, \nu\rangle, \ w_t|_\Sigma \in L_2(0,T;L_2(\Gamma)), \tag{7.6}$$

continuously.

(b) For $g \equiv 0$, $f \equiv 0$, the map $w_0 \to w(t)$ defines a strongly continuous semigroup e^{At} with generator A on the space $\mathcal{H} \equiv H^1(\Omega)/const$, given by $H^1(\Omega)$ quotient the constant functions, topologized by the gradient norm. More precisely, a suitable translation $(A - k^2 I)$ generates a s.c. contraction semigroup on $\mathcal{H}$, so that $\|e^{At}\|_{\mathcal{H}} \le e^{k^2 t}$, $t \ge 0$.

(c) The same semigroup conclusion holds true on the space $\mathcal{H}$, if the dissipative term w_t is omitted (Neumann B.C.).

We refer to [**L-T-Z.2**, Section 7 and Appendix C]. Theorem 2.6.1 is a special case of Theorem 7.2.

8. Global Uniqueness Theorems. Continuous Observability. Stabilization

We begin with the main case of interest in the present paper.

The case with pure homogeneous Neumann B.C. on Σ. We consider the following over-determined problem with Γ_0 as yet unspecified and $\Gamma_1 = \Gamma \setminus \Gamma_0$:

$$
\begin{cases}
iw_t + \Delta w = F(w) & \text{in } (0,T] \times \Omega \equiv Q; & (8.1a)\\
\langle Dw, \nu\rangle \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma; & (8.1b)\\
w|_{\Sigma_1} \equiv 0 & \text{in } (0,T] \times \Gamma_1 \equiv \Sigma_1. & (8.1c)
\end{cases}
$$

Thus, $f \equiv 0$ here. As a corollary of estimate (2.1.13) of Theorem 2.1.3, once extended as in Theorem 7.1, we obtain the following global uniqueness theorem.

Theorem 8.1. *Assume hypotheses (A.1) = (1.4), (A.2) = (1.5), (A.3) = (2.1.9) on $F(w)$.*

(a) Let, at first, w be a solution of Eqn. (8.1a) in the class $H^{2,2}(Q)$ in (2.1.1) and of the Neumann B.C. (8.1b) on Σ. Then, with reference to the boundary terms $\tilde{B}_\Sigma(w)$ and $B_\Sigma(w)$:

$$\tilde{B}_\Sigma(w) = B_\Sigma(w) = 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi}[2\tau^2|Dd|^2 + \Phi]|w|^2 \langle Dd, \nu\rangle d\Sigma - 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi}[\xi_t \eta - \xi \eta_t]\langle Dd, \nu\rangle d\Sigma - 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi}|\langle Dw, \mu\rangle|^2 \langle Dd, \nu\rangle d\Sigma. \tag{8.2}$$

Furthermore, assume the geometrical condition $\langle Dd, \nu\rangle \equiv 0$ on Γ_0 in (1.11). Then we obtain

$$C \int_0^T \int_{\Gamma_1} \left[|w|^2 + |w_t|^2 + |\langle Dw, \mu\rangle|^2\right] d\Sigma_1 \geq \tilde{B}_\Sigma(w), \tag{8.3}$$

regardless of the choice of Φ: either $\Phi \equiv 0$ or else $\Phi = \tau \Delta d$, see (4.3b).

(b) Let now w be a solution of (8.1a) in the finite energy class (2.2.1) satisfying the Neumann B.C. (8.1b). Assume the additional assumption (A.4) = (2.2.2), so that Theorem 7.1 holds true. Let, still, $\langle Dd, \nu\rangle = 0$ on Γ_0. Then, the following inequality holds true:

$$C \int_0^T \int_{\Gamma_1} \left[|w|^2 + |w_t|^2 + |\langle Dw, \mu\rangle|^2\right] d\Sigma_1 \geq \tilde{B}_\Sigma(w) \geq k_{\varphi,\tau}[\mathbb{E}(T) + \mathbb{E}(0)]. \tag{8.4}$$

(c) **(Global uniqueness)** *Let now w be a solution in the finite energy class (2.2.1) of the over-determined problem (8.1a-b-c). Furthermore, assume the geometrical condition $\langle Dd, \nu\rangle \equiv 0$ on Γ_0 in (1.11), as well as assumption (A.4) = (2.2.2). Then, in fact, $w \equiv 0$ in Q. [This conclusion improves upon Corollary 2.1.2 by enlarging the class of solutions from $H^{2,2}(Q)$-solution in (2.1.1) to the finite energy class of solutions (2.2.1).]*

Proof. (a) We return to $\tilde{B}_\Sigma(w)$ given by (2.1.14) and see that (8.1b) yields $\tilde{B}_\Sigma(w) = B_\Sigma(w)$. Moreover, passing to $B_\Sigma(w)$ as given by (2.1.5), we see that its 2^{nd}, 4^{th}, 5^{th} integral terms vanish, while $|Dw|^2 = |\langle Dw, \nu\rangle|^2 + |\langle Dw, \mu\rangle|^2$. This leads to identity (8.2). Under the additional assumption $\langle Dd, \nu\rangle \equiv 0$ on Γ_0, then (8.2) leads to (8.3) at once.

(b) Here, the starting point is that, under present assumptions, the extension Theorem 7.1 holds true. Thus, estimate (2.1.13) of Theorem 2.1.3 holds true also for finite energy solutions in the class (2.2.1). Then (2.1.13) and (8.3) yield (8.4).

(c) Now all boundary terms in (8.4) vanish, so that $\tilde{B}_\Sigma(w) = 0$, since $w|_{\Sigma_1} \equiv 0$, so that $\langle Dw, \mu\rangle \equiv 0$ on Σ_1, for the tangential vector field μ. But then $\tilde{B}_\Sigma(w) = 0$ implies $\mathbb{E}(0) = 0$ by (8.4), hence $w_0 = 0$ by definition (1.9), as desired. Since the problem is forward well-posed as a s.c. semigroup on $H^1(\Omega)/\text{const}$, see Theorem 7.2(b), it follows that $w \equiv 0$ in $\mathbb{R}_t^+ \times \Omega$. [Or else, we invoke inequality (6.12b) with: $s = 0$, $\mathbb{E}(0) = 0$, and $G(T) = 0$ by (6.13), (8.1b). Hence, the Gronwall inequality yields $\mathbb{E}(t) \equiv 0$ in $\mathbb{R}_t^+ \times \Omega$.] $\square$

The case of homogeneous Dirichlet B.C. on Σ. We consider the following over-determined problem with Γ_0 as yet unspecified and $\Gamma_1 = \Gamma \setminus \Gamma_0$:

$$\begin{cases} iw_t + \Delta w = F(w) & \text{in } (0,T] \times \Omega \equiv Q; & (8.5a) \\ w|_\Sigma = 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma; & (8.5b) \\ \langle Dw, \nu \rangle \equiv 0 & \text{in } (0,T] \times \Gamma_1 \equiv \Sigma_1. & (8.5c) \end{cases}$$

Again, $f \equiv 0$ here. As a corollary of Theorem 2.1.3, that is, of estimate (2.1.13), we obtain the following global uniqueness/continuous observability theorem. A big difference with respect to the Neumann B.C. case in (8.1a-b-c), there is no need of the regularizing treatment addressed in Section 7, leading to Theorem 7.1 (whose proof is closely based on the proof of [**L-T-Z.2**, Section 7] translated into the Riemannian setting). Thus, we do not need to inherit the additional assumption (A.4) = (2.2.2) on which Section 7 rests (not critically, however). Thus, the situation now is more favorable. We obtain:

THEOREM 8.2. *Assume hypotheses (A.1) = (1.4), (A.2) = (1.5), as well as assumption (A.3) = (2.1.9) on $F(w)$.*

(a) Let w be a solution of Eqn. (8.5a) in the class $H^{2,2}(Q)$ in (2.1.1), and of the Dirichlet B.C. (8.5b) on Σ. Then, with reference to the boundary terms $\tilde{B}_\Sigma(w)$ and $B_\Sigma(w)$ in (2.1.14) and (2.1.5), we have:

$$\tilde{B}_\Sigma(w) = B_\Sigma(w) + C \int_0^T \int_\Gamma |\langle Dw, \nu\rangle|^2 |\langle R_1, \nu\rangle| d\Sigma \tag{8.6}$$

$$= 2\tau \int_0^T \int_\Gamma e^{2\tau\varphi} |\langle Dw, \nu\rangle|^2 \langle Dd, \nu\rangle d\Sigma \tag{8.7}$$

$$+ C \int_0^T \int_\Gamma |\langle Dw, \nu\rangle|^2 |\langle R_1, \nu\rangle| d\Sigma.$$

(b) **(Continuous observability inequality)** *Let now w be a solution of Eqn. (8.5a) in the finite energy class (2.2.1) satisfying the Dirichlet B.C. (8.5b) and $w_0 \in H^1(\Omega)$. Furthermore, assume the geometrical condition $\langle Dd, \nu\rangle \le 0$ on Γ_0 as in (1.10). Finally, we recall that without loss of generality we can achieve the additional property that $\langle R_1, \nu\rangle \equiv 0$ on Γ_0, see (2.1.11b). Then the following continuous observability inequality holds true:*

$$\int_0^T \int_{\Gamma_1} |\langle Dw, \nu\rangle|^2 d\Sigma_1 \ge \tilde{B}_\Sigma(w) \ge k_{\varphi,\tau}[\mathbb{E}(T) + \mathbb{E}(0)]. \tag{8.8}$$

(c) **(Global uniqueness)** *Let now w be a solution of the over-determined problem (8.5a-b-c) in the finite energy class (2.2.1). Further, assume the properties in part (b): that is, let $\langle Dd, \nu\rangle \le 0$ on Γ_0 as in (1.10), and, moreover, w.l.o.g. we may achieve the additional property that $\langle R_1, \nu\rangle \equiv 0$ on Γ_0, see (2.1.11b). Then, in fact, $w \equiv 0$ in Q.*

Proof. (a) By (8.5b) and hence $\langle Dw, \mu\rangle \equiv 0$ on Σ, μ a tangential vector field, we see from (2.1.14) that (8.6) holds true. Moreover, all terms in $B_\Sigma(w)$ given by (2.1.5) vanish by (8.5b), except the last two terms, where we use—as usual [**T-Y.1**, Eqn. (8.6)]—as a consequence of (8.5b), that $Dw = \langle Dw, \nu\rangle \nu$, hence:

$$\text{(8.9)} \qquad \text{on } \Sigma: \quad \begin{aligned} &\langle Dd, D\bar{w}\rangle = \langle D\bar{w}, \nu\rangle \langle Dd, \nu\rangle; \\ &\langle Dd, Dw\rangle = \langle Dw, \nu\rangle\ \langle Dd, \nu\rangle;\ |Dw| = |\langle Dw, \nu\rangle|. \end{aligned}$$

Then, conclusion (8.7) follows at once from (8.6) and (8.9) and (2.1.5).

