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SUB-HARMONICS OF FIRST ORDER HAMILTONIAN SYSTEMS AND THEIR ASYMPTOTIC BEHAVIORS

XIANGJIN XU
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD, 21218, USA

Abstract. *In this paper some new existence results for sub-harmonics are proved for first order Hamiltonian systems with super-quadratic potentials by using two new estimates on C^0 bound for the periodic solutions. Applying the uniform estimates on the sub-harmonics, the asymptotic behaviors of sub-harmonics is studied when the systems have globally super-quadratic potentials.*

1. **Introduction and Main Results.** In this paper we study the existence of sub-harmonic solutions (i.e. kT-periodic solutions) and the asymptotic behavior of the sub-harmonics for the following first order Hamiltonian system

$$(HS) \qquad -J\dot{u} - B(t)u = \nabla H(t,u), \quad u \in \mathbf{R}^{2N}, t \in \mathbf{R}, \tag{1}$$

where $B(t)$ is a given continuous T-periodic and symmetric $2N \times 2N$ -matrix function, $H \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is T-periodic in t, $\nabla H := \nabla_u H(t,u)$ and $\mathbf{J}$ is the standard symplectic matrix $\begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$. The first result on subharmonics of the system (HS) was obtained by Rabinowitz in his pioneering work [13] for constant matrix B and certain conditions on H. Since then there are many papers on the existence of sub-harmonics of the system (HS), such as [4], [5], [6], [11], [15], [16], [18], [19]. Especially, in [5] and [15], the authors studied the asymptotic behaviors of subharmonics of the system (HS) under certain conditions on $B(t)$ and $H(t,u)$.

In this paper, we firstly obtain two a priori estimates on the C^0 bound for the periodic solutions of the modified systems of the system (HS) following the ideas in [17], [18] and [19]. Applying these estimates to the system (HS) and the ideas from [13], we have the following existence results on subharmonics:

Theorem 1.1. *For $T > 0$, suppose $H(t,u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies:*
(H1) there are constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu H(t,u) \le (\nabla H(t,u), u), \quad \forall |u| \ge r_0,$$

(H2) $H(t,u) \ge 0$, for all $(t,u) \in S_T \times \mathbf{R}^{2N}$,

$$(H3) \quad \limsup_{|u|\to\infty} \frac{H_t(t,u)}{|u|^\mu H(t,u)} = 0, \ \text{or} \ \liminf_{|u|\to\infty} \frac{H_t(t,u)}{|u|^\mu H(t,u)} = 0, \qquad \text{uniformly in } t.$$

(H4) $H(t,u) = o(|u|^2)$, uniformly in t as $u \to 0$,
then there is a sequence $\{k_i\} \subset \mathbf{N}$ such that $k_i \to \infty$ as $i \to \infty$, and corresponding distinct k_iT-periodic solutions $\{u_{k_i}\}$ of the system (HS).

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Remark 1. *$(H1)$, $(H2)$ and $(H4)$ are usual conditions when one studies the existence of periodic solutions for non-autonomous Hamiltonian systems possessing super-quadratic potentials. There are many papers (cf.* [4], [5], [6], [11], [15]*) studying the existence of subharmonics of the system (HS) with some growth conditions on $H(t,u)$ at infinity of u variable. It is well known that for autonomous Hamiltonian systems the existence of infinite many periodic solutions was proved in Rabinowitz's pioneering work* [12] *under condition $(H1)$ only. Here the condition $(H3)$ is on $H_t(t,u)/(|u|^\mu H(t,u))$, which measures how far the system (HS) is away from the autonomous system , or one may consider the system (HS) with condition $(H3)$ as a perturbed system with a large perturbation. Especially from condition (H3), one can add any super-quadratic autonomous potential to the system (HS) and the results will still hold. Such a condition on $H_t(t,u)/H(t,u)$ as $(H3)$ was first introduced in* [10] *to study the existence of periodic solutions of the system (HS).*

Using the same conditions on $H(t,u)$ as in [13], we have a slightly more general existence result on subharmonics for the system (HS) as following:

Theorem 1.2. *Suppose $H(t,u) \in C^1(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (H1), (H2), (H4) and:*
(H5) There is a constant $c > 0$, such that

$$|\nabla H(t,u)| \leq c(\nabla H(t,u), u), \forall u \geq r_0.$$

then there is a sequence $\{k_i\} \subset \mathbf{N}$ such that $k_i \to \infty$ as $i \to \infty$, and corresponding distinct k_iT-periodic solutions $\{u_{k_i}\}$ of the system (HS).

