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Subharmonics for First Order Convex Nonautonomous Hamiltonian Systems

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In this paper new estimates on the C^0-norm of solutions are shown for first order convex Hamiltonian systems possessing super-quadratic potentials. Applying these estimates, some new results on the existence of subharmonics are obtained, which generalize the main results in Ekeland and Hofer [5], and a question about *a priori* estimates on subharmonics raised by Ekeland and Hofer [5] is answered when the convex Hamiltonian systems have globally super-quadratic potentials. Using the uniform estimates on the subharmonics, the behavior of convergence of subharmonics is studied too.

KEY WORDS: Subharmonics; Hamiltonian system; convex; superquadratic potential.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study periodic solutions of the following Hamiltonian system

$$\text{(HS)} \qquad \mathcal{J}\dot{u} + \nabla H(t, u) = 0, \qquad (t, u) \in S_T \times \mathbf{R}^{2N}, \tag{1}$$

where $\mathcal{J}$ denotes the standard symplectic matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

and $H \in C^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is T-periodic in t-variable, and $\nabla H(t, u)$ denotes the partial gradient with respect to the u-variable.

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Here we are interested in the existence of kT-periodic solutions of (HS) for $k \in \mathbf{N}$, which solve the boundary-value problem

$$(\mathrm{HS})_k \begin{cases} \mathcal{J}\dot{u} + \nabla H(t, u) = 0, & (t, u) \in S_T \times \mathbf{R}^{2N}, \\ u(0) = u(kT). \end{cases}$$

The kT-periodic solutions of $(\mathrm{HS})_k$, for $k \geqslant 2$, are called *subharmonics*. Here as Ekeland and Hofer [5] we define the following notations. Given an integer $j \in \mathbf{Z}$ and a kT-periodic solution u_k of $(\mathrm{HS})_k$, define $j * u_k$, the phase shift, as

$$(j * u_k)(t) = u_k(t + jT).$$

Since H is T-periodic in t, whenever u_k solves $(\mathrm{HS})_k$, so do its phase shifts $j * u_k$, for all $j \in \mathbf{Z}$. For two solutions u_k of $(\mathrm{HS})_k$ and u_h of $(\mathrm{HS})_h$, we say they are *geometrically distinct* if

$$j * u_k \neq i * u_h, \qquad \forall i, j \in \mathbf{Z}.$$

We say that kT is a *simple* period of u_k if

$$j * u_k \neq u_k, \qquad \forall j \neq 0 \bmod k.$$

Rabinowitz [9] first showed the existence of infinitely many distinct subharmonics for the first order Hamiltonian systems (HS). After that, there are many other people studied the existence of subharmonics, for example, Clarke and Ekeland [1], Ekeland and Hofer [5], Felmer [6], Liu [7], Tanaka [11], and Xu [12–15]. Especially in Ekeland and Hofer [5], the authors proved the following results:

Theorem 1.1 (Ekeland and Hofer). *Suppose that* $H \in C^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ *satisfies:*

- *(H0)* *$H(t, u)$ is T-periodic in t and strictly convex in u, i.e., for any $u \neq 0$, there is some function $a(t, u) > 0$ such that $Hess_u(H(t, u)) \geqslant a(t, u)\, Id_{2N}$.*
- *(H1)* *there are constants $\mu > 2$ and $r_0 > 0$ such that*

$$0 < \mu H(t, u) \leqslant (\nabla H(t, u), u), \qquad \forall\, |u| \geqslant r_0.$$

- *(H2)* *$H(t, u) \geqslant 0$, $\forall (t, u) \in \mathbf{R} \times \mathbf{R}^{2N}$, and $H(t, u) = o(|u|^2)$, as $u \to 0$.*
- *$(H3)_p$* *There is a constant $c > 0$, such that*

$$|\nabla H(t, u)| \leqslant c\, |u|^{\mu - 1}, \qquad \forall u \geqslant r_0.$$

Then there is a solution u_k of $(HS)_k$ for each $k \in \mathbf{N}$ such that u_k, $k \in \mathbf{N}$, are pairwise geometrically distinct. Furthermore, for almost all H satisfies above conditions, there exists a sequence of $\{u_k\}_{k \in \mathbf{N}}$ such that each u_k is a solution of $(HS)_k$ with simple kT-period.

Remark 1.1. In Section 6 of Ekeland and Hofer [5], the authors asked the following question: *what a priori estimates can be given on the subharmonics $\{u_k\}$?* For the subharmonics $\{u_k\}$, the authors show that there exists a uniform bound:

$$\frac{1}{kT}\int_0^{kT} H(t, u_k(t))\, dt \leqslant C, \qquad \text{for all} \quad k \geqslant 1. \tag{2}$$

In the autonomous case, (2) implies that the amplitudes $|u_k(t)|$ themselves are uniformly bounded since H satisfies the superquadratic condition and $H(u_k(t)) = \textit{constant}$. And the authors stated: *No such conditions seem possible in the nonautonomous case.*

In this paper, we first generalize the main results of Ekeland and Hofer [5] on the existence of subharmonics for system (HS) by loosing the growth condition $(\mathrm{H3})_p$ on the potential H, and obtain the following:

Theorem 1.2. *Suppose that $H \in C^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (H0), (H1), (H2), and:*

(H3) There is a constant $c > 0$, such that

$$|\nabla H(t, u)| \leqslant c(\nabla H(t, u), u), \qquad \forall u \geqslant r_0.$$

Then there is a solution u_k of $(HS)_k$ for each $k \in \mathbf{N}$ such that u_k, $k \in \mathbf{N}$, are pairwise geometrically distinct. Furthermore, for almost all H satisfies above conditions, there exists a sequence of $\{u_k\}_{k \in \mathbf{N}}$ such that each u_k is a solution of $(HS)_k$ with simple kT-period.

