



Periodic solutions for non-autonomous Hamiltonian systems possessing super-quadratic potentials

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1. Introduction

We consider the non-autonomous Hamiltonian system

$$\mathcal{J}\dot{u} + \nabla H(t, u) = 0, \quad (t, u) \in S_T \times \mathbf{R}^{2N},$$

where $H \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is T -periodic in t -variable, and $\mathcal{J} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ denotes the standard symplectic matrix and ∇ denotes the gradient with respect to the u -variable. We are interested in the existence of T -periodic solutions of (1). For the autonomous case, i.e. H is independent of t , in his pioneer work [8] Rabinowitz first proved the existence of at least one periodic solution for (1). Many works have been done on this topics, such as [2–13]. We refer to [2,3,11] for further references. At this paper, we first consider the case that H is symmetric in the u -variable. In [2] when H is super-quadratic at infinitely and satisfies

(H2) _{p} There are constants $c, d > 0$ and $p \in (1, 2)$, such that

$$|\nabla H(t, u)|^p \leq c(\nabla H(t, u), u) + d, \quad \forall u \in \mathbf{R}^{2N},$$

Barsch and Willem proved there exist T -periodic solutions with arbitrarily large L^∞ -norm when H is symmetric in the u -variable. We will explain the symmetry

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in Section 3. In this paper we have the following results:

Theorem 1.1. For $T > 0$, $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$, $S_T = \mathbf{R}/(T\mathbf{Z})$, satisfies the following conditions:

(G) There exists an admissible representation $\varrho: G \rightarrow O(2N)$ on $V = \mathbf{R}^{2N}$ which satisfies $\varrho(g)^t J \varrho(g) = J$ for every $g \in G$, such that H is invariant with respect to this action.

(H1) There are constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu H(t, u) \leq z \nabla H(t, u), \quad \forall |u| \geq r_0.$$

(H2) There are constants $c, d > 0$, such that

$$|\nabla H(t, u)| \leq c(\nabla H(t, u), u) + d, \quad \forall u \in \mathbf{R}^{2N}.$$

Then (1) possesses a sequence of T -periodic solutions with unbounded C^0 -norm.

Theorem 1.2. For $T > 0$, $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (G), (H1) and

$$(H3) \quad \limsup_{|u| \rightarrow \infty} \frac{H_t(t, u)}{|u|^\mu H(t, u)} = 0, \quad \text{or} \quad \liminf_{|u| \rightarrow \infty} \frac{H_t(t, u)}{|u|^\mu H(t, u)} = 0, \quad \text{uniformly in } t.$$

Then (1) possesses a sequence of T -periodic solutions with unbounded C^0 -norm.

When H does not have any symmetric conditions, In [3] Li and Szulkin and in [4] Li and Willem proved the existence of at least one T -periodic solution of (1) provided some certain conditions on H near $u = 0$, (H1) and (H2)_p. In [6] Long and the author proved the same result as [3] by loosing the growth condition (H2)_p to (H2), and also proved the existence of at least one T -periodic solution of (1) under a strong form of condition (H3) as following

(H3)' there exists a constant \bar{a} such that

$$\liminf_{|z| \rightarrow \infty} \frac{H_t(t, u)}{H(t, u)} \geq \bar{a} > -\frac{2}{T}, \quad \text{uniformly in } t.$$

In this paper, as symmetric case we have the following results.

Theorem 1.3. For $T > 0$, $H = \frac{1}{2}(B(t)u, u) + \bar{H}(t, u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$, satisfies (H1), (H3) and

(H4) $\bar{H}(t, u) = o(|u|^2)$, uniformly in t as $u \rightarrow 0$,

Then (1) has at least one non-trivial T -periodic solution in each of the following two cases:

(i) The boundary value problem

$$\mathcal{J}\dot{u} + B(t)u = 0, \quad u(0) = u(T)$$

has only the trivial solution.

(ii) There is a constant $\rho > 0$ such that $\bar{H}(t, u) > 0$ (or $\bar{H}(t, u) < 0$) for all u satisfying $0 < |u| < \rho$.

Theorem 1.4. For $T > 0$, $H = \frac{1}{2}(B(t)u, u) + \bar{H}(t, u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$, satisfies (H1), (H2) and (H4), Then (1) has at least one non-trivial T -periodic solution in each of case (i) (ii) in above theorem.

Here is the outline of our paper. In Section 2, we first truncate the H by a increasing sequence $\{H_n\}$ such that H_n satisfies the growth condition $(H2)_p$ for some $p \in (1, 2)$, i.e., for the modified systems we may use the results in [2,3] to get the T -periodic solutions. Then we give two new estimates for the bounded C^0 -norm of periodic solutions of (1) when the corresponding critical values of periodic solutions are bounded. In Section 3, we study the symmetric Hamiltonian systems. Since the solutions in [2] are gotten by minimax procedure, we show the corresponding critical values are bounded. Using our estimates obtaining in Section 2 and the properties of $\{H_n\}$, we know the solutions of modified systems is the solutions of (1) for n large enough. In Section 4, we study the Hamiltonian systems without symmetry. Using the same way as in Section 3, we prove Theorem 1.3 and Theorem 1.4. We also study the sub-harmonic solutions of system (1) by following the main idea in [11].