(b) Theorem 2.1.1 (or Theorem 5.1) and Theorem 2.1.3 continue to hold true for finite energy solutions in the class (2.2.1) if these satisfy the Dirichlet B.C. (8.5b), as explained at the outset of Section 7. (Ultimately, this is due to the trace regularity result of [**L-T.2**], [**L-T.4**, Chapter 10, Section 9] translated in the Riemannian setting. Moreover, since $\langle Dd, \nu\rangle \le 0$ on Σ_0 and w.l.o.g. (see (2.1.11b)) $\langle R_1, \nu\rangle = 0$ on Σ_0, then (8.7) combined with the basic estimate (2.1.13) of Theorem 2.1.3 yields (8.8).

(c) The additional assumption (8.5c) used in (8.8) implies $\mathbb{E}(0) = 0$ or $w_0 = 0$. Since problem (8.5) is forward well-posed as a s.c. semigroup on $\mathcal{H} \equiv H^1(\Omega)/\text{const}$ (Theorem 7.2(c)), then it follows that $w \equiv 0$ in $R_t^+ \times \Omega$. □

Remark 8.1. Theorem 8.2, parts (b) and (c) hold true with $\Gamma_1 \equiv \Gamma$, in which case the assumptions $\langle Dd, \nu\rangle \le 0$ on Γ_0 and w.l.o.g. (see (2.1.11b), $\langle R_1, \nu\rangle \equiv 0$ on Γ_0 are empty. □

While in equality (8.8) is the Continuous Observability Inequality (COI) for the Dirichlet problem (8.5), by contrast inequality (8.4) is not yet the COI of the Neumann problem (8.1), because of the presence of the term $|\langle Dw, \mu\rangle|^2$ on Γ_1.

To control the tangential gradient $|\langle Dw, \mu\rangle| = |D_{\tan} w|^2$, we shall invoke the following result from [**Tr.1**], [**T-Y.1**, p. 658], which uses micro-local analysis.

THEOREM 8.3. *Let $f \in L_2(Q)$ and let w be a solution of Eqn. (1.1) in the class (2.1.1).*

(a) Given a non-empty portion Γ_1 of Γ of positive measure, given $T > 0$ and given $0 < \epsilon < T$ and $\epsilon_0 > 0$ arbitrarily small, there exists a constant $C_{\epsilon,\epsilon_0,T} > 0$ such that

$$\text{(8.10)} \qquad \begin{aligned} &\int_\epsilon^{T-\epsilon} \int_{\Gamma_1} |\langle Dw, \mu\rangle|^2 d\Gamma_1 dt \\ &\quad \le C_{\epsilon,\epsilon_0,T} \Big\{ \int_0^T \int_{\Gamma_1} |w_t|^2 d\Gamma_1 dt + \int_0^T \int_\Gamma |\langle Dw, \nu\rangle|^2 d\Gamma\, dt \\ &\qquad + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|f\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q)} \Big\}. \end{aligned}$$

Remark 8.2. Inequality (8.10) will suffice for our purposes but, in fact, on the RHS of (8.10), w_t could be penalized in $L_2(0, T; H^{-1}(\Gamma_1))$. Refer to [**L-T-Z.2**, Remark 8.2]. □

THEOREM 8.4. **(Continuous observability inequality, Neumann case)** *Assume the hypotheses of Theorem 8.1b: that is, hypotheses (A.1) = (1.4), (A.2) = (1.5), (A.3) = (2.1.9), and (A.4) = (2.2.2) on $F(w)$, and, finally, the geometric condition $\langle Dd, \nu\rangle \equiv 0$ on Γ_0, as in (1.11).*

Let w be the finite energy solution in the class (2.1.2) of Eqn. (8.1a) satisfying the Neumann B.C. (8.1b) and $w_0 \in H^1(\Omega)$. Then the following COI holds true:

There exists a constant $C_T > 0$ such that

$$\int_0^T \int_{\Gamma_1} |w_t|^2 d\Sigma_1 \geq C_T[\mathbb{E}(T) + \mathbb{E}(0)]. \tag{8.11}$$

Proof. Step 1. Under the present hypothesis, we first establish the weaker conclusion: there exists $C_T > 0$ such that

$$\int_0^T \int_{\Gamma_1} [|w|^2 + |w_t|^2] d\Sigma_1 + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \geq C_T[\mathbb{E}(0) + \mathbb{E}(T)]. \tag{8.12}$$

To this end, we substitute (8.10) with $f \equiv 0$ in estimate (8.3), this time rewritten over $[\epsilon, T-\epsilon]$ rather than over $[0,T]$, thereby obtaining by virtue of (8.1b):

$$C\left\{ \int_\epsilon^{T-\epsilon} \int_{\Gamma_1} [|w|^2 + |w_t|^2] d\Gamma_1 dt + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \right\} \geq k_{\varphi,\tau}\mathbb{E}(\epsilon). \tag{8.13}$$

Next, we recall from (6.32), where $G(T) = 0$ by (6.13) and (8.1b), that

$$\mathbb{E}(\epsilon) \geq \frac{\mathbb{E}(0) + \mathbb{E}(T)}{2}\, e^{-C_T T}. \tag{8.14}$$

Substituting (8.14) into the RHS of (8.13) yields (8.12), as desired.

Step 2. To eliminate the interior l.o.t. in estimate (8.12), we apply the by now standard compactness/uniqueness argument [**L-T.1**], [**Lio.1**], [**Lit.1**]. To this end, we invoke the global uniqueness Theorem 8.1(c). This way, (8.13) yields

$$\int_0^T \int_{\Gamma_1} [|w|^2 + |w_t|^2] d\Sigma_1 \geq C_T[\mathbb{E}(T) + \mathbb{E}(0)]. \tag{8.15}$$

Step 3. Indeed, we can likewise absorb the lower-order boundary term $\int_{\Sigma_1} |w|^2 d\Sigma_1$, again by using a compactness/uniqueness argument, which invokes the global uniqueness Theorem 8.1(c). This way, (8.15) yields (8.11). Steps 2 and 3 can be combined. □

Sketch of proof of Theorem 2.6.1. (a) Part (a) follows as in [**L-T-Z.2**, Appendix C.2: see in particular, Eqn. (C.24) and Eqn. (C.28) with $r_1 \equiv 0$ for dissipativity and Eqn. (C.34) for maximality]. [We refer also the more general Theorem 7.3 above.]

(b) Estimate (2.6.5) follows by returning to Theorem 2.1.3, extended as in Section 7, where we use the B.C. (2.6.1c) in $\tilde{B}_\Sigma(w)$ in (2.1.14), and where we invoke Theorem 8.3 for the tangential gradient: precisely as in the proof of Theorem 8.4. The passage from (2.6.5) to (2.6.6) is standard. □

9. Replacement of Assumption (A.2) = (1.5) by Virtue of Two Vector Fields

Orientation. This section is the counterpart, in the case of non-conservative Schrödinger equations, of the Euclidean case, in [**L-T-Z.2**] as extended to the present Riemannian setting $\{M, g\}$. In turn, these efforts in the case of Schrödinger equations are the counterpart of the prior treatments of second-order hyperbolic equations in [**L-T-Z.1**, Section 10] (Euclidean setting) and [**T-Y.1**, Section 10] (Riemannian setting). The *common goal is to dispense with the working assumption* (A.2) = (1.5). This is done, as in all the aforementioned references, by writing Ω as the overlapping union of two "suitable" subdomains Ω_1 and Ω_2, in correspondence of two functions d_k, $k = 1, 2$, each strictly convex in Ω and thus satisfying (1.4), where now, however, each d_k has *no critical point* on Ω_k, $k = 1, 2$. This way, two vector fields are employed.

9.1. Basic Setting Using Two Conservative Vector Fields as in (1.4). Statement of Main Results. Postulated setting. We divide the original open bounded set Ω of M into two overlapping subdomains Ω_1 and Ω_2: $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$, chosen (in infinite many ways) as to fulfill all the conditions, in particular (a)–(b) below (following the setting of Section 1), as well as the conclusion (9.1.14) of Proposition 9.1.1. We assume that:

(a) there exist two functions $d_k : \overline{\Omega} \to \mathbb{R}^+$ of class C^3, $k = 1, 2$, which are strictly convex in the metric g. This means that the Hessian $D^2 d_k$ of d_k (a two-order tensor) satisfies $D^2 d_k(X, X) > 0$, $\forall\, x \in \overline{\Omega} \subset M$, $\forall\, X \in M_x =$ the tangent space at x. By compactness of Ω, we can always achieve that: There exists a constant $\rho > 0$ such that

$$D^2 d_k(X, \overline{X}) \equiv \langle D_X(Dd_k), \overline{X}\rangle_g \geq \rho |X|_g^2, \quad \forall\, x \in \Omega,\ \forall\, X \in T_x M,\ k = 1, 2; \tag{9.1.1}$$

and by translation we can achieve that

$$\min_{\overline{\Omega}} d_k(x) \geq m > 0, \qquad k = 1, 2; \tag{9.1.2}$$

(b) each function d_k has no critical points on $\overline{\Omega}_k$, $\quad k = 1, 2$:

$$\inf_{\overline{\Omega}_k} |Dd_k| \geq 2p > 0. \tag{9.1.3}$$

Next, following Section 1, we define the functions

$$\varphi_k(x, t) = d_k(x) - c\left(t - \frac{T}{2}\right), \quad x \in \Omega,\ 0 \leq t \leq T;\ k = 1, 2, \tag{9.1.4}$$

where, for $T > 0$ preassigned arbitrary, the constant $c = c_T$ is selected as to satisfy

$$cT^2 > 4 \max_{\overline{\Omega}} d_k(x) + 4\delta, \quad k = 1, 2, \tag{9.1.5}$$

for some $\delta > 0$ suitably small and kept fixed henceforth. Such functions φ_k have thus the following *properties*:

(i)

$$\varphi_k(x, 0) = \varphi_k(x, T) = d_k(x) - c\,\frac{T^2}{4} \leq -\delta, \text{ uniformly in } x \in \overline{\Omega}; \tag{9.1.6}$$

(ii) there are t_0, t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that

$$\min_{x \in \overline{\Omega}, t \in [t_0, t_1]} \varphi_k(x, t) \geq \sigma > 0, \quad 0 < \sigma < m, \tag{9.1.7}$$

since $\varphi_k(x, \frac{T}{2}) = d_k(x) \geq m > 0$ for all $x \in \Omega$.