Theorem 1.3. *Suppose that $H \in C^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ satisfies (H0), (H1), (H2), and*

(H4) $$\limsup_{|u| \to \infty} \frac{H_t(t, u)}{|u|^{\mu}\, H(t, u)} = 0, \text{ or } \liminf_{|u| \to \infty} \frac{H_t(t, u)}{|u|^{\mu}\, H(t, u)} = 0, \text{ uniformly in } t.$$

Then the results of above Theorem also hold.

Remark 1.2. In Ekeland and Hofer [5], the potential H is required to have growth rate $|u|^{\mu}$ at infinity by condition (H1) and $(\mathrm{H3})_p$. Here we use a much weaker condition (H3) (or (H4)), which include those potentials H growing like $\exp(|u|^p)$ at infinity. The condition (H3), which is first introduced in Rabinowitz [10], inquires the angle between $\nabla H(t, u)$ and u is $O(1/|u|)$ at infinity. And the condition (H4) is on $H_t(t, u)/(|u|^{\mu} H(t, u))$, which measures how far the system (HS) is away from the autonomous system, or one may consider the system (HS) with condition (H3) as a perturbed system with a large perturbation. Especially from condition (H3), one can add any superquadratic autonomous potential to the system (HS) and the results will still hold. Such a condition on $H_t(t, u)/H(t, u)$ as (H4) was first introduced in Long and Xu [8] and Xu [14] to study the existence of periodic solutions of the system (HS).

Next we answer the question about *a priori* estimates on subharmonics raised by Ekeland and Hofer [5] when the system (HS) has globally superquadratic potential H, which means the superquadratic condition (H1) holds for all $u \neq 0$. We have the following result:

Theorem 1.4. *Suppose H satisfies (H0), the globally superquadratic condition (H1), (H2), and (H3) (or (H4)), then there exists subharmonics* $\{u_k\}$ *of the system (HS) such that* u_k, $k \in \mathbf{N}$, *are pairwise geometrically distinct. Furthermore there is a uniform bound M for* $\{\|u_k\|_{C^0}\}_{k \in \mathbf{N}}$.

Here is the outline of this paper. In Section 2, we introduce the modified systems $(\mathrm{HS})^n$ with potentials H_n having growth rate $|u|^{\lambda\mu}$ at infinity, which makes us be able to apply the existence results of Ekeland and Hofer [5] to the modified systems. We prove two new *a priori* estimates on the C^0 norm of solutions for the modified systems.

In Section 3, we first construct the modified systems $(\mathrm{HS})^n$ to approximate the system (HS) as done in Ekeland [2] and Ekeland and Hofer [4], which satisfy the assumptions of the Theorems in Ekeland and Hofer [5]. Then we obtain the existence of subharmonics $\{u_k^n\}_{k \in \mathbf{R}}$ for each modified system $(\mathrm{HS})^n$ by applying the Theorems in Ekeland and Hofer [5] to the modified system. Applying the estimates of Section 2 to the subharmonics $\{u_k^n\}_{k \in \mathbf{R}}$, we obtain the existence of subharmonics $\{u_k\}_{k \in \mathbf{R}}$ for the system (HS). At the end of Section 3, we answer the question of Ekeland and Hofer in [5] for the system (HS) with globally superquadratic potentials H.

In Section 4, we consider the system (HS) with potential $H(t, u) = \frac{1}{2}(Au, u) + W(t, u)$, where A is a $2N \times 2N$ symmetric matrix, $W(t, u)$ satisfies globally superquadratic condition. We study the convergence of the subharmonics $\{u_k\}_{k \in \mathbf{R}}$ for the system (HS). When

$\sigma(\mathcal{J}A) \cap i\mathbf{R} \neq \emptyset$, following the ideas in Rabinowitz [9] and Felmer [6], we have $\|u_k\|_{C^1} \to 0$ as $k \to \infty$. When $\sigma(\mathcal{J}A) \cap i\mathbf{R} \neq \emptyset$, following the ideas in Tanaka [11] and using the uniform estimates on the subharmonics $\{u_k\}_{k \in \mathbf{R}}$ for the system (HS), we show that there exists a nontrivial homoclinic orbit u_∞ of the system (HS) such that $u_k(t) \to u_\infty(t)$ in $C^1_{\text{loc}}(\mathbf{R}, \mathbf{R}^{2N})$ as $k \to \infty$.