2. Two estimates

In this section, we consider the Hamiltonian system

$$\mathcal{J}\dot{u} + \nabla H(t, u) = 0, \quad (t, u) \in S_T \times \mathbf{R}^{2N}.$$

Let $X := H^{1/2}(S_T, \mathbf{R}^{2N})$ be the Sobolev space of T -periodic \mathbf{R}^{2N} -valued functions with inner product $(\cdot, \cdot)_X$ and norm $\|\cdot\|$.

Set $\alpha_0 = \min_{|u|=r_0, t \in S_T} H(t, u)$, $\beta_0 = \max_{|u| \leq r_0, t \in S_T} |H(t, u)|$. Condition (H1) imply that for some $\beta_3 \geq 0$

$$\begin{aligned} \alpha_0 |u|^\mu &\leq H(t, u), & \forall |u| \geq r_0, \\ \alpha_0 |u|^\mu &\leq H(t, u) + \beta_0 \leq \frac{1}{\mu}(\nabla H(t, u)u + \beta_3), & \forall u \in \mathbf{R}^{2N}. \end{aligned}$$

Modifying [5] (cf. appendix of [5]), choose $\sigma \in (0, 1)$, such that $\mu\sigma > 2$, we truncate H as following proposition:

Proposition 2.1. Assume conditions (H1) then there exist two sequences $\{K_n\}$ and $\{K'_n\}$ in \mathbf{R} and a sequence of functions $\{H_n\}$ such that

- (i) $0 < K_0 < K_n < K_{n+1}$, $\forall n \in \mathbf{N}$, and $K_n \rightarrow \infty$, as $n \rightarrow \infty$, where $K_0 = \max\{1, r_0, \beta_0/\alpha_0(1 - \sigma)\}$; and $K_n < K'_n$, $\forall n \in \mathbf{R}$.
- (ii) for any given $t \in S_T$, $H_n(t, u) \in C^1(\mathbf{R}^{2N}, \mathbf{R})$, for every $n \in \mathbf{N}$.
- (iii) $H_n(t, u) = H(t, u)$, $\forall |u| \leq K_n$, for every $n \in \mathbf{N}$; and for some $\lambda \in (\sigma, 1)$, such that $H_n(t, u) = (\tau_n + 1)|u|^{\mu\lambda}$, $\forall |u| \geq K'_n$, for every $n \in \mathbf{N}$.
- (iv) $H_n(t, u) \leq H_{n+1}(t, u) \leq H(t, u)$, $\forall (t, u) \in S_T \times \mathbf{R}^{2N}$.
- (v) $0 < \mu\sigma H_n(t, u) \leq \nabla H_n(t, u)u$, $\forall |u| \geq r_0$, for every $n \in \mathbf{N}$.

Note that in [5] the truncating functions are constructed for autonomous Hamiltonian functions. In fact the proof also works for time dependent $H(t, u)$.

Define the functional $I_n : X \rightarrow \mathbf{R}$ by

$$I_n(u) = \frac{1}{2} \int_0^T (-\mathcal{J}\dot{u} \cdot u) dt - \int_0^T H_n(t, u) dt.$$

It is well known that $I_n \in C^1(X, \mathbf{R})$ and finding T -periodic solutions of (1) with H replaced by H_n is equivalent to finding critical points of I_n in X .

The periodic solutions of (1) are obtained as critical points of the functional

$$I(u) = \frac{1}{2} \int_0^T -\mathcal{J}\dot{u} \cdot u dt - \int_0^T H(t, u) dt$$

defined on the Hilbert space $H^{1/2}(S_T, \mathbf{R}^{2N})$ and $I(u)$ is C^1 at $u \in H^1(S_T, \mathbf{R}^{2N})$. For any periodic solution u of (1) we have

$$-\frac{1}{2} \int_0^T \mathcal{J}\dot{u} \cdot u dt = \int_0^T \nabla H(t, u) \cdot u dt.$$

Now we will prove two new estimates for T -periodic solutions of non-autonomous Hamiltonian systems with growth condition (H2) or (H3).

Remark 2.1. For autonomous Hamiltonian system possessing super-quadratic potentials, such estimate is trivial since we have $H(u(t)) = \text{constant}$ when $u(t)$ is a solution of autonomous system:

Lemma 2.1. *Suppose $H(t, u)$ satisfies (H1), (H2) and $u(t)$ is a critical point of I_n such that $I_n(u) \leq N$, then we have the following estimate:*

$$\|u\|_{C^0} \leq M$$

where M is independent of u and n .

