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Homoclinic orbits for first order hamiltonian systems possessing super-quadratic potentials

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1. Introduction

Let $H(t,u): \mathbf{R} \times \mathbf{R}^{2N} \to \mathbf{R}$ be a continuously differentiable function and consider the Hamiltonian system

$$\mathscr{J}\dot{u} + \nabla H(t,u) = 0, \quad (t,u) \in S_T \times \mathbf{R}^{2N},$$

where $H \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is T-periodic in the t-variable, and $\mathscr{J} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ denotes the standard symplectic matrix and ∇ denotes the gradient with respect to the u-variable. Suppose that $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u)$ where $L \in C^1(\mathbf{R}, \mathbf{R}^{4N^2})$ is a $2N \times 2N$ symmetric matrix valued function and $W \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is globally super-quadratic in the u-variable, i.e., the potential H satisfies

(H1) There is a constant $\mu > 2$ such that

$$0 < \mu W(t,u) \leqslant z\nabla W(t,u), \quad \forall |u| > 0.$$

Recall that a solution u of the system (1) is said to be homoclinic to 0 if $u \neq 0$ and $u \to 0$ as $|t| \to \infty$. We are interested in the existence of homoclinic orbits of the system (1). In recent years there have been many papers devoted to the existence of homoclinic orbits for the system (1)(e.g., [1,3–5,8,9]). In most of these papers, there is a condition on the growth of $H(t,u)$ at infinity such as

$(\mathrm{H4})_p$ There are constants $c, R > 0$ and $p \in (1,2)$, such that

$$|\nabla W(t,u)|^p \leqslant c(\nabla W(t,u), u), \quad \forall |u| \geqslant R.$$

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Condition $(H4)_p$ implies that $|\nabla W(t,u)|$ grows at infinity no faster than $|u|^{1/(p-1)}$. For second-order Hamiltonian systems, in [2] the authors proved the existence of homoclinic orbits by assuming only super-quadratic condition (H1) for the potentials. On the other hand, the existence of T-periodic solutions for the system (1) has been proved in [7] and [10] recently under some conditions weaker than $(H4)_p$ on the growth rate of the potential at infinity such as

(H4) There is a constant $c > 0$, such that

$$|\nabla W(t,u)| \leqslant c(\nabla W(t,u),u), \quad \forall |u| \geqslant 1.$$

or

$$(H5) \quad \limsup_{|u|\to\infty} \frac{W_t(t,u)}{|u|^\mu W(t,u)} = 0, \quad \text{or } \liminf_{|u|\to\infty} \frac{W_t(t,u)}{|u|^\mu W(t,u)} = 0, \text{ uniformly in } t.$$

It is natural to ask the following question: Can one relax the condition $(H4)_p$, as has been done for the problem of the existence of T-periodic solutions of the system (1)?

Here, we answer the above question. Using the ideas for the problem on the existence of T-periodic solutions of the system (1) in paper [10], we first prove two new estimates on the bound of C^0-norm of homoclinic orbits of the system (1) under the conditions (H1) and (H4) (or (H5)), then we will show that we can relax condition $(H4)_p$ exactly as in paper [10]. The main purpose of the present paper is to establish the existence of homoclinic orbits of the system (1) both with and without symmetry hypotheses.

Here is the outline of our present paper. In Section 2, we prove two new estimates on the bound of C^0-norm of homoclinic orbits of the system (1) when the corresponding critical values of homoclinic orbits are bounded. In Section 3, we study the symmetric Hamiltonian systems. We first obtain an improvement of Theorem 2.1 in [1] by some precise computations. Since the solutions in [1] are obtained by minimax procedure, we show that the corresponding critical values are bounded. Using our estimates obtained in Section 2, we know the solutions of modified systems to be the solutions of the system (1) for a large enough n, then we get the main results of this section, Theorems 3.1 and 3.2. In Section 4, we study the Hamiltonian systems without symmetry. Using the same idea which is used in Section 3, we get the main results of this section Theorems 4.1 and 4.2.

2. Two estimates

In this section, we first truncate the potential W by an increasing sequence $\{W_n\}$ such that each W_n satisfies the growth condition $(H4)_p$ for some $p \in (1,2)$. Then we give the function space E and the new norm $||\cdot||_E$ according to the self-adjoint operator $A = -(\mathscr{J}\frac{\mathrm{d}}{\mathrm{d}t} + L(t))$ as was done in Section 2 of [3,4] and the corresponding functional I_n with respect to the system (1) replacing W by W_n. Finally, we prove two new estimates on the bound of C^0-norm of homoclinic orbits of the system (1) under the conditions (H1) and (H4) (or (H5)) when the corresponding critical values of homoclinic orbits are bounded.

2.1. Truncation of $W(t,u)$

Set $a=\min_{|u|=1,t\in S_T} W(t,u)$, $b=\max_{|u|\leqslant 1,t\in S_T}|W(t,u)|$. Condition (H1) implies that for some $\beta_3\geqslant 0$

$$\begin{cases} a|u|^\mu \leqslant W(t,u) & \forall |u|\geqslant 1,\\ b|u|^\mu \geqslant W(t,u) & \forall |u|\leqslant 1,\\ a|u|^\mu \leqslant W(t,u)+b\leqslant \frac{1}{\mu}(\nabla W(t,u)u+\beta_3) & \forall u\in \mathbf{R}^{2N}. \end{cases}$$

As was done in [6] (cf. Appendix of [6]), choose $\sigma\in(0,1)$, such that $\mu\sigma>2$, we truncate W as the following proposition:

Proposition 2.1. *Assume that for condition* (H1) *there exist two sequences* $\{K_n\}$ *and* $\{K_n'\}$ *in* $\mathbf{R}$ *and a sequence of functions* $\{W_n\}$ *such that*

(i) $0<K_0<K_n<K_{n+1},\forall n\in\mathbf{N}$, *and* $K_n\to\infty$, *as* $n\to\infty$, *where* $K_0=\max\{1, b/a(1-\sigma)\}$; *and* $K_n<K_n'$, $\forall n\in\mathbf{R}$.
(ii) *For any given* $t\in S_T$, $W_n(t,u)\in C^1(\mathbf{R}^{2N},\mathbf{R})$, *for every* $n\in\mathbf{N}$.
(iii) $W_n(t,u)=W(t,u),\forall|u|\leqslant K_n$, *for every* $n\in\mathbf{N}$; *and for some* $\lambda\in(\sigma,1)$, *such that* $W_n(t,u)=(\tau_n+1)|u|^{\mu\lambda},\forall|u|\geqslant K_n'$, *for every* $n\in\mathbf{N}$.
(iv) $W_n(t,u)\leqslant W_{n+1}(t,u)\leqslant W(t,u),\forall(t,u)\in S_T\times\mathbf{R}^{2N}$.
(v) $0<\mu\sigma W_n(t,u)\leqslant\nabla W_n(t,u)u,\forall|u|>0$, *for every* $n\in\mathbf{N}$.