2. TWO ESTIMATES

In this section, we consider the modified systems

$$(\text{HS})^n \qquad \mathcal{J}\dot{u} + \nabla H_n(t, u) = 0, \qquad (t, u) \in S_T \times \mathbf{R}^{2N}.$$

Here we define $\{H_n\}$ as

Definition 2.1. For H satisfying (H1), $\sigma \in (0, 1]$ such that $\mu\sigma > 2$, and two sequences $\{K_n\}$ and $\{K'_n\}$ in $\mathbf{R}$, $\{H_n\}$ satisfies

- (i) K_n is monotonous increasing to infinity, as $n \to \infty$ and $K_n < K'_n$, $\forall n \in \mathbf{N}$.
- (ii) for any given $t \in S_T$, $H_n(t, \cdot) \in C^2(\mathbf{R}^{2N}, \mathbf{R})$, for every $n \in \mathbf{N}$.
- (iii) $H_n(t, u) = H(t, u)$, $\forall |u| \leqslant K_n$, for every $n \in \mathbf{N}$; and for some $\lambda \in (\sigma, 1]$, such that $H_n(t, u) = c_n |u|^{\mu\lambda}$, $\forall |u| \geqslant K'_n$, for every $n \in \mathbf{N}$.
- (iv) $0 < \sigma\mu H_n(t, u) \leqslant (\nabla H_n(t, u), u)$, $\forall |u| \geqslant r_0$, for every $n \in \mathbf{N}$.

Now we prove two new estimates on the C^0 norm of solutions of the modified systems $(\text{HS})^n$ for potential H with growth condition (H3) or (H4).

Lemma 2.1. *Suppose that H satisfies (H1) and (H3), $\{H_n\}$ satisfies Definition 2.1, and $u(t)$ is a solution of $(HS)^n$ such that*

$$\int_0^T (\nabla H_n(t, u), u)\, dt \leqslant C, \qquad \int_0^T H_n(t, u)\, dt \leqslant C,$$

then we have the following estimate

$$\|u\|_{C^0} \leqslant M$$

where M is independent of u and n.

Proof. Integrating (iv) of Definition 2.1 gives

$$H_n(t, u) \geqslant a\, |u|^{\mu\sigma} - b, \qquad \forall u \in \mathbf{R}^{2N},$$

where a and b are independent of n using (iii) of Definition 2.1. Hence we have

$$C \geqslant \int_0^T H_n(t, u)\, dt \geqslant aT \int_0^T |u|^{\mu\sigma}\, dt - bT \geqslant aT(\min_{t \in S_T} |u(t)|)^{\mu\sigma} - bT.$$

Therefore we have $\min_{t \in S_T} |u(t)| \leqslant C_0$ where C_0 is independent of u and n. Without loss generality, we may assume $|u(t)|$ obtains its minimum at $t = 0$, and $|u(0)| \geqslant r_0$,

$$|u(t)| - |u(0)| = \int_0^t \frac{d}{ds} |u(s)|\, ds \leqslant \int_0^t |\dot{u}(s)|\, ds = \int_0^t |\nabla H_n(t, u(s))|\, ds.$$

From (H3) and (iii) of Definition 2.1 we have

$$|\nabla H_n(t, u)| \leqslant c(\nabla H_n(t, u), u), \qquad \text{for} \quad r_0 \leqslant |u| \leqslant K_n.$$

We first show for large enough n,

$$\|u\|_{C^0} \leqslant K_n.$$

If not, by passing a subsequence, for each $n \in \mathbf{N}$, there exists $u_n(t)$ and $t_n \in S_T$, such that $|u_n(t_n)| = K_n$ and $|u_n(t)| \leqslant K_n$ for $t \in [0, t_n)$. Let

$$d = \max_{\{(t, u) \in S_T \times \mathbf{R}^{2N} \mid H(t, u) < 0\}} \{|(\nabla H_n(t, u), u)|\},$$

we then have

$$\begin{aligned} |u_n(t_n)| &\leqslant \int_0^{t_n} |\nabla H_n(s, u_n(s))|\, ds + |u_n(0)| \\ &\leqslant c \int_0^{t_n} (\nabla H_n(s, u_n(s)), u_n(s))\, ds + |u_n(0)| \\ &\leqslant c \int_0^T (\nabla H_n(s, u_n(s)), u_n(s))\, ds + c\, dT + C_0 \\ &\leqslant cC + c\, dT + C_0, \end{aligned}$$

where c, d, C, and C_0 are independent of u and n. But then $K_n \to \infty$, as $n \to \infty$, which leads a contradiction. Hence there exists $m \in \mathbf{N}$, depending only on H and C such that for any $n \geqslant m$, $\|u\|_{C^0} \leqslant K_n$ holds. Repeating this argument, we find that for any $n \geqslant m$,

$$|u(t)| \leqslant cC + c\, dT + C_0, \qquad \forall t \in S_T.$$

For $k < m$, from (iii) of Definition 2.1, we have

$$|\nabla H_k(t, u)| \leqslant c_k(\nabla H_k(t, u), u) \qquad \forall |u| \geqslant r_0,$$

for some suitable constant c_k. By the same argument,

$$\begin{aligned} |u(t)| &\leqslant |u(0)| + \int_0^t c_k(\nabla H_k(s, u(s)), u(s))\, ds + d_k T \\ &\leqslant c_k C + d_k T + C_0, \qquad \forall t \in S_T \end{aligned}$$

where c_k and d_k are determined by H_k for $k = 1, 2,..., m-1$. Therefore

$$\|u\|_{C^0} \leqslant \max\{cC + c\, dT + C_0, c_k C + d_k T + C_0, k = 1, 2,..., m-1\} = M,$$

which completes the proof. □

Lemma 2.2. *Suppose that $H(t, u)$ satisfies (H1) and (H4), $\{H_n\}$ satisfies Definition 2.1, and $u(t)$ is a solution of system $(HS)^n$ such that*

$$\int_0^T \nabla H_n(t, u)\, u\, dt \leqslant C, \qquad \int_0^T H_n(t, u)\, dt \leqslant C.$$

Then we have the following estimate

$$\|u\|_{C^0} \leqslant M$$

where M is independent of u and n.