Proof. Since u is a critical point of I_n and each $H_n(t, u)$ satisfies (iii) and (v) of Proposition 2.1, we have

$$\begin{aligned} I_n(u) &= \frac{1}{2} \int_0^T -\mathcal{J}\dot{u} \cdot u dt - \int_0^T H_n(t, u) dt \\ &= \frac{1}{2} \int_0^T \nabla H_n(t, u) \cdot u dt - \int_0^T H_n(t, u) dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu\sigma}\right) \int_0^T \nabla H_n(t, u) \cdot u dt - C_1 \\ &\geq \left(\frac{\mu\sigma}{2} - 1\right) \int_0^T H_n(t, u) dt - C_2, \end{aligned}$$

where C_1, C_2 are independent of n . From above we have

$$\int_0^T \nabla H_n(t, u) u dt \leq M_1, \quad \int_0^T H_n(t, u) dt \leq M_2$$

for some constants M_1 and M_2 independent of n . Now integrating (v) of Proposition 2.1 we yields

$$H_n(t, u) \geq a|u|^{\mu\sigma} - b, \quad \forall u \in \mathbf{R}^{2N},$$

where a and b are independent of n since

$$H_n(t, u) = H(t, u), \quad \forall |u| \leq r_0 \text{ and } \forall n \in \mathbf{N}.$$

Hence we have

$$N \geq I_n(u) \geq \left(\frac{\mu\sigma}{2} - 1\right) a \int_0^T |u|^{\mu\sigma} dt - C_3 \geq C_4 \left(\min_{t \in S_T} |u(t)|\right)^{\mu\sigma} - C_3$$

for some n -independent constants C_3 and C_4 . Then we have

$$\min_{t \in S_T} |u(t)| \leq N_1$$

where N_1 is independent of n . Without loss generality, we may assume $|u(t)|$ obtains its minimum at $t = 0$,

$$\begin{aligned} |u(t)| - |u(0)| &= \int_0^t \frac{d}{ds} |u(s)| ds = \int_0^t u(s) \cdot \dot{u}(s) / |u(s)| ds \\ &\leq \int_0^t |u(s) \cdot \dot{u}(s)| / |u(s)| ds \leq \int_0^t |\dot{u}(s)| ds \\ &= \int_0^t |\nabla H_n(t, u(s))| ds. \end{aligned}$$

From (H2) and (iii) of Proposition 2.1 we have

$$|\nabla H(t, u)| \leq c(\nabla H(t, u), u) + d, \quad \text{for } |u| \leq K_n$$

We first show for large enough n ,

$$\|u\|_{C^0} \leq K_n.$$

If not, by passing a subsequence, for each $n \in \mathbf{N}$, there exists $u_n(t)$ and $t_n \in S_T$, such that $|u_n(t_n)| = K_n$ and $|u_n(t)| \leq K_n$ for $t \in [0, t_n)$. Hence we have

$$\begin{aligned} K_n = |u_n(t_n)| &\leq \int_0^{t_n} |\nabla H_n(s, u_n(s))| ds + |u_n(0)| \\ &\leq c \int_0^{t_n} (\nabla H_n(s, u_n(s)), u_n(s)) ds + dT + |u_n(0)| \\ &\leq c \int_0^T (\nabla H_n(s, u_n(s)), u_n(s)) ds + N_2 + N_1 \leq cM_1 + N_1 + N_2, \end{aligned}$$

where c, M_1, N_1 and N_2 are n -independent constants. But we have $K_n \rightarrow \infty$, as $n \rightarrow \infty$, this leads a contradiction. Hence there exists $m \in \mathbf{N}$, which is determined by $H(t, u)$

and N only, for any $n \geq m$, $\|u\|_{C^0} \leq K_n$ holds. Repeating about computation, for any $n \geq m$, we have

$$|u(t)| \leq cM_1 + N_1 + N_2, \quad \forall t \in S_T.$$

For $k < m$, from (iii) of Proposition 2.1, we have

$$|\nabla H_k(t, u)| \leq (\nabla H_k(t, u), u) + D_k, \quad \forall (t, u) \in \mathbf{R} \times \mathbf{R}^{2N}$$

for some suitable constant D_k . Hence we have

$$\begin{aligned} \|u(t)\| &\leq |u(0)| + \int_0^t (\nabla H_k(s, u(s)), u(s)) ds + D_k T \leq M_1 + N_1 + D_k T + C_k, \\ &\forall t \in S_T, \end{aligned}$$

where C_k and D_k are determined by H_k for $k = 1, 2, \dots, m - 1$. Then we have

$$\|u\|_{C^0} \leq \max\{cM_1 + N_1 + N_2, M_1 + N_1 + D_k T + C_k, k = 1, 2, \dots, m - 1\} = M.$$

Hence our Lemma holds. \square

Lemma 2.2. *Suppose $H(t, u)$ satisfies (H1), (H3) and $u(t)$ is a critical point of I_n such that $I_n(u) \leq N$, then we have the following estimate:*

$$\|u\|_{C^0} \leq M,$$

where M is independent of u and n .

Proof. As the first part Proof of Lemma 2.1, we have

$$\int_0^T \nabla H_n(t, u) u dt \leq M_1, \quad \int_0^T H_n(t, u) dt \leq M_2$$

for some constants M_1 and M_2 independent of n . And $\min_{t \in S_T} |u(t)| \leq N_1$ where N_1 is independent of n . Now integrating (H1) we yields

$$H(t, u) \geq a|u|^\mu - b, \quad \forall u \in \mathbf{R}^{2N},$$

where a and b are independent of n . Since $H(t, u)$ satisfies (H3), define

$$\sigma(r) = \sup_{u \geq r, t \in S_T} \frac{H_t(t, u)}{|u|^\mu H(t, u)}$$

and

$$\delta(r) = \inf_{u \geq r, t \in S_T} \frac{H_t(t, u)}{|u|^\mu H(t, u)}$$

then (H3) means

$$\lim_{r \rightarrow \infty} \sigma(r) = 0 \quad \text{or} \quad \lim_{r \rightarrow \infty} \delta(r) = 0.$$

Case I: suppose we have $\lim_{r \rightarrow \infty} \sigma(r) = 0$.

