Advanced Nonlinear Studies **6** (2006), 399–410

Homoclinic Orbits For First Order Hamiltonian Systems With Convex Potentials

Xiangjin Xu

Mathematical Sciences Research Institute
17 Gauss Way, Berkeley, CA 94720
Current address:
Department of Mathematics, University of Virginia
P.O. Box 400137, Charlottesville, VA 22904
e-mail: xx8n@virginia.edu

Received in revised form 16 January 2006
Communicated by Yiming Long

Abstract

In this paper new estimates on the C^0-norm of homoclinic orbit are shown for first order convex Hamiltonian systems possessing super-quadratic potentials. Applying these estimates, some new results on the existence of infinitely many geometrically distinct homoclinic orbits are proved, which generalize the main results in [2] and [10].

2000 Mathematics Subject Classification. 37J45, 34C25, 47J30.
Key words. Homoclinic orbit, convex Hamiltonian system, super-quadratic potential.

1 Introduction

Let $H(t,u) \in C^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ and consider the Hamiltonian system

$$(HS) \qquad \mathcal{J}\dot{u} + \nabla H(t,u) = 0, \quad (t,u) \in S_T \times \mathbf{R}^{2N},$$

with $H(t,u) = \frac{1}{2}(Au,u) + W(t,u)$, where $\mathcal{J} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$, $A = A^*$ are $2N \times 2N$ symmetric matrices, and $W(t,u)$ is T-periodic in t-variable and satisfies globally super-quadratic in the u-variable, i.e., the potential W satisfies

(H1). There is a constant $\mu > 2$ such that

$$0 < \mu W(t,u) \leq u\nabla W(t,u), \quad \forall |u| > 0.$$

Recall that a solution u of the system (HS) is said to be homoclinic to 0 if $u \neq 0$ and $u(t) \to 0$ as $|t| \to \infty$. In recent years there have been many papers devoted to the existence of homoclinic orbits for the system (HS) (e.g., [1]- [4], [6]- [11]) by variational approach. In these papers, a condition is required on the growth of $H(t,u)$ at infinity as

$(H4)_p$. There are constants $c > 0$ and $p \geq \mu$, such that

$$|W(t,u)| \leq c|u|^p, \qquad for \quad |u| \quad large.$$

In the author's recent paper [12] and [15], the existence of homoclinic orbits for the system (HS) with (or without) symmetry was studied under some weaker conditions as

(H4). There are constant $c, R > 0$, such that

$$|\nabla W(t,u)| \leq c(\nabla W(t,u), u), \forall |u| \geq R.$$

or

$$(H5).\ \limsup_{|u|\to\infty} \frac{W_t(t,u)}{|u|^\mu W(t,u)} = 0,\ or\ \liminf_{|u|\to\infty} \frac{W_t(t,u)}{|u|^\mu W(t,u)} = 0,\ uniformly.$$

When $W(t,u)$ is further assumed to be strictly convex in u variable, the existence of infinitely many geometrically distinct homoclinic orbits for the system (HS) was proved under condition $(H4)_p$ in [10] by using the dual action principle of Clarke and a precise studying at the level c and the level $2c$ of the functional f.

In this paper, we shall show the existence of infinitely many geometrically distinct homoclinic orbits for the convex Hamiltonian systems with potentials satisfying (H4) or (H5). The reason that we can't apply directly the results in [12] to the convex Hamiltonian systems (HS) here is that in [10], the critical points are obtained through the dual action principle of Clarke and we can't get the uniformly bounded estimates on the critical values as required in [12]. In [14], we study the periodic solutions for the convex Hamiltonian systems (HS), where we need to deal with the same problem for the periodic solutions. Combining the ideas in [12] and [14], we prove two new estimates for C^0 bound for the homoclinic orbits with respect to bounded potentials under condition (H4) or (H5); we also proved the same estimates for C^0 bound for periodic solutions in [8] and [13]. Using the ideas from [12], [14] and [15], we have the following results:

Theorem 1.1 *Suppose $H(t,u)$ satisfies $(H1)$, $(H4)$ (or $(H5)$), and*

(A) $\mathcal{J}A$ is a constant matrix, all eigenvalues of which have non-zero real part,

(H2) $H(t,u)$ is T-periodic in t-variable and strictly convex in u-variable,

(H3) for some constant $c > 0$, $W(t,u) \geq c|u|^\mu$ for all $(t,u) \in \mathbf{R} \times \mathbf{R}^{2N}$.