Since $T > 0$ is arbitrarily small but fixed, we see from (9.1.5) that w.l.o.g. we may take $c \geq 1$, so that $c \geq \sqrt{c}$. Accordingly, we define the functions

$$\varphi_k^*(x, t) \equiv d_k(x) - \sqrt{c}\left(t - \frac{T}{2}\right)^2, \ x \in \Omega;\ 0 \leq t \leq T;\ k = 1, 2; \tag{9.1.8}$$

so that, by (9.1.4), (9.1.8), we have

$$\varphi_k^*(x, t) \geq \varphi_k(x, t),\ x \in \Omega;\ 0 \leq t \leq T;\ k = 1, 2. \tag{9.1.9}$$

Furthermore, we define the sets (subsets of the cylinder $Q \equiv \Omega \times [0,T]$):

$$(9.1.10) \quad Q_k(\sigma) \equiv \{(x,t) \in \Omega \times [0,T] : \varphi_k(x,t) \geq \sigma > 0\};$$

$$(9.1.11) \quad Q_k^*(\sigma^*) \equiv \{(x,t) \in \Omega \times [0,T] : \varphi_k^*(x,t) \geq \sigma^* > 0\};\ 0 \leq \sigma^* < \sigma < m,$$

with constant σ^* selected as to satisfy $0 < \sigma^* < \sigma < m$.

Recalling (9.1.7), (9.1.10), and (9.1.11), we obtain

(i)

$$(9.1.12a) \qquad \Omega_k \times [t_0, t_1] \subset Q_k(\sigma) \subset Q_k^*(\sigma^*) \subset \Omega \times [0,T] \equiv Q;$$

(ii)

(9.1.12b) by (9.1.6), at $t = 0$ and $t = T$: no point of Ω belongs to $Q_k(\sigma)$.

After the above setting, we may state the following result:

Proposition 9.1.1. *Consider the case (the worst case scenario of Remark 1.3) where—in the setting of the original Section 1—the given strictly convex function $d(x)$ (that is, satisfying (1.4)) has a critical point x_0 on the boundary Γ_0: $Dd(x_0) = 0$, $x_0 \in \Gamma_0$. Assume further that such $d(x)$ is given by*

$$(9.1.13) \qquad d(x) = \rho^2(x); \quad \rho(x) = dist_g(x, x_0), \quad x_0 \in \Gamma_0,$$

as is the case for numerous illustrations given in Section 10. Let $d_1(x) \equiv d(x) + m$, $m > 0$, in Ω so that property (9.1.2) holds true in Ω. Let $Q_1^*(\sigma^*)$ be the corresponding set defined in (9.1.11) for some $0 < \sigma^* < m$.

Then, the following property holds true: by taking σ^* sufficiently close to m, we may always achieve that

$$(9.1.14) \qquad \begin{cases} \text{for any } (x,t) \in Q_1^*(\sigma^*) \text{ (defined by (9.1.11)), we then have} \\ \text{that } |Dd_1(x)| \geq p > 0, \end{cases}$$

for some constant $p > 0$. Moreover, we may then define a subset $\Omega_1 \subset \Omega$ such that the required property (9.1.3) on Ω_1 holds true (in addition to (9.1.1) and (9.1.2)).

Remark 9.1.1. Proposition 9.1.1 extends with cosmetic modifications to the case where $d(x)$ in (9.1.13) is taken to be $d(x) = h(\rho(x))$ for a strictly increasing function $h(r)$, $h'(r) > 0$ with $h(0) = 0$. □

Proof. Let $d_1(x) \equiv d(x) + m$ on Ω. For, as yet, any constant σ^*, $0 < \sigma^* < m$, we consider the boundary surface of the corresponding set $Q_1^*(\sigma^*)$ defined by (9.1.11); that is, the level set

$$(9.1.15) \qquad \varphi_1^*(x,t) \equiv \rho^2(x) - \sqrt{c}\left(t - \frac{T}{2}\right)^2 \equiv \sigma^* > 0.$$

For $t = \frac{T}{2}$, this level set yields a surface $\mathcal{S}$ in Ω of all points $x \in \Omega$, where distance $\rho(x) = \mathrm{dist}_g(x, x_0)$ from the aforementioned critical point $x_0 \in \Gamma_0$ is $\sqrt{\sigma^*}$:

$$(9.1.16) \qquad \mathcal{S} \equiv \{x \in \Omega :\ \rho(x) = \mathrm{dist}_g(x, x_0) = \sqrt{\sigma^*} > 0\}.$$

On the other hand, for $t \neq \frac{T}{2}$, $0 < t \leq T$, the points x on the boundary surface of the set $Q_1^*(\sigma^*)$ defined by (9.1.15) are further away from the point $x_0 \in \Gamma_0$, at the larger distance

$$(9.1.17) \qquad \rho(x) = \mathrm{dist}_g(x, x_0) = \left\{\sigma^* + \sqrt{c}\left(t - \frac{T}{2}\right)^2\right\}^{\frac{1}{2}} > \sqrt{\sigma^*}$$

from it. This means that the orthogonal projections of the time-space set $Q_1^*(\sigma^*)$ (along the time axis) onto Ω is a subset $\mathcal{O}_1(\sigma^*)$

$$\text{Proj}_t[Q_1^*(\sigma^*)] = \mathcal{O}_1(\sigma^*) \equiv x \in \Omega : \ \text{dist}_g(x, x_0) \geq \sqrt{\sigma^*}\} \tag{9.1.18}$$

of Ω, having the following two properties: (i) the surface $\mathcal{S}$ is part of boundary of $\mathcal{O}_1(\sigma^*)$ near x_0; (ii) all of the points of $\mathcal{O}_1(\sigma^*)$ lie on the side of $\mathcal{S}$ further away from x_0. With $\sigma^* < m$ as selected in (9.1.11), we then define the sought-after set Ω_1 by

$$\Omega_1 \equiv \{x \in \Omega : \ \rho(x) = \text{dist}_g(x, x_0) \geq \sqrt{m}\} \subset \mathcal{O}_1(\sigma^*). \tag{9.1.19}$$

On Ω_1 we have (recall $Dd_1 = \nabla_g d_1 = Dd = \nabla_g d$) from (9.1.13):

$$\text{on } \Omega_1 : \quad |Dd_1(x)| \ = \ 2 \ \text{dist}_g(x, x_0)|D(\text{dist}_g(x, x_0))| \tag{9.1.20}$$

$$\geq \ 2\sqrt{m} \equiv 2p, \tag{9.1.21}$$

by invoking, in the last step, (9.1.19) Moreover, in the last step from (9.1.20) to (9.1.21) we have used that $|D(\text{dist}_g(x, x_0)| \geq 1$. One way to see this is as follows: in a geodesic frame of a neighborhood of x, one can choose the geodesic γ joining x and x_0 as one axis of the above geodesic frame. This then gives $D_\gamma(\text{dist}_g(x, x_0)) \equiv 1$. In (9.1.21), the constant p is defined by $p = \sqrt{m}$. Thus, by choosing σ^* sufficiently close to m, we see via (9.1.18)–(9.1.21), that we can have $|Dd_1(x)| \geq p$ on $\mathcal{O}_1(\sigma^*)$, since Dd_1 is continuous. By definition of $\mathcal{O}_1(\sigma^*)$ in (9.1.18), this conclusion is precisely (9.1.14), as desired. □

The main result of the present paper is the following Theorem 9.1.2, which extends all previous results: Theorem 2.1.1, Corollary 2.1.2, Theorem 2.1.3 of Section 2.

THEOREM 9.1.2. *All main results of Section 2—Theorem 2.1.1, Corollary 2.1.2, Theorem 2.1.3—continue to hold true with assumption (A.2) = (1.5) removed and replaced by the setting of the present Section 9 based on hypotheses (a) = (9.1.1), (b) = (9.1.3), so that under (9.1.13), Proposition 9.1.1, that is, property (9.1.14) holds true.*

Remark 9.1.2. In (9.1.13) we could use, more generally, $d(x) = h(\rho(x))$, where $h(r)$ has the properties: $h(0) = 0$, $h'(r) > 0$, $r > 0$ (strict monotonicity). □

9.2. Cut-off Functions χ_k and Corresponding Sub-problems for $w_k = \chi_k w$, $k = 1, 2$. Cut-off functions χ_k. Let $\chi_k(t, x)$ be smooth functions, $k = 1, 2$. At this stage, it is not important to specify how the function χ_k is constructed. Eventually, in the case of purely Neumann B.C. associated with (1.1)—as in (1.11)—the function $\chi_k(t, x)$ will be the complicated function constructed in [**L-T-Z.1**, Section 10.2], which has the important feature, among others, of being *only* time-dependent (but not space-dependent) on a *small interior layer of the boundary* Γ. This latter goal is dictated by the Neumann B.C. and would not be necessary when dealing with the Dirichlet B.C. (1.10). At any rate, we only assume here at present that such cut-off functions fulfill the requirement

$$|\chi_k| \leq \text{const, and } \chi_k(t, x) \equiv 1 \text{ on } Q_k(\sigma), \ k = 1, 2, \tag{9.2.1}$$

which is one of the properties satisfied by the cut-off functions in [**L-T-Z.1**, Section 10.2].