By the definition of $\sigma(r)$, we have $\sigma(r)$ is decreasing to 0. Since $H_n(t, u) > 0$ for $|u| \geq r_0$, we have

$$\begin{aligned} M_2 &\geq \int_0^T H_n(t, u) dt \\ &= \int_{\{H_n(t, u) \geq 0\}} H_n(t, u) dt + \int_{\{H_n(t, u) < 0\}} H_n(t, u) dt \\ &\geq \int_{\{H_n(t, u) \geq 0\}} H_n(t, u) dt - T \sup_{u \leq r_0, t \in S_T} |H(t, u)|. \end{aligned}$$

Hence we have

$$\int_{\{H_n(t, u) \geq 0\}} H_n(t, u) dt \leq M_2 + T \sup_{u \leq r_0, t \in S_T} |H(t, u)| = M_3,$$

where M_3 is a constant independent of n and u . Fix a large $R > r_0$, such that

$$a - \sigma(R)M_3 > 0.$$

Firstly we show $|u|_{C^0} \leq K_n$ for large n . If not, by passing a subsequence we may assume for each n , there exists $u_n(t)$, a_n and b_n such that

$$(a_n, b_n) \subset \{t \in S_T \mid R < |u_n(t)| < K_n\}$$

and $|u_n(a_n)| = R$, $|u_n(b_n)| = K_n$. Here we have

$$\begin{aligned} &H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \\ &= \int_{a_n}^{b_n} \frac{d}{dt} H_n(t, u(t)) dt = \int_{a_n}^{b_n} \nabla H_n(t, u_n(t)) \cdot \dot{u}_n(t) dt + \int_{a_n}^{b_n} H_t(t, u_n(t)) dt \\ &= \int_{a_n}^{b_n} H_t(t, u_n(t)) dt \leq \int_{a_n}^{b_n} \sigma(|u_n(t)|) |u_n(t)|^\mu H(t, u_n(t)) dt \\ &\leq \sigma(R) K_n^\mu \int_{a_n}^{b_n} H(t, u_n(t)) dt \leq \sigma(R) K_n^\mu \int_{\{H_n(t, u_n(t)) \geq 0\}} H(t, u_n(t)) dt \\ &\leq \sigma(R) M_3 K_n^\mu. \end{aligned}$$

Hence we have

$$H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \leq \sigma(R) M_3 K_n^\mu.$$

On the other hand, we have

$$\begin{aligned} &H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \geq a |u_n(b_n)|^\mu - b - \max_{|u| \leq R, t \in S_T} |H(t, u)| \\ &= a K_n^\mu - \left(b + \max_{|u| \leq R, t \in S_T} |H(t, u)| \right). \end{aligned}$$

Combine above two formulas, we have

$$(a - \sigma(R)M_3)K_n^\mu \leq b + \max_{|u| \leq R, t \in S_T} |H(t, u)|.$$

Since $a - \sigma(R)M_3 > 0$ and $K_n \rightarrow \infty$ as $n \rightarrow \infty$, the left-hand side tends to infinite, but the right-hand side is a constant independent of u and n . This leads a contradiction. Hence there exists $m \in \mathbb{N}$, which is determined by $H(t, u)$ and N only, such that for any $n \geq m$, if $u(t)$ is a critical point of I_n such that $I_n(u) \leq N$, we have $|u|_{C^0} \leq K_n$.

For $n \geq m$, if $\|u\|_{C^0}$ does not have an n -independent upper bound M_0 , then following the above proof with change K_n by M_n where $M_n \rightarrow \infty$ as $n \rightarrow \infty$, we can get the contradiction too. For $n < m$, as the Proof in last part of Lemma 2.1, we have

$$|u(t)| \leq M_1 + N_1 + D_k T + C_k, \quad \forall t \in S_T,$$

where C_k and D_k are determined by H_k , $k = 1, 2, \dots, m - 1$.

Hence we have

$$\|u\|_{C^0} \leq \max\{M_0, M_1 + N_1 + D_k T + C_k, k = 1, 2, \dots, m - 1\} = M.$$

Case II: suppose we have $\lim_{r \rightarrow \infty} \delta(r) = 0$.

We need only to modify the proof of Case I a little. By the definition of $\delta(r)$, we have $\delta(r)$ is increasing to 0. Fix a large $R > r_0$ such that

$$a + \delta(R)M_3 > 0.$$

Firstly we show $|u|_{C^0} \leq K_n$ for large n . If not, by passing a subsequence we may assume for each n , there exists $u_n(t)$, a_n and b_n such that

$$(a_n, b_n) \subset \{t \in S_T | R < |u_n(t)| < K_n\}$$

and $|u_n(a_n)| = K_n$, $|u_n(b_n)| = R$. Here we have

$$\begin{aligned} H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) &= \int_{a_n}^{b_n} H_t(t, u_n(t)) dt \\ &\geq \int_{a_n}^{b_n} \delta(|u_n(t)|) |u_n(t)|^\mu H(t, u_n(t)) dt \\ &\geq \delta(R) K_n^\mu \int_{\{H_n(t, u_n(t)) \geq 0\}} H(t, u_n(t)) dt \\ &\geq \delta(R) M_3 K_n^\mu. \end{aligned}$$