Then there are infinitely many homoclinic orbits of system (HS), geometrically distinct in the following sense $u \neq v$ if and only if $u(\cdot) \neq v(\cdot + nT)$ for all $n \in \mathbf{Z}$.

Remark 1.1 In [10], the systems are restricted to those of which the potential grows as $|u|^\mu$ at infinity from the condition $(H4)_p$. Here by our condition (H4) or (H5), the same results hold for systems with a much larger class potentials, including those with growth rates $\exp(|u|^\alpha)$ at infinity. Especially from condition (H5), one can add any super-quadratic autonomous potential to the system (HS) and the results will still hold.

We organize this paper as following: In Section 2, we prove two new estimates on the bound of C^0-norm of homoclinic orbits of the system (HS) under the conditions $(H1)$ and $(H4)$(or $(H5)$). In Section 3, we firstly study the homoclinc orbits of a sequence of modified systems and show there is a uniform C^0 bound for all orbits, then we obtain our main results, and we also get some results when the quadratic term of the potential depends on t-variable.

2 Two estimates

In this section, we shall study the C^0 bound of homoclinic orbits for a sequence of modified systems

$$(HS)^n \qquad \mathcal{J}\dot{u} + \nabla H_n(t,u) = 0, \quad (t,u) \in S_T \times \mathbf{R}^{2N}.$$

where the potentials $\{H_n = \frac{1}{2}(Au,u) + W_n(t,u)\}$ satisfy

Proposition 2.1 *H satisfies (H1), $\sigma \in (0,1]$ such that $\mu\sigma > 2$, and there exist two sequences $\{K_n\}$ and $\{K'_n\}$ in $\mathbf{R}$ such that $\{H_n\}$ satisfies*

(i) $0 < K_0 < K_n < K_{n+1}, \forall n \in \mathbf{N}$, and $K_n \to \infty$ as $n \to \infty$, and $K_n < K'_n$ for each $n \in \mathbf{N}$.

(ii) for any given $t \in S_T$, $H_n(t,u) \in C^2(\mathbf{R}^{2N}, \mathbf{R})$, for every $n \in \mathbf{N}$.

(iii) $H_n(t,u) = H(t,u), \forall |u| \le K_n$, for every $n \in \mathbf{N}$; and for some $\lambda \in [\sigma, 1]$, such that $W_n(t,u) = (\tau_n + 1)|u|^{\mu\lambda}, \forall |u| \ge K'_n$, for every $n \in \mathbf{N}$.

(iv) $0 < \mu\sigma W_n(t,u) \le (\nabla W_n(t,u), u), \forall |u| > 0$, for every $n \in \mathbf{N}$.

Remark 2.1 Proposition 2.1 was first proved in [7], where the author studied the multiplicity of periodic solutions of perturbed superquadratic Hamiltonian systems. In fact, in [7] the author proved also that one can choose $\{H_n(t,u)\}$ monotone increasing as $n \to \infty$. Here we don't need this monotone property. Later, this proposition was widely used in [8], [12]-[15] to studying the existence and multiplicity of periodic solutions and homoclinic orbits for first order Hamiltonian systems with general superquadratic potentials.

Here, combining the ideas from [12], where we study the existence of periodic solutions for the convex Hamiltonian system, and [14], where we study the existence of homoclinic orbits for the general Hamiltonian systems, we prove two new estimates on the C^0 bound of homoclinic orbits of the modified systems $(HS)^n$ under the condition (H4) or (H5).

Lemma 2.1 *Suppose $H(t,u)$ satisfies (H1) and (H4), $\{H_n\}$ satisfies Proposition 2.1, and $u(t)$ is a solution of system $(HS)^n$ such that*

$$\int_{\mathbf{R}} \nabla W_n(t,u) u dt \le C, \qquad \int_{\mathbf{R}} W_n(t,u) dt \le C. \tag{2.1}$$

Then we have the following estimate

$$||u||_{C^0} \leq M$$

where M *is independent of* u *and* n .