Note that in [6] the truncating functions are constructed for autonomous Hamiltonian functions. In fact, the proof also works for time dependent $W(t,u)$.

2.2. Setting for functional I_n

Let $A=-(\mathscr{J}(\mathrm{d}/\mathrm{d}t)+L(t))$ be the self-adjoint operator acting on $L^2(\mathbf{R},\mathbf{R}^{2N})$ with the domain $\mathscr{D}(A)=H^1(\mathbf{R},\mathbf{R}^{2N})$ and $\sigma(A)$ be the spectrum of A. We use the norm $||\cdot||_E$, which is defined as

$$||u||_E=(||A|^{1/2}u|_2^2+|u|_2^2)^{1/2},$$

instead of the norm $||\cdot||_\mu$ in [3] or [4], and the Banach space E, which is the completion of the set $\mathscr{D}(A)=H^1(\mathbf{R},\mathbf{R}^{2N})$ under the norm $||\cdot||_E$, instead of the space E_μ in [3] or [4]. As in Section 2 of [3] or [4], we have these facts: E has the direct sum decomposition $E=E^-\oplus E^+$, and E is embedded continuous in L^v for any $v\in[2,\infty)$ and compactly in L^v_{loc} for any $v\in[2,\infty)$. Using our norm $||\cdot||_E$ instead of $||\cdot||_\mu$, the reader can check the details of these facts following the Proofs in Section 2 in [3] or [4]. Setting

$$\Phi_n(u)=\int_{\mathbf{R}} W_n(t,u)\,\mathrm{d}t,$$

it is easy to check if $\Phi_n\in C^1(E,\mathbf{R})$ since each W_n satisfies (H3) and (iii) of Proposition 2.1, and E is embedded continuous in L^v for any $v\in[2,\infty)$; furthermore, we have

$$\Phi_n'(u)v=\int_{\mathbf{R}}\nabla W_n(t,u)v\,\mathrm{d}t,\quad \forall u,v\in E.$$

Consider the functional

$$I_n(u) = \frac{1}{2}\int_{\mathbf{R}} (-\mathscr{J}\dot{u}\cdot u - L(t)u\cdot u)\,\mathrm{d}t - \int_{\mathbf{R}} W_n(t,u)\,\mathrm{d}t$$
$$= \frac{1}{2}(||u^+||_E^2 - ||u^-||_E^2) - \Phi_n(u)$$

for $u = u^+ + u^- \in E = E^+ \oplus E^-$. We have $I_n \in C^1(E,\mathbf{R})$, while any critical point of I_n corresponds to a homoclinic orbit of the system (1) replacing W by W_n.

2.3. Two estimates

Now we will prove two new estimates on the bound of C^0-norm for homoclinic orbits of the system (1) under the growth conditions (H1) and (H4) (or (H5)).

Lemma 2.1. *Suppose that $W(t,u)$ satisfies* (H1), (H4) *and $u(t)$ is a critical point of I_n such that $I_n(u) \leqslant N$, then we have the following estimate*:

$$||u||_{C^0} \leqslant M$$

where M is independent of u and n.

Proof. Since u is a critical point of I_n and each $W_n(t,u)$ satisfies (iii) and (v) of Proposition 2.1, we have

$$I_n(u) = \frac{1}{2}\int_{\mathbf{R}} -(\mathscr{J}\dot{u} + L(t)u)u\,\mathrm{d}t - \int_{\mathbf{R}} W_n(t,u)\,\mathrm{d}t$$
$$= \frac{1}{2}\int_{\mathbf{R}} \nabla W_n(t,u)\cdot u\,\mathrm{d}t - \int_{\mathbf{R}} W_n(t,u)\,\mathrm{d}t$$
$$\geqslant \left(\frac{1}{2} - \frac{1}{\mu\sigma}\right)\int_{\mathbf{R}} \nabla W_n(t,u)\cdot u\,\mathrm{d}t$$
$$\geqslant \left(\frac{\mu\sigma}{2} - 1\right)\int_{\mathbf{R}} W_n(t,u)\,\mathrm{d}t$$

From the above, we have

$$\int_{\mathbf{R}} \nabla W_n(t,u)u\,\mathrm{d}t \leqslant M_1, \quad \int_{\mathbf{R}} W_n(t,u)\,\mathrm{d}t \leqslant M_2 \tag{2}$$

for some constants M_1 and M_2 independent of n. Now integrating (v) of Proposition 2.1 we have

$$W_n(t,u) \geqslant a|u|^{\mu\sigma}, \quad \forall |u| \geqslant 1,$$
$$W_n(t,u) \leqslant b|u|^{\mu\sigma}, \quad \forall |u| \leqslant 1,$$

where $a = \min_{|u|=1, t\in S_T} W(t,u)$ and $b = \max_{|u|\leqslant 1, t\in S_T} |W(t,u)|$ are independent of n since

$$W_n(t,u) = W(t,u), \quad \forall |u| \leqslant K_1 \quad \text{and} \quad \forall n \in \mathbf{N}.$$

We first show that for a large enough n,

$$||u||_{C^0} \leqslant K_n$$

If not, by passing a subsequence, without the loss of generality, for each $n \in \mathbf{N}$, there exists $u_n(t)$ and $t_n \in \mathbf{R}^+$, such that $|u_n(t_n)| = K_n$, $|u_n(0)| = 1$ and $1 \leqslant |u_n(t)| \leqslant K_n$ for $t \in [0, t_n)$. Since

$$M_2 \geqslant \int_{\mathbf{R}} W_n(t, u_n)\,\mathrm{d}t \geqslant \int_0^{t_n} W_n(t, u_n)\,\mathrm{d}t \geqslant a \int_0^{t_n} |u_n|^{\mu}\,\mathrm{d}t \geqslant a \int_0^{t_n} |u_n|\,\mathrm{d}t$$

we have $\int_0^{t_n} |u_n|\,\mathrm{d}t \leqslant M_2/a$. Hence, we have

$$\begin{aligned}
K_n - 1 &= |u_n(t_n)| - |u_n(0)| \\
&= \int_0^{t_n} \frac{\mathrm{d}}{\mathrm{d}s} |u_n(s)|\,\mathrm{d}s \\
&= \int_0^{t_n} u_n(s) \cdot \dot{u}_n(s)/|u_n(s)|\,\mathrm{d}s \\
&\leqslant \int_0^{t_n} |\dot{u}(s)|\,\mathrm{d}s \quad (\text{since } |u_n(s)| \leqslant K_n) \\
&\leqslant \int_0^{t_n} (|L(t)u_n(s)| + |\nabla W(s, u_n(s))|)\,\mathrm{d}s \quad (\text{by (H4)}) \\
&\leqslant ||L||_{L^\infty} \int_0^{t_n} |u_n(s)|\,\mathrm{d}s + c \int_0^{t_n} (\nabla W(s, u_n(s)), u_n(s))\,\mathrm{d}s \\
&\leqslant N_1 M_2 + c \int_{\mathbf{R}} (\nabla W_n(s, u_n(s)), u_n(s))\,\mathrm{d}s \\
&\leqslant N_1 M_2 + c M_1,
\end{aligned}$$

where c, N_1, M_1 and M_2 are n-independent constants. But we have $K_n \to \infty$, *as* $n \to \infty$, which leads to a contradiction. Hence, there exists $m \in \mathbf{N}$, which is determined only by $W(t, u)$ and N, for any $n \geqslant m$; if u is a critical point of I_n with $I_n(u) \leqslant N$, then $||u||_{C^0} \leqslant K_n$ holds.