Hence we have

$$H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \geq \delta(R) M_3 K_n^\mu.$$

On the other hand, we have

$$\begin{aligned} H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) &\leq \max_{|u| \leq R, t \in S_T} |H(t, u) - a|u_n(a_n)|^\mu + b \\ &= \left(b + \max_{|u| \leq R, t \in S_T} |H(t, u)| \right) - aK_n^\mu. \end{aligned}$$

Combine above two formulas, we have

$$(a + \delta(R)M_3)K_n^\mu \leq b + \max_{|u| \leq R, t \in S_T} |H(t, u)|.$$

Since $a + \delta(R)M_3 > 0$ and $K_n \rightarrow \infty$ as $n \rightarrow \infty$, the left-hand side tends to infinite, but the right-hand side is a constant independent of u and n . This leads a contradiction. Hence $|u|_{C^0} \leq K_n$ holds for large n . Using the same discuss as those in Case I, we have

$$\|u\|_{C^0} \leq M.$$

Combine above two Cases, we know our Lemma holds. \square

3. Symmetric Hamiltonian systems

In this section, we consider the Hamiltonian system

$$\mathcal{J}\dot{u} + \nabla H(t, u) = 0, \quad (t, u) \in S_T \times \mathbf{R}^{2N},$$

where $H(t, u)$ being T -periodic in t -variable and is symmetric in the u -variable, i.e., there is a compact Lie group G acting on \mathbf{R}^{2N} via a representation $\varrho : G \rightarrow O(2N)$ and H is invariant under this action: $H(t, gu) = H(t, u)$ for every $t \in \mathbf{R}$, $g \in G$, $u \in \mathbf{R}^n$. We let V denote the vector space \mathbf{R}^{2N} considered as a G -space. If the action is symplectic, i.e., $\varrho(g)^t \mathcal{J} \varrho(g) = J$ for every $g \in G$, then every periodic solution u of (1) gives rise to a G -orbit gu , $g \in G$, of periodic solutions of (1).

Definition 3.1. We call V (or ϱ) admissible if given $k \geq 1$ and an open bounded G -invariant neighborhood $\mathcal{O} \subset V^k$ of 0 in V^k , any continuous map $f : \bar{\mathcal{O}} \rightarrow V^{k-1}$ which commutes with the action has a zero in $\partial\mathcal{O}$, where G acts on V^k via $g(v_1, \dots, v_k) := (gv_1, \dots, gv_k)$.

In [2], the authors have the following main result:

Theorem 3.1 (Theorem 4.1 in [2]). *If $H \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (G), (H1), and (H2)_p, then (1) possesses an unbounded sequence of classical solutions.*

In [2] the solutions are obtained as the critical points of the functional

$$I(u) = \frac{1}{2} \int_0^T -\mathcal{J}\dot{u} \cdot u \, dt - \int_0^T H(t, u) \, dt$$

by using the following critical point theorem.

Theorem 3.2 (Theorem 3.1 in [2]). *Let E be a G -Hilbert space and $\phi \in C^1(E, \mathbf{R})$ be a G -invariant functional satisfying*

(A1). *There exists an admissible representation V of G such that $E = \bigoplus_{j \in \mathbf{Z}} E^j$ is a G -Hilbert space with $E^j = V$ as a representation of G , for every $j \in \mathbf{Z}$.*

(A2). *There exists $a \in \mathbf{R}$ such that for each $k \geq 1$*

$$\inf_{R>0} \sup_{u \in E_k, \|u\| \geq R} \phi(u) < a,$$

where $E_k = \bigoplus_{j \leq k} E^j$.

(A3). $b_k = \sup_{r>0} \inf_{u \in E_{k-1}^\perp, \|u\|=r} \phi(u) \rightarrow \infty$, as $k \rightarrow \infty$.

(A4). $d_k = \sup_{u \in E_k} \phi(u) < \infty$.

(A5). *Every sequence $u_n \in F_n = E_{-n-1}^\perp = \bigoplus_{j \geq -n} E^j$ such that $\phi(u_n) \geq a$ is bounded and $(\phi|_{F_n})'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ contains a subsequence which converges in E to a critical point of ϕ .*

Then ϕ has an unbounded sequence of critical values. In fact, for each $k \geq 1$ with $b_k > a$, there exists a critical value $c_k \in [b_k, d_k]$.

Now we will prove our Theorems 1.1 and 1.2 by truncating the potential $H(t, u)$ with $H_n(t, u)$ as Proposition 2.1 in Section 2 to obtain a sequence of new systems such that the new systems satisfy the conditions of Theorem 4.1 in [2]. Replacing H by H_n , we study a sequence of new systems

$$\mathcal{J}\dot{u} + \nabla H_n(t, u) = 0. \tag{2}$$

From Proposition 2.1, we know each H_n satisfies (H1) and (G).

Remark 3.1. For condition (G), we need only to let

$$H_n(t, u) = \int_G H_n(t, gu) dg$$

where dg is the standard Haar measure on compact Lie group G .