Proof. Integrating (iv) of Proposition 2.1 we yield

$$\begin{aligned} W_n(t,u) &\geq a|u|^{\mu\sigma}, \quad \forall |u| \geq 1, \\ W_n(t,u) &\leq b|u|^{\mu\sigma}, \quad \forall |u| \leq 1 \end{aligned}$$

Where $a = \min_{|u|=1, t\in S_T} W(t,u)$ and $b = \max_{|u|\leq 1, t\in S_T} |W(t,u)|$ are independent of n. We first show that for large enough n,

$$||u||_{C^0} \leq K_n.$$

If not, by passing a subsequence, without loss generality, for each $n \in \mathbf{N}$, there exists $u_n(t)$ and $t_n \in \mathbf{R}^+$, such that $|u_n(t_n)| = K_n$, $|u_n(0)| = 1$ and $1 \leq |u_n(t)| \leq K_n$ for $t \in [0, t_n)$. Since

$$C \geq \int_{\mathbf{R}} W_n(t,u_n)dt \geq \int_0^{t_n} W_n(t,u_n)dt \geq a\int_0^{t_n} |u_n|^{\mu} dt \geq a\int_0^{t_n} |u_n|dt$$

we have $\int_0^{t_n} |u_n|dt \leq C/a$. Hence we have

$$\begin{aligned} K_n - 1 &= |u_n(t_n)| - |u_n(0)| = \int_0^{t_n} \frac{d}{ds}|u_n(s)|ds \\ &= \int_0^{t_n} (u_n(s), \dot{u}_n(s))/|u_n(s)|ds \\ &\leq \int_0^{t_n} |\dot{u}(s)|ds \qquad (since\ |u_n(s)| \leq K_n) \\ &\leq \int_0^{t_n} (|Au_n(s)| + |\nabla W(s,u_n(s))|)ds \quad (by\ (H4)) \\ &\leq ||A||_{L^\infty}\int_0^{t_n} |u_n(s)|ds + c\int_0^{t_n} (\nabla W(s,u_n(s)), u_n(s))ds \\ &\leq N_1 C + c\int_{\mathbf{R}} (\nabla W_n(s,u_n(s)), u_n(s))ds \\ &\leq N_1 C + cC \end{aligned}$$

where c, N_1 and C are n-independent constants. But we have $K_n \to \infty,\ as\ n \to \infty$. This leads to a contradiction. Hence there exists $m \in \mathbf{N}$, which is determined by $W(t,u)$ and N only, such that for any $n \geq m$, if u is a critical point of the modified system $(HS)^n$ with (2.1), then $||u||_{C^0} \leq K_n$ holds.

Hence for any critical point u of the modified system $(HS)^n$ with (2.1), if $n \geq m$, repeating the about computation, we have

$$|u(t)| \leq N_1 C + cC + 1, \quad \forall\ t \in \mathbf{R}.$$

For $k < m$, from (iii) of Proposition 2.1, we have

$$|\nabla W_k(t,u)| \leq c_k(\nabla W_k(t,u), u), \quad \forall\, |u| > 1$$

for some suitable constant c_k, which is determined by W_k for $k = 1, 2, \cdots, m-1$. Hence by the same argument as above we have

$$|u(t)| \leq N_1 C + c_k C + 1, \quad \forall\, t \in \mathbf{R}.$$

Then we have

$$||u||_{C^0} \leq \max\{N_1 C + cC + 1,\ N_1 C + c_k C + 1,\ k = 1, 2, \cdots, m-1\} = M.$$

Hence our Lemma holds.

Q.E.D.

Lemma 2.2 *Suppose $H(t,u)$ satisfies (H1) and (H5), $\{H_n\}$ satisfies Proposition 2.1, and $u(t)$ is a solution of system $(HS)^n$ such that*

$$\int_{\mathbf{R}} \nabla W_n(t,u) u dt \leq C, \qquad \int_{\mathbf{R}} W_n(t,u) dt \leq C.$$

Then we have the following estimate

$$||u||_{C^0} \leq M$$

where M is independent of u and n .