Hence, for any critical point u of I_n with $I_n(u) \leqslant N$, if $n \geqslant m$, repeating the above computation, we have

$$|u(t)| \leqslant N_1 M_2 + c M_1 + 1, \quad \forall\, t \in \mathbf{R}.$$

For $k < m$, from (iii) of Proposition 2.1, we have

$$|\nabla W_k(t, u)| \leqslant c_k (\nabla W_k(t, u), u), \quad \forall\, |u| > 1$$

for some suitable constant c_k, which is determined by W_k for $k = 1, 2, \ldots, m-1$. Hence, by the same argument as above, we have

$$|u(t)| \leqslant N_1 M_2 + c_k M_1 + 1, \quad \forall\, t \in \mathbf{R}.$$

Then, we have

$$||u||_{C^0} \leqslant \max\{N_1 M_2 + c M_1 + 1,\ N_1 M_2 + c_k M_1 + 1,\ k = 1, 2, \ldots, m - 1\} = M$$

Hence, our Lemma holds. □

Lemma 2.2. *Suppose that $W(t,u)$ satisfies* (H1), (H5) *and $u(t)$ is a critical point of I_n such that $I_n(u) \leqslant N$, then we have the following estimate*:

$$||u||_{C^0} \leqslant M$$

where M is independent of u and n.

Proof. As the first part of the Proof of Lemma 2.1, we have

$$\int_{\mathbf{R}} \nabla W_n(t,u)u \, \mathrm{d}t \leqslant M_1, \quad \int_{\mathbf{R}} W_n(t,u) \, \mathrm{d}t \leqslant M_2$$

for some constants M_1 and M_2 independent of n. Now integrating (v) of Proposition 2.1 we have

$$W_n(t,u) \geqslant a|u|^{\mu\sigma}, \quad \forall |u| \geqslant 1,$$
$$W_n(t,u) \leqslant b|u|^{\mu\sigma}, \quad \forall |u| \leqslant 1,$$

where $a = \min_{|u|=1, t \in S_T} W(t,u)$ and $b = \max_{|u| \leqslant 1, t \in S_T} |W(t,u)|$ are independent of n. Since $W(t,u)$ satisfies (H5), define

$$\sigma(r) = \sup_{|u| \geqslant r, t \in S_T} \frac{W_t(t,u)}{|u|^{\mu} W(t,u)}$$

and

$$\delta(r) = \inf_{|u| \geqslant r, t \in S_T} \frac{W_t(t,u)}{|u|^{\mu} W(t,u)}.$$

Then, (H5) means

$$\lim_{r \to \infty} \sigma(r) = 0 \quad \text{or} \quad \lim_{r \to \infty} \delta(r) = 0.$$

Case I: Suppose that we have

$$\lim_{r \to \infty} \sigma(r) = 0.$$

By the definition of $\sigma(r)$, we have $\sigma(r)$ which decreases to 0. Fix a large $R > 1$, such that

$$a - \sigma(R) M_2 > 0.$$

Firstly, we show $|u|_{C^0} \leqslant K_n$ for large n. If not, by passing a subsequence we may assume that for each n, there exist $u_n(t)$, a_n and b_n such that

$$(a_n, b_n) \subset \{t \in \mathbf{R} | R < |u_n(t)| < K_n\}$$

and $|u_n(a_n)| = R$, $|u_n(b_n)| = K_n$. We have

$$M_2 \geqslant \int_{\mathbf{R}} W_n(t,u_n) \, \mathrm{d}t \geqslant \int_{a_n}^{b_n} W(t,u_n) \, \mathrm{d}t \geqslant \int_{a_n}^{b_n} a|u_n(t)|^{\mu} \, \mathrm{d}t \geqslant (b_n - a_n)a.$$

Hence, $b_n - a_n \leqslant M_2/a$. Here, we have

$$\begin{aligned}
&H(b_n,u_n(b_n)) - H(a_n,u_n(a_n)) \\
&\quad = \int_{a_n}^{b_n} \frac{\mathrm{d}}{\mathrm{d}t} H_n(t,u_n(t))\,\mathrm{d}t \quad (\text{since } |u_n(t)| \leqslant K_n) \\
&\quad = \int_{a_n}^{b_n} \nabla H_n(t,u_n(t))\dot{u}_n(t)\,\mathrm{d}t + \int_{a_n}^{b_n} H_t(t,u_n(t))\,\mathrm{d}t \\
&\quad = \int_{a_n}^{b_n} \frac{1}{2} L'(t)u_n u_n\,\mathrm{d}t + \int_{a_n}^{b_n} W_t(t,u_n)\,\mathrm{d}t \\
&\quad \leqslant \frac{1}{2}||L'||_{L^\infty} \int_{a_n}^{b_n} |u_n|^2\,\mathrm{d}t + \int_{a_n}^{b_n} \sigma(|u_n|)|u_n|^\mu W(t,u_n)\,\mathrm{d}t \\
&\quad \leqslant N_1 M_2 K_n^2 + \sigma(R)K_n^\mu \int_{a_n}^{b_n} W(t,u_n)\,\mathrm{d}t \\
&\quad \leqslant N_1 M_2 K_n^2 + \sigma(R)K_n^\mu \int_{\mathbf{R}} W_n(t,u_n)\,\mathrm{d}t \\
&\quad \leqslant N_1 M_2 K_n^2 + \sigma(R) M_2 K_n^\mu.
\end{aligned}$$

Hence, we have

$$H(b_n,u_n(b_n)) - H(a_n,u_n(a_n)) \leqslant N_1 M_2 K_n^2 + \sigma(R) M_2 K_n^\mu$$

On the other hand, we have

$$\begin{aligned}
&H(b_n,u_n(b_n)) - H(a_n,u_n(a_n)) \\
&\quad = \frac{1}{2}(L(b_n)u_n(b_n),u_n(b_n)) + W(b_n,u_n(b_n)) - H(a_n,u_n(a_n)) \\
&\quad \geqslant a|u_n(b_n)|^\mu - ||L||_{L^\infty}|u_n(b_n)|^2 - \max_{|u|\leqslant R, t\in S_T} |H(t,u)| \\
&\quad = aK_n^\mu - ||L||_{L^\infty}K_n^2 - \max_{|u|\leqslant R, t\in S_T} |H(t,u)|
\end{aligned}$$

Combining the above two formulas, we have

$$(a - \sigma(R)M_2)K_n^\mu - N_2 K_n^2 \leqslant \max_{|u|\leqslant R, t\in S_T} |H(t,u)|$$

Since $\mu > 2$, $a - \sigma(R)M_2 > 0$ and $K_n \to \infty$ as $n \to \infty$, the left-hand side tends to infinity, but the right-hand side is a constant independent of u and n. This leads to a contradiction. Hence, there exists $m \in \mathbf{N}$, which is determined only by $H(t,u)$ and N, such that for any $n \geqslant m$, if $u(t)$ is a critical point of I_n such that $I_n(u) \leqslant N$, we have $|u|_{C^0} \leqslant K_n$.