Remark 2. *In* [13]*, $B(t)$ was further assumed to be a constant matrix and the same result as Theorem 1.2 was proved there.*

In the second part of this paper, we study the asymptotic behaviors of subharmonics. Firstly we show there is a uniform C^1 bound for the subharmonics of the system (HS) when the system has the globally super-quadratic potential

(H1)' there is constant $\mu > 2$ such that

$$0 < \mu H(t,u) \leq z\nabla H(t,u), \quad \forall |u| > 0,$$

Theorem 1.3. *Suppose $H(t,u)$ satisfies the globally super-quadratic condition $(H1)'$, $(H2)$- $(H4)$, then there exists infinitely many distinct sub-harmonics $\{u_k\}$ of the system (HS) and there is a uniform bound M for $\{||u_k||_{C^1}\}_{k\in\mathbf{N}}$.*

Using the uniform bound on the subharmonics from Theorem 1.3 and the ideas in [5] and [13], when $B(t)$ satisfies condition:

$(B)^c$ $\sigma(A) \cap \mathbf{R} \neq \emptyset$, where $A = -(\mathbf{J}(d/dt) + B(t))$ and $\sigma(A)$ is the spectrum of the self-adjoint operator A on $W^{1/2}(S_T, \mathbf{R}^{2N})$.
we have the following result:

Theorem 1.4. *Assume $B(t)$ satisfies $(B)^c$ and $H(t,u)$ satisfies the conditions of Theorem 1.3, then there is a sequence of pairwise geometrically distinct subharmonics $\{u_k(t)\}_{k\in\mathbf{N}} \subset C^1(\mathbf{R}, \mathbf{R}^{2N})$ of the system (HS) such that $||u_k||_{C^1} \to 0$ as $k \to \infty$.*

Using the uniform bound on the subharmonics from Theorem 1.3 and the ideas in [15], when $B(t)$ satisfies condition:

(B) $\sigma(A) \cap \mathbf{R} = \emptyset$, where $A = -(\mathbf{J}(d/dt) + B(t))$ and $\sigma(A)$ is the spectrum of the self-adjoint operator A on $W^{1/2}(S_T, \mathbf{R}^{2N})$.
we have the following result:

THEOREM 1.5. *Assume $B(t)$ satisfies condition (B) and $H(t,u)$ satisfies the conditions of Theorem 1.3, then there is a sequence of pairwise geometrically distinct subharmonics $\{u_k(t)\}_{k\in\mathbf{N}} \subset C^1(\mathbf{R},\mathbf{R}^{2N})$ of the system (HS) such that*

(i) *there are constants $m, M > 0$ independent of $k \in \mathbf{N}$ such that*

$$m \le \int_0^{kT} [\frac{1}{2}(-\mathbf{J}\dot{u}_k, u_k) - H(t,u_k)]\, dt \le M;$$

(ii) *moreover $\{u_k(t)\}_{k\in\mathbf{N}}$ is compact in the following sense: for any sequence of integers $k_n \to \infty$, there exists a subsequence $\{k_{n_i}\}_{i\in\mathbf{N}}$ and a nontrivial homoclinic orbit $u_\infty(t)$ emanating from 0 such that*

$$u_{k_{n_i}}(t) \to u_\infty(t) \text{ in } C^1_{loc}(\mathbf{R},\mathbf{R}^{2N}), \text{ as } i \to \infty.$$

REMARK 3. *in* [5], [13] *and* [15], *the authors dealt with the system (HS) with a constant matrix B and conditions on $H(t,u)$ as these in Theorem 1.2. Here we use the functional setting for the linearized system as used in* [3], *by which one can let $B(t)$ depend on t variable. And applying the uniform C^1 bound on the subharmonics of the system (HS), one can directly apply those results in* [5], [13] *and* [15] *to our situation.*

We organize this paper as following: in Section 2, two a priopi estimates on C^0 bound of the periodic solutions of the modified systems of the system (HS) are proved. In Section 3, Theorem 1.1 and Theorem 1.2 are proved by applying these estimates. In Section 4, the asymptotic behaviors of sub-harmonic solutions of the system (HS) are studied and Theorem 1.3 - 1.5 are proved.

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2. **Two Estimates.** In this Section we study the C^0 bound of the periodic solutions of the following modified systems

$$(HS)^n \qquad \mathbf{J}\dot{u} + \nabla H_n(t,u) = 0, \quad (t,u) \in S_T \times \mathbf{R}^{2N}.$$

Here $\{H_n\}$ satisfy

PROPOSITION 2.1. *For H satisfying $(H1)$, $\sigma \in (0,1]$ such that $\mu\sigma > 2$, and two sequences $\{K_n\}$ and $\{K'_n\}$ in $\mathbf{R}$, $\{H_n\}$ satisfies*

(i) *K_n is monotonous increasing to infinity, as $n \to \infty$, and $K_n < K'_n$, $\forall n \in \mathbf{N}$.*
(ii) *for any given $t \in S_T$, $H_n(t,\cdot) \in C^2(\mathbf{R}^{2N},\mathbf{R})$, for every $n \in \mathbf{N}$.*
(iii) *$H_n(t,u) = H(t,u), \forall |u| \le K_n$, for every $n \in \mathbf{N}$; and for some $\lambda \in [\sigma,1]$, such that $H_n(t,u) = c_n|u|^{\mu\lambda}, \forall |u| \ge K'_n$, for every $n \in \mathbf{N}$.*
(iv) *$0 < \sigma\mu H_n(t,u) \le (\nabla H_n(t,u),u), \forall |u| \ge r_0$, for every $n \in \mathbf{N}$.*

REMARK 4. *The truncated results on $H(t,u)$ as Proposition 2.1 was first proved by Long in* [7] *and* [8], *where Long got a better monotone truncated sequence $\{H_n\}$ on H for $H(t,u)$ C^2 in u variable. In next two sections, we don't need the monotone property and we can choose $\lambda = \sigma = 1$ when we define the truncated sequence $\{H_n\}$ as in* [13] *by*

$$H_n(t,u) = \chi_n(|u|)H(t,u) + (1 - \chi_n(|u|))c_n|u|^\mu$$

where $\chi_n(s) = 1$ *for* $s \le K_n$, $\chi_n(s) = 0$ *for* $s \ge K_n'$, *and* $\chi_n'(s) < 0$ *for* $s \in (K_n, K_n')$, *and* c_n *(depend on n) be sufficiently large constant. In this truncation,* H *need only be* C^1.