Proof. As the first part of Proof of Lemma 2.1, we have $\min_{t \in S_T} |u(t)| \leqslant C_0$ and

$$H(t, u) \geqslant a\, |u|^\mu - b, \qquad \forall u \in \mathbf{R}^{2N},$$

where a and b are independent of n. Define

$$\sigma(r) = \sup_{|u| \geqslant r,\, t \in S_T} \frac{H_t(t, u)}{|u|^\mu\, H(t, u)}, \qquad \text{and} \qquad \delta(r) = \inf_{|u| \geqslant r,\, t \in S_T} \frac{H_t(t, u)}{|u|^\mu\, H(t, u)}.$$

Then (H4) means

$$\lim_{r \to \infty} \sigma(r) = 0 \qquad \text{or} \qquad \lim_{r \to \infty} \delta(r) = 0.$$

Case I. Suppose we have $\lim_{r\to\infty}\sigma(r)=0$.

Since $H_n(t,u)>0$ for $|u|\geqslant r_0$, we have

$$C\geqslant\int_0^T H_n(t,u)\,dt\geqslant\int_{\{H_n(t,u)\geqslant 0\}} H_n(t,u)\,dt-T\sup_{H(t,u)<0,\,t\in S_T}|H(t,u)|.$$

Hence we have

$$\int_{\{H_n(t,u)\geqslant 0\}} H_n(t,u)\,dt\leqslant C+T\sup_{H(t,u)<0,\,t\in S_T}|H(t,u)|=\bar{C},$$

where $\bar{C}$ is a constant independent of n and u. By the definition of $\sigma(r)$, we have that $\sigma(r)$ is decreasing to 0. Fix a large $R>r_0$, such that

$$a-\sigma(R)\,\bar{C}>0.$$

First, we show $|u|_{C^0}\leqslant K_n$ for large n. If not, by passing a subsequence we may assume for each n, there exists $u_n(t)$, a_n, and b_n such that

$$(a_n,b_n)\subset\{t\in S_T\mid R<|u_n(t)|<K_n\}$$

and $|u_n(a_n)|=R$, $|u_n(b_n)|=K_n$. Therefore we have

$$\begin{aligned}H(b_n,u_n(b_n))-H(a_n,u_n(a_n))&=\int_{a_n}^{b_n}\frac{d}{dt}H_n(t,u(t))\,dt\\&=\int_{a_n}^{b_n}[(\nabla H_n(t,u_n(t)),\dot{u}_n(t))+H_t(t,u_n(t))]\,dt\\&=\int_{a_n}^{b_n}H_t(t,u_n(t))\,dt\\&\leqslant\int_{a_n}^{b_n}\sigma(|u_n(t)|)\,|u_n(t)|^{\mu}\,H(t,u_n(t))\,dt\\&\leqslant\sigma(R)\,K_n^{\mu}\int_{a_n}^{b_n}H(t,u_n(t))\,dt\\&\leqslant\sigma(R)\,\bar{C}K_n^{\mu}.\end{aligned}$$

On the other hand,

$$\begin{aligned}H(b_n,u_n(b_n))-H(a_n,u_n(a_n))&\geqslant a\,|u_n(b_n)|^{\mu}-b-\max_{|u|\leqslant R,\,t\in S_T}|H(t,u)|\\&=aK_n^{\mu}-(b+\max_{|u|\leqslant R,\,t\in S_T}|H(t,u)|).\end{aligned}$$

If we combine these two formulas, we get that

$$(a-\sigma(R)\,\bar{C})\,K_n^{\mu}\leqslant b+\max_{|u|\leqslant R,\,t\in S_T}|H(t,u)|.$$

Since $a-\sigma(R)\,\bar{C}>0$ and $K_n\to\infty$ as $n\to\infty$, the left side tends to infinity, but the right side is a constant independent of u and n. This leads a contradiction. Hence there exists $m\in\mathbf{N}$, which is determined by $H(t,u)$ and N only, such that for any $n\geqslant m$, if $u(t)$ is a critical point of I_n such that $I_n(u)\leqslant N$, we have $|u|_{C^0}\leqslant K_n$.

For $n\geqslant m$, if $\|u\|_{C^0}$ doesn't have an n-independent upper bound M_0, then following the above proof with K_n replaced by M_n where $M_n\to\infty$ as $n\to\infty$, we also get a contradiction. For $n<m$, as the proof in last part of Lemma 2.1, we have

$$|u(t)|\leqslant c_kC+d_kT+C_0,\qquad \forall t\in S_T,$$

where c_k and d_k are determined by H_k, $k=1,2,\ldots,m-1$.

Hence we have

$$\|u\|_{C^0}\leqslant\max\{M_0,\,c_kC+d_kT+C_0,\,k=1,2,\ldots,m-1\}=M.$$

Case II. Suppose we have $\lim_{r\to\infty}\delta(r)=0$.

