PROCEEDINGS OF THE FOURTH INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 24 – 27, 2002, Wilmington, NC, USA pp. **1–8**

MULTIPLE SOLUTIONS OF SUPER-QUADRATIC SECOND ORDER DYNAMICAL SYSTEMS

XIANGJIN XU
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD, 21218 USA

ABSTRACT. *In this paper the existence of periodic solutions of large norm for the super-quadratic second order dynamical systems* $A\ddot{x} = -\nabla V(x)$ *is proved. And some results for perturbed systems are also gained.*

1. **Introduction.** This paper consider the existence of periodic solutions of the second order dynamical systems

$$A\ddot{x} = -\nabla V(x), \tag{1}$$

where $x \in \mathbf{R}^N$, A is a nonsingular $N \times N$ symmetric matrix, but not necessarily positive definite. Without loss of generality, we assume $A = \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1)$ with the number of 1 equal to $N_1 > 0$ and the number of -1 equal to $N_2 = N - N_1$. In [2], Han stated that Professor L.Nirenberg asked the following question: whether under suitable conditions, infinitely many periodic solutions of (1) with prescribed period exist - as is known in the case $A = Id$. We have the following results:

Theorem 1. *Suppose* $V \in C^1(\mathbf{R}^N)$ *satisfies:*
(V1) there exists $\mu > 2$ *such that* $0 < \mu V(x) \le (x, \nabla V(x))$ *for* $|x|$ *large.*
(V2) there exist $a, b > 0$, *such that*

$$|\nabla V(x)| \le c(\nabla V(x), x) + d, \qquad \forall x \in \mathbf{R}^N.$$

Then for any $T > 0$ *and* $R > 0$, *(1) has a* T*-period solution* $x(t)$ *with* $\max_{t\in[0,T]} |x(t)| \ge R$. *If A is positive definite, (V2) does not need.*

Theorem 2. *Suppose* $V \in C^1(\mathbf{R}^N)$ *satisfies (V1) and*
(V3) let Π *be the projection operator of* $\mathbf{R}^N$ *onto the negative eigenspace of* A, *assume* $(\Pi x, \nabla V(x)) \ge -\alpha V(x)$ *for some* $\alpha > 0$ *and for* $|x|$ *large.*
then the results of Theorem 1 also holds.

Remark 1. Theorem 1.1 and Theorem 1.2 treat the case of a dynamical system which has attraction in certain directions and repulsion in the other directions. The condition $(V2)$ means that the direction difference between $\nabla V(x)$ and x is not too large. The condition $(V3)$ introduced in [2], says that the repulsion is not to be too strong. When the matrix A is positive definite, the results without condition $(V2)$ and $(V3)$ is well known, cf. [4], [7]. Our theorems improve the main result

---

1991 *Mathematics Subject Classification.* 37J45,(34C25,47J30).
Key words and phrases. periodic solution, second order dynamical systems.

in [2], where the author further required $V(x) \geq 0$ for all $x \in \mathbf{R}^N$ and proved the existence of one nonconstant solution essentially.

Following the idea of [6], where study the first order autonomous Hamiltonian systems using S^1 index theory $(i(\cdot), \mathcal{B})$ introduced in [1], we firstly introduce a simple system and study the structure of its solutions and the increasing estimates of the critical values of this new system using S^1 index theory, then we prove Theorem 1.1 and Theorem 1.2 by comparing the critical values between the original system and the new system. And we also obtain some results on perturbed systems by the same argument and the increasing estimates gained in [3].

This paper grow from part of my master degree thesis [8]. It is a pleasure to thank my advisor, Professor Yiming Long, for his constant encouragement and helpful suggestions when I studied in Nankai Institute of Mathematics.