Dynamical systems for $w_k \equiv \chi_k w$. Let $w \in C^2(\mathbb{R}_t \times \Omega)$, $\Omega \subset M$, be a solution of (1.1). We introduce new variables on $[0,T] \times \Omega$:

$$w_k(t,x) \equiv \chi_k(t,x)w(t,x); \quad f_k(t,x) \equiv \chi_k(t,x)f(t,x), \quad k=1,2. \tag{9.2.2}$$

We then readily see that each term w_k satisfies the following problem:

$$\begin{cases} iw_{k,t} + \Delta w_k = F(w_k) + f_k + \mathcal{K}_k w; \quad i=\sqrt{-1},\ k=1,2; & \text{(9.2.3a)} \\ w_k(0,\ \cdot\ ) \equiv w_{k,0} = \chi_k(0,\ \cdot\ )w(0,\ \cdot\ ); & \text{(9.2.3b)} \end{cases}$$

$$\mathcal{K}_k = \left[i\,\frac{d}{dt} + \Delta - F, \chi_k \right] = \text{commutator active only on } (\text{supp } \chi_k), \tag{9.2.3c}$$

where F is defined in (1.2a) and satisfies estimate (1.2b), so that we now obtain

$$\begin{aligned} &|F(w_k) + f_k + \mathcal{K}_k w|^2 \\ &\quad\leq\ c_T \left\{ |Dw_k|^2 + |w_k|^2 + |f_k|^2 + (|Dw|^2 + |w|^2)|_{\text{supp}\chi_k} \right\},\ (t,x) \in Q. \end{aligned} \tag{9.2.4}$$

Preliminary estimate: Counterpart of Corollary 4.2, Eqn. (4.13). As constructed above, each problem w_k in (9.2.3), $k=1,2$, satisfies the counterpart of Theorem 4.1, in particular estimate (4.10). We take (4.10) as our present starting point. Our next result is the counterpart of [**L-T-Z.2**, Proposition 9.2.1] in the case of the Schrödinger equation in the Euclidean setting. In turn, this result was the counterpart of [**L-T-Z.1**, Proposition 10.3.1], in the case of the generalized wave equation in the Euclidean setting, or [**T-Y.1**, Proposition 10.2.1] in the case of the Riemannian wave equation.

PROPOSITION 9.2.1. *Let $w \in C^2(\mathbb{R}_t \times \Omega)$, $\Omega \subset M$, be a solution of (1.1). Let the setting of Section 9.1 based on assumptions (a) = (9.1.1), (b) = (9.1.3) be in force, so that, under (9.1.13), Proposition 9.1.1 holds true. Then, each problem (9.2.3), $k=1,2$, satisfies the following pointwise inequality for $\epsilon > 0$ small and $0 < \delta_0 < 1$:*

(9.2.5)

$$\begin{aligned} &B_\Sigma(w_k) + \left(1 + \frac{1}{\epsilon}\right) \left\{ C \int_Q e^{2\tau\varphi_k} |f_k|^2 dQ + Ce^{2\tau\sigma} \int_0^T \mathbb{E}(t)dt \right\} \\ &\quad\geq\ \left[\delta_0 \left(2\rho\tau - \frac{\epsilon}{2} \right) - \left(1 + \frac{1}{\epsilon}\right) C_T \right] \int_0^T \int_\Omega e^{2\tau\varphi_k} \left[|Dw_k|^2 + |w_k|^2 \right] d\Omega\, dt \\ &\quad\quad + [4\rho(1-\delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)] \int_{Q_k(\sigma)} e^{2\tau\varphi_k} |w_k|^2 dx\, dt \\ &\quad\quad - (C_{d_k,T})\tau e^{-2\tau\sigma} [\mathbb{E}(T) + \mathbb{E}(0)]. \end{aligned}$$

Remark 9.2.1. For τ sufficiently large as to make $[4\rho(1-\delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)]$, we reach one of our goals and drop the integral term involving $|w_k|^2$ in (9.2.5). ☐

Proof. Step 1. Here we shall show that w_k satisfies the following estimate (counterpart of (4.13))

$$(9.2.6)\quad \left(1+\frac{1}{\epsilon}\right)\theta_k^2\Big|i\frac{\partial}{\partial t}w_k+\Delta w_k|^2-\frac{\partial \mathcal{M}_k}{\partial t}+\text{div } V_k$$

$$\geq \quad \delta_0\left(2\rho\tau-\frac{\epsilon}{2}\right)\theta_k^2|Dw_k|^2+B_k(\tau)\theta_k^2|w_k|^2, \quad \forall\ (t,x)\in Q,$$

where $0<\delta_0<1$ is a suitable constant, and where the coefficient $B_k(\tau)$ in front of the lower-order term is defined by (recall (4.13)):

$$(9.2.7)\qquad B_k(\tau)\equiv[4\rho\tau^3(1-\delta_0)|Dd_k|^2+\mathcal{O}(\tau^2)],$$

and satisfies the estimate

$$(9.2.8\text{a})\qquad B_k(\tau)\ \geq\ [4\tau^3\rho p^2(1-\delta_0)+\mathcal{O}(\tau^2)]\quad \text{in } Q_k^*(\sigma^*);$$

$$(9.2.8\text{b})\qquad B_k(\tau)\ =\ \mathcal{O}(\tau^3)\qquad \text{in } [Q_k^*(\sigma^*)]^c,$$

recalling the set (9.1.11), where $[Q_k^*(\sigma^*)]^c$ is the complement of $Q_k^*(\sigma^*)$ in $[0,T]\times\Omega\equiv Q$.

Indeed, we first write the present version of estimate (4.10), our present starting point, for w_k, involving also $v_k=\theta_k w_k$ (see (3.2)):

$$(9.2.9)\qquad \left(1+\frac{1}{\epsilon}\right)\theta_k^2\left|i\frac{\partial}{\partial t}w_k+\Delta w_k\right|^2-\frac{\partial \mathcal{M}_k}{\partial t}+\text{div } V_k$$

$$\geq\quad [4\tau\rho-\epsilon]|Dv_k|^2+[4\tau^3\rho|Dd_k|^2+\mathcal{O}(\tau^2)]\theta_k^2|w_k|^2.$$

Next, recalling estimate (4.18) on the passage from $|Dv_k|^2$ to $|Dw_k|^2$:

$$(9.2.10)\qquad [4\tau\rho-\epsilon\|Dv_k|^2\ \geq\ \left[2\tau\rho-\frac{\epsilon}{2}\right]\delta_0\theta_k^2|Dw_k|^2$$

$$-\ 4\tau^3\rho\delta_0|Dd_k|^2\theta_k^2|w_k|^2+\epsilon\delta_0\tau^2|Dd_k|^2\theta_k^2|w_k|^2,$$

and using (9.2.10) on the RHS of (9.2.9), we obtain

$$(9.2.11)\qquad \text{RHS of (9.2.9)}\ \geq\ \left[2\tau\rho-\frac{\epsilon}{2}\right]\delta_0\theta_k^2|Dw_k|^2+[4\tau^3\rho|Dd_k|^2(1-\delta_0)$$

$$+\quad \mathcal{O}(\tau^2)]\theta_k^2|w_k|^2.$$

Finally, recalling the definition (9.2.7) of B_k as well as $|Dd_k|\geq p>0$ from (9.1.3), we see that (9.2.11) yields inequality (9.2.6), with B_k subject to (9.2.8a) on $Q_k^*(\sigma^*)$ and (9.2.8b) on its complement, as desired.

Step 2. Integrating the pointwise estimate (9.2.6) on all of $(0,T]\times\Omega\equiv Q$, we obtain via (9.2.3a),

$$(9.2.12)\qquad \left(1+\frac{1}{\epsilon}\right)\int_0^T\int_\Omega e^{2\tau\varphi_k}|F(w_k)+f_k+\mathcal{K}_k w|^2dQ$$

$$+\ (C_{d_k,T})\tau e^{-2\tau\delta}[\mathbb{E}_k(T)+\mathbb{E}_k(0)]+B_\Sigma(w_k)$$

$$\geq\quad \delta_0\left(2\rho\tau-\frac{\epsilon}{2}\right)\int_0^T\int_\Omega e^{2\tau\varphi_k}|Dw_k|^2dQ$$

$$+\ [4\rho(1-\delta_0)p^2\tau^3+\mathcal{O}(\tau^2)]\int_0^T\int_\Omega\theta_k^2|w_k|^2dQ,$$

counterpart of (5.5).

Step 3. Regarding the LHS of (9.2.12), the following estimate holds true:

$$\begin{aligned}(9.2.13)\quad & \int_0^T \int_\Omega e^{2\tau\varphi_k}|F(w_k) + f_k + \mathcal{K}_k w|^2 dQ \\ & \leq \; C\int_Q e^{2\tau\varphi_k}|f_k|^2 dQ + c_T \int_Q e^{2\tau\varphi_k}[|Dw_k|^2 + |w_k|^2]dQ \\ & \qquad + ce^{2\tau\sigma}\int_0^T \mathbb{E}(t)dt.\end{aligned}$$

In fact, we recall (9.2.4) and obtain

(9.2.14)

$$\begin{aligned}&\text{LHS of (9.2.13)} \\ &\leq \; c_T \left\{ \int_Q e^{2\tau\varphi_k}|f_k|^2 dQ + \int_{Q_k(\sigma)} e^{2\tau\varphi_k}[|Dw_k|^2 + |w_k|^2 + |Dw|^2 + |w|^2]dx\,dt \right. \\ &\qquad \left. + \int_{[Q_k(\sigma)]^c} e^{2\tau\varphi_k}[|Dw_k|^2 + |w_k|^2 + |Dw|^2 + |w|^2]dx\,dt \right\},\end{aligned}$$

since $Q = Q_k(\sigma) \cup [Q_k(\sigma)]^c$ by definition. On $Q_k(\sigma)$: we have $\chi_k \equiv 1$, by (9.2.1), hence $w_k \equiv w$. On $[Q_k(\sigma)]^c$: we use $|Dw_k| \leq c|Dw|$, $|w_k| \leq c|w|$ by (9.2.1) and, moreover, $\varphi_k \leq \sigma$ by the definition (9.1.10) of $Q_k(\sigma)$, thus $e^{2\tau\varphi_k} \leq e^{2\tau\sigma}$. This way, (9.2.14) leads to (9.2.13) via (1.1).