Proof. As above proof, we have

$$\begin{aligned} W_n(t,u) &\geq a|u|^{\mu\sigma}, \quad \forall |u| \geq 1, \\ W_n(t,u) &\leq b|u|^{\mu\sigma}, \quad \forall |u| \leq 1. \end{aligned}$$

Since $W(t,u)$ satisfies $(H5)$, defining

$$\sigma(r) = \sup_{|u| \geq r, t \in S_T} \frac{W_t(t,u)}{|u|^\mu W(t,u)}$$

and

$$\delta(r) = \inf_{|u| \geq r, t \in S_T} \frac{W_t(t,u)}{|u|^\mu W(t,u)}$$

then $(H5)$ means

$$\lim_{r \to \infty} \sigma(r) = 0 \ or \ \lim_{r \to \infty} \delta(r) = 0.$$

Case I: Suppose that we have

$$\lim_{r \to \infty} \sigma(r) = 0.$$

By the definition of $\sigma(r)$, we have $\sigma(r)$ is decreasing to 0. Fix a large $R > 1$ such that

$$a - \sigma(R)C > 0.$$

Firstly we show $|u|_{C^0} \le K_n$ for large n. If not, by passing a subsequence we may assume for each n, there exist $u_n(t)$, a_n and b_n such that

$$(a_n, b_n) \subset \{t \in \mathbf{R} | R < |u_n(t)| < K_n\}$$

and $|u_n(a_n)| = R$, $|u_n(b_n)| = K_n$. Since we have

$$C \ge \int_{\mathbf{R}} W_n(t, u_n)dt \ge \int_{a_n}^{b_n} W(t, u_n)dt \ge \int_{a_n}^{b_n} a|u_n(t)|^\mu dt \ge \int_{a_n}^{b_n} a|u_n(t)|^2 dt$$

hence $\int_{a_n}^{b_n} a|u_n(t)|^2 dt \le C/a$. Here we have

$$\begin{aligned}
& H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \\
= \; & \int_{a_n}^{b_n} \frac{d}{dt} H_n(t, u_n(t))dt \qquad (since\ |u_n(t)| \le K_n) \\
= \; & \int_{a_n}^{b_n} \nabla H_n(t, u_n(t)) \cdot \dot{u}_n(t)dt + \int_{a_n}^{b_n} H_t(t, u_n(t))dt \\
= \; & \int_{a_n}^{b_n} \frac{1}{2}(\frac{d}{dt}A)u_n \cdot u_n dt + \int_{a_n}^{b_n} W_t(t, u_n)dt \\
\le \; & ||A'||_{L^\infty} \int_{a_n}^{b_n} a|u_n(t)|^2 dt + \int_{a_n}^{b_n} \sigma(|u_n|)|u_n|^\mu W(t, u_n)dt \\
\le \; & N_1 C + \sigma(R) K_n^\mu \int_{a_n}^{b_n} W(t, u_n)dt \\
\le \; & N_1 C + \sigma(R) K_n^\mu \int_{\mathbf{R}} W_n(t, u_n)dt \\
\le \; & N_1 C + \sigma(R) C K_n^\mu.
\end{aligned}$$

Hence we have

$$H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \le N_1 C + \sigma(R) C K_n^\mu.$$

On the other hand, we have

$$\begin{aligned}
& H(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \\
= \; & \frac{1}{2}(Au_n(b_n), u_n(b_n)) + W(b_n, u_n(b_n)) - H(a_n, u_n(a_n)) \\
\ge \; & a|u_n(b_n)|^\mu - ||A||_{L^\infty} |u_n(b_n)|^2 - \max_{|u| \le R, t \in S_T} |H(t, u)| \\
= \; & a K_n^\mu - ||A||_{L^\infty} K_n^2 - \max_{|u| \le R, t \in S_T} |H(t, u)|.
\end{aligned}$$

Combine above two formulas, we have

$$(a-\sigma(R)C)K_n^{\mu}-||A||_{L^\infty}K_n^2\leq N_1C+\max_{|u|\leq R,t\in S_T}|H(t,u)|.$$

Since $\mu>2$, $a-\sigma(R)C>0$ and $K_n\to\infty$ as $n\to\infty$, the left side tends to infinity, but the right side is a constant independent of u and n. This leads to a contradiction. Hence there exists $m\in\mathbf{N}$, which is determined by $H(t,u)$ and N only, such that for any $n\geq m$, if $u(t)$ is a critical point of the modified system $(HS)^n$ with (2.1), we have $|u|_{C^0}\leq K_n$.

For $n\geq m$, if the C^0-norm of critical points u, which satisfy (2.1), doesn't have an n-independent upper bound M_0, then repeating the above proof by replacing K_n by M_n with $M_n\to\infty$ as $n\to\infty$, we can also get the contradiction. For $n<m$, as the proof in last part of Lemma 2.1, we have

$$|u(t)|\leq N_1C+c_kC+1,\ \forall\, t\in\mathbf{R}$$

where c_k is determined by W_k for $k=1,2,\cdots,m-1$.