2. **Proofs.** For any given $T > 0$, let $X = W^{1,2}(S_T, \mathbf{R}^N)$. Define

$$I(x) = \int_0^T (A\dot{x}, \dot{x})dt - \int_0^T V(x)dt, \qquad \forall x \in X.$$

It is well known that $I \in C^1(X, \mathbf{R})$. We define

$$\begin{aligned} X^+ &= \{x \in X | x = \sum_{|j| \neq 0} c_j e^{ijt},\ c_j \in \mathbf{C}^{N_1}, c_{-j} = \bar{c}_j\} \\ X^0 &= \{x \in X | x = constants\} \\ X^- &= \{x \in X | x = \sum_{|j| \neq 0} c_j e^{ijt},\ c_j \in \mathbf{C}^{N_2}, c_{-j} = \bar{c}_j\} \end{aligned}$$

We have $X = X^+ \oplus X^0 \oplus X^-$.

The minimax procedure that we will use in the proofs takes advantage of the S^1 index theory $(i(\cdot), \mathcal{B})$ introduced in [1], where $\mathcal{B}$ denotes the family of closed S^1 invariant subset of $X - 0$ and $i(\cdot)$ is the S^1 cohomological index on the invariant subset, since there is a natural S^1 invariance possessed by the functional I. One can find the details of the index theory in [1].

For $m \in \mathbf{N}$, we define

$$H_m = (\text{span}\{e^{ijt}v_k, e^{-ijt}v_k | j \leq [m/N_1],\ k \leq m - N_1 j\} \cap X^+) \oplus X^0 \oplus X^-,$$

where $\{v_k\}$ is the standard base of $\mathbf{C}^{N_1}$ and $[a]$ denotes the greatest integer not greater than a. Then H_m is an invariant subspace of X. (V1) implies for certain constants a_1 and a_2

$$V(x) \geq a_1|x|^\mu - a_2.$$

By the Hölder inequality,

$$I(x) \leq ||x^+||^2 - a_1||x||^\mu_{L^2} + a_2 \leq ||x^+||^2 - a_1||x^+||^\mu_{L^2} + a_2. \tag{2}$$

Since $H_m \cap X^+$ is m dimensional and $\mu > 2$,(2) shows that for any given constant $M(0)$ there is an $R_m > 0$ such that

$$I(x) \leq -TM(0),$$

for all $x \in H_m$ such that $||x|| \geq R_m$. Let $D_m = B_{R_m} \cap H_m$. Let G_m denote the class of mappings $h \in C(D_m, X)$ which satisfy:

(g_1) h is equivariant,

(g_2) $h(x) = x$ for $x \in (\partial B_{R_m} \cap H_m) \oplus X^0$,

(g_3) $P^- h(x) = \alpha(x)x^- + \phi(x)$, where $\phi(x)$ is a compact map and $\alpha \in C(D_m, [1, \bar{\alpha}])$, constant $\bar{\alpha}$ depending on h.

Since $h(x) = x \in G_m$ for all $m \in \mathbf{N}$, $G_m \neq \emptyset$. For $j \in \mathbf{N}$, define

$$\Gamma_j = \{\overline{h(D_m - Y)} | m \geq j, h \in G_m, Y \in \mathcal{B}, and\ i(Y) \leq m - j\}.$$

For Γ_j we have the following properties:
(i) (Monotonicity): $\Gamma_{j+1} \subset \Gamma_j$,
(ii) (Excision): If $B \in \Gamma_j$ and $Z \in \mathcal{B}$ with $i(Z) \leq s < j$, then $B - Z \in \Gamma_{j-s}$,
(iii) (Invariance): If $\varphi \in C(X, X)$ and satisfies $(g_1) - (g_3)$ for $m \geq j$, $B \in \Gamma_j$ implies $\varphi(B) \in \Gamma_j$.

The next result which was proved in [6] is crucial for later estimates.