Let $X := W^{1/2,2}(S_T, \mathbf{R}^{2N})$, and define the functional $I_n : X \to \mathbf{R}$ by

$$I_n(u) = \frac{1}{2}\int_0^T (-Ju' - B(t)u) \cdot u\, dt - \int_0^T H_n(t,u)\, dt.$$

It is well known that $I_n \in C^1(X, \mathbf{R})$, and the periodic solutions of $(HS)^n$ are obtained as critical points of the functional I_n. Using the ideas in [17], [18], [19], we prove the following two Lemmas on the C^0 bound of the periodic solutions of the following modified systems $(HS)^n$.

LEMMA 2.1. *Suppose that* H *satisfies (H1) and* $(H5)$, $\{H_n\}$ *satisfies Proposition 2.1, and* $u(t)$ *is a T-periodic solution of* $(HS)^n$ *such that*

$$\int_0^T (\nabla H_n(t,u), u)\, dt \le C, \qquad \int_0^T H_n(t,u)\, dt \le C,$$

then there is a constant M *independent of* u *and* n *and depend on* C *only such that*

$$||u||_{C^0} \le M.$$

Proof. Integrating (iv) of Proposition 2.1 gives

$$H_n(t,u) \ge a|u|^{\mu\sigma} - b, \quad \forall u \in \mathbf{R}^{2N},$$

where a and b are independent of n by (iii) of Proposition 2.1. Hence we have

$$C \ge \int_0^T H_n(t,u)\, dt \ge aT\int_0^T |u|^{\mu\sigma}\, dt - bT \ge aT(\min_{t\in S_T} |u(t)|)^{\mu\sigma} - bT.$$

Therefore we have $\min_{t\in S_T} |u(t)| \le C_0$ where C_0 is independent of u and n. Without loss generality, we may assume $|u(t)|$ obtains its minimum at $t = 0$, and $|u(0)| \ge r_0$,

$$|u(t)| - |u(0)| = \int_0^t \frac{d}{ds}|u(s)|\, ds \le \int_0^t |\dot{u}(s)|\, ds \le \int_0^t (|B(s)u(s)| + |\nabla H_n(s,u(s))|)\, ds$$

From $(H5)$ and (iii) of Proposition 2.1 we have

$$|\nabla H_n(t,u)| \le c(\nabla H_n(t,u), u), \quad for \quad r_0 \le |u| \le K_n$$

We first show for large enough n,

$$||u||_{C^0} \le K_n$$

If not, by passing a subsequence, for each $n \in \mathbf{N}$, there exists $u_n(t)$ and $t_n \in S_T$, such that $|u_n(t_n)| = K_n$ and $|u_n(t)| \le K_n$ for $t \in [0, t_n)$. Let

$$d = \max_{\{(t,u)\in S_T\times\mathbf{R}^{2N} | H(t,u)<0\}} \{|(\nabla H_n(t,u), u)|\},$$

we then have

$$\begin{aligned}
|u_n(t_n)| &\le \int_0^{t_n} |\nabla H_n(s, u_n(s))|\, ds + ||B||_\infty \int_0^{t_n} |u(s)|\, ds + |u_n(0)| \\
&\le c\int_0^{t_n} (\nabla H_n(s,u_n(s)), u_n(s))\, ds + c\int_0^{t_n} |u(s)|^{\sigma\mu}\, ds + |u_n(0)| \\
&\le c\int_0^{T} (\nabla H_n(s,u_n(s)), u_n(s))\, ds + c\int_0^{t_n} |H(s,u(s))|\, ds + c\, dt + C_0 \\
&\le cC + cdT + C_0,
\end{aligned}$$

where c(may different in each step), d, C and C_0 are independent of u and n. But then $K_n \to \infty$, *as* $n \to \infty$, which leads to a contradiction. Hence there exists $m \in \mathbf{N}$, depending only on H and C such that for any $n \geq m$, $||u||_{C^0} \leq K_n$ holds. Repeating this argument, we find that for any $n \geq m$,

$$|u(t)| \leq cC + cdT + C_0, \ \forall\, t \in S_T.$$

For $k < m$, from (iii) of Proposition 2.1, we have

$$|\nabla H_k(t,u)| \leq c_k(\nabla H_k(t,u),u)\ \forall\ |u| \geq r_0,$$

for some suitable constant c_k. By the same argument,

$$|u(t)| \leq c_kC + d_kT + C_0, \ \forall\, t \in S_T$$

where c_k and d_k are determined by H_k for $k = 1, 2, \cdots, m-1$. Therefore

$$||u||_{C^0} \leq \max\{cC + cdT + C_0,\ c_kC + d_kT + C_0,\ k = 1, 2, \cdots, m-1\} = M,$$

which completes the proof. Q.E.D.