Hence we have

$$||u||_{C^0}\leq\max\{M_0,\ N_1C+c_kC+1,\ k=1,2,\cdots,m-1\}=M.$$

Case II: Suppose that we have $\lim_{r\to\infty}\delta(r)=0$.

We need only to modify the proof of **Case I** a little. By the definition of $\delta(r)$, we have $\delta(r)$ is increasing to 0. Fix a large $R>1$ such that

$$a+\delta(R)C>0.$$

We have

$$(a+\delta(R)C)K_n^{\mu}-||A||_{L^\infty}K_n^2\leq\max_{|u|\leq R,t\in S_T}|H(t,u)|$$

where $a+\delta(R)\bar{C}>0$ and $K_n\to\infty$ as $n\to\infty$. Using the same argument as in Case I, we have

$$||u||_{C^0}\leq M.$$

By combining these two cases, we obtain the Lemma. Q.E.D.

3 Proof of Theorem 1.1

In order to prove our Theorem 1.1, we first modify the H by a sequence $\{H_n\}$ which satisfies Proposition 2.1 and $(H2)$ such that we can apply the results in [10] to the modified system $(HS)^n$, which are the systems replacing H by H_n in the system (HS). We do the truncation the same way as done in page 185 of [5], where the authors dealt with the truncation for H under autonomous case. Here we do the same truncation to $H(t,\cdot)$ for any fixed $t\in S_T$ as they did to $H(\cdot)$, and we can also let $\sigma=1$ in Proposition 2.1 from the construction in [5]. Then we have a sequence of modified systems:

$$(HS)^n\qquad \mathcal{J}\dot{u}+\nabla H_n(t,u)=0,\quad (t,u)\in S_T\times\mathbf{R}^{2N}$$

where each $H_n = \frac{1}{2}(Au, u) + W_n(t, u)$ satisfies (A), $(H1)$, $(H2)$, $(H3)$ and $(H4)_p$.

Now following the setting in [2] and [10], we state the following Lemmas without proofs; one can find the further details and proofs in [2] and [10]. Before studying the non-linear equation $(HS)^n$, we first study the linear operator: $u \mapsto -\mathcal{J}u' - Au$. We have:

Lemma 3.1 (Lemma 1 in [10]) *Assuming (A) and choosing $\beta \in (1,2)$, the operator $D : W^{1,\beta} \to L^\beta$*

$$u \mapsto -\mathcal{J}u' - Au$$

is a bicontinuous bijection, whose inverse L is a convolution kernel:

$$x = -\mathcal{J}u' - Au \quad \Leftrightarrow \quad Lx(t) = u(t) = (\mathcal{L} * x)(t)$$

with

$$\mathcal{L}(t) = e^{tE}(\chi_{\mathbf{R}_+}(t)P_s + \chi_{\mathbf{R}_-}(t)P_u).$$

Here $*$ is the convolution operator, P_s and P_u are the projections on the stable and unstable spaces of the hyperbolic flow e^{tE}, and χ_I is the characteristic function of set I.

The variational formulation used here is inspired by Clarke's dual action principle. As in [10], define the Legendre transform G of W by

$$G(t, x) = \sup_{u \in \mathbf{R}^{2N}} [(x, u) - W(t, u)].$$

Now we define the functional f.

Lemma 3.2 (Lemma 2 in [10]) *Suppose (A), $(H1)$, $(H2)$, $(H3)$ and $(H4)_p$ are true for H. Define $\beta \in (1,2)$ by $\frac{1}{\mu} + \frac{1}{\beta} = 1$. Consider the functional f_n on $L^\beta(\mathbf{R}, \mathbf{R}^{2N})$*

$$f(x) = \int_{\mathbf{R}} G_n(t, x)dt - \frac{1}{2}\int_{\mathbf{R}} (x, Lx)dt.$$

f is well defined and of class C^1. Denote

$$\mathcal{C} = \{x \in L^{\mu/(\mu-1)} | x \neq 0 \text{ and } f'(x) = 0\},$$

if $x \in \mathcal{C}$, we have $u(t) = Lx(t)$ is a classical solution of the system (HS).