Lemma 1. *Let $h \in G_m$, $j \leq m$, $\rho < R_m$, and $\Theta = \{x \in D_m | h(x) \in \partial B_\rho \cap V_{j-1}^\perp\}$, then Θ is compact and $i(\Theta) \geq m - j + 1$. If $i(Y) \leq m - j$, and $W = \overline{\Theta - Y}$, then $\overline{h(D_m - Y)} \cap \partial B_\rho \cap V_{j-1}^\perp \supset h(w) \neq \emptyset$.*

And we will use the following calculus lemma:

Lemma 2. *For $m \geq 1$, $x = (x_1, \cdots, x_{N_1}) \in (H_m \cap X^+)^\perp$, we have*

$$[m/N_1]||x||_{L^2} \leq ||\dot{x}||_{L^2} \tag{3}$$

$$||x_l||_{L^\infty} \leq (\frac{8}{T([m/N_1] - 1})^{1/2}||\dot{x}_l||_{L^2}, \qquad \forall l = 1, \cdots, N_1. \tag{4}$$

Proof. For $x \in (H_m \cap X^+)^\perp$, we have the Fourier series expansion,

$$x(t) = \sum_{|j| \geq [m/N_1]} c_j e^{ijt},\ c_j \in \mathbf{C}^{N_1},\ c_{-j} = \bar{c}_j.$$

$$||\dot{x}||_{L^2}^2 = \frac{T}{2} \sum_{|j| \geq [m/N_1]} |c_j|^2 j^2 \geq \frac{T}{2}[m/N_1]^2 \sum_{|j| \geq [m/N_1]} |c_j|^2 = [m/N_1]^2 ||x||_{L^2}.$$

By the same computation as above, we get that for every $l = 1, \cdots, N_1$,

$$\begin{aligned} ||x_l||_{L^\infty} &\leq 2 \sum_{|j| \geq [m/N_1]} (|c_j||j|)\frac{1}{j} \\ &\leq 2(\sum_{|j| \geq [m/N_1]} |c_j|^2 j^2)^{1/2} (\sum_{|j| \geq [m/N_1]} \frac{1}{j^2})^{1/2} \\ &\leq (\frac{8}{T([m/N_1] - 1})^{1/2} ||\dot{x}_l||_{L^2}. \end{aligned}$$

Q.E.D.

As [6], we first study a simple system. There is $M \in C^2(\mathbf{R}, \mathbf{R})$ satisfying:
(M1) $M(|x|) \geq V(x)$ for all $x \in \mathbf{R}^N$ and $M(s) - M(0) = o(s^2)$ at $s = 0$,
(M2) $M(s)$ is strictly monotonically increasing in s and tend to infinity as $s \to \infty$,
(M3) $N(s) = M'(s)/s$ is strictly monotone and tends to infinity as $s \to \infty$,
(M4) $\frac{1}{3}sM'(s) - M(s)$ is strictly monotone increasing in s, and $sM'(s) \geq \mu(M(s) - M(0))$ for all s.
We refer to [6] for the construction of M . Define

$$J(x) = \int_0^T (A\dot{x}, \dot{x})dt - \int_0^T M(|x|)dt, \qquad \forall x \in X.$$

We have $J \in C^1(X, \mathbf{R})$.

Now we show I, J satisfy (PS) condition.

Lemma 3. *If V satisfies (V1) and (V2), I satisfies (PS) condition.*

Proof. Let (x_n) be a sequence in X such that $|I(x_n)| < M$ and $I'(x_n) \to 0$, as $n \to \infty$. Denote $\epsilon_n = ||I'(x_n)||_{W^{1,2}}$. Let $x_n = (x_n^1, x_n^2) \in \mathbf{R}^{N_1} \oplus \mathbf{R}^{N_2}$. We have

$$\begin{aligned} I'(x_n)x_n^1 &= \int_0^T |\dot{x}_n^1|^2 dt - \int_0^T \nabla V(x_n) x_n^1 dt, \\ I'(x_n)x_n^2 &= -\int_0^T |\dot{x}_n^2|^2 dt - \int_0^T \nabla V(x_n) x_n^2 dt. \end{aligned}$$

Since $x_n \in W^{1,2} \hookrightarrow C^0$, we have

$$\begin{aligned} \epsilon_n ||x_n^1||_{W^{1,2}} &\geq ||\dot{x}_n^1||_{L^2}^2 - ||x_n^1||_{C^0} \int_0^T |\nabla V(x_n)| dt, \\ -\epsilon_n ||x_n^2||_{W^{1,2}} &\leq -||\dot{x}_n^2||_{L^2}^2 + ||x_n^2||_{C^0} \int_0^T |\nabla V(x_n)| dt. \end{aligned}$$