LEMMA 2.2. *Suppose that $H(t,u)$ satisfies $(H1)$ and $(H3)$, $\{H_n\}$ satisfies Proposition 2.1, and $u(t)$ is a T-periodic solution of system $(HS)^n$ such that*

$$\int_0^T \nabla H_n(t,u)u\,dt \leq C, \qquad \int_0^T H_n(t,u)\,dt \leq C.$$

then there is a constant M independent of u and n and dependent on C only such that

$$||u||_{C^0} \leq M.$$

Proof. Since $H_n(t,u) > 0$ for $|u| \geq r_0$, we have

$$C \geq \int_0^T H_n(t,u)\,dt \geq \int_{\{H_n(t,u)\geq 0\}} H_n(t,u)\,dt - T \sup_{H(t,u)<0,t\in S_T} |H(t,u)|.$$

Hence we have

$$\int_{\{H_n(t,u)\geq 0\}} H_n(t,u)\,dt \leq C + T \sup_{H(t,u)<0,t\in S_T} |H(t,u)| = \bar{C},$$

where $\bar{C}$ is a constant independent of n and u. Define

$$\sigma(r) = \sup_{|u|\geq r,t\in S_T} \frac{H_t(t,u)}{|u|^\mu H(t,u)}, \quad \textit{and} \quad \delta(r) = \inf_{|u|\geq r,t\in S_T} \frac{H_t(t,u)}{|u|^\mu H(t,u)}.$$

Then $(H3)$ means

$$\lim_{r\to\infty} \sigma(r) = 0 \textit{ or } \lim_{r\to\infty} \delta(r) = 0.$$

Case I: Suppose we have $\lim_{r\to\infty}\sigma(r) = 0$.

By the definition of $\sigma(r)$, we have that $\sigma(r)$ is decreasing to 0. Fix a large $R > r_0$, such that

$$a - \sigma(R)\bar{C} > 0.$$

First, we show $|u|_{C^0} \leq K_n$ for large n. If not, by passing to a subsequence we may assume for each n, there exists $u_n(t)$, a_n and b_n such that

$$(a_n, b_n) \subset \{t \in S_T | R < |u_n(t)| < K_n\}$$

and $|u_n(a_n)| = R$, $|u_n(b_n)| = K_n$. Denote

$$\hat{H}_n(t,u) = \frac{1}{2}(B(t)u,u) + H_n(t,u); \quad \hat{H}(t,u) = \frac{1}{2}(B(t)u,u) + H(t,u)$$

Therefore we have

$$\begin{aligned}
& \hat{H}(b_n, u_n(b_n)) - \hat{H}(a_n, u_n(a_n)) \\
= & \int_{a_n}^{b_n} \frac{d}{dt}\hat{H}_n(t, u_n(t))\,dt \\
= & \int_{a_n}^{b_n} [(\nabla \hat{H}_n(t, u_n(t)), \dot{u}(t)) + \hat{H}_t(t, u_n(t))]\,dt \\
= & \int_{a_n}^{b_n} H_t(t, u_n(t)) + \frac{1}{2}(B_t(t)u_n(t), u_n(t))]\,dt \\
\leq & \int_{a_n}^{b_n} \sigma(|u_n(t)|)|u_n(t)|^{\mu} H(t, u_n(t))\,dt + ||B_t||_\infty \int_{a_n}^{b_n} |u_n(t)|^2\,dt \\
\leq & \sigma(R)K_n^{\mu} \int_{a_n}^{b_n} H(t, u_n(t))\,dt + c\int_{a_n}^{b_n} H(t, u_n(t))\,dt \\
\leq & \sigma(R)\bar{C}K_n^{\mu} + c\bar{C}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \hat{H}(b_n, u_n(b_n)) - \hat{H}(a_n, u_n(a_n)) \\
\geq & a|u_n(b_n)|^{\mu} - b - ||B||_\infty |u_n(b_n)|^2 - \max_{|u|\leq R, t\in S_T} |\hat{H}(t,u)| \\
= & aK_n^{\mu} - ||B||_\infty K_n^2 - (b + \max_{|u|\leq R, t\in S_T} |H(t,u)|)
\end{aligned}$$

Combine these two formulas, we get that

$$(a - \sigma(R)\bar{C})K_n^{\mu} - ||B||_\infty K_n^2 \leq b + c\bar{C} + \max_{|u|\leq R, t\in S_T} |H(t,u)|$$

Since $a - \sigma(R)\bar{C} > 0$, $\mu > 2$, and $K_n \to \infty$ as $n \to \infty$, the left side tends to infinity, but the right side is a constant independent of u and n. This leads to a contradiction. Hence there exists $m \in \mathbf{N}$, which is determined by $H(t,u)$ and C only, such that for any $n \geq m$, $|u|_{C^0} \leq K_n$.