Now define $\Gamma = \{\gamma \in C^0([0,1], L^{\mu/(\mu-1)}) | \gamma(0) = 0, f(\gamma(1)) < 0\}$. We are in a minimax situation:

Lemma 3.3 (Lemma 5 in [10]) *Assume (A), $(H1)$, $(H2)$, $(H3)$ and $(H4)_p$ are true for H. Then Γ is non-empty, and $c = \inf_{\gamma \in \Gamma}(\max f(\gamma(\cdot))) > 0$. $\mathcal{C}$ has therefore at least one element x, with $f(x) \le c$. Moreover, if $\mathcal{C}/\mathbf{Z}$ is finite, then there is $x \in \mathcal{C}$ with $f(x) = c$.*

These Lemmas were stated in [10] and proved in [2]. In [10], for the convex Hamiltonian system with potential H satisfying (A), $(H1)$, $(H2)$, $(H3)$ and $(H4)_p$, by studying the level $2c$ of the functional f for the system (HS), the author proved the following main result:

Theorem 3.1 **(Multiplicity Theorem in [10])** *Assuming H satisfies (A), $(H1)$, $(H2)$, $(H3)$ and $(H4)_p$, there are infinitely many homoclinic orbits of system (HS), geometrically distinct in the sense that $u \neq v$ if and only if $u(\cdot) \neq v(\cdot + nT)$ for all $n \in \mathbf{Z}$.*

To prove Theorem 1.1, we shall apply Theorem 3.1 to the modified systems $(HS)^n$, and using the Lemmas in last section and the estimates on the homoclinic orbits of the modified system $(HS)^n$, which is based on the construction of the homoclinic orbits in [10]. We show that for n large enough, the homoclinic orbits of the modified system $(HS)^n$ are exactly the homoclinic orbits of system (HS).

Proof of Theorem 1.1 . From Proposition 2.1 and condition (A) and (H3), for the each modified system $(HS)^n$, the conditions of Theorem 3.1 are satisfied. Then we have infinitely many geometrically distinct homoclinic orbits $\{u_{n,k}\}_{k\in\mathbf{N}}$ for every modified system $(HS)^n$.

Next we study the bound of $\{||u_{n,k}||_{C^0}\}_{n,k\in\mathbf{N}}$. Define the Legendre transform $\{G_n\}$ of $\{W_n\}$ by

$$G_n(t,x) = \sup_{u\in\mathbf{R}^{2N}}[(x,u) - W_n(t,u)].$$

From Proposition 2.1, we have $W_n(t,u) \geq a|u|^\mu$ for all $|u| \geq 1$, where we may let $a = \min_{|u|=1,t\in S_T} W(t,u)$ as in Lemma 2.1. Define $W_0(t,u) = \min\{a,c\}|u|^\mu$, where c comes from condition (H3), which implies $W_n(t,u) \geq W_0(t,u), \forall |u| \geq 0$, for every $n \in \mathbf{N}$. By duality, the assumptions on $\{W_n\}$ imply the following properties of $\{G_n\}$:

(G1). $G_n(t,x) \leq G_0(t,x), \forall |x| \geq 0$, for every $n \in \mathbf{N}$, where $G_0(t,x)$ is the Legendre transform of $W_0(t,u)$.

(G2). $\frac{\mu}{\mu-1}G_n(t,x) \geq (\nabla G_n(t,x),x) > 0, \forall |x| > 0$, for every $n \in \mathbf{N}$.

For each integer $n \in \mathbf{N}\cup\{0\}$ as in Lemma 3.2, with $\beta = \mu/(\mu-1)$, we define the functional f_n on $L^{\mu/(\mu-1)}(\mathbf{R},\mathbf{R}^{2N})$

$$f_n(x) = \int_{\mathbf{R}}[G_n(t,x) - \frac{1}{2}(x,Lx)]dt,$$

where $Lx(t)$ is defined in Lemma 3.1. From Lemma 3.2, we know f_n is well defined and of class C^1 for each $n \in \mathbf{N}\cup\{0\}$. Let

$$\mathcal{C}_n = \{x \in L^{\mu/(\mu-1)} | x \neq 0 \ and \ f_n'(x) = 0\}.$$

If $x \in \mathcal{C}$, we have $u(t) = Lx(t)$ is a classical solution of the modified system $(HS)^n$.