Since (V2), we have

$$||\dot{x}_n||_{L^2}^2 \leq \epsilon_n ||x_n||_{W^{1,2}} + 2||x_n||_{C^0}[c \int_0^T (\nabla V(x_n), x_n) dt + dT]$$

On the other hand, we have

$$\begin{aligned} M + \epsilon_n ||x_n||_{W^{1,2}} &\geq I(x_n) - \frac{1}{2}(I'(x_n), x_n) \\ &= \frac{1}{2}\int_0^T (\nabla V(x_n), x_n) dt - \int_0^T V(x_n) dt \\ &\geq (\frac{1}{2} - \frac{1}{\mu}) \int_0^T (\nabla V(x_n), x_n) dt + c_1 \\ &\geq (\frac{\mu}{2} - 1) \int_0^T V(x_n) dt + c_2 \\ &\geq c_3 ||x_n||_{L^\mu}^\mu + c_4 \end{aligned} \tag{5}$$

Thus by the Hölder inequality and Sobolev embedding Theorem, we have

$$||x_n||_{L^2}^2 \leq M_1(1 + ||x_n||_{W^{1,2}}^{2/\mu})$$

$$||x_n||_{C^0} \leq K||x||_{W^{1,2}}$$

$$\int_0^T (\nabla V(x_n), x_n) dt \leq M_2(1 + \epsilon_n ||x_n||_{W^{1,2}})$$

Hence we have

$$||x_n||_{W^{1,2}}^2 \leq M_3(1 + ||x_n||_{W^{1,2}}^{2/\mu} + ||x_n||_{W^{1,2}} + \epsilon_n ||x_n||_{W^{1,2}}^2)$$

Since $\epsilon_n \to 0$, as $n \to \infty$, we have $||x_n||_{W^{1,2}}$ bounded. From the form of I' we know (PS) condition holds. Q.E.D.

From (M3) and (M4), $M(|x|)$ satisfies (V1) and (V2), we have

Lemma 4. *J satisfies (PS) condition.*

Lemma 5. *If V satisfies (V1) and (V3), I satisfies (PS) condition.*

Proof. Let (x_n) be a sequence in X such that $|I(x_n)| < M$ and $I'(x_n) \to 0$, as $n \to \infty$. Denote $\epsilon_n = ||I'(x_n)||_{W^{1,2}}$. Let $x_n = (x_n^1, x_n^2) \in \mathbf{R}^{N_1} \oplus \mathbf{R}^{N_2}$. We have

$$\begin{aligned} I'(x_n)x_n^1 &= ||\dot{x}_n^1||_{L^2}^2 - \int_0^T \nabla V(x_n)x_n dt + \int_0^T \nabla V(x_n)x_n^2 dt, \\ I'(x_n)x_n^2 &= -\int_0^T |\dot{x}_n^2|^2 dt - \int_0^T \nabla V(x_n)x_n^2 dt. \end{aligned}$$

From (5) and (V3), we have

$$\begin{aligned} \epsilon_n ||x_n^1||_{W^{1,2}} &\geq ||\dot{x}_n^1||_{L^2}^2 - C_1(1 + \epsilon_n ||x_n||_{W^{1,2}}) + C_2(1 + \epsilon_n ||x_n||_{W^{1,2}}), \\ -\epsilon_n ||x_n^2||_{W^{1,2}} &\leq -||\dot{x}_n^2||_{L^2}^2 + C_2(1 + \epsilon_n ||x_n||_{W^{1,2}}), \end{aligned}$$

for some constants C_1 and C_2. Thus we have

$$||x_n||_{W^{1,2}}^2 \leq C_3(1 + ||x_n||_{W^{1,2}}^{2/\mu} + ||x_n||_{W^{1,2}} + \epsilon_n ||x_n||_{W^{1,2}}^2)$$

for some constant C_3. Since $\epsilon_n \to 0$, as $n \to \infty$, we have $||x_n||_{W^{1,2}}$ bounded. From the form of I' we know (PS) condition holds. Q.E.D.