For $n \geq m$, if $||u||_{C^0}$ doesn't have an n-independent upper bound M_0, then following the above proof with K_n replaced by M_n where $M_n \to \infty$ as $n \to \infty$, we also get a contradiction. For $n < m$, as the proof in last part of Lemma 2.1, we have

$$|u(t)| \leq c_k C + d_k T + C_0,\ \forall\, t \in S_T,$$

where c_k and d_k are determined by H_k, $k = 1, 2, \cdots, m-1$.

Hence we have

$$||u||_{C^0} \leq \max\{M_0,\ c_k C + d_k T + C_0,\ k = 1, 2, \cdots, m-1\} = M$$

Case II: Suppose we have $\lim_{r\to\infty} \delta(r) = 0$.

We need only to modify the proof of Case I a little. We have

$$(a + \delta(R)\bar{C})K_n^{\mu} - ||B||_\infty K_n^2 \leq b + c\bar{C} + \max_{|u|\leq R, t\in S_T} |H(t,u)|$$

where $a + \delta(R)\bar{C} > 0$, $\mu > 2$, and $K_n \to \infty$ as $n \to \infty$. Using the same argument as in Case I, we have

$$||u||_{C^0} \leq M$$

By combining these two cases, we prove this Lemma. Q.E.D.

REMARK 5. Here we proved the Lemmas for general potentials H, which doesn't assume any condition nearby $u = 0$ on the potential $H(t, u)$. Such kind estimates was first proved in [18] when the author studied the existence of periodic solutions of the first order Hamiltonian systems possessing super-quadratic potentials. In this paper $H(t, u) \geq 0$ is satisfied, which implies $d = d_k = 0$ and $\bar{C} = C$, then the bound M is independent of the period T and depends only on C and $H(t, u)$ from the proofs of the Lemmas. This is one key observation to apply these estimates to get the existence of subharmonics and to get the uniform estimates for subharmonics $\{u_k\}$ in the next sections.

3. **Existence of subharmonics.** We first show that there exists at least one nonzero T-periodic solution of (HS) under the conditions of Theorem 1.1 and Theorem 1.2. as did in [13]. We truncate the potential $H(t, u)$ by $\{H_n(t, u)\}$ as done in [13] satisfying Proposition 2.1 with $\lambda = \sigma = 1$ to get a sequence of modified systems, and we define $I_n(u)$ for these new systems.

We use Theorem 1.4 in [1] to obtain the existence of the nonzero critical point of I_n. One may check the details of the proof in [13], here we only give a sketched proof. Let $X := H^{\frac{1}{2}}(S_T, \mathbf{R}^{2N})$, By [9] and standard spectral theory, there exists a decomposition $X = X^+ \oplus X^0 \oplus X^-$ according to the self-adjoint operator B by extending the bilinear form

$$(Bu, v) = \frac{1}{2}\int_0^T -\mathbf{J}\dot{u} \cdot v\, dt - \int_0^T (B(t)u, v)\, dt$$

with $\dim X^0 = \ker B < \infty$, $\dim X^+ = \dim X^- = \infty$.

We verify the conditions of Theorem 1.4 in [1] for I_n, set $X_1 = X^+$, $X_2 = X^0 \oplus X^-$ and

$$\begin{aligned} I_n(u) &= \frac{1}{2}\int_0^T -\mathbf{J}\dot{u} \cdot u\, dt - \frac{1}{2}\int_0^T (B(t)u, u)\, dt - \int_0^T H_n(t, u)\, dt \\ &= \frac{1}{2}(||u^+||^2 - ||u^-||^2) - \int_0^T H_n(t, u)\, dt. \end{aligned}$$

As the proof of Theorem 6.10 in [14], we have $I_n \in C^1(X, \mathbf{R})$ and I_n satisfies (I1)-(I3) of Theorem 1.4 in [1]. To verify (I4), we construct $S = \partial B_\rho \cap X_1$ which is the same as that in [1]. To obtain Q with r_1 and r_2 independent of n, following the proof of Theorem 1.4 in [13], we let $e \in \partial B_1 \cap X_1$ and $u = u^0 + u^- \in X_2$, then

$$\begin{aligned} I_n(u + se) &= s^2 - ||u^-||^2 - \int_0^T H_n(t, u)\, dt \\ &\leq s^2 - ||u^-||^2 - a_3(||u^0||^\mu + s^\mu) + a_4 \end{aligned}$$

with n-independent constants a_3 and a_4 which are determined by (iv) of Proposition 2.1. Choose r_1 so that

$$\phi(s) = s^2 - a_3 s^\mu + a_4 \leq 0 \tag{2}$$

for all $s \geq r_1$. Choose r_2 large enough as [13], we have $I_n \leq 0$ on ∂Q with $Q = \{se | 0 \leq s \leq r_1\} \oplus (B_{r_2} \cap X_2)$. So from Theorem 1.4 in [1], I_n possesses a nonzero critical point u_n with $I_n(u_n) \geq \alpha_n > 0$.