And from (iii) of Proposition 2.1, we have

$$H_n(t, u) = (\tau_n + 1)|u|^{\lambda\mu}, \quad \text{for } |u| \geq K'_n,$$

let $p_n = \lambda\mu/(\lambda\mu - 1) \in (1, 2)$, we can check H_n satisfies (H2) $_{p_n}$ for some C_n and D_n which are determined by $H_n(t, u)$. Hence from Theorem 4.1 of [2], we know (2) has a sequence of classic solutions $\{u_k^n\}$ with unbounded critical values and we also have $I_n(u_k^n) \in [b_k^n, d_k^n]$ where b_k^n, d_k^n are defined as those in Theorem 3.1 in [2]. Now we will study b_k^n, d_k^n and a^n which obtain from Theorem 3.1 in [2] corresponding to I_n .

Lemma 3.1. *Given $n \in \mathbf{N}$, for each $k \geq 1$, $E_k = \bigoplus_{j \leq k} E^j$,*

$$\lim_{R \rightarrow \infty} \sup_{u \in E_k, \|u\| \geq R} I_n(u) = \inf_{R>0} \sup_{u \in E_k, \|u\| \geq R} I_n(u) = -\infty.$$

Proof. For any given $n \in \mathbf{N}$, integrating (v) of Proposition 2.1, we have

$$H_n(t, u) \geq a|u|^{\mu\sigma} - b, \quad \forall u \in \mathbf{R}^{2N},$$

where a and b are independent of n . Hence by Sobolev embedding theorem we have

$$\begin{aligned} I_n(u) &= \frac{1}{2} \int_0^T -\mathcal{J}\dot{u} \cdot u \, dt - \int_0^T H_n(t, u) \\ &\leq \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - aT \|u\|_{L^{\sigma\mu}}^{\sigma\mu} + bT \\ &\leq \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - a_1 \|u\|_{L^2}^{\sigma\mu} - a_2 \|u\|_{L^{\sigma\mu}}^{\sigma\mu} + b_1 \\ &\leq \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - a_1 \|u^+\|_{L^2}^{\sigma\mu} - a_2 \|u\|_{L^{\sigma\mu}}^{\sigma\mu} + b_1 \\ &\leq \left(\frac{1}{2} \|u^+\|^2 - a_1 \|u^+\|_{L^2}^{\sigma\mu} \right) - \frac{1}{2} \|u^-\|^2 - a_2 \|u\|_{L^{\sigma\mu}}^{\sigma\mu} + b_1, \end{aligned}$$

where a_1, a_2 and b_1 are some suitable positive constants. Since for $u \in E_k, u^+ \in \bigoplus_{j=1}^k E^j$ which is finite dimensional space, $\|u^+\|$ and $\|u^+\|_{L^2}$ are equivalent norms for $u^+ \in \bigoplus_{j=1}^k E^j$. Hence for $\sigma\mu > 2, \frac{1}{2} \|u^+\|^2 - a_1 \|u^+\|_{L^2}^{\sigma\mu}$ is bounded for any $u^+ \in \bigoplus_{j=1}^k E^j$. This implies

$$\lim_{R \rightarrow \infty} \sup_{u \in E_k, \|u\| \geq R} I_n(u) = -\infty. \quad \square$$

Lemma 3.2. *Given $k \in \mathbf{N}, \{d_k^n\}$ is decreasing as $n \rightarrow \infty$ and bounded by d_k^1 and $d_k = \sup_{u \in E_k} I(u)$.*

Proof. From (iv) of Proposition 2.1, for any $n \in \mathbf{N}$ we have

$$H_n(t, u) \leq H_{n+1}(t, u) \leq H(t, u), \quad \forall (t, u) \in S_T \times \mathbf{R}^{2N}.$$

This implies

$$I_n(u) \geq I_{n+1}(u) \geq I(u), \quad \forall u \in H^{1/2}(S_T, \mathbf{R}^{2N}).$$

By the definition of d_k^n we have

$$d_k^1 \geq d_k^n \geq d_k^{n+1} \geq d_k, \quad \forall n \in \mathbf{N}$$

hence our lemma holds. \square

Proof of Theorems 1.1 and 1.2. From Lemma 3.1, for any given $n \in \mathbf{N}$, we can choose an a^n negative enough such that $b_k^n > a^n$ for every $k \in \mathbf{N}$, since from the definition of b_k^n , they are increasing to ∞ as $k \rightarrow \infty$. Then we know from Theorem 3.1 in [2] that I_n has an unbounded sequence of critical values $\{c_k^n\}$, for each $k \in \mathbf{N}$, and $c_k^n \in [b_k^n, d_k^n]$, that means that there exist a sequence of critical points $\{u_k^n(t)\}$ such that $I_n(u_k^n) = c_k^n$. From Lemma 3.2 we have $I_n(u_k^n) = c_k^n \leq d_k^n \leq d_k^1$, for all $n, k \in \mathbf{N}$.

Hence for any given $k \in \mathbf{N}$, since $H(t, u)$ satisfies (H1), (H2) (or (H3)), and u_k^n is a critical point of I_n such that $I_n(u_k^n) \leq d_k^1$ holds for all $k \in \mathbf{N}$. From Lemma 2.1 (or Lemma 2.2) of Section 2, we have constant M_k , which is dependent on d_k^1 and $H(t, u)$ only, such that $\|u_k^n\|_{C^0} \leq M_k$ holds for all $n \in \mathbf{N}$.