For $n \geqslant m$, if the C^0-norm of critical points u satisfying $I_n(u) \leqslant N$ does not have an n-independent upper bound M_0, then repeating the above proof by replacing K_n by M_n with $M_n \to \infty$ as $n \to \infty$, we can also get the contradiction. For $n < m$, as the Proof in last part of Lemma 2.1, we have

$$|u(t)| \leqslant N_1M_2 + c_kM_1 + 1, \quad \forall\ t \in \mathbf{R},$$

where c_k is determined by W_k for $k = 1, 2, \ldots, m-1$.

Hence, we have

$$||u||_{C^0} \leqslant \max\{M_0,\ N_1M_2 + c_kM_1 + 1,\ k = 1, 2, \ldots, m-1\} = M$$

Case II: Suppose that we have

$$\lim_{r\to\infty} \delta(r) = 0.$$

We need only to modify the proof of Case I slightly. By the definition of $\delta(r)$, we have $\delta(r)$ which increases to 0. Fix a large $R > 1$ such that

$$a + \delta(R)M_2 > 0.$$

Firstly we show that $|u|_{C^0} \leqslant K_n$ for large n. If not, by passing a subsequence we may assume that for each n there exist $u_n(t)$, a_n and b_n such that

$$(a_n, b_n) \subset \{t \in \mathbf{R} | R < |u_n(t)| < K_n\}$$

and $|u_n(a_n)| = K_n$, $|u_n(b_n)| = R$. As in Case I, we know that $b_n - a_n \leqslant M_2/a$ and we have

$$\begin{aligned}
&H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \\
&\quad = \int_{a_n}^{b_n} \frac{1}{2} L'(t) u_n(t) u_n(t)\, \mathrm{d}t + \int_{a_n}^{b_n} W_t(t, u_n)\, \mathrm{d}t \\
&\quad \geqslant -\frac{1}{2} ||L'||_{L^\infty} \int_{a_n}^{b_n} |u_n|^2\, \mathrm{d}t + \int_{a_n}^{b_n} \delta(R)|u_n|^\mu W(t, u_n)\, \mathrm{d}t \\
&\quad \geqslant -N_1M_2K_n^2 + \delta(R)K_n^\mu \int_{a_n}^{b_n} W(t, u_n)\, \mathrm{d}t \\
&\quad \geqslant -N_1M_2K_n^2 + \delta(R)K_n^\mu \int_{\mathbf{R}} W_n(t, u_n)\, \mathrm{d}t \\
&\quad \geqslant -N_1M_2K_n^2 + \delta(R)M_2K_n^\mu.
\end{aligned}$$

Hence, we have

$$H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \geqslant -N_1M_2K_n^2 + \delta(R)M_2K_n^\mu.$$

On the other hand, we have

$$\begin{aligned}
&H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \\
&\quad \leqslant \max_{|u| \leqslant R, t \in S_T} |H(t, u)| - a|u_n(a_n)|^\mu + ||L||_{L^\infty} |u_n(b_n)|^2 \\
&\quad = \max_{|u| \leqslant R, t \in S_T} |H(t, u)| - aK_n^\mu + ||L||_{L^\infty} K_n^2.
\end{aligned}$$

Combining the above two formulas, we have

$$(a + \delta(R)M_2)K_n^{\mu} - N_2K_n^2 \leqslant \max_{|u| \leqslant R, t \in S_T} |H(t,u)|.$$

Since $\mu > 2$, $a + \delta(R)M_2 > 0$ and $K_n \to \infty$ as $n \to \infty$, the left-hand side tends to infinity, but the right-hand side is a constant independent of u and n. This leads to a contradiction. Hence, $|u|_{C^0} \leqslant K_n$ holds for large n. Using the same discussion as that in Case I, we have

$$||u||_{C^0} \leqslant M$$

Combining the above two cases, we know that our Lemma holds. □

Remark 2.1. In our Proofs of the above two Lemmas, we use (2) only to induce the corresponding estimates in fact. When we study the convex Hamiltonian systems, since the critical points are obtained through the Clarke duality principle, we can only obtain some estimates as (2) for the critical points. Hence, we can get the same estimates on the bound of C^0-norm of the solutions of the convex Hamiltonian systems possessing super-quadratic potentials. We will deal with the convex Hamiltonian systems in another paper.

3. Symmetric Hamiltonian systems

In this section, we consider the Hamiltonian system

$$\mathscr{J}\dot{u} + \nabla H(t,u) = 0, \quad (t,u) \in S_T \times \mathbf{R}^{2N},$$

where $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u)$ is the T-periodic in the t-variable and is symmetric in the u-variable, i.e., there is a compact Lie group G acting on $\mathbf{R}^{2N}$ via a representation $\varrho : G \to O(2N)$ and H is invariant under this action: $H(t,gu) = H(t,u)$ for every $t \in \mathbf{R}$, $g \in G$, $u \in \mathbf{R}^n$. We let V denote the vector space $\mathbf{R}^{2N}$ considered as a G-space.

Definition 3.1. We call V (or ϱ) *admissible* if a given $k \geqslant 1$ and an open bounded G-invariant neighborhood $\mathscr{O} \subset V^k$ of 0 is in V^k, and any continuous map $f : \bar{\mathscr{O}} \to V^{k-1}$ which commutes with the action has a zero in $\partial\mathscr{O}$, where G acts on V^k via $g(v_1,\ldots,v_k) := (gv_1,\ldots,gv_k)$. We call ϱ *symplectic* if $\varrho(g)^t \mathscr{J} \varrho(g) = \mathscr{J}$ is satisfied for every $g \in G$.

If the action is symplectic, every homoclinic orbit u of the system (1) gives rise to a *G-orbit* $\{gu | g \in G\}$ of homoclinic orbits of the system (1). For each $k \in \mathbf{Z}$, let $(k * u)(t) := u(t + kT)$, which defines a representation of $\mathbf{Z}$ in E. Since $H(t,u)$ is T-periodic in t-variable, we have that each I_n is $\mathbf{Z}$-invariant. Hence, each I_n is also invariant with respect to the representation of $\mathbf{Z} \times G$ in E given by

$$((k,g) * u)(t) := (gu)(t + kT).$$

Now, let $\mathscr{O}(u) = \mathscr{O}_{\mathbf{Z}\times G}(u) \equiv \{(k,g) * u | k \in \mathbf{Z}, g \in G\}$ be the *orbit* of $u \in E$. If u is a critical point of I_n, $\mathscr{O}(u)$ will be called the *critical orbit* of u. Two homoclinic orbits

u, v of the system (1) are said to be *geometrically distinct* if they are not in the same critical orbit, i.e., $\mathcal{O}(u) \neq \mathcal{O}(v)$.