Now we need to find an n-independent upper bound for $\{\|u_n\|_{C^0}\}$. In Theorem 1.4 in [1], the critical value c can be characterized as the minimax of I_n over an appropriate class of sets (cf.[1]). Observe that Q is one of such sets and $H_n(t,u)$ satisfies $(H2)$, therefor we have

$$I_n(u_n) = c_n \le \sup_{u\in Q} I_n(u) \le \sup_{||u^0+u^-||\le r_2, s\in[0,r_1]} (s^2 - ||u^-||^2 - \int_0^T H_n(t,u)\,dt) \le r_1^2.$$

Since u_n is a critical point of I_n and each $H_n(t,u)$ satisfies Proposition 2.1, we have

$$\begin{aligned} I_n(u_n) &= \frac{1}{2}\int_0^T (-\mathbf{J}\dot{u_n}, u_n)\,dt - \int_0^T H_n(t,u_n)\,dt \\ &= \frac{1}{2}\int_0^T \nabla H_n(t,u_n)u_n\,dt - \int_0^T H_n(t,u_n)\,dt \\ &\ge (\frac{1}{2}-\frac{1}{\mu})\int_0^T \nabla H_n(t,u_n)u_n\,dt - C_1 \\ &\ge (\frac{\mu}{2}-1)\int_0^T H_n(t,u_n)\,dt - C_2 \end{aligned}$$

Where C_1, C_2 are independent of n. From above we have

$$\int_0^T \nabla H_n(t,u_n)u_n\,dt \le C, \qquad \int_0^T H_n(t,u_n)\,dt \le C$$

for some constant C independent of n. Hence from Lemma 2.1 and Lemma 2.2, we have an n-independent constant M such that

$$||u_n||_{C^0} \le M, \quad for\ all\ n \in \mathbf{N}.$$

On the other hand, we have

$$H_n(t,u) = H(t,u), \quad for \quad |u| < K_n.$$

Hence for large $n \in \mathbf{N}$ such that $K_n > M$, u_n is a nonzero T-periodic solution of the system (HS).

Next we show that the system (HS) has infinitely many distinct sub-harmonics. We will follow the ideas in Proof of Theorem 1.36 in [13]. For a given $k \in \mathbf{N}$, we make the change of variables $s = k^{-1}t$. Thus if $u(t)$ is a kT-periodic solution of the system (HS), $\eta(s) = u(ks)$ satisfies

$$\mathbf{J}\frac{d\eta}{ds} + k(B(ks)\eta + \nabla H(ks,\eta)) = 0 \tag{3}$$

Since $k\bar{H}(ks,u)$ satisfies the conditions of our Theorem too, there is a solution $\eta_k(s)$ of (3), which is a critical point of

$$I^k(\eta) = \frac{1}{2}\int_0^T -\mathbf{J}\dot{\eta}\cdot\eta\,ds - k\int_0^T (B(ks)\eta,\eta)\,ds - k\int_0^T H(ks,\eta)\,ds.$$

Note that $\eta_1(ks)$ also satisfies (3), then if $\eta_1(ks) = \eta_k(s)$, we have $c_k = I^k(\eta_k) = kI^1(\eta_1) = kc_1$.

Next we show that $c_k = I^k(\eta_k)$ is bounded from above and the upper bound is independent of k. In the proof for the existence of one solution, we have $c_k \le r_1^2(k)$ and the parameter $r_1(k)$ is determined by condition (2). The corresponding condition satisfied by $r_1(k)$ is

$$\phi_k(s) = s^2 - ka_3s^\mu + ka_4 \le 0,$$

for all $s \geq r_1(k)$. It follows that we can let

$$r_1(k) \leq \max\left(\left(\frac{2}{ka_3}\right)^{\frac{1}{\mu-2}}, \left(\frac{2a_4}{a_3}\right)^{\frac{1}{\mu}}\right) \leq \left(\frac{2}{a_3}\right)^{\frac{1}{\mu-2}} + \left(\frac{2a_4}{a_3}\right)^{\frac{1}{\mu}}. \tag{4}$$

Now, for any given $m \in \mathbf{N}$, if for some $k > m$, $\eta_k(s) = \eta_m(s)$ holds for all $s \in \mathbf{R}$, we have that $\eta_k(s)$ as kT-periodic function is $\frac{k}{l}$ folds of $\eta_l(s)$ as lT-period function and $\eta_m(s)$ as mT-periodic function is $\frac{m}{l}$ folds of $\eta_l(s)$ as lT-period function, for some $l \in \mathbf{N}$ such that $l|k$ and $l|m$ and some corresponding $\eta_l(s)$. Hence we have

$$c_k = I^k(\eta_k) = \frac{k}{l}I^l(\eta_l), \quad c_m = I^m(\eta_m) = \frac{m}{l}I^l(\eta_l),$$

that means

$$c_k = \frac{k}{m}c_m.$$

On the other hand, we have $c_m > 0$ and $\{c_k\}$ is bounded by a k-independent constant from (4). This implies that there are at most finitely many $k > m$ such that $\eta_k(s) = \eta_m(s)$ for any given $m \in \mathbf{N}$. Hence Theorem 1.1 and Theorem 1.2 are proved.