Now we define two minimax sequences of I and J:

$$c_j = \inf_{B \in \Gamma_j} \sup_{x \in B} I(x), \qquad d_j = \inf_{B \in \Gamma_j} \sup_{x \in B} J(x). \tag{6}$$

By (M1) we have $c_j \geq d_j$, for all $j \in \mathbf{N}$. By Lemma 2.1, we have $c_{j+1} \geq c_j$, $d_{j+1} \geq d_j \geq d_1$. Set $\bar{M}(s) = M(s) - M(0)$ and let

$$\bar{J}(x) = \int_0^T (A\dot{x}, \dot{x})dt - \int_0^T \bar{M}(x)dt, \qquad \forall x \in X.$$

we have

$$d_j = \inf_{B \in \Gamma_j} \sup_{x \in B} \bar{J}(x) - TM(0) = \bar{d}_j - TM(0).$$

As Lemma 1.33 and Lemma 1.40 of [6] we have

Lemma 6. *(1)* $\bar{d}_{j+1} \geq \bar{d}_j \geq \bar{d}_1 > 0$,
(2) $\bar{d}_j$ *is a critical value of* $\bar{J}$,
(3) Any critical points of $\bar{J}$, *corresponding to* $\bar{d}_j$ *lie in* $X - X^0$,
(4) If $\bar{d}_{j+l} = \cdots = \bar{d}_{j+1} = d$ *and* $\mathcal{K} = (\bar{J}')^{-1}(0) \cap \bar{J}^{-1}(\bar{d})$, *then* $i(\mathcal{K}) \geq l$.

Next we will make a closer study of the critical value $\bar{d}_j$ of $\bar{J}$. Let $x = (x_1, x_2) \in \mathbf{R}^{N_1} \oplus \mathbf{R}^{N_2}$ be a corresponding critical point. Then x is a classical solution of

$$\begin{cases} \ddot{x}_1 = -\frac{\partial}{\partial x_1} M(|x|) = -M'(|x|)\frac{x_1}{|x|}, \\ -\ddot{x}_2 = -\frac{\partial}{\partial x_2} M(|x|) = -M'(|x|)\frac{x_2}{|x|}. \end{cases} \tag{7}$$

Condition (M1) guarantees that there are no problems with the right hand side when $x(t_0) = 0$. We first prove the following Lemma:

Lemma 7. *For* $f \in C([0,T], \mathbf{R})$, *if* $f(t) \geq 0$ *and* $f(t_0) = 0$ *only when* $x(t_0) = 0$, *the boundary value problem*

$$\begin{cases} \ddot{x}(t) = f(t)x(t), \qquad \forall t \in [0,T] \\ x(0) = x(T), \dot{x}(0) = \dot{x}(T). \end{cases} \tag{8}$$

has only solution 0.

In the proof, we will use the following Theorem:

Theorem 3 (Theorem 3 of Chapter 1 in [5]). *On interval (a, b), $u(t)$ satisfies*

$$\ddot{u} + g(t)\dot{u} + h(t)u \geq 0$$

where $h(t) \leq 0$ and g and h are bounded on every closed subinterval of (a, b). If u get the not negative maximum M at an inter point c, we have

$$u(t) = M, \qquad \forall t \in (a, b).$$

Proof. Let $x(t)$ be a solution of (8). From the boundary values we have a $t_0 \in [0, T]$ such that $\dot{x}(t_0) = 0$. Without loss generality, let $t_0 = 0$. For $b \in (0, T]$, we have

$$x(b) - x(0) = \int_0^b \dot{x}(t)dt = \int_0^b \int_0^t \ddot{x}(s)dsdt = \int_0^b \int_0^t f(s)x(s)dsdt. \tag{9}$$

Case 1: If $x(0) < 0$, there exist $a > 0$ such that for all $\tau \in [0, a)$, $x(\tau) < 0$. From (9) we know $x(\tau) < x(0)$ for all $\tau \in [0, a)$, hence we know $x(t)$ is a strictly monotonically decreasing function. It is a contradiction.