Step 4. Regarding the last integral term on the RHS of (9.2.12), we recall that $\theta_k = e^{2\tau\varphi_k}$, where $\varphi_k \leq \varphi_k^*$ on Q by (9.1.9), and where $\varphi_k^* \leq \sigma^*$ on $[Q_k^*(\sigma^*)]^c$ by (9.1.11). Thus, $\varphi_k \leq \sigma^*$ on $[Q_k^*(\sigma^*)]^c$. By definition, $Q_k^*(\sigma^*) \cup [Q_k^*(\sigma^*)]^c \equiv Q$. Hence,

$$\begin{aligned}(9.2.15)\quad & [4\rho(1-\delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)]\int_Q \theta_k^2 |w_k|^2 dQ \\ & \geq \; [4\rho(1-\delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)]\int_{Q_k^*(\sigma^*)} e^{2\tau\varphi_k}|w_k|^2 dx\,dt \\ & \qquad - c\tau^3 e^{2\tau\sigma^*}\int_{[Q_k^*(\sigma^*)]^c} |w_k|^2 dx\,dt \\ (9.2.16)\quad & \geq \; [4\rho(1-\delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)]\int_{Q_k(\sigma)} e^{2\tau\varphi_k}|w_k|^2 dx\,dt \\ & \qquad - c\tau^3 e^{2\tau\sigma^*}\int_Q |w|^2 dx\,dt,\end{aligned}$$

where in the last step, we have also used $Q_k(\sigma) \subset Q_k^*(\sigma^*)$, by (9.1.12a) and $|\chi_k| \leq$ const, by (9.2.1), $k = 1, 2$.

Step 5. We return to inequality (9.2.12). On its LHS we use estimate (9.2.13); on its RHS we invoke estimate (9.2.16). This way we obtain

$$\begin{aligned}(9.2.17)\qquad &\left(1+\frac{1}{\epsilon}\right)\Bigg\{C\int_Q e^{2\tau\varphi_k}|f_k|^2dQ + c_T\int_Q e^{2\tau\varphi_k}[|Dw_k|^2+|w_k|^2]dQ \\ &\quad + \ c\tau^3e^{2\tau\sigma^*}\int_Q |w|^2dQ + Ce^{2\tau\sigma}\int_0^T \mathbb{E}(t)dt\Bigg\} \\ &\quad + (C_{d_k,T})\tau e^{-2\tau\delta}[\mathbb{E}_k(T)+\mathbb{E}_k(0)] + B_\Sigma(w_k) \\ &\geq \ \delta_0\left(2\rho\tau-\frac{\epsilon}{2}\right)\int_0^T\int_\Omega e^{2\tau\varphi_k}|Dw_k|^2dQ \\ &\quad + [4\rho(1-\delta_0)p^2\tau^3+\mathcal{O}(\tau^2)]\int_{Q_k(\sigma)} e^{2\tau\varphi_k}|w_k|^2dx\,dt.\end{aligned}$$

But $\tau^3e^{2\tau\sigma^*}\leq ce^{2\tau\sigma}$ since $\sigma^*<\sigma$ by (9.1.11): hence, on the LHS of (9.2.17), the third integral term containing $\int_Q|w|^2dQ$ is absorbed by the fourth integral term containing $\int_0^T\mathbb{E}(t)dt$:

$$(9.2.18)\qquad C\tau^3e^{2\tau\sigma^*}\int_Q|w|^2dQ\leq C_1e^{2\tau\sigma}\int_0^T\mathbb{E}(t)dt,$$

$\mathbb{E}(t)$ defined in (1.9). Moreover,

$$(9.2.19)\qquad [\mathbb{E}_k(T)+\mathbb{E}_k(0)]\leq c[\mathbb{E}(T)+\mathbb{E}(0)],\ \text{since } 0\leq\chi_k\leq c.$$

This way, by the last two estimates, we see that (9.2.17) yields (9.2.5) and Proposition 9.2.1 is established. □

Remark 9.2.2 (Alternative path over the use of property (9.1.14) of Proposition 9.1.1)**.** Alternatively over the use of property (9.1.14) of Proposition 9.1.1, one may introduce an additional cut-off function $\psi_k(x)$ such that

$$\psi_k(x)=\begin{cases}1 & \text{on } \Omega_k\\ 0 & \text{outside } \tilde\Omega_k\end{cases}\qquad k=1,2,$$

where $\tilde\Omega_k$ is a slightly enlarged set over Ω_k for which $|Dd_k(x)|\geq p>0$ on $\tilde\Omega_k$. This can always be achieved, for instance, if (9.1.14) holds true, since Dd_k is continuous. Then, $w_k(t,x)$ and $f_k(t,x)$ in (9.2.2) are replaced by

$$\tilde w_k(t,x)=\psi_k(x)\chi_k(t,x)w(t,x)=\psi_k(x)w_x(t,x);$$

$$\tilde f_k(t,x)=\psi_k(x)\chi_k(t,x)w(t,x)=\psi_k(x)f_k(t,x).$$

This yields corresponding $\tilde w_k$-problems in place of the (9.2.3)-problem for w_k. Regarding the critical term $B_k(\tau)$ in (9.2.7), we distinguish two cases: (i) on $\tilde\Omega_k$ we proceed as before in (9.2.8a), since $|Dd_k(x)|\geq p>0$ on Ω_k; (ii) in the complementary part $\Omega/\tilde\Omega_k$, we have that $w_k\equiv 0$.

Finally, given $x\in\Omega$, then $x\in\Omega_k$ for some k, by $\Omega=\Omega_1\cup\Omega_2$, and so $|\nabla d_k(x)|\geq p>0$ for this k. Thus, $Dd_1(x)$ and $Dd_2(x)$ do *not* vanish simultaneously, we can combine the two cases. □

9.3. Carleman Estimate, First Version, for the w-Problem. Building upon Proposition 9.2.1, we obtain the counterpart of Theorem 5.1, as in [**L-T-Z.2**, Theorem 9.3.1].

THEOREM 9.3.1. *Let $T > 0$ and assume the setting of Section 9.1: in particular, assumptions (a) = (9.1.1) and (b) = (9.1.2), so that under (9.1.13), Proposition 9.1.1 holds true. Let $w \in C^2(\mathbb{R}_t \times \Omega)$, $\Omega \subset M$, be a solution of Eqn. (1.1) [and no B.C.] under the standing assumption (1.2b) on F and (1.3) on f. Then: the following one-parameter family of estimates hold true for all $\tau > 0$ sufficiently large and all $\epsilon > 0$ small:*

$$\text{(9.3.1a)} \qquad B_\Sigma(w) + C\int_Q |f|^2 dQ \geq \left[\delta_0\left(2\rho\tau - \frac{\epsilon}{2}\right) - \left(1 + \frac{1}{\epsilon}\right)C_T\right] e^{2\tau\sigma}\int_{t_0}^{t_1} \mathbb{E}(t)dt$$
$$- \; Ce^{2\tau\sigma}\int_0^T \mathbb{E}(t)dt - (C_{d,T})\tau e^{-2\tau\delta}[\mathbb{E}(T) + \mathbb{E}(0)].$$

$$\text{(9.3.1b)} \qquad B_\Sigma(w) = \sum_{k=1}^{2} B_\Sigma(w_k); \; C_{d,T} = \sum_{k=1}^{2} C_{d_k,T}.$$

Proof. Step 1. We recall property (9.1.7): $\varphi_k \geq \sigma > 0$ in $[t_0, t_1] \times \Omega$; and, moreover, that $[t_0, t_1] \times \Omega_k \subset Q_k(\sigma)$ by (9.1.12a), where $w_k \equiv w$ on $Q_k(\sigma)$, since $\chi_k \equiv 1$ here by (9.2.1). Thus we estimate the first integral terms on the RHS of (9.2.5) for $k = 1, 2$:

$$\text{(9.3.2)} \qquad \sum_{k=1}^{2}\int_0^T\int_\Omega e^{2\tau\varphi_k}[|Dw_k|^2 + |w_k|^2]d\Omega\, dt$$
$$\geq \sum_{k=1}^{2}\int_{t_0}^{t_1}\int_{\Omega_k} e^{2\tau\sigma}[|Dw_k|^2 + |w_k|^2]d\Omega_k\, dt$$
$$= e^{2\tau\sigma}\sum_{k=1}^{2}\int_{t_0}^{t_1}\int_{\Omega_k}[|Dw|^2 + |w|^2]d\Omega_k\, dt$$
$$\geq e^{2\tau\sigma}\int_{t_0}^{t_1}\int_\Omega[|Dw|^2 + |w|^2]d\Omega\, dt,$$

where the last step follows, since the integral terms over Ω_k, $k = 1, 2$, collect also the contributions on the non-empty set $\Omega_1 \cap \Omega_2$, as assumed.

Step 2. With τ sufficiently large as to have $[4\rho(1-\delta_0)p^2\tau^3 + \mathcal{O}(\tau^2)] > 0$, so that Remark 9.2.1 applies, we sum up Eqn. (9.2.5) of Proposition 9.2.1 for $k = 1, 2$, and obtain, by virtue of (9.3.2):

$$\begin{aligned}(9.3.3)\quad &\left(1+\frac{1}{\epsilon}\right)\left\{C\sum_{k=1}^{2}\int_Q e^{2\tau\varphi_k}|f_k|^2dQ + Ce^{2\tau\sigma}\int_0^T \mathbb{E}(t)dt\right\} + \sum_{k=1}^{2} B_\Sigma(w_k)\\ &\geq \left[\delta_0\left(2\rho\tau - \frac{\epsilon}{2}\right) - \left(1+\frac{1}{\epsilon}\right)C_T\right] e^{2\tau\sigma}\int_{t_0}^{t_1}\int_\Omega [|Dw|^2+|w|^2]d\Omega\, dt\\ &\quad - \tau\left(\sum_{k=1}^{2} C_{d_k,T}\right) e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)],\end{aligned}$$

and (9.3.1) is established, via (1.9) for $\mathbb{E}(t)$. □

Final Remark. To obtain the desired Theorem 9.1.1, one proceeds precisely as in [**L-T-Z.1**, Sections 10.4–10.7], which admit a faithful counterpart *mutatis mutandis* from second-order hyperbolic equations to Schrödinger equations. Therefore, this part of the argument will not be repeated.