We need only to modify the proof of Case I a little. We have

$$(a+\delta(R)\,\bar{C})\,K_n^{\mu}\leqslant b+\max_{|u|\leqslant R,\,t\in S_T}|H(t,u)|,$$

where $a+\delta(R)\,\bar{C}>0$ and $K_n\to\infty$ as $n\to\infty$. Using the same argument as in Case I, we have

$$\|u\|_{C^0}\leqslant M.$$

By combining these two cases, we obtain the Lemma. □

Remark 2.1. Here we proved these two Lemmas for general potentials H, where we don't make any assumptions on the potential H nearby $u=0$ and convexity. In next part of this paper $H(t,u)\geqslant 0$ is satisfied, which implies $d=d_k=0$ and $\bar{C}=C$, then the bound M is independent of the period T and depends on C and H only from the proofs of two Lemmas. This is one key observation to get *a priori* estimates for subharmonics $\{u_k\}$ and answer the question of Ekeland and Hofer [5].

Remark 2.2. When $H(t,u)=\frac{1}{2}(Au,u)+W(t,u)$, where A is a $2N\times 2N$ symmetric matrix, and $W(t,u)$ satisfies (H1) and (H3) (or (H4)), notice that in the Proof of Lemma 2.1, the quadratic part doesn't show up, and in the Proof of Lemma 2.2, the quadratic part add a CK_n^2 term only, Lemmas 2.1 and 2.2 still hold for such potentials.

3. PROOFS OF MAIN THEOREMS

In this section, following the ideas in Xu [12–16], and using the estimates obtained in the last section, we first prove Theorems 1.2 and 1.3.

Proof of Theorems 1.2 and 1.3. We first modify the H by a sequence $\{H_n\}$ which satisfies Definition 2.1 and (H0). We do the truncation the same as done in p. 533 of Ekeland [2] or p. 185 of Ekeland and Hofer [4]. In Ekeland [2] or Ekeland and Hofer [4], the authors dealt with the truncation for H under autonomous case. Here we do the same truncation to $H(t,\cdot)$ for any fixed $t\in S_T$ as they did to $H(\cdot)$. Hence for the new systems $(\mathrm{HS})^n$, the conditions of Theorem 1.1 are satisfied, then we have subharmonics $\{u_{n,k}\}_{k\in\mathbf{N}}$ for each system $(\mathrm{HS})^n$.

Next we study the bound of $\{\|u_{n,k}\|_{C^0}\}_{n\in\mathbf{N}}$ for any fixed $k\in\mathbf{N}$. As in Ekeland and Hofer [5], define the Legendre transform $\{G_n\}$ of $\{H_n\}$ by

$$G_n(t,x)=\sup_{u\in\mathbf{R}^{2N}}[(x,u)-H_n(t,u)],$$

and by duality, the assumptions on $\{H_n\}$ imply the following properties of $\{G_n\}$:

(G1) $\frac{\mu\sigma}{\mu\sigma-1}G_n(t,x)\geqslant(\nabla G_n(t,x),x)>0$, $\forall|x|\geqslant r_0$, for every $n\in\mathbf{N}$.

(G2) $G_n(t,x)\leqslant\frac{1}{a}|x|^{\frac{\mu\sigma}{\mu\sigma-1}}+b=G_0(t,x)$, $\forall x\in\mathbf{R}^{2N}$.

For each integer $n,k\in\mathbf{N}$ define

$$\Phi_{n,k}(x)=\frac{1}{kT}\int_0^{kT}\left[\frac{1}{2}\left(\mathcal{J}x(t),\int_0^t x(s)\,ds\right)+G_n(t,-\mathcal{J}x(t))\right]dt.$$

If $\Phi_{n,k}$ has a critical point $x_{n,k}$, then the system $(\mathrm{HS})^n$, obtained from the system (HS) by replacing H by H_n, has a kT-periodic solution $u_{n,k}(t)$ with $\dot u_{n,k}(t)=x_{n,k}(t)$. For fixed $k\in\mathbf{N}$, define

$$\begin{aligned}c_{n,k}&=\Phi_{n,k}(x_{n,k})\\&=\frac{1}{kT}\int_0^{kT}\left[\frac{1}{2}\left(\mathcal{J}x_{n,k}(t),\int_0^t x_{n,k}(s)\,ds\right)+G_n(t-\mathcal{J}x_{n,k}(t))\right]dt\end{aligned}$$

and by duality, there are the equivalent formula for $\{c_{n,k}\}$:

$$c_{n,k} = \frac{1}{kT} \int_0^{kT} \left[\frac{1}{2} (\mathscr{J} \dot{u}_{n,k}(t), u_{n,k}(t)) - H_{n,k}(t, u_{n,k}(t)) \right] dt.$$

Since $\{H_n(t, u)\}$ satisfy (iv) of Definition 2.1 and $u_{n,k}$ is a kT-periodic solution of the system $(\mathrm{HS})^n$, we have

$$\begin{aligned} c_{n,k} &= \frac{1}{kT} \int_0^{kT} \left[\frac{1}{2} (\nabla H_n(t, u_{n,k}(t)), u_{n,k}(t))\, dt - H_{n,k}(t, u_{n,k}(t)) \right] dt \\ &\geqslant \frac{1}{kT} \int_0^{kT} \left(\frac{1}{2} - \frac{1}{\mu\sigma} \right) (\nabla H_n(t, u_{n,k}(t)), u_{n,k}(t))\, dt + L \\ &\geqslant \frac{1}{kT} \int_0^{kT} \left(\frac{\mu\sigma}{2} - 1 \right) H_{n,k}(t, u_{n,k}(t))\, dt + L, \end{aligned} \tag{3}$$

where L is a constant coming from (H1) or (iv) of Definition 2.1. Note that $L = 0$ if H satisfies the globally superquadratic condition which means (H1) holds for $r_0 = 0$.