Hence for large $n \in \mathbf{N}$ such that $K_n > M_k$, we have

$$\|u_k^n\|_{C^0} \leq M_k < K_n.$$

On the other hand, we have

$$H_n(t, u) = H(t, u), \quad \forall |u| < K_n.$$

This implies that $u_k^n(t)$ is a classic solution of (1) when $\|u_k^n\|_{C^0} < K_n$.

Hence for any given $k \in \mathbf{N}$, there exists a large enough $n \in \mathbf{N}$ such that $u_1^n(t), \dots, u_k^n(t)$ are classic T -periodic solutions of (1), i.e., Theorems 1.1 and 1.2 hold. \square

4. Hamiltonian systems without symmetry

In this section we will study the Hamiltonian systems without symmetry

$$J\dot{u} + B(t)u + \nabla \bar{H}(t, u) = 0, \quad (t, u) \in \mathbf{R} \times \mathbf{R}^{2N}, \tag{3}$$

let $H(t, u) = \frac{1}{2}(B(t)u, u) + \bar{H}(t, u)$, where $B(t)$ is a given continuous T -periodic and symmetric $2N \times 2N$ -matrix value function and $\bar{H}(t, u)$ is T -periodic in t -variable. In [6] there is the following result:

Theorem 4.1 (Theorem 1.1 in [6]). *For $T > 0$, $\bar{H} \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (H1), (H4) and (H3'), then (3) has a non-zero T -periodic solution in each case (i), (ii) in Theorem 1.3.*

We will use our estimates in Section 2 to replace those estimates in Proof of [6]. Here we will only give the sketch proof of Theorems 1.3 and 1.4.

Sketch Proof of Theorems 1.3 and 1.4. As Proof in [6], we first truncate the potential $H(t, u)$ by $\{H_n(t, u)\}$ obtaining from Proposition 2.1 in Section 2 to get a sequence of new systems. As in Section 2, we define $I_n(u)$ for these new systems. Use the same discuss as in Proof of Theorem 1.1 in [6], we have an n -independent constant $N > 0$, for each $n \in \mathbf{N}$, there exists a non-trivial critical point u_n of I_n , such that $I_n(u_n) \leq N$ holds for all $n \in \mathbf{N}$.

Since $H(t, u)$ satisfies (H1) and (H3) (or (H2)), by Lemma 2.2 (or Lemma 2.1), we have a n -independent constant M such that

$$\|u_n\|_{C^0} \leq M, \quad \text{for all } n \in \mathbf{N}.$$

On the other hand, we have

$$H_n(t, u) = H(t, u), \quad \text{for } |u| < K_n.$$

Hence for large $n \in \mathbf{N}$ such that $K_n > M$, u_n is a non-trivial T -periodic solution of (3), i.e. Theorems 1.3 and 1.4 hold. \square

In the following part we will study the sub-harmonic solutions (i.e. kT -periodic solutions) for non-autonomous systems (3). In [9] Rabinowitz first studied the existence

of sub-harmonic solutions for systems (3) under some certain conditions. One can also find some results on sub-harmonic solutions at [12,13]. Here we have the following result:

Theorem 4.2. For $T > 0$, $H \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (H1), (H3), (H4) and (H5) $H(t, u) \geq 0$, for all $(t, u) \in S_T \times \mathbf{R}^{2N}$, then there is a sequence $\{k_i\} \subset \mathbf{N}$ such that $k_i \rightarrow \infty$ as $i \rightarrow \infty$, and corresponding distinct $k_i T$ -periodic solutions $\{u_{k_i}\}$ of system (3).

Proof. We first show that there exists at least one non-zero T -periodic solution of (3) under our conditions. We truncate the potential $H(t, u)$ by $\{H_n(t, u)\}$ obtaining from Proposition 2.1 in Section 2 to get a sequence of new systems. As in Section 2, we define $I_n(u)$ for these new systems.

We will use Theorem 1.4 in [1] to obtain the existence of the non-zero critical point of I_n . Let $X := H^{1/2}(S_T, \mathbf{R}^{2N})$, by [7] and standard spectral theory, there exists a decomposition $X = X^+ \oplus X^0 \oplus X^-$ according to the selfadjoint operator B by extending the bilinear form

$$B(u, v) = \frac{1}{2} \int_0^T -\mathcal{J}\dot{u} \cdot v \, dt - \int_0^T (B(t)u, v) \, dt$$

with $\dim X^0 = \ker B < \infty$, $\dim X^+ = \dim X^- = \infty$.