In this section, we will show the following results:

Theorem 3.1. *Suppose that* $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies*

(L) *$L(t)$ depends on t with period T, and there is $\alpha > 0$ such that $(0,\alpha) \cap \sigma(A) = \emptyset$, where $A = -(\mathcal{J}(\mathrm{d}/\mathrm{d}t) + L(t))$ and $\sigma(A)$ is the spectrum of A.*

(G) *There is an admissible symplectic representation ρ of a compact Lie group G on $\mathbf{R}^{2N}$ such that H is invariant with respect to this action.*

(H1) *There is a constant $\mu > 2$ such that*

$$0 < \mu W(t,u) \leqslant z\nabla W(t,u), \quad \forall |u| > 0,$$

(H2) *$\nabla W(t,u)$ is locally Lipschitzian continuous in u-variable.*

(H3) *$\nabla W(t,u) = o(|u|)$, uniformly in t as $u \to 0$,*

(H4) *There is a constant $c > 0$, such that*

$$|\nabla W(t,u)| \leqslant c(\nabla W(t,u), u), \quad \forall |u| \geqslant 1.$$

Then (1) *has infinitely many geometrically distinct homoclinic orbits.*

Theorem 3.2. *Suppose that* $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (L), (G), (H1)–(H3) *and*

(H5) $\displaystyle \limsup_{|u|\to\infty} \frac{W_t(t,u)}{|u|^{\mu} W(t,u)} = 0,$ *or* $\displaystyle \liminf_{|u|\to\infty} \frac{W_t(t,u)}{|u|^{\mu} W(t,u)} = 0,$ *uniformly in t.*

Then (1) *has infinitely many geometrically distinct homoclinic orbits.*

When $W(t,u)$ does not depend on the t-variable, since (H5) is satisfied naturally, we have the following result:

Theorem 3.3. *Suppose* $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (L), (G), (H1)–(H3) *and $W(u)$ is independent of t-variable, then* (1) *has infinitely many geometrically distinct homoclinic orbits.*

In paper [1], the authors prove the following main result:

Theorem 3.4 (Theorem 2.1 in Arioli and Szulkin [1]). *If* $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (G), (H1), (H3), $(H4)_p$ *and*

(L)′ *L is a constant symmetric $2N \times 2N$ matrix and $\sigma(\mathcal{J}L) \cap i\mathbf{R} = \emptyset$.*

(H2)′ *There are $\bar{c}, \varepsilon_0 > 0$ and $p > 2$ such that*

$$|\nabla W(t,u+v) - \nabla W(t,u)| \leqslant \bar{c}|v|(1 + |u|^{p-1})$$

for all t and all u, v with $|v| \leqslant \varepsilon_0$.

(H6) *There exist $c, r_0 > 0$ such that*

$$|\nabla W(t,u)|^2 \leqslant c(\nabla W(t,u) \cdot u, \quad \forall |u| \leqslant r_0$$

then (1) *has infinitely many geometrically distinct homoclinic orbits.*

Remark 3.1. Condition (H2)$'$ implies that $\nabla W(t,u)$ is locally Lipschitzian continuous in u-variable, which means that $\nabla W(t,u)$ is Lipschitzian continuous in u -variable for any compact set in $\mathbf{R}^{2N}$, i.e., condition (H2)$'$ implies condition (H2).

Firstly, we can replace $(L)'$ by (L) as was done in [3,4]. In fact, we need only to use the function space E and the new norm $||\cdot||_E$ that we obtained in Section 2 instead of those in the first part in Section 3 of [1].

Secondly, we need to show that the condition (H6) is not necessary after we modify the Proofs in Section 3 of [1] slightly. Condition (H6) is used in the Proof of Lemma 3.4 in [1] initially. Now we show that this Lemma is also true without the condition (H6). Recall that a sequence $\{u_n\}$ is called $(PS)_c$-sequence if $I(u_n)\to c$ and $I'(u_n)\to 0$ as $n\to\infty$.

Lemma 3.1 (Lemma 3.4 in Arioli and Szulkin [1]). *Assume W satisfies* (H1), (H3) *and* $(\mathrm{H4})_p$, *let $\{u_n\}\subset E$ be a $(PS)_d$-sequence, then $\{u_n\}$ is bounded and $d\geqslant 0$.*

Proof. From (H3), for any $\varepsilon_0>0$, there is a $\delta_0>0$, such that

$$|\nabla W(t,u)|\leqslant \varepsilon_0|u| \quad \forall |u|\leqslant\delta_0, \quad uniformly\ in\ t.$$

By (H1) and $\{u_n\}$ being a $(PS)_d$-sequence, for large n we have

$$\begin{aligned} d+1+||u_n||_E &\geqslant I(u_n)-\frac{1}{2}(I'(u_n),u_n)\\ &=\frac{1}{2}\int_{\mathbf{R}}\nabla W(t,u_n)\mathrm{d}t-\int_{\mathbf{R}}W(t,u_n)\,\mathrm{d}t\\ &\geqslant\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbf{R}}\nabla W(t,u_n)u_n\,\mathrm{d}t\\ &\geqslant\left(\frac{\mu}{2}-1\right)\int_{\mathbf{R}}W(t,u_n)\,\mathrm{d}t. \end{aligned}$$

From $(\mathrm{H4})_p$, we have

$$|\nabla Wt,u)|^p\leqslant c_1(\varepsilon_0)\nabla W(t,u)u, \quad \forall |u|\geqslant\delta_0,$$

where

$$c_1(\varepsilon_0)=\max\left\{c,\ \sup_{\delta_0\leqslant|u|\leqslant 1,t\in S_T}\ \frac{|\nabla W(t,u)|^p}{\nabla W(t,u)\cdot u}\right\}.$$

Then,

$$\begin{aligned} d+1+||u_n||_E &\geqslant\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbf{R}}\nabla W(t,u_n)u_n\,\mathrm{d}t\\ &\geqslant\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{|u_n(t)|\geqslant\delta_0}\nabla W(t,u_n)u_n\,\mathrm{d}t\\ &\geqslant\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{|u_n(t)|\geqslant\delta_0}\frac{1}{c_1(\varepsilon_0)}|\nabla W(t,u_n)|^p\,\mathrm{d}t. \end{aligned}$$

This implies that

$$\int_{|u_n(t)|\geqslant\delta_0}|\nabla W(t,u_n)|^p\,\mathrm{d}t\leqslant\frac{2c_1(\varepsilon_0)\mu}{\mu-2}(d+1+||u_n||_E).$$