In [6], by using the iterated Maslov-type index theory and estimating the Maslov-type indices of the critical points of the direct variational calculus, Liu studied the existence of subharmonic solutions for the system (HS) under some conditions on H. Replacing the condition (H4) in [6] by our condition (H3), we have the following result:

THEOREM 3.1. *Suppose $H \in C^2(S_T \times \mathbf{R}^{2N}, \mathbf{R})$ and satisfies (H1)-(H4), and*

(H6) $B(t)$ is a symmetric continuous matrix with period T, $|B|_{C^0} \leq \omega$ for some $\omega > 0$, and $B(t)$ is a semi-positive definite matrix for all $t \in [0,T]$.
then for each integer $1 \leq k < \frac{2\pi}{\omega T}$ there is a kT-periodic nonconstant solution u_k of the system (HS). If all $\{u_k\}$ are non-degenerate, u_j and u_{pj} are geometrically distinct for $p > 1$.

Especially for $B(t) \equiv 0$, for each integer $k \geq 1$ there is a kT-periodic nonconstant solution u_k of the system (HS). If all $\{u_k\}$ are non-degenerate, u_j and u_{pj} are geometrically distinct for $p > 1$.

If one doesn't assume non-degenerate condition on $\{u_k\}$, u_j and u_{pj} are geometrically distinct for $p > 2n + 1$.

The proof of this Theorem follows the proof of Theorem 1.1 in [6]. The only difference is how to remove the growth restriction on H under our condition (H3), and notice that apply our Lemma 2.2, we get a K-independent upper bound for $\{u_K^k\}$ for each $k \in \mathbf{N}$, where u_K^k is the kT-periodic solution obtained in [6] for the modified system $(HS)_K$. The Theorem is proved.

4. Asymptotic behaviors of subharmonics.

In this section the asymptotic behaviors of the sub-harmonics of the system (HS) are studied when the potential $H(t,u)$ of the system (HS) satisfies the globally super-quadratic condition $(H1)'$. We firstly show that there is a C^1 uniform bound for sub-harmonics $\{u_k\}$ as stated in Theorem 1.3.

Proof of Theorem 1.3. From the proofs of Theorem 1.1, we need only to show there exist a uniform bound for all $\{||u_{n,k}||_{C^0}\}_{n,k\in\mathbf{N}}$ where $\{u_{n,k}\}_{n,k\in\mathbf{N}}$ are those sub-harmonics that we obtain for $(HS)^n$.

Let $c_{n,k} = I_n^k(u_{n,k})$, where I_n^k is the functional defined by H_n for kT-periodic solution. From the Proof in Section 3, we have $c_{n,k} \le r_1^2(k)$, where

$$r_1(k) \le \left(\frac{2}{a_3}\right)^{\frac{1}{\mu-2}} + \left(\frac{2a_4}{a_3}\right)^{\frac{1}{\mu}}$$

which are independent of n and k. Hence from the globally super-quadratic condition $(H1)'$, we have

$$\begin{aligned} C \ge I_{n,k}(u_{n,k}) &= \frac{1}{2}\int_0^{kT} (-\mathbf{J}\dot{u}_{n,k} - B(t)u_{n,k}) \cdot u_{n,k}\,dt - \int_0^{kT} H(t,u_{n,k})\,dt \\ &= \frac{1}{2}\int_0^{kT} \nabla H(t,u_{n,k}) \cdot u_{n,k}\,dt - \int_0^{kT} H(t,u_{n,k})\,dt \\ &\ge (\frac{1}{2} - \frac{1}{\mu})\int_0^{kT} \nabla H(t,u_{n,k}) \cdot u_{n,k}\,dt \\ &\ge (\frac{\mu}{2} - 1)\int_0^{kT} H(t,u_{n,k})\,dt \end{aligned}$$

which implies

$$\int_0^{kT} \nabla H_n(t,u_{n,k})u_{n,k}\,dt \le C, \qquad \int_0^{kT} H_n(t,u_{n,k})\,dt \le C.$$

Here C is independent of k. From Lemma 2.1, Lemma 2.2 and Remark 2.1, we know there is a constant M independent of n and k such that

$$||u_{n,k}||_{C^0} \le M.$$

And $u_{n,k}$ satisfies the equation of (HS), then there is a uniform C^1 bound M for all sub-harmonics $\{u_{n,k}\}_{n,k\in\mathbf{N}}$.

Q.E.D.

Now we study the asymptotic behaviors of sub-harmonics $\{u_k\}_{k\in\mathbf{N}}$ as $k \to \infty$ under some further conditions as in Theorem 1.4 and Theorem 1.5. Here we apply the uniform C^1 estimates on subharmonics as in Theorem 1.3 to make use of those argument in [5], [13] and [15]. Since in those paper, $B(t)$ was further assumed to be constant matrix, we need make a new functional setting as done in [3] as following.

Let $A = -(\mathbf{J}\frac{d}{dt} + B(t))$ is the self-adjoint operator acting on $L^2(\mathbf{R},\mathbf{R}^{2N})$ with the domain $\mathrm{D}(A) = H^1(\mathbf{R},\mathbf{R}^{2N})$ and $\sigma(A)$ is the spectrum of A. We use the norm $||\cdot||_E$, which is defined as