We verify the conditions of Theorem 1.4 in [1] for I_n , set $X_1 = X^+$, $X_2 = X^0 \oplus X^-$ and

$$\begin{aligned} I_n(u) &= \frac{1}{2} \int_0^T -\mathcal{J}\dot{u} \cdot u \, dt - \int_0^T H_n(t, u) \, dt \\ &= \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \int_0^T \tilde{H}_n(t, u) \, dt. \end{aligned}$$

As the proof of Theorem 6.10 in [11], we have $I_n \in C^1(X, \mathbf{R})$ and I_n satisfies (I1)–(I3) of Theorem 1.4 in [1]. To verify (I4), we construct $S = \partial B_\rho \cap X_1$ which is the same as that in [1]. To obtain Q with r_1 and r_2 independent of n , following the proof of Theorem 1.4 in [9], we let $e \in \partial B_1 \cap X_1$ and $u = u^0 + u^- \in X_2$, then

$$\begin{aligned} I_n(u + se) &= s^2 - \|u^-\|^2 - \int_0^T \tilde{H}_n(t, u) \, dt \\ &\leq s^2 - \|u^-\|^2 - a_3(\|u^0\|^{\sigma\mu} + s^{\sigma\mu}) + a_4 \end{aligned}$$

with n -independent constants a_3 and a_4 which are determined by (v) of Proposition 2.1 in Section 2. Choose r_1 so that

$$\phi(s) = s^2 - a_3 s^\mu + a_4 \leq 0 \tag{4}$$

for all $s \geq r_1$. Choose r_2 large enough as [9], we have $I_n \leq 0$ on ∂Q with $Q = \{se \mid 0 \leq s \leq r_1\} \oplus (B_{r_2} \cap X_2)$. So from Theorem 1.4 in [1], I_n possesses a non-zero critical point u_n with $I_n(u_n) \geq \alpha_n > 0$.

Now we need to find an n -independent upper bound for $\{\|u_n\|_{C^0}\}$. In Theorem 1.4 in [1], the critical value c can be characterized as the minimax of I_n over an appropriate

class of sets (cf. [1]). Observe that Q is one of such sets and $H_n(t, u)$ satisfies (H5), therefore we have

$$I_n(u_n) = c_n \leq \sup_{u \in Q} I_n(u) \leq \sup_{\|u^0 + u^-\| \leq r_2, s \in [0, r_1]} \left(s^2 - \|u^-\|^2 - \int_0^T H_n(t, u) dt \right) \leq r_1^2.$$

Hence from Lemma 2.2 of Section 2, we have an n -independent constant M such that

$$\|u_n\|_{C^0} \leq M, \quad \text{for all } n \in \mathbf{N}.$$

On the other hand, we have

$$H_n(t, u) = H(t, u), \quad \text{for } |u| < K_n.$$

Hence for large $n \in \mathbf{N}$ such that $K_n > M$, u_n is a non-zero T -periodic solution of (3).

Now we show that (3) has infinitely many sub-harmonic solutions. We will follow the idea of Proof of Theorem 1.36 in [9]. For a given $k \in \mathbf{N}$, we make the change of variables $s = k^{-1}t$. Thus if $u(t)$ is a kT -periodic solution of (3), $\eta(s) = u(ks)$ satisfies

$$\mathcal{J} \frac{d\eta}{ds} + k(B(ks)\eta + \nabla \bar{H}(ks, \eta)) = 0. \tag{5}$$

Since $k\bar{H}(ks, u)$ satisfies the conditions of our Theorem, there is a solution $\eta_k(s)$ of (5), which is a critical point of

$$I_k(\eta) = \frac{1}{2} \int_0^T -\mathcal{J} \dot{\eta} \cdot \eta ds - k \int_0^T H(ks, \eta) ds.$$

Note that $\eta_1(ks)$ also satisfies (5), then if $\eta_1(ks) = \eta_k(s)$, we have $c_k = I_k(\eta_k) = kI_1(\eta_1) = kc_1$.

Next we show that c_k is bounded from above and the upper bound is independent of k . In the proof for the existence of one solution, we have $c_k \leq r_1^2(k)$ and the parameter $r_1(k)$ is determined by condition (4). The corresponding condition satisfied by $r_1(k)$ is

$$\phi_k(s) = s^2 - ka_3s^{\sigma\mu} + ka_4 \leq 0$$

for all $s \geq r_1(k)$. It follows that we can let

$$r_1(k) \leq \max \left(\left(\frac{2}{ka_3} \right)^{1/(\sigma\mu-2)}, \left(\frac{2a_4}{a_3} \right)^{1/\sigma\mu} \right) \leq \left(\frac{2}{a_3} \right)^{1/(\sigma\mu-2)} + \left(\frac{2a_4}{a_3} \right)^{1/\sigma\mu}. \tag{6}$$

Now, for any given $m \in \mathbf{N}$, if for some $k > m$, $\eta_k(s) = \eta_m(s)$ holds for all $s \in \mathbf{R}$, we have that $\eta_k(s)$ as kT -periodic function is k/l folds of $\eta_l(s)$ as lT -period function and $\eta_m(s)$ as mT -periodic function is m/l folds of $\eta_l(s)$ as lT -period function, for some $l \in \mathbf{N}$ such that $l|k$ and $l|m$ and some corresponding $\eta_l(s)$. Hence we have

$$c_k = I_k(\eta_k) = \frac{k}{l} I_l(\eta_l), \quad c_m = I_m(\eta_m) = \frac{m}{l} I_l(\eta_l),$$

that means

$$c_k = \frac{k}{m} c_m.$$

On the other hand, we have $c_m > 0$ and $\{c_k\}$ is bounded by a k -independent constant from (6). This implies that there are at most finitely many $k > m$ such that $\eta_k(s) = \eta_m(s)$ for any given $m \in \mathbf{N}$. Hence our theorem holds. \square

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