On the other hand, for large n, let $u_n=u_n^++u_n^-\in E=E^+\oplus E^-$, using Hölder inequality and Sobolev embedding theorem since E is embedded continuous in L^ν for any $\nu\in[2,\infty)$, we have

$$\begin{aligned}||u_n^+||_E&\geqslant(I'(u_n),u_n^+)\\&=\frac{1}{2}||u_n^+||_E^2-\int_{\mathbf{R}}\nabla W(t,u_n)u_n^+\,\mathrm{d}t\\&\geqslant\frac{1}{2}||u_n^+||_E^2-\int_{\mathbf{R}}|\nabla W(t,u_n)|\,|u_n^+|\,\mathrm{d}t\\&\geqslant\frac{1}{2}||u_n^+||_E^2-\left(\int_{|u_n(t)|\leqslant\delta_0}+\int_{|u_n(t)|>\delta_0}\right)|\nabla W(t,u_n)|\,|u_n^+|\,\mathrm{d}t\\&\geqslant\frac{1}{2}||u_n^+||_E^2-\int_{|u_n(t)|\leqslant\delta_0}\varepsilon_0|u_n|\,|u_n^+|\,\mathrm{d}t\\&\quad-\left(\left(\int_{|u_n(t)|>\delta_0}|\nabla W(t,u_n)|^p\,\mathrm{d}t\right)^{1/p}\left(\int_{|u_n(t)|>\delta_0}|u_n^+|^q\,\mathrm{d}t\right)^{1/q}\right)\\&\geqslant\frac{1}{2}||u_n^+||_E^2-C\varepsilon_0||u_n||_E||u_n^+||_{L^2}-DN(\varepsilon_0)||u_n^+||_E(d+1+||u_n||_E)^{1/q}\end{aligned}$$

and

$$\begin{aligned}||u_n^-||_E&\geqslant-(I'(u_n),u_n^-)\\&=\frac{1}{2}||u_n^-||_E^2+\int_{\mathbf{R}}\nabla W(t,u_n)u_n^-\,\mathrm{d}t\\&\geqslant\frac{1}{2}||u_n^-||_E^2-\int_{\mathbf{R}}|\nabla W(t,u_n)|\,|u_n^-|\,\mathrm{d}t\\&\geqslant\frac{1}{2}||u_n^-||_E^2-\left(\int_{|u_n(t)|\leqslant\delta_0}+\int_{|u_n(t)|>\delta_0}\right)|\nabla W(t,u_n)|\,|u_n^-|\,\mathrm{d}t\\&\geqslant\frac{1}{2}||u_n^-||_E^2-\int_{|u_n(t)|\leqslant\delta_0}\varepsilon_0|u_n|\,|u_n^-|\,\mathrm{d}t\\&\quad-\left(\left(\int_{|u_n(t)|>\delta_0}|\nabla W(t,u_n)|^p\,\mathrm{d}t\right)^{1/p}\left(\int_{|u_n(t)|>\delta_0}|u_n^-|^q\,\mathrm{d}t\right)^{1/q}\right)\\&\geqslant\frac{1}{2}||u_n^-||_E^2-C\varepsilon_0||u_n||_E||u_n^-||_{L^2}-DN(\varepsilon_0)||u_n^-||_E(d+1+||u_n||_E)^{1/q}\end{aligned}$$

where $1/p + 1/q = 1$, $N(\varepsilon_0)$ is a constant dependent on ε_0 and C, D are the Sobolev embedding constants. Hence, we have

$$\begin{aligned}2||u_n||_E &\geqslant ||u_n^+||_E + ||u_n^-||_E \\ &\geqslant \frac{1}{2}(||u_n^+||_E^2 + ||u_n^-||_E^2) - C\varepsilon_0||u_n||_E(||u_n^+||_{L^2} + ||u_n^-||_{L^2}) \\ &\quad - DN(\varepsilon_0)(||u_n^+||_E + ||u_n^-||_E)(d + 1 + ||u_n||_E)^{1/q} \\ &\geqslant \frac{1}{2}||u_n||_E^2 - 2C^2\varepsilon_0||u_n||_E^2 - 2DN(\varepsilon_0)||u_n||_E(d + 1 + ||u_n||_E)^{1/q}.\end{aligned}$$

This implies that

$$2 \geqslant \left(\frac{1}{2} - 2C^2\varepsilon_0\right)||u_n||_E - 2D(d + 1 + ||u_n||_E)^{1/q}.$$

Fix a small enough ε_0, such that $\frac{1}{2} - 2C^2\varepsilon_0 > 0$, this implies that $\{||u_n||_E\}$ is bounded.

Hence, we have

$$0 \leqslant \lim_{n\to\infty}|(I'(u_n), u_n)| \leqslant \lim_{n\to\infty}||I'(u_n)||\,||u_n||_E = 0.$$

This implies that

$$d = \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty}(I(u_n) - \frac{1}{2}(I'(u_n), u_n)) \geqslant \liminf_{n\to\infty}\left(\frac{\mu}{2} - 1\right)\int_{\mathbf{R}} W(t, u_n)\mathrm{d}t \geqslant 0.$$

Hence, our Lemma is proved. □

Now replacing the Lemma 3.4 in [1] by our Lemma 3.1, we have Lemma 3.6, Lemma 3.7, Theorem 3.8 and Lemma 3.9 in [1] without using condition (H6). Hence, we do not need the condition (H6) in the Proofs of Theorem 2.1 in [1]. We summarize the above results as the following Theorem.

Theorem 3.5. *Suppose that H satisfies* (L), (G), (H1), (H2)′, (H3) *and* $(\mathrm{H4})_p$, *then* (1) *has infinitely many geometrically distinct homoclinic orbits.*

Now we will prove our Theorems 3.1 and 3.2 by truncating the potential $W(t,u)$ with $W_n(t,u)$ as Proposition 2.1 in Section 2 to obtain a sequence of new systems such that the new systems satisfy the conditions of the above Theorem. Replacing W by W_n, we study a sequence of new systems

$$\mathscr{J}\dot{u} + L(t)u + \nabla W_n(t,u) = 0. \quad \forall (t,u) \in S_T \times \mathbf{R}^{2N}. \tag{3}$$

Proof of Theorems 3.1 and 3.2. From Proposition 2.1, we know that each W_n satisfies (H1) and (G). For condition (G), we need only to let

$$W_n(t,u) = \int_G W_n(t, gu)\,\mathrm{d}g$$

where dg is the standard Haar measure on compact Lie group G. And from (iii) of Proposition 2.1, we have

$$W_n(t,u) = (\tau_n + 1)|u|^{\lambda\mu}, \quad \text{for } |u| \geqslant K_n'.$$

Let $p_n = \lambda\mu/(\lambda\mu - 1) \in (1,2)$, we can check if H_n satisfies $(H2)_{p_n}$ for some c_n which is determined by $W_n(t,u)$.