Define

$$\Gamma_n = \{\gamma \in C^0([0,1], L^{\mu/(\mu-1)}) | \gamma(0) = 0, f_n(\gamma(1)) < 0\}.$$

From Lemma 3.3, we know that Γ is non-empty, and

$$c_n = \inf_{\gamma \in \Gamma} (\max f_n(\gamma(\cdot))) > 0, \forall n \in \mathbf{N}.$$

From (G1), we have

$$c_n \leq c_0, \qquad \forall n \in \mathbf{N}.$$

From the Remark at the end of [10], we know that for each integer $n \in \mathbf{N} \cup \{0\}$, $(\mathcal{C}_n \cap f_n^{2c_n+1})/\mathbf{Z}$ is infinite; here we choose $\epsilon = 1$ in the results of that Remark. Hence we may assume that for every system $(HS)^n$, the homoclinic orbits $u_{n,k} = Lx_{n,k}$, which obtained from Theorem 3.1, satisfy

$$f_n(x_{n,k}) \leq 2c_n + 1 \leq 2c_0 + 1, \qquad \forall n, k \in \mathbf{N}.$$

Since $x_{n,k}$ is a critical point for f_n on $L^{\mu/(\mu-1)}$, we have

$$u_{n,k}(t) = Lx_{n,k}(t) = \nabla G_n(t, x_{n,k}(t)), \quad \forall t \in \mathbf{R}.$$

Using the Lgendre reciprocity formula

$$\nabla G_n(t,x) = u, \quad iff \quad \nabla W_n(t,u) = x,$$

together with (G2) and (H1), we have

$$\begin{aligned} 2c_0 + 1 &\geq \int_{\mathbf{R}} [G_n(t, x_{n,k}) - \frac{1}{2}(x_{n,k}, Lx_{n,k})]dt \\ &\geq [\frac{\mu-1}{\mu} - \frac{1}{2}] \int_{\mathbf{R}} (x_{n,k}, \nabla G_n(t, x_{n,k}(t)))dt \\ &= [\frac{1}{2} - \frac{1}{\mu}] \int_{\mathbf{R}} (u_{n,k}, \nabla W_n(t, u_{n,k}(t)))dt \\ &\geq [\frac{\mu}{2} - 1] \int_{\mathbf{R}} W_n(t, u_{n,k}(t))dt. \end{aligned}$$

Hence we have

$$\int_{\mathbf{R}} \nabla W_n(t, u_{n,k}) u_{n,k} dt \leq C, \qquad \int_{\mathbf{R}} W_n(t, u_{n,k}) dt \leq C \qquad \forall n, k \in \mathbf{N}$$

where C is independent of n, k. Now applying Lemma 2.1 and Lemma 2.2 to the homoclinic orbit sequences $\{u_{n,k}\}_{n,k \in \mathbf{N}}$, we have a uniform bound M such that

$$||u_{n,k}||_{C^0} \leq M, \qquad \forall n, k \in \mathbf{N}.$$

Since $K_n \to \infty$ as $n \to \infty$, we have $K_n > M$ for n large enough. For such a system $(HS)^n$, from (iii) of Proposition 2.1, which says

$$H_n(t,u) = H(t,u), \qquad \forall |u| \leq K_n$$

we know that the homoclinic orbits of system $(HS)^n$, each of which satisfies

$$|u_{n,k}(t)| < K_n, \qquad \forall t \in \mathbf{R},$$

are exactly the homoclinic orbits of system (HS). Q.E.D.

Notice that condition (A) is only used in Lemma 3.1 in the proof. For the potential $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u)$, we replace condition (A) by:

(L). *$L(t)$ depends on t with period T, and there is $\alpha > 0$ such that $(0,\alpha) \cap \sigma(A) = \emptyset$ where $A = -(\mathcal{J}\frac{d}{dt} + L(t))$ is the selfadjoint operator acting on $L^2(\mathbf{R},\mathbf{R}^{2N})$ with the domain $\mathrm{D}(A) = H^1(\mathbf{R},\mathbf{R}^{2N})$ and $\sigma(A)$ is the spectrum of A.*

or

($\bar{\mathbf{L}}$). *$L(t)$ depends on t with period T, and there is $\alpha > 0$ such that $(-\alpha,0) \cup \sigma(A) = \emptyset$ where $A = -(\mathcal{J}\frac{d}{dt} + L(t))$ is the selfadjoint operator acting on $L^2(\mathbf{R},\mathbf{R}^{2N})$ with the domain $\mathrm{D}(A) = H^1(\mathbf{R},\mathbf{R}^{2N})$ and $\sigma(A)$ is the spectrum of A.*