In Ekeland and Hofer [5], all solutions $\{x_{n,k}\}$ are obtained via the mountain pass theorem, i.e.,

$$c_{n,k} = \Phi_{n,k}(x_{n,k}) = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \Phi_{n,k}(\gamma(t)),$$

where Γ is the set of all continuous paths $\gamma \colon [0, 1] \to X_{kT}$, such that $\gamma(0) = 0$ and $\gamma(1) = x_0$ for some suitable x_0 such that $\Phi_{n,k}(x_0) \leqslant 0$. Define

$$\Phi_{0,k}(x) = \frac{1}{kT} \int_0^{kT} \left[\frac{1}{2} \left(\mathscr{J} x(t), \int_0^t x(s)\, ds \right) + G_0(t, -\mathscr{J} x(t)) \right] dt.$$

Since (G2) holds, for fixed $k \in \mathbf{N}$, we have $\Phi_{n,k}(x) \leqslant \Phi_{0,k}(x)$ for any path x hold for all $n \in \mathbf{N}$. Taking the path γ_0,

$$\gamma_0(s) = (s\dot{y}_k, s\dot{q}_k),$$

with $y_k(t) = (0, \ldots, 0, R) \sin \frac{2\pi}{kT} t$, and $q_k(t) = (0, \ldots, 0, R) \cos \frac{2\pi}{kT} t$, where $R > 0$ large enough such that $\Phi_{0,k}(\dot{y}_k, \dot{q}_k) \leqslant 0$. Then we have

$$c_{n,k} \leqslant \max_{0 \leqslant s \leqslant 1} \Phi_{n,k}(\gamma_0(s)) \leqslant \max_{0 \leqslant s \leqslant 1} \Phi_{0,k}(\gamma_0(s)).$$

Define $\alpha = \frac{\mu\sigma}{\mu\sigma - 1} \in (1, 2)$, then we have

$$\Phi_{0,k}(\gamma_0(s)) = \frac{1}{a} \left(\frac{2\pi Rs}{kT} \right)^{\alpha} + b - \frac{kT}{2\pi} \left(\frac{2\pi Rs}{kT} \right)^2,$$

also,

$$
\begin{aligned}
c_{n,k} &\leqslant \max_{0\leqslant s\leqslant 1} \Phi_{0,k}(\gamma_0(s)) \\
&\leqslant \max_{0\leqslant s\leqslant \infty} \left[\frac{1}{a} s^{\alpha}+b-\frac{kT}{2\pi} s^2\right] \\
&=\frac{1}{a}\left(1-\frac{\alpha}{2}\right)\left(\frac{\alpha\pi}{akT}\right)^{\frac{\alpha}{2-\alpha}}+b.
\end{aligned}
\tag{4}
$$

If we combine (3) and (4), we get

$$\int_0^{kT} (\nabla H_n(t, u_{n,k}(t)), u_{n,k}(t))\, dt \leqslant D(kT)^{\frac{-2}{2-\alpha}}+kT(b+D_1 L)=C_k,$$

$$\int_0^{kT} H_n(t, u_{n,k}(t))\, dt \leqslant D(kT)^{\frac{-2}{2-\alpha}}+kT(b+D_1 L)=C_k.$$

Hence if $k\in \mathbf{N}$ is fixed, applying Lemma 2.1 (or Lemma 2.2) to $\{u_{n,k}\}_{n\in\mathbf{N}}$, we get a constant M_k, which is dependent on C_k and H only, such that $\|u_{n,k}\|_{C^0}\leqslant M_k$ holds for all $n\in\mathbf{N}$. Now for large $n\in\mathbf{N}$ such that $K_n > M_k$, we have

$$\|u_{n,k}\|_{C^0}\leqslant M_k < K_n.$$

On the other hand, we have

$$H_n(t, u)=H(t, u), \qquad \forall\, |u|<K_n.$$

This implies that $u_{n,k}(t)$ is a solution of system $(\mathrm{HS})_k$ when $\|u_{n,k}\|_{C^0}<K_n$.

Hence for any given $k\in\mathbf{N}$, there exists a large enough $n\in\mathbf{N}$ such that $u_{n,1},\ldots, u_{n,k}$ are subharmonics of the system (HS), i.e., Theorems 1.2 and 1.3 are proved. □

In next part we answer the question about *a priori* estimates on subharmonics raised by Ekeland and Hofer in [5] when the system (HS) has globally superquadratic potential H, which means the superquadratic condition (H1) holds for all $u\neq 0$.

Proof of Theorem 1.4. From the proofs of Theorems 1.1 and 1.2, we need only to show there exist a uniform bound for $\{\|u_{n,k}\|_{C^0}\}_{n,k\in\mathbf{N}}$ where $\{u_{n,k}\}_{n,k\in\mathbf{N}}$ are those subharmonics that we obtain for $(\mathrm{HS})^n$.