Now we need only to show that each W_n satisfies condition (H2)$'$. Since $\nabla W(t,u)$ is locally Lipschitzian continuous in u-variable and

$$W_n(t,u) = (\tau_n + 1)|u|^{\lambda\mu}, \quad \text{for } |u| \geqslant K_n',$$

we have

$$A = \sup_{|u| \leqslant K_n', |v| \leqslant 1} \frac{|\nabla W_n(t,u+v) - \nabla W_n(t,u)|}{|v|(1 + |u|^{\lambda\mu-1})} < \infty$$

$$B = \sup_{|u| \geqslant K_n', |v| \leqslant 1} \lambda\mu(\tau_n + 1) \frac{||u+v|^{\lambda\mu-2}(u+v) - |u|^{\lambda\mu-2}u|}{|v|(1 + |u|^{\lambda\mu-1})} < \infty.$$

The first one holds since $\nabla W(t,u)$ is locally Lipschitzian continuous in u-variable, and the second one holds since

$$\nabla W(t,u) = \lambda\mu(\tau_n + 1)|u|^{\lambda\mu-2}u \in C^1(\mathbf{R}^{2N}, \mathbf{R}^{2N}), \quad \text{for } |u| \geqslant K_n'.$$

Hence, (H2)$'$ holds for some constant $\bar{c}_n = \max\{A, B\}$.

From the above Theorem 3.5, we know that each system (3) has a sequence of classic solutions $\{u_k^n\}$ with unbounded critical values $\{d_k^n\}$. As shown in Section 6 of [1], we know that

$$d_k^n = I_n(u_k^n) = \inf_{I^*(A) \geqslant k} \sup_{u \in A} I_n(u)$$

where the pseudo-index $I^*(A)$ was defined as Definition 4.4 (p. 303) in [1].

Given $k \in \mathbf{N}$, from (iv) of Proposition 2.1, for any $n \in \mathbf{N}$ we have

$$W_n(t,u) \leqslant W_{n+1}(t,u) \leqslant W(t,u), \quad \forall (t,u) \in S_T \times \mathbf{R}^{2N}.$$

This implies that

$$I_n(u) \geqslant I_{n+1}(u) \geqslant I(u), \quad \forall u \in H^{1/2}(\mathbf{R}, \mathbf{R}^{2N}).$$

By the definition of $\{d_k^n\}$ we have

$$d_k^1 \geqslant d_k^n \geqslant d_k^{n+1}, \quad \forall n \in \mathbf{N}.$$

Hence, for any given $k \in \mathbf{N}$, since $W(t,u)$ satisfies (H1), (H4) (or (H5)), and u_k^n is a critical point of I_n such that $I_n(u_k^n) \leqslant d_k^1$ holds for all $k \in \mathbf{N}$. From Lemma 2.1 (or Lemma 2.2) of Section 2, we have constant M_k, which is dependent only on d_k^1 and $W(t,u)$, such that $||u_k^n||_{C^0} \leqslant M_k$ holds for all $n \in \mathbf{N}$.

Hence, for large $n \in \mathbf{N}$ such that $K_n > M_k$, we have

$$||u_k^n||_{C^0} \leqslant M_k < K_n.$$

On the other hand, we have

$$W_n(t,u) = W(t,u), \quad \forall |u| < K_n.$$

This implies that $u_k^n(t)$ is a homoclinic orbit of the system (1) when $||u_k^n||_{C^0} < K_n$.

Hence, for any given $k \in \mathbf{N}$, there exists a large enough $n \in \mathbf{N}$ such that $u_1^n(t), \ldots, u_k^n(t)$ are classic homoclinic orbits of the system (1). i.e., Theorems 3.1 and 3.2 hold. □

Remark 3.2. In our proof, we use the monotone property of $\{W_n(t,u)\}$ only to get the upper bound of the critical values $\{d_k^n\}$ for each $k \in \mathbf{N}$. In fact, the monotone property is not necessary since we can construct that each $W_n(t,u)$ satisfies condition (H1) and (iii) of Proposition 2.1, which ensures that

$$W_n(t,u) \geqslant a|u|^\mu, \quad \forall |u| \geqslant 1.$$

Then without using the monotone property, we can also obtain the upper bound of the critical values $\{d_k^n\}$ for each $k \in \mathbf{N}$. When we deal with the convex Hamiltonian systems in future, we will use this idea.

For our Theorem 3.3, we need only to check condition (H5) which is always satisfied when the potential W is independent of the t-variable, which comes from $(\mathrm{d}/\mathrm{d}t)W(u) = 0$.

On the other hand, in our Theorem 3.3, the potential $H(t,u) = \frac{1}{2}L(t)uu + W(u)$ depends on the t-variable and the functional $I(u)$ is not invariant with respect to the representation of $\mathbf{R} \times G$ given by $((s,g)u)(t) = g(u(t+s))$ for $(s,g) \in \mathbf{R} \times G$, which is not like the one mentioned in Remark 2.3 of [1] for autonomous Hamiltonian systems. Our Theorem 3.3 makes sense when $L(t)$ is not a constant matrix.

4. Hamiltonian systems without symmetry

In this section, we will study the homoclinic orbits of the Hamiltonian system

$$\mathscr{J}\dot{u} + \nabla H(t,u) = 0, \quad (t,u) \in \mathbf{R} \times \mathbf{R}^{2N}. \tag{4}$$

Let $H(t,u) = \frac{1}{2}(L(t)u,u) + W(t,u)$, where $L(t)$ is a given continuous T-periodic and symmetric $2N \times 2N$-matrix-value function and $W(t,u)$ is T-periodic in the t-variable. In this section, we improve those results in [4,5,9] and get the following results:

Theorem 4.1. *Suppose that* $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (L), (H1), (H3) *and* (H4), *then* (1) *has at least one homoclinic orbit.*

Theorem 4.2. *Suppose that* $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (L), (H1), (H3) *and* (H5), *then* (1) *has at least one homoclinic orbit.*

Especially when W does not depend on t-variable, since condition (H5) is satisfied naturally, we have the following result:

Theorem 4.3. *Suppose that* $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (L), (H1), (H3) *and* W *does not depend on t-variable*, *then* (1) *has at least one homoclinic orbit.*

Remark 4.1. As said in [3,4], when the condition (L) is replaced by

($\bar{L}$) $L(t)$ depends on t with period T, and there is $\alpha > 0$ such that $(-\alpha, 0) \cup \sigma(A) = \emptyset$ where $A = -(\mathscr{J}(\mathrm{d}/\mathrm{d}t) + L(t))$ is the self-adjoint operator acting on $L^2(\mathbf{R}, \mathbf{R}^{2N})$ with the domain $\mathscr{D}(A) = H^1(\mathbf{R}, \mathbf{R}^{2N})$ and $\sigma(A)$ is the spectrum of A.

and $-W(t,u)$ satisfies other conditions in our Theorems, wherein the above results are still valid.

In [4] there is the following result:

Theorem 4.4 (Theorem 1.1 in Ding and Willem [4]). *Suppose that* $H(t,u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (L), (H1), $(H4)_p$ *and*
(H3)'. *There is an* $a > 0$ *such that for all* (t,u) *with* $|u| \leqslant 1$, *there is*

$$|\nabla W(t,u)| \leqslant a|u|^{\mu-1}$$

then (1) *has at least one homoclinic orbit.*

In fact, we can replace (H3)$'$ by (H3) in the Proof of [4]. In the Proof of [4], there are only three places using the condition (H3)$'$.