In [3] and [4], the same results on the linear operator $D : u \mapsto -\mathcal{J}u' - Au$ are studied, and our Lemma 2.1 and Lemma 2.2 hold for the potentials $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u)$. Hence we have the following results:

Theorem 3.2 *Suppose $H(t,u) = \frac{1}{2}L(t)u \cdot u + W(t,u)$, where $L \in C^1(\mathbf{R},\mathbf{R}^{4N^2})$ is a $2N \times 2N$ symmetric matrix valued function and $W \in C^2(\mathbf{R} \times \mathbf{R}^{2N},\mathbf{R})$ satisfies (L)(or $(\bar{L})$), $(H1)$, $(H2)$, $(H3)$ and $(H4)$ (or $(H5)$), there are infinitely many homoclinic orbits of system (HS), geometrically distinct in the sense that $u \neq v$ if and only if $u(\cdot) \neq v(\cdot + nT)$ for all $n \in \mathbf{Z}$.*

Acknowledgement: The first version of this paper was done in department of mathematics, Johns Hopkins University, where the author studied for his Ph.D. The final version of this paper was completed during the Fall 2005, while the author was visiting in MSRI, Berkeley, as a MSRI Postdoc. He wishes to thank his teachers and friends in Hopkins for their helps and discussions in his research, and the MSRI for the warm and stimulating hospitality. Finally but not the least, the author'd like to thank the referee and editor for their careful reading and valued suggestions on this paper.

References

[1] G. Arioli and A. Szulkin, *Homoclinic solutions of Hamiltonian systems with symmetry*, J. Diff. Eq. **158** (1999), 291-313.

[2] V. Coti-Zelati, Ekeland, I. and E. Sere,*A variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann. **288** (1990), 133 - 160.

[3] Y. H. Ding and M. Girardi, *Infinitely many homoclinic orbits of a Hamiltonian system with symmetry*, Nonlinear Analysis TMA, **38** (1999), 391-415.

[4] Y. H. Ding and M. Willem, *Homoclinic orbits of a hamiltonian system*, Z. Angew. Math. Phys. **50** (1999),759-778.

[5] Ekeland, I. and Hofer, *H. Periodic solutions with prescribed minimal period for convex autonomous Hamiltonian systems*, Invent. Math. **81** (1985), 155-188.

[6] H. Hofer and K. Wysocki, *First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems*, Math. Ann. **228** (1990), 483-503.

[7] Long, Yiming, *Periodic solutions of perturbed superquadratic Hamiltonian systems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) bf 17, no. 1 (1990), 35–77.

[8] Long, Yiming and Xu, Xiangjin, *Periodic solutions for a class of nonautonomous Hamiltonian systems*, Nonlinear Anal. **41** (2000), no. 3-4, Ser. A: Theory Methods, 455–463.

[9] P.H. Rabinowitz, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh. **114** (1990), 33-38.

[10] E. Sere, *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Z. **209** (1992), 27 - 42.

[11] Tanaka, Kazunaga, *Homoclinic orbits in a first order superquadratic Hamiltonian system: convergence of subharmonic orbits*, J. Differential Equations **94** (1991), no. 2, 315–339.

[12] Xu, Xiangjin, *Homoclinic Orbits for First Order Hamiltonian Systems possessing super-quadratic potentials*, Nonlinear Analysis, TMA **51** (2002) 197 - 214.

[13] Xu, Xiangjin, *Periodic solutions for non-autonomous Hamiltonian systems possessing super-quadratic potentials*, Nonlinear Anal. **51**, no.6 (2002), 941–955.

[14] Xu, Xiangjin, *Subharmonics for first order convex nonautonomous Hamiltonian systems*, Journal of Dynamics and Differential Equations **15**, No. 1 (2003), 107-123.

[15] Xu, Xiangjin, *Subharmonics of first order Hamiltonian systems and their asymptotic behaviors*, Discrete and Continuous Dynamical Systems: Series B, **3**, no. 4 (2003), 643-654.