Since H satisfies (H0), we have

$$H(t, u)\geqslant a(t, u)\,|u|^2\geqslant A(|u|)\,|u|^2, \qquad \forall (t, u)\in S_T\times\mathbf{R}^{2N},$$

where $A(r)=\min_{t\in S_T, |u|=r} a(t,u)>0$ for $r>0$, from (H2) we have $A(r)\to 0$, as $r\to 0$. By duality, G satisfies

$$G(t,x)\leqslant B(|x|)\,|x|^2, \qquad \forall (t,x)\in S_T\times \mathbf{R}^{2N},$$

where $B(r)\to\infty$ as $r\to 0$ and $B(r)=o(\frac{1}{r})$ near $r=0$.

Hence we have

$$H_n(t,u)\geqslant \max\{a\,|u|^{\mu\sigma}-b,\ A(|u|)\,|u|^2\},$$

and by the duality,

$$G_n(t,x)\leqslant \min\left\{\frac{1}{a}\,|x|^{\mu\sigma}+b,\ B(|x|)\,|x|^2\right\}=G_0(t,x).$$

Using this G_0, we get

$$c_{n,k}\leqslant \max_{0\leqslant s\leqslant 1} \Phi_{0,k}(\gamma_0(s)).$$

Here

$$\Phi_{0,k}(\gamma_0(s))=\min\left\{\frac{1}{a}\left(\frac{2\pi Rs}{kt}\right)^{\alpha}+b,\ B\left(\frac{2\pi Rs}{kT}\right)\left|\frac{2\pi Rs}{kT}\right|^2\right\}-\frac{kT}{2\pi}\left(\frac{2\pi Rs}{kT}\right)^2.$$

Since $\frac{1}{a}\,t^{\alpha}+b-\frac{kT}{2\pi}\,t^2$ obtains its maximum at $t=(\frac{\alpha\pi}{akT})^{1/(2-\alpha)}$, which is close to $t=0$, and for such a value t, $B(t)\,|t|^2<\frac{1}{a}\,t^{\alpha}+b$ for large $k\in\mathbf{N}$. Hence we need to estimate the maximum of

$$f(t)=B(t)\,t^2-\frac{kT}{2\pi}\,t^2.$$

Since $B(t)>\frac{kT}{2\pi}$ implies $t\leqslant\frac{2c_1\pi}{kT}$ for some constant c_1 depending on B only. Hence from $B(t)=o(1/t)$, we have $B(t)\leqslant\frac{c_2}{t}$ for $t\leqslant\frac{2c_1\pi}{kT}$ where c_2 depending on B only. Hence we have

$$f(t)\leqslant c_2 t-\frac{kT}{2\pi}\,t^2\leqslant\frac{c_2^2}{2kT},$$

which yields

$$c_{n,k}\leqslant\frac{c_2^2}{2kT},$$

where c_2 is dependent on H only.

Now H satisfies the globally super-quadratic potential, which implies $L=0$ as said in the proof of Theorems 1.2 and 1.3. We have

$$\int_0^{kT} (\nabla H_n(t, u_{n,k}(t)), u_{n,k}(t))\,dt \leqslant C, \qquad \int_0^{kT} H_n(t, u_{n,k}(t))\,dt \leqslant C,$$

where $C=\frac{c_2^2 D}{2}$ depends on H only. Then from Lemma 2.1 (or Lemma 2.2), and the Remark 2.1, we know that there is a uniform bound M for $\{\|u_{n,k}\|_{C^0}\}_{n,k\in\mathbf{N}}$, and M is dependent on H only. □

Remark 3.1. Notice that for large k, $c_{n,k}$ is small from above proof, and the dependence of the related constants, we can find that $\|u_k\|_{C^0}\to 0$ as $k\to\infty$. We will study the convergence of the subharmonics with more general potential H in next section, which includes above as a special case.

4. CONVERGENCE OF SUBHARMONICS

In this section, we study the convergence of the subharmonics $\{u_k\}_{k\in\mathbf{R}}$ for the system (HS), which potential $H(t,u)$ has the form

$$H(t,u)=\tfrac{1}{2}(Au,u)+W(t,u),$$

satisfying

(A) A is a $2N\times 2N$ symmetric matrix such that

$$\sigma(\mathcal{J}A)\cap i\mathbf{R}\neq\varnothing$$

or

(A)^c A is a $2N\times 2N$ symmetric matrix such that

$$\sigma(\mathcal{J}A)\cap i\mathbf{R}\neq\varnothing$$

and $W(t,u)$ is T-periodic in t variable and globally superquadratic in u variable; more precisely $W(t,u)$ satisfies (H0), (H1) with $r_0=0$, (H2), (H3) (or (H4)) and

(H5) there are $\alpha\geqslant\mu$ and $c>0$ such that

$$W(t,u)\geqslant c\,|u|^{\alpha}, \qquad \forall (t,u)\in\mathbf{R}\times\mathbf{R}^{2N}.$$

Following the ideas in Felmer [6] and Rabinowitz [10], we have

Theorem 4.1. *Assume A satisfies $(A)^c$ and $W(t,u)$ satisfies above conditions, then there is a sequence of pairwise geometrically distinct subharmonics $\{u_k(t)\}_{k\in\mathbf{N}}\subset C^1(\mathbf{R},\mathbf{R}^{2N})$ of the system (HS) such that $\|u_k\|_{C^1}\to 0$ as $k\to\infty$.*

Remark 4.1. In [6] and [10], without assuming the convex condition on the potential $W(t,u)$, the above result without pairwise geometrically distinct for each $k\in\mathbf{N}$ was proved under conditions (A)^c, (H1) with $r_0=0$, (H2), (H3), and (H5) in Rabinowitz [10], and the existence of subharmonics was proved under conditions (A)^c, (H1) with $r_0=0$, (H2), a stronger condition similar to (H4) and (H5) in Xu [12] and [14]. Here we study the convex Hamiltonian systems, we obtain pairwise geometrically distinct subharmonics.