The first place is in the Proof of Lemma 4.2 (p. 773) in [4]. Here, we need only to notice that for integrating condition (H1), we have

$$W(t,u) \leqslant b|u|^{\mu}, \quad \forall |u| \leqslant 1$$

for some constant b as in the beginning of Section 2. Combining with condition $(H4)_p$, we have

$$|W(t,u)| \leqslant d(|u|^{\mu} + |u|^{q})$$

where $q = p/(p-1) > 2$. Then the Proof of Lemma 4.2 in [4] follows.

The second place is in the Proof of Lemma 4.5 (p. 774) in [4]. The condition (H3)$'$ is used to prove that when $\{u_n\}$ is a $(PS)_c$-sequence, $\{||u_n||_E\}$ is bounded, which is exactly what we have shown in our Lemma 3.1.

The third place where the authors used the condition (H3)$'$ in [4] is in page 777 in [4] to get

$$|\dot{u}|^2 \leqslant d_1(|u|^2 + |u|^{2q-2}).$$

Using condition (H3), for any $\varepsilon > 0$, we have $\delta(\varepsilon) > 0$ such that

$$|\nabla W(t,u)| < \varepsilon|u|, \quad \forall |u| \leqslant \delta(\varepsilon)$$

and from $(H4)_p$, we have

$$|\nabla W(t,u)| < a(\varepsilon)|u|^{q-1}, \quad \forall |u| \geqslant \delta(\varepsilon).$$

Then, we have

$$\begin{aligned} |\dot{u}| &= |L(t)u + \nabla W(t,u)| \\ &\leqslant |L(t)u| + |\nabla W(t,u)| \\ &\leqslant ||L||_{L^\infty}|u| + \varepsilon|u| + a(\varepsilon)|u|^{q-1}. \end{aligned}$$

This implies that

$$|\dot{u}|^2 \leqslant d_1(\varepsilon)(|u|^2 + |u|^{2q-2})$$

since $q > 2$. We summarize the above results as the following Theorem.

Theorem 4.5. *Suppose that* $H(t,u)\in C^1(S_T\times \mathbf{R}^{2N},\mathbf{R})$ *satisfies* (L), (H1), (H3) and $(\text{H4})_p$, *then* (1) *has at least one homoclinic orbit.*

Now we will prove our Theorems 4.1 and 4.2 by truncating the potential $W(t,u)$ with $W_n(t,u)$ as Proposition 2.1 in Section 2 to obtain a sequence of new systems such that the new systems satisfy the conditions of the above Theorem. Replacing W by W_n, we study a sequence of new systems

$$\mathscr{J}\dot{u}+L(t)u+\nabla W_n(t,u)=0,\quad \forall\ (t,u)\in S_T\times \mathbf{R}^{2N}.$$

Proof of Theorems 4.1 and 4.2. We first truncate the potential $W(t,u)$ by $\{W_n(t,u)\}$ obtained from Proposition 2.1 in Section 2 to get a sequence of new systems. As in Section 2, we define $I_n(u)$ for these new systems. For every $n\in\mathbf{N}$, there are $b_n,\rho_n>0$ such that $I_n|_{S_\rho\cap E^+}\geqslant b_n$ as Lemma 4.2 in [4], and there are $y_0^n\in S_1\cup E^+$ and $R>\rho_n$ such that $I_n|_{\partial M}\leqslant 0$, where

$$M:=\{u=x+\lambda y_0^n|x\in E^-,||u||\langle R,\lambda\rangle 0\},$$
$$\partial M\ =\{u=x+\lambda y_0^n|x\in E^-,||u||=R \text{ and } \lambda\geqslant 0 \text{ or } ||u||\leqslant R \text{ and } \lambda=0\}$$

as Lemma 4.3 in [4]. The reason that we can choose the same R and M for all I_n is that $\{I_n(u)\}$ is a decreasing sequence for every $u\in E$ since our truncation sequence $\{W_n\}$ satisfies (iv) of Proposition 2.1. From Theorem 3.1 in [4], we know that the critical values $c_n\in[b_n,\sup_{\bar{M}} I_n]$. Hence, we have an n-independent constant $\sup_{\bar{M}} I_1>0$, for each $n\in\mathbf{N}$, for which there exists a nontrivial critical point u_n of I_n, such that $I_n(u_n)\leqslant \sup_{\bar{M}}I_n\leqslant \sup_{\bar{M}} I_1$ holds for all $n\in\mathbf{N}$.

Since $W(t,u)$ satisfies (H1) and (H4) (or (H5)), by Lemma 2.1 (or Lemma 2.2), we have an n-independent constant M such that

$$||u_n||_{C^0}\leqslant M,\quad \text{for all } n\in\mathbf{N}.$$

On the other hand, we have

$$W_n(t,u)=W(t,u),\quad \text{for } |u|<K_n.$$

Hence, for large $n\in\mathbf{N}$ such that $K_n>M$, u_n is a homoclinic orbit of (4), i.e. Theorems 4.1 and 4.2 hold. □

For our Theorem 4.3, we need only to check if condition (H5) is always satisfied when the potential W is independent of t-variable, which comes from $(\mathrm{d}/\mathrm{d}t)W(u)=0$.

References

[1] G. Arioli, A. Szulkin, Homoclinic solutions of Hamiltonian systems with symmetry, J. Diff. Eq. 158 (1999) 291–313.

[2] V. Coti-Zelati, P.H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing super-quadratic potentials, J. Amer. Math. Soc. 4 (1991) 693–727.

[3] Y.H. Ding, M. Girardi, Infinitely many homoclinic orbits of a Hamiltonian system with symmetry, Nonlinear Anal. TMA Ser. A. 38 (1999) 391–415.

[4] Y.H. Ding, M. Willem, Homoclinic orbits of a Hamiltonian system, Z. angew. Math. Phys. 50 (1999) 759–778.

[5] H. Hofer, K. Wysocki, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, Math. Ann. 228 (1990) 483–503.
[6] Y. Long, Periodic solutions of perturbed super-quadratic Hamiltonian systems, Ann. Scuola Norm. Sup. Pisa (1990) 35–77.
[7] Y. Long, Xiangjin Xu, Periodic solutions for a class of non-autonomous Hamiltonian systems, Nonlinear Anal. TMA 41 (2000) 455–463.
[8] P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh. 114 (1990) 33–38.
[9] K. Tanaka, Homoclinic orbits in a first order super-quadratic Hamiltonian system: convergence of subharmonic orbits, J. Diff. Eq. 94 (1991) 315–339.
[10] Xiangjin Xu, Periodic solutions of non-autonomous Hamiltonian systems possessing super-quadratic potentials, Nonlinear Anal. PII: S0362-546X(01)00870-7, in this Volume.