Case 2: If $x(0) \geq 0$, Since $x(0) \geq 0$, we know x get the not negative maximum M at an interior point c. Let $g(t) = 0$, and $h(t) = f(t)$. We know the conditions of above Theorem are satisfied, hence we have

$$x(t) = M, \qquad \forall t \in [0, T].$$

Since $f(t) = 0$ only if $x(t) = 0$, we get $M = 0$. Q.E.D.

By the Lemma 7, each solution $x(t)$ of (7) has the form $(x_1(t), 0)$, since $\frac{M'(|x(t)|)}{|x(t)|}$ satisfies the conditions of Lemma 7. Hence we reduce (7) to the following second order Hamiltonian systems

$$\ddot{x}_1 = -M'(|x_1|)\frac{x_1}{|x_1|}, \tag{10}$$

As in [3], we have the following increasing estimates for the critical values $\{\bar{d}_j\}$ of (7):

Lemma 8. *$\bar{d}_j \to \infty$ as $j \to \infty$.*

Proof. Since the critical values $\{\bar{d}_j\}$ of (7) are the same as those getting for (10) by the minimax procedure on the corresponding invariant sets, we only need to prove the Lemma for (10).

For $A \in \Gamma_j$, there are $h \in G_m$, $Y \in \mathcal{B}$ and $i(Y) \leq m - j$ such that $A = h(\overline{D_m - Y})$. By Lemma 2.1, there exist $y \in \overline{D_m - Y}$ such that

$$x = (x_1, \cdots, x_{N_1}) = h(y) \in \partial B_{R_0}(X^+) \cap (H_j \cap X^+)^{\perp}.$$

By (3) of Lemma 2, we have

$$R_0^2 = ||x||_X^2 \leq (1 + [j/N_1]^{-2})||\dot{x}||_{L^2}^2. \tag{11}$$

Hence we have

$$R_0^2 \geq ||\dot{x}||_{L^2}^2 \geq \frac{1}{2}R_0^2 \quad and \quad ||\dot{x}_l||_{L^2} \leq R_0, \quad for \quad l = 1, \cdots, N_1.$$

By (4) of Lemma 2, we have

$$||x_l||_{L^\infty} \leq (\frac{8}{T([j/N_1] - 1)})^{1/2}||\dot{x}_l||_{L^2} \leq R_0(\frac{8}{T([j/N_1] - 1)})^{1/2}.$$

Fix $R_0 = (\frac{8}{T([j/N_1]-1)})^{-1/2}$, we have $||x||_{L^\infty} \leq N_1$. Hence from (11) we have

$$\begin{aligned}\bar{J}(x) &= \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \int_0^T M(|x|)dt \\ &\geq \frac{[j/N_1]^2}{2(1+[j/N_1]^2)}\frac{T}{8}([j/N_1]-1) - T\max_{|s|\leq N_1} M(s) \\ &\geq \frac{T}{32}([j/N_1]-1) - T\max_{|s|\leq N_1} M(s).\end{aligned}$$

By the definition of $\bar{d}_j$, we have

$$\bar{d}_j \geq \frac{T}{32}([j/N_1]-1) - T\max_{|s|\leq N_1} M(s).$$

Hence the Lemma holds. Q.E.D.

With the increasing estimates for $\bar{d}_j$ and d_j, now we study the minimax values c_j.

Lemma 9. *If $c_j > Ta_2$,*
(i) c_j is a critical value of I,
(ii) any corresponding critical point lies in $X - X^0$,
(iii) if $c_{j+l} = \cdots = c_{j+1} = c > Ta_2$, $i(I^{-1}(c)\cap(I')^{-1}(0)) \geq l$.

Proof. Note that

$$\sup_{x\in X^0} I(x) = T\sup_{x\in X^0}(-V(x)) \leq T\sup_{x\in X^0}(a_2 - a_1|x|^\mu).$$

Thus if $c_j > Ta_2$, similar to the proof of Lemma 6, using standard equivariant deformation lemma, where we need I satisfying (PS) condition, we get statement (i)-(iii). Q.E.D.