$$||u||_E = (||A|^{\frac{1}{2}}u|_2^2 + |u|_2^2)^{\frac{1}{2}},$$

instead of the norm $||\cdot||_\mu$ in [3], and the Banach space E, which is the completion of the set $\mathrm{D}(A) = H^1(\mathbf{R},\mathbf{R}^{2N})$ under the norm $||\cdot||_E$, instead of the space E_μ in [3]. As Section 2 in [3], we have these facts: E has the direct sum decomposition $E = E^- \oplus E^+$, and E is embedded continuous in L^ν for any $\nu \in [2,\infty)$ and compactly in L^ν_{loc} for any $\nu \in [2,\infty)$. Notice that using our norm $||\cdot||_E$ instead of $||\cdot||_\mu$, the reader can check the details of these facts following the Proofs in Section 2 in [3]. Set

$$\Phi_n(u) = \int_{\mathbf{R}} H_n(t,u)\,dt$$

It is easy to check $\Phi_n \in C^1(E,\mathbf{R})$ since each H_n satisfies $(H1)$ and (iii) of Proposition 2.1, and E is embedded continuous in L^ν for any $\nu \in [2,\infty)$. We will use this functional setting to replace those in [5], [13] and [15], where only deal with the case that $B(t)$ is a constant matrix.

In [5] and [13], the authors show that the sub-harmonics converge to 0 when the constant matrix B satisfies:
$(A)^c$ *$\sigma(\mathbf{J}B) \cap i\mathbf{R} \neq \emptyset$, where $\sigma(A)$ is the spectrum of the matrix A.*
Notice that it is the same condition as our condition $(B)^c$ when $B(t)$ is a constant matrix. From Theorem 1.3 we have the uniform C^1 bound M for all sub-harmonics $\{u_{n,k}\}_{n,k\in\mathbf{N}}$, we can see all sub-harmonics of the system (HS) are the same as those of the modified system $(HS)_n$ for some large n. Now fixed a large n, from Proposition 2.1, the modified system $(HS)_n$ satisfies the conditions on H in [5] and [13], and we use the functional setting on $A = -(\mathbf{J}\frac{d}{dt} + B(t))$ instead of those used in [5] and [13] for constant matrix B, we prove Theorem 1.4.

REMARK 6. *In* [13]*, the author proved that the subharmonics converge to* 0 *when* $\sigma(\mathbf{J}B) \subset i\mathbf{R}$*, and in* [5]*, the author proved the subharmonics converge to* 0 *under condition* $(A)^c$*.*

In [15], the author shows that there exists a nontrivial homoclinic orbit for the system (HS) as the limit of the subharmonics u_k when the constant matrix A satisfies:
(A) *A is a $2N \times 2N$ symmetric matrix such that $\sigma(\mathbf{J}A) \cap i\mathbf{R} = \emptyset$.*
Notice that it is the same condition as our condition (B) when $B(t)$ is a constant matrix. Now as above fixed a large n, we see that the modified system $(HS)_n$ satisfies the conditions of Theorem 0.1 in [15], and we use the functional setting on $A = -(\mathbf{J}\frac{d}{dt} + B(t))$ instead of those used in [15] for constant matrix A, and we prove Theorem 1.5.

In the last part of this section, we study when the sub-harmonics $\{u_k(t) = \eta_k(t/k)\}$ obtained in Theorem 1.1 and Theorem 1.2 are uniformly bounded in C^1. Let Tl_k^{-1} denote the minimal period of $\eta_k(t)$. Then $u_k(t) = \eta(kt)$ has minimal period Tkl_k^{-1}.

THEOREM 4.1. *Under the conditions (H1),(H2), (H3)(or (H5)),(H4), if kl_k^{-1} don't tend to infinity along some subsequence, then the functions $u_k(t)'s$ are uniformly bounded in $||\cdot||_{C^1}$.*

Proof. We have

$$\begin{aligned} c_k &= I^k(u_k) - \frac{1}{2}((I^k)'(u_k), u_k) \\ &= \int_0^{kT} \left[\frac{1}{2}(u_k(s), \nabla H(s, u_k)) - H(s, u_k)\right] ds \\ &= k^{-1}l_k \int_0^{kl_k^{-1}T} \left[\frac{1}{2}(u_k(s), \nabla H(s, u_k(s))) - H(s, u_k(s))\right] ds \end{aligned}$$

From $(H1)$, we have

$$\begin{aligned} c_k &\geq k^{-1}l_k \left\{ (\frac{1}{2} - \frac{1}{\mu}) \int_0^{kl_k^{-1}T} (u_k(s), \nabla H(s, u_k(s)))\, ds - a_1 kl_k^{-1}T \right\} \\ &\geq k^{-1}l_k \left\{ (\frac{\mu}{2} - 1) \int_0^{kl_k^{-1}T} H(s, u_k(s))\, ds - a_2 kl_k^{-1}T \right\} \end{aligned}$$

for some constants a_1 and a_2 independent of k. Since kl_k^{-1} is bounded, we have

$$\int_0^{kl_k^{-1}} \nabla H(t, u_k) u_k \, dt \le C, \quad \int_0^{kl_k^{-1}T} H(t, u_k) \, dt \le C$$

for some constant C independent of k. From Lemma 2.1 and Lemma 2.2, we have $\|u_k\|_\infty \le M$, for some constant M independent of k. And u_k satisfies the equation of (HS), then there is a uniform C^1 bound M for all sub-harmonics $\{u_k\}_{k\in\mathbf{N}}$.

Q.E.D.

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E-mail address: xxu@math.jhu.edu