10. Assumption (A.1): Classes of Strictly Convex Functions on $\{M, g\}$

In this section, we provide classes of examples of strictly convex functions on the Riemannian manifold $\{M, g\}$; that is, satisfying assumption (A.1) = (1.4). Two main cases will be emphasized.

Subsection 10.1 is devoted to the case arising from Schrödinger equations with variable coefficient (in space) principal part, which are defined on a Euclidean bounded domain $\Omega \subset \mathbb{R}^n$. Here, then, the Riemannian manifold is $\{\mathbb{R}^n, g\}$, where the metric g is derived from the coefficients $a_{ij}(x)$ of the basic differential operator; in fact, $[g_{ij}(x)] = [a_{ij}(x)]^{-1}$. For this case—a primary reason for studying Schrödinger equations on a Riemannian manifold—several classes of examples of strictly convex functions were already given explicitly in prior references [**T-Y.1**], [**L-T-Y.1**], [**Y.1**], [**Y.2**].

Subsection 10.2 considers instead additional genuine Riemannian manifolds $\{M, g\}$, $M \neq \mathbb{R}^n$.

10.1. Variable Coefficient Schrödinger Equations Defined on a Bounded Euclidean Domain. Schrödinger equations on a Euclidean domain.

In this section, $\tilde{\Omega}$ is an open bounded domain in $\mathbb{R}^n$, with boundary $\partial\tilde{\Omega} = \tilde{\Gamma}$ of class, say, C^2. Let $x = [x_1, x_2, \ldots, x_n]$, and let

$$(10.1.1)\qquad \mathcal{A}w = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right);\ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2,\quad x\in\tilde{\Omega},$$

be a second-order differential operator, with real coefficients $a_{ij} = a_{ji}$ of class C^1, see Remark 10.1.1 below, satisfying the uniform ellipticity condition for some positive constant $a > 0$. Thus, we can extend $a_{ij}(x)$ smoothly to all of $\mathbb{R}^n$, so that the matrices

$$(10.1.2)\quad A(x) = (a_{ij}(x));\quad G(x) = [A(x)]^{-1} = (g_{ij}(x)),\ i,j = 1,\ldots,n,\ x\in\mathbb{R}^n$$

are positive definite on any $x \in \mathbb{R}^n$.

Let $\tilde{F}$ be a linear, first-order differential operator: $\tilde{F}(w) = \tilde{R}(t,x) \cdot \nabla_0 w + r(t,x)w$, satisfying

$$|\tilde{F}(w)| \leq C_T[|\nabla_0 w|^2 + |w|^2], \quad t,x \in \tilde{Q} = (0,T] \times \tilde{\Omega} \text{ a.e.,} \tag{10.1.3}$$

where ∇_0 is the gradient in $\mathbb{R}^n$ and " $\cdot$ " is the $\mathbb{R}^n$-inner product. The corresponding Schrödinger equation on $\tilde{\Omega}$ is

$$iw_t + \mathcal{A}w = \tilde{F}(w) \quad \text{in } \tilde{Q} \tag{10.1.4}$$

to be supplemented by the initial condition and by boundary conditions.

Riemannian metric. Let $\mathbb{R}^n$ have the usual topology with natural coordinate system $x = [x_1, x_2, \ldots, x_n]$. For each $x \in \mathbb{R}^n$, define the inner product and the norm on the tangent space $\mathbb{C}^n_x = \mathbb{C}^n$ by

$$g(X,Y) = \langle X,Y \rangle_g = \sum_{i,j=1}^{n} g_{ij}(x)\alpha_i\overline{\beta}_j; \tag{10.1.5}$$

$$|X|_g = [\langle X,Y \rangle_g]^{\frac{1}{2}}, \quad X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{C}^n_x. \tag{10.1.6}$$

It is easily checked that $(\mathbb{R}^n, g)$ is a complete Riemannian manifold with the Riemannian metric $g = \sum_{i,j=1}^n g_{ij} dx_i dx_j$. (If $A(x) = I$, i.e., $\mathcal{A} = -\Delta$, then $G(x) = I$ and g is the Euclidean $\mathbb{R}^n$-metric.) One also has [**Y.2**, Eqn. (2.2.11), p. 393]

$$\mathcal{A}w = -\Delta_g w + (Df)(w); \quad f(x) = \frac{1}{2} \ln \, \det[a_{ij}(x)], \tag{10.1.7}$$

where Δ_g is the corresponding Laplace-Beltrami operator; that is, under the change of metric, from the original Euclidean metric to g, we have that the second-order elliptic operator (10.1.1) becomes Δ_g on $(\mathbb{R}^n, g)$, modulo a first-order term. Thus, Eqn. (10.1.4) is turned into Eqn. (1.1), where (1.2b) is satisfied.

Remark 10.1.1. Let the coefficients a_{ij} in (10.1.1) be of class C^1, as assumed. Then the entries g_{ij} in (10.1.2) are of class C^1 as well. Thus, the connection coefficients (Christoffel symbols) Γ^{ℓ}_{ik}, see [**Do.1**, p. 54], are of class C^0. The geodesic solutions to a corresponding second-order, nonlinear ordinary differential equation [**Do.1**, p. 62] are then of class C^2. Thus, the square of the distance function $\text{dist}^2_g(x, x_0)$ is in C^2. Typically, but by no means always, the required strictly convex function is taken to be $\text{dist}^2_g(x, x_0)$, under suitable assumptions on the sectional curvature. See below. We also notice that in our case, where the manifolds are complete, the geodesics exist globally. □

Classes of examples of strictly convex functions $d(x)$ in the Riemannian metric g. Several classes of $(a_{ij}(x))$ yielding strictly convex functions $d(x)$ in the metric g induced by $(g_{ij}(x)) = (a_{ij}(x))^{-1}$ on all of $\mathbb{R}^n$ are given in [**L-T-Y.1**], [**T-Y.1**], [**Y.1**], [**Y.2**]. Thus, *any* (sufficiently smooth) open bounded domain may be taken. Often, Green-Wu's [**G-W.1**] theorem is invoked, see Theorem 10.2.2.2 below. This assortment of examples can also be derived from the more systematic treatment of Section 10.2 to follow.

10.2. The General Riemannian Case $\{M, g\}$. Orientation. In this subsection, we provide classes of examples of strictly convex functions (that is, satisfying (A.1)) on a general Riemannian manifold $\{M, g\}$. To this end, we shall proceed according to the following strategy:

(i) At first, in Subsection 10.2.1, we provide strictly convex functions $d(x)$ on three canonical Riemannian manifolds: the sphere (S^n, can); the Euclidean space $(\mathbb{R}^n, \mathrm{can})$; the hyperbolic space (H^n, can), with canonical metrics. These are the typical manifolds with constant sectional curvature K: respectively, $K > 0$, $K = 0$, $K < 0$. In these three canonical cases, the required strictly convex function $d(x)$ is constructed as a composition $d(x) = h(\rho(x))$ of s suitable function $h(\,\cdot\,)$ and the underlying distance function $\rho(x) = \mathrm{dist}_g(x, x_0)$, $x_0 \in M$.

(ii) Subsequently, the Hessian Comparison Theorem is invoked and used to further enlarge the class of examples in point (i), by making a comparison with the three canonical cases.

10.2.1. *Riemannian Manifolds with Constant Sectional Curvature: ($\mathbf{S}^n$,can), ($\mathbf{R}^n$,can), ($\mathbf{H}^n$,can).* Let ($\mathbf{S}^n$,can), ($\mathbf{R}^n$,can), ($\mathbf{H}^n$,can) be the sphere, the Euclidean space, the hyperbolic space, with canonical metrics. These are the typical manifold with constant sectional curvature. We shall use the notation (M, g) to denote any one of them with everywhere constant sectional curvature K. Given $x_0 \in M$, denote $\rho = \rho(x) = \mathrm{dist}(x, x_0)$ the distance function from x_0 to x with respect to the metric g. From [**W-S-Y.1**], we have

$$D^2\rho(Y,Y) = \begin{cases} \sqrt{K}\cot\left(\sqrt{K}\rho(x)\right)|Y|_g^2, & K > 0; \quad (10.2.1.1a) \\ \frac{1}{\rho(x)}|Y|_g^2, & K = 0; \quad (10.2.1.1b) \\ \sqrt{-K}\coth\left(\sqrt{-K}\rho(x)\right)|Y|_g^2, & K < 0, \quad (10.2.1.1c) \end{cases}$$

for all $Y \in T_xM$ and $Y \perp (D\rho)(x)$.

Next, let $h(t)$ be a C^∞ function on $\mathbf{R}$. Let X be a vector field on (M, g). Write $X =< X, D\rho >_g D\rho + Y$, where $Y \perp D\rho$. We have

$$D^2h(\rho)(X,X) = \langle D_X(Dh(\rho)), X\rangle_g. \tag{10.2.1.2}$$

We compute $D(h(\rho)) = h'(\rho)D\rho$ and hence:

$$\begin{aligned} D_X(Dh(\rho)) &= D_X(h'(\rho)D\rho) = X(h'(\rho))D\rho + h'(\rho)D_X(D\rho) \\ &= h''(\rho)\langle D\rho, X\rangle D\rho + h'(\rho)D_X(D\rho). \end{aligned} \tag{10.2.1.3}$$

Substituting (10.2.1.3) in (10.2.1.2) yields

$$D^2h(\rho)(X,X) = h'(\rho)D^2\rho(X,X) + h''(\rho)\langle X, D\rho\rangle_g^2 \tag{10.2.1.4}$$

$$= h'(\rho)D^2\rho(Y,Y) + h''(\rho)\langle X, D\rho\rangle_g^2, \tag{10.2.1.5}$$

where in the last step we have used the identity $D^2\rho(X,X) = D^2_\rho(Y,Y)$, where $X = \langle X, D\rho\rangle_g D\rho + Y$, $Y \perp D\rho$. Indeed, we preliminarily have

$$\begin{aligned} D^2\rho(X,X) &= D^2\rho(\langle X, D\rho\rangle D\rho + Y, \langle X, D\rho\rangle D\rho + Y\rangle \\ &= D^2\rho(Y,Y) + 2D^2\rho(Y, D\rho)\langle X, D\rho\rangle + D^2\rho(D\rho, D\rho)\langle X, D\rho\rangle^2, \end{aligned}$$

where $D_{D\rho}(D\rho) = 0$ so that $D^2\rho(D\rho, D\rho) = \langle D_{D\rho}(D\rho), D\rho\rangle = 0$ and $D^2\rho(D\rho, Y) = \langle D_{D\rho}(D\rho), Y\rangle \equiv 0$, and the claim is proved.