Proof of Theorem 1.1 and Theorem 1.2. Since $c_j \geq d_j \to \infty$ as $j \to \infty$ and the definition of $\bar{d}_j$, the requirement that $c_j > Ta_2$ is satisfied for all large j. Then following the above Lemma, we prove Theorem 1 and Theorem 2. Q.E.D.

Remark 2. From the study of (7), $N_1 > 0$ is necessary for our Theorems, which is missed in [2]. And it is natural to ask the following question: Do the results of Theorem 1 and Theorem 2 hold with only (V1) as the case $A = Id$? And from the proofs, we use condition $(V2)$ or $(V3)$ only to show I satisfies (PS) condition.

In the following part, we will deal with the following perturbation problems:

$$A\ddot{x} + \nabla V(x) = f(t). \tag{12}$$

We have the following result:

Theorem 4. *Suppose $V \in C^1(\mathbf{R}^N, \mathbf{R})$ and satisfying (V1) and (V2) (or (V3)), then for any given $T, R > 0$ and T-periodic function $f \in L^2([0,T], \mathbf{R}^{N_1})$, (12) possesses a T-periodic solution $x(t)$ with $\max_{t\in[0,T]}|x(t)| \geq R$.*

Proof. We first consider the system

$$A\ddot{x} + \nabla M(|x|) = f(t). \tag{13}$$

where $M(|x|)$ satisfies (M1)-(M4). As the proof for Theorem 1.1 and Theorem 1.2, studying the solutions of (13), by using Lemma 7 we know each solution of (13) has the form $x(t) = (x_1(t), 0)$. Using the setting of invariant sets similarly to [3], as the proof of Theorem 1.2 in [3], we know the critical values of the variational functional of (13) are unbounded. Using the same argument, we have the result of the theorem by comparing the critical values between the original system (12) and the new system (13). Q.E.D.

As section 6 in [3], for the following general forced systems

$$A\ddot{x} + U_x(t,x) = 0, \tag{14}$$

Using the result of increasing estimates in Theorem 6.2 in [3] and the argument of the above proof, we have the following result:

Theorem 5. *Let $U \in C^1(S_T \times \mathbf{R}^N, \mathbf{R})$, where $S_T = \mathbf{R}/(T\mathbf{Z})$, satisfies*

(UI) $(U_x(t,x), \Pi x)$ is independent of t, where Π is the projection operator of $\mathbf{R}^N$ onto the negative eigenspace of A.

(UII) There exist $V : \mathbf{R}^N \to \mathbf{R}$ satisfying (V1) and (V2) (or (V3)) and constants $C > 0$, $1 \le \sigma \le \mu/2$, such that

$$|U(t,x) - V(x)| \le C(1 + |x|^{\sigma-1}).$$

Then (14) possesses infinitely many distinct T-periodic solutions.

REFERENCES

[1] E.R. Fadell and P.H. Rabinowtz, Generalized cohomological index theories for Lie grovp actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. 45 (1978), 139-174.

[2] Z. Han, Periodic Solutions for a Class of Nonautonomovs Dynamical Systems of Second Order. J. Diff. Eq, 90 (1991), 408-417.

[3] Y. Long, Multiple solutions of perturbed superquadratic second order Hamiltonian systems. Trans. A.M.S. 311 (1989), 749-780.

[4] J. Mawhin and M.Willem, Critical point theory and Hamiltonian systems. Applied Mathematical Science. Vol. 74. Springer-Verlag, 1989.

[5] M.H. Protter and H.F. Weinberger, "Maximum Principles in Differential Equations", Prentice-Hall, 1967.

[6] P.H.Rabinowitz, Periodic solutions of Large Norm of Hamiltonian systems, J. Diff. Eq. 50, (1983)33-48.

[7] P.H.Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations", CBMS 65, A.M.S., Providence, R.I., 1986.

[8] Xiangjin Xu, Periodic solutions of Hamiltonian systems and differential systems. Thesis for Master. Nankai University. April 1999.

email address: xxu@math.jhu.edu