We can choose special C^∞ function $h(t)$ to get strictly convex function $d(x) = h(\rho(x))$.

Example 10.2.1.1. Square of distance function: $d(x) = \frac{1}{2}\,\mathbf{dist}^2(x, x_0) = \frac{1}{2}\rho^2(x)$**.** Now we have $h(t) = \frac{1}{2}t^2$. From (10.2.1.5) with such $h(\,\cdot\,)$ we obtain, invoking also (10.2.1.1):

$$D^2 d(X,X) = \rho(x) D^2\rho(Y,Y) + \langle X, D\rho\rangle_g^2$$

$$= \begin{cases} (\sqrt{K}\rho(x))\cot(\sqrt{K}\rho(x))|Y|_g^2 + \langle X, D\rho\rangle_g^2, & K>0; \quad (10.2.1.6a)\\ |Y|_g^2 + \langle X, D\rho\rangle_g^2, & K=0; \quad (10.2.1.6b)\\ (\sqrt{-K}\rho(x))\coth(\sqrt{-K}\rho(x))|Y|_g^2 + \langle X, D\rho\rangle_g^2, & K<0. \quad (10.2.1.6c)\end{cases}$$

Next, we restrict $\rho(x) = \operatorname{dist}(x, x_0)$ as follows:
(10.2.1.7)

$$\rho(x) = \operatorname{dist}(x,x_0) \le \frac{\beta}{\sqrt{K}} < \frac{\pi}{2\sqrt{K}}, \text{ or } 0 < \sqrt{K}\rho(x) \le \beta < \frac{\pi}{2} \text{ on } (\mathbf{S}^n, \text{can});$$

so that $\frac{\sqrt{K}\rho}{\sin(\sqrt{K}\rho)} \ge 1$ and $\cos(\sqrt{K}\rho(x)) \ge \cos\beta > 0$. Then (10.2.1.6a) yields

$$\begin{aligned} Dd^2(X,X) &\ge \cos(\sqrt{K}\rho)[|Y|_g^2 + \langle X, D\rho\rangle_g^2] \\ &\ge (\cos\beta)|X|_g^2, \quad K>0, \quad \cos\beta>0, \end{aligned} \tag{10.2.1.8}$$

under (10.2.1.7), since $X = \langle X, D\rho\rangle_g D\rho + Y$, $Y \perp D\rho$. Similarly, for this same orthogonality relation, (10.2.1.5b) yields

$$D^2 d(X,X) = |X|_g^2, \qquad K = 0. \tag{10.2.1.9}$$

Finally, as to (10.2.1.6c), since $f(x) = \frac{x}{\tan h(x)} \to +\infty$ as $x \to +\infty$, and $f(x) \to 1$ as $x \to 0$ by L'Hopital's Rule, we have $\min_{x\ge 0} f(x) = f(0) = 1$. Hence, (10.2.1.6c) yields once more by the orthogonality property

$$D^2 d(X,X) \ge |Y|_g^2 + \langle X, D\rho\rangle_g^2 = |X|_g^2, \quad K<0. \tag{10.2.1.10}$$

Conclusion. The function $d(x) = \frac{1}{2}\rho^2(x) = \frac{1}{2}\operatorname{dist}^2(x, x_0)$ is strictly convex on $(\mathbb{R}^n, \text{can})$, where $K = 0$, and on $(\mathbf{H}^n, \text{can})$, where $K < 0$, by (10.2.1.9) and (10.2.1.10); moreover, $d(x) = \frac{1}{2}\rho^2(x)$ is strictly convex also on $(\mathbf{S}^n, \text{can})$, where $K > 0$, by (10.2.1.8), *provided that it satisfies* (10.2.1.7). □

Example 10.2.1.2. ([**Y.2**, Proposition 2.1] As in [**Y.2**], we choose the C^∞ function $h(t)$ as

$$h(t) = \begin{cases} -\cos(\sqrt{K}t), & K>0; \quad (10.2.1.11a)\\ \frac{1}{2}t^2, & K=0; \quad (10.2.1.11b)\\ \cosh(\sqrt{-K}t), & K<0. \quad (10.2.1.11c)\end{cases}$$

For $d(x) = h(\rho(x))$, from (10.2.1.5), using first (10.2.1.5), and then (10.2.1.1a-b-c), we obtain

$$D^2h(\rho)(X,X) \;=\; D^2d(X,X) = h'(\rho)D^2\rho(Y,Y) + h''(\rho)\langle X, D\rho\rangle_g^2$$

$$= \begin{cases} K\cos(\sqrt{K}\rho(x))|X|_g^2, & K > 0; \quad (10.2.1.12a) \\ |X|_g^2, & K = 0; \quad (10.2.1.12b) \\ -K\cosh(\sqrt{-K}\rho(x))|X|_g^2, & K < 0, \quad (10.2.1.12c) \end{cases}$$

where in the last step we have used once more the orthogonality relation used in obtaining (10.2.1.8).

¿From (10.2.1.12a), under the same restriction as in (10.2.1.7), so that $\cos(\sqrt{K}\rho(x)) \geq \cos\beta > 0$, for some $0 < \beta < \frac{\pi}{2}$, we obtain (10.2.1.13a), while for (10.2.1.13c) we use $\cosh x \geq 1$. We obtain from (10.2.1.12):

$$D^2d(X,X) \begin{cases} \geq (\cos\beta)|X|_g^2, & K > 0, \cos\beta > 0; \quad (10.2.1.13a) \\ = |X|_g^2, & K = 0; \quad (10.2.1.13b) \\ \geq (-K)|X|_g^2, & K < 0. \quad (10.2.1.13c) \end{cases}$$

Conclusion. From the above we can see that: $d(x) = h(\rho(x))$ is a strictly convex function on $(\mathbf{R}^n, \text{can})$ where $K = 0$; and $(\mathbf{H}^n, \text{can})$, where $K < 0$, by (10.2.1.13b) and (10.2.1.13c); moreover, $d(x) = h(\rho(x))$ is strictly convex also on $(\mathbf{S}^n, \text{can})$, where $K > 0$, by (10.2.1.13a) *provided that it satisfies* (10.2.1.7). □

10.2.2. *Two General Results of the Existence of Strictly Convex Functions.* We report here two known results: one of recent origin and valid on 2-dimensional manifolds dim $n = 2$, and one very established and well known.

THEOREM 10.2.2.1 ([**B-G-L.1**] Strictly convex function in 2-D case via curvature flows). *Let $\bar{\Omega}$ be a two-dimensional, smooth, compact, Riemannian surface whose boundary $\partial\Omega$ has positive second fundamental form. Assume there are no closed geodesics in Ω, then there exists a C^2 strictly convex function in $\bar{\Omega}$.*

This theorem is proved in [**B-G-L.1**] by using a nonlinear parabolic equation which arises in a quite unrelated geometric problem of curve-shortening flows.

THEOREM 10.2.2.2 (Theorem 1(a) in [**G-W.1**]). *Complete non-compact manifold with positive sectional curvature] If M is a complete, non-compact, Riemannian manifold of everywhere positive sectional curvature, then there exists a C^∞ Lipschitz continuous strictly convex exhaustion function on M.*

Here a function $f : M \to \mathbf{R}$ is an exhaustion function if for every $\lambda \in \mathbf{R}$, $f^{-1}((-\infty, \lambda])$ is a compact subset of M.

10.2.3. *The Hessian Comparison Theorem.*

THEOREM 10.2.3.1. **(**[**S-Y.1**, Theorem 1.1]**)**. *Let M_1 and M_2 be two n-dimensional, complete, Riemannian manifolds. Assume that $\gamma_j : [0, a] \to M_j$ $(j = 1, 2)$ are two geodesics parameterized by arc length, and γ_j does not intersect the cut* *locus of $\gamma_j(0)$. Let ρ_j be the distance function from $\gamma_j(0)$ on M_j, and let K_j be the sectional curvature of M_j. Assume that at $\gamma_1(t)$ and $\gamma_2(t)$, $0 \le t \le a$, we have*

$$K_1\left(X_1, \frac{\partial}{\partial\gamma_1}\right) \ge K_2\left(X_2, \frac{\partial}{\partial\gamma_2}\right),$$

where X_j is any unit vector in $T_{\gamma_j(t)}M_j$ perpendicular to $\frac{\partial}{\partial\gamma_j}$. Then

$$D^2\rho_1(X_1, X_1) \le D^2\rho_2(X_2, X_2),$$

where $X_j \in T_{\gamma_j(a)}M_j$ with $< x_j, \frac{\partial}{\partial\gamma_j} > (\gamma_j(a)) = 0$ and $|X_j| = 1$.

Using the Hessian Comparison Theorem, we obtain the following results:

PROPOSITION 10.2.3.2 (Square of distance function). *If all sectional curvatures of (M, g) have an upper bound K, then:*

(i) if $K > 0$: $d(x) = \frac{1}{2}\mathrm{dist}^2(x, x_0)$ is a strictly convex function on M when $\mathrm{dist}(x, x_0) < \frac{\pi}{2\sqrt{K}}$, $\forall\ x_0 \in M$;

(ii) if $K \le 0$: $d(x) = \frac{1}{2}\mathrm{dist}^2(x, x_0)$ is a strictly convex function on M, where M is simply connected.

PROPOSITION 10.2.3.3 ($h(t)$ as in Example 10.2.1.2). *Let $h(t)$ be defined by (10.2.1.11). If all sectional curvatures of (M, g) have an upper bound K, then:*

(i) if $K > 0$: $d(x) = h(\rho(x))$ is a strictly convex function on M when $\mathrm{dist}(x, x_0) < \frac{\pi}{2\sqrt{K}}$, $\forall\ x_0 \in M$;

(ii) if $K \le 0$: $d(x) = h(\rho(x))$ is a strictly convex function on M, where M is simply connected.

Proof. (i) If $K > 0$, we apply the Hessian Comparison Theorem to $d(x) = \frac{1}{2}\mathrm{dist}^2(x, x_0)$ (or $h(\rho(x))$) for (M, g) and the sphere $(\mathbf{S}^n, \mathrm{can})$ with sectional curvature $= K$, and obtain the result.

(ii) If $K \le 0$ and M is simply connected, we have that M is homeomorphic to $\mathbb{R}^n$ (see Remark 10.2.3.1 below), apply the Hessian Comparison Theorem to $d(x) = \frac{1}{2}\mathrm{dist}^2(x, x_0)$ (or $h(\rho(x))$) for (M, g) and the Euclidean space $(\mathbf{R}^n, \mathrm{can})$ with sectional curvature $= 0$ and obtain the result. $\square$

Remark 10.2.3.1. When $K \le 0$, the reason why we require M to be simply connected is in order to eliminate examples such as a flat torus or H_n/G [ ]. Indeed, when M is simply connected and has $K \le 0$, then M is homeomorphic to $\mathbb{R}^n$ by the exponential map $\exp_{x_0} : T_{x_0}M \equiv \mathbb{R}^n \to M$. $\square$

10.3. Examples of Strictly Convex Functions $d(x)$ Satisfying the Geometrical Condition (2.1.7): $\langle Dd, \nu\rangle|_{\Gamma_0} = \frac{\partial d}{\partial\nu}|_{\Gamma_0} = 0$.

10.3.1. *Geodesic Flat Boundary.* Let (M, g) be a complete Riemannian manifold with metric g. Let Ω be an open, bounded, connected subset of M with smooth boundary (say, of class C^2) $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$.

DEFINITION 10.3.1.1. *An open subset Γ_0 of the boundary Γ is called "locally geodesic flat" if for any two close points $x_1, x_2 \in \Gamma_0$, the unique geodesic $\gamma(t)$ joining x_1 and x_2 with respect to the metric g is contained in Γ_0.*

Now assume that Ω is chosen as to satisfy Proposition 10.2.3.2 or 10.2.3.3, and Γ_0 be a locally geodesic flat connected part of boundary $\partial\Omega$. In order to define the strictly convex function $d(x)$, we need only to choose a suitable point x_0. Choose

$x_0 \in M \backslash \overline{\Omega}$ such that x_0 is on a locally geodesic flat hypersurface S of M which includes Γ_0 as a subset.

Define first $\rho(x) = \text{dist}(x, x_0)$ $\forall x \in \Omega$, and then define $d(x) = h(\rho(x))$ as in Proposition 10.2.3.2 or 10.2.3.3. We then know that $d(x)$ is a strictly convex function on Ω.

Next we check that $\frac{\partial d}{\partial \nu}|_{\Gamma_0} = 0$: Using a geodesic frame on a neighborhood of S, one has $\frac{\partial \rho(x)}{\partial \nu} = 0$, $\forall\, x \in S$. Hence

$$\frac{\partial d(x)}{\partial \nu} = \frac{\partial h(\rho(x))}{\partial \nu} = h'(\rho)\frac{\partial \rho(x)}{\partial \nu} = 0, \qquad \forall x \in \Gamma_0,$$

as desired.

Remark 10.3.1.1. In the Euclidean setting $\mathbb{R}^n$, we can choose Γ_0 to be an open subset of a hyperplane and define $d(x) = \|x - x_0\|^2$ with x_0 on the hyperplane but away from Γ_0 to get $\frac{\partial d}{\partial \nu} = 2(x - x_0) \cdot \nu = 0$ on Γ_0 [**L-T-Z.1**, p. 288]. The present concept of "geodesic flat boundary" is a natural generalization of the Euclidean case. $\square$

10.3.2. *Strictly Convex Function $d(x)$, as a Perturbation of d_0 with $\frac{\partial d_0}{\partial \nu}|_{\Gamma_0} \leq 0$.* Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection D. Let Ω be an open, bounded, connected subset of M with smooth boundary (say, of class C^2) $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. The portion Γ_0 of $\partial\Omega$ is defined as follows: Let $\ell : M \to \mathbf{R}$ be a function of class C^2. Then we define

$$\Gamma_0 = \{x \in \partial\Omega : \ \ell(x) = 0\} \ \textit{with}\ D\ell = \text{grad}_g \ell \neq 0, \ \textit{on}\ \Gamma_0. \tag{10.3.2.1}$$

THEOREM 10.3.2.1. ([**T-Y.1**, Theorem B.1]). *In above setting, assume that:*
(i)(Convexity of ℓ near Γ_0) $D^2\ell(X, X)(x) \geq 0$, $\quad \forall\, x \in \Gamma_0$, $\forall\, X \in T_xM$;
(ii) there exists a function $d_0 : \bar{\Omega} \to \mathbf{R}$ of class C^2, such that

(ii_1)

$$D^2 d_0(X, X)(x) \geq \rho_0 |X|_g^2, \ \textit{where}\ \rho_0 > 0, \quad \forall\, x \in \Gamma_0, \quad \forall\, X \in T_xM;$$

(ii_2)

$$\left.\frac{\partial d_0}{\partial \nu}\right|_{\Gamma_0} = \langle Dd_0, \nu\rangle_g \leq 0, \qquad \textit{on}\ \Gamma_0.$$

Then, there exists a function $d : M \to \mathbf{R}$ of class C^2 (which is explicitly constructed in a layer (collar) of Γ_0, the critical set, Remark 1.4), such that it satisfies the following two conditions:

(a) $\frac{\partial d}{\partial \nu}|_{\Gamma_0} = \langle Dd, \nu\rangle_g = 0$, $\qquad$ *on* Γ_0;

(b) $D^2 d(X, X)(x) \geq (\rho_0 - \epsilon)|X|_g^2$, $\quad \forall\, x \in \Gamma_0$, $\quad \forall\, X \in T_xM$,

where $\epsilon > 0$ is arbitrarily small.

The function $d(x)$ is explicitly constructed near Γ_0, within Ω, as a perturbation of the original function d_0 assumed in (ii) above. For details of proof, we refer to [**T-Y.1**, Appendix B].

Appendix A: The Most General Form Allowed for the Coefficient $P(t, x)$ of the Gradient Term Dw in (1.1), through Changes of Variables and Time and Space Scaling

This is a summarized counterpart of [**L-T-Z.2**, Appendix A]. The purpose is likewise two-fold: (i) to give the most general form allowed here by the coefficient

$P(t,x)$ in (1.2b); (ii) to ascertain what version we can assume in the text without loss of generality. The same suitable changes of variables and time-space scaling displayed in [**L-T-Z.1**, Appendix A] in the Euclidean setting continue to hold true in the present Riemannian setting.

(i) **Case dim** $\Omega \equiv 1$**.** In this case, we have the following claim [**L-T-Z.1**, Theorem A.1].

THEOREM A.1. *The one-dimensional Schrödinger equation*

$$iw_t + \Delta w = F(w) = q_1(t,x)Dw + q_0(t,x)w; \ \ \Delta w = D_{\partial/\partial x}(Dw),$$

under the assumptions

$$q_0, q_1, q_{1t}, |Dq_1| \in L_\infty(Q) \tag{A.1}$$

is transformed into the new form

$$i\tilde{w}_t + \Delta\tilde{w} = \tilde{q}_0(t,x)\tilde{w}; \quad \tilde{q}_0 \equiv q_0 - \frac{i}{2}\,\pi_t + \frac{1}{4}\,(D\pi)^2 - \frac{1}{2}\,\Delta\pi \in L_\infty(Q), \tag{A.2}$$

with no first-order term*, by means of the transformation*

$$\tilde{w}(t,x) \equiv e^{-\frac{1}{2}\pi(t,x)}w(t,x); \quad \pi(t,x) = \int_0^x q_1(t,\xi)d\xi, \tag{A.3}$$

so that $D\pi = q_1$*,* $\Delta\pi = Dq_1$*.* A-fortiori*, assumption (A.3) is then satisfied.*

(ii) **Case dim** $\Omega \geq 2$**.** In this case, as in [**L-T-Z.2**, Appendix A, Theorem A.2], and with the same proof, we have the following result:

THEOREM A.2. *Let dim* $\Omega \geq 2$*, and consider the Schrödinger equation (1.1), as specialized with vector field* $P(t,x) = D\pi(t,x) - iR_1(t,x)$*:*

$$iw_t + \Delta w = \langle (D\pi(t,x) - iR_1(t,x), Dw\rangle + q_0(t,x)w, \tag{A.4}$$

$\pi(t,x)$ *being a real-valued function on* M *and* $R_1(t,x)$ *being a real-valued vector field on* M*, subject to the assumptions*

$$\pi_t, \ |D\pi|, \ \Delta\pi, \ |R_1|, \ q_0 \in L_\infty(Q). \tag{A.5}$$

(i) **Equation***. Then, after performing the change of variable*

$$\tilde{w}(t,x) = e^{-\frac{1}{2}\pi(t,x)}w(t,x), \tag{A.6}$$

the original Eqn. (A.4) is transformed into the new form

$$i\tilde{w}_t + \Delta\tilde{w} = -i\langle R_1(t,x), D\tilde{w}\rangle + \tilde{q}_0(t,x)\tilde{w}, \tag{A.7}$$

that is, with the purely imaginary vector field $-iR_1(t,x)$ *to take the inner product against the first-order (in space) vector field* $D\tilde{w}$*.*

(ii) **Boundary conditions.** *Moreover, the transformation* $w \to \tilde{w}$ *in (A.6) changes the boundary conditions*

$$\langle Dw, \nu\rangle + \beta w = 0; \quad \textit{respectively, } \langle Dw, \nu\rangle + w_t = g \textit{ on } \Sigma, \tag{A.8}$$

for w*, into the boundary conditions*

$$\langle D\tilde{w}, \nu\rangle + \left(\frac{1}{2}\langle D\pi, \nu\rangle + \beta\right)\tilde{w} \equiv 0; \tag{A.9}$$

respectively,

$$\langle D\tilde{w}, \nu\rangle + \tilde{w}_t + \left(\frac{1}{2}\langle D\pi, \nu\rangle + \pi_t\right)\tilde{w} = e^{-\frac{\pi}{2}}g,$$

for $\tilde{w}$*.*

Time/space scaling are given in [**L-T-Z.2**, Appendix A, Proposition A.6].

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