Following the ideas in Tanaka [11], we have

Theorem 4.2. *Assume A satisfies (A) and $W(t,u)$ satisfies above conditions, then there is a sequence of pairwise geometrically distinct subharmonics $\{u_k(t)\}_{k\in\mathbf{N}}\subset C^1(\mathbf{R},\mathbf{R}^{2N})$ of the system (HS) such that*

(i) there are constants m, $M>0$ independent of $k\in\mathbf{N}$ such that

$$m\leqslant\int_0^{kT}\left[\tfrac{1}{2}(-\mathcal{J}\dot{u}_k,u_k)-H(t,u_k)\right]dt\leqslant M;$$

(ii) moreover $\{u_k(t)\}_{k\in\mathbf{N}}$ is compact in the following sense: for any sequence of integers $k_n\to\infty$, there exists a subsequence $\{k_{n_i}\}_{i\in\mathbf{N}}$ and a nontrivial homoclinic orbit $u_\infty(t)$ emanating from 0 such that

$$u_{k_{n_i}}(t)\to u_\infty(t)\ \text{ in } C^1_{\mathrm{loc}}(\mathbf{R},\mathbf{R}^{2N}),\qquad \text{as}\quad i\to\infty.$$

Remark 4.2. In Tanaka [11], without assuming the convex condition on the potential $W(t,u)$, the above result without pairwise geometrically distinct for each $k\in\mathbf{N}$ was proved under conditions (A)^c, (H1) with $r_0=0$, (H2), (H3), and (H5). Here we study the convex Hamiltonian systems, we obtain pairwise geometrically distinct subharmonics. When $W(t,u)$ satisfies (H3), our result on the existence of the homoclinic orbit is a special case of Tanaka [11], but when $W(t,u)$ satisfies (H4), our result on the existence of the homoclinic orbit is new. In Xu [13] and [16], one can find the existence of homoclinic orbits for Hamiltonian system (HS) under general setting.

To prove the existence of pairwise geometrically distinct subharmonics $\{u_k(t)\}_{k\in\mathbf{N}}$ for the system (HS) with potential $H(t,u)=\frac{1}{2}(Au,u)+W(t,u)$, we need only modify the argument of Section 3 a little by adding the quadratic part of $H(t,u)$ to the quadratic term in the variational functional $\Phi(u)$. As Section 3, define the Legendre transform $\{V_n\}$ of $\{W_n\}$ by

$$V_n(t,x)=\sup_{u\in\mathbf{R}^{2N}}[(x,u)-W_n(t,u)].$$

For each integer n, $k\in\mathbf{N}$ define

$$\Phi_{n,k}(x)=\frac{1}{kT}\int_0^{kT}\left[\frac{1}{2}\left(\mathcal{J}x(t)+\int_0^t x(s)\,ds,\int_0^t x(s)\,ds\right)+V_n(t,-\mathcal{J}x(t))\right]dt.$$

If $\Phi_{n,k}$ has a critical point $x_{n,k}$, then the system $(\mathrm{HS})^n$, obtained from the system (HS) by replacing W by W_n, has a kT-periodic solution $u_{n,k}(t)$ with $\dot{u}_{n,k}(t)=x_{n,k}(t)$. For fixed $k\in\mathbf{N}$, define

$$\begin{aligned}c_{n,k}&=\Phi_{n,k}(x_{n,k})\\&=\frac{1}{kT}\int_0^{kT}\left[\frac{1}{2}\left(\mathcal{J}x_{n,k}(t)+\int_0^t x(s)\,ds,\int_0^t x_{n,k}(s)\,ds\right)+V_n(t,-\mathcal{J}x_{n,k}(t))\right]dt\end{aligned}$$

and by duality, there are the equivalent formula for $\{c_{n,k}\}$:

$$c_{n,k}=\frac{1}{kT}\int_0^{kT}\left[\frac{1}{2}(\mathcal{J}\dot{u}_{n,k}(t),u_{n,k}(t))-H_{n,k}(t,u_{n,k}(t))\right]dt.$$

Then using the same argument as Section 3, we obtain the existence of subharmonics $\{u_k(t)\}_{k\in\mathbf{N}}$ for the system (HS) and $\{u_k(t)\}_{k\in\mathbf{N}}$ has uniform C^0 upper bound since $W(t,u)$ satisfies the globally superquadratic condition. $\{u_k(t)\}_{k\in\mathbf{N}}$ has uniform C^1 upper bound since each $u_k(t)$ satisfies the equation, so we can regard $\{u_k(t)\}_{k\in\mathbf{N}}$ as the subharmonics of some modified system $(\mathrm{HS})^n$ for a fixed large $n\in\mathbf{N}$.

Now for the convergent result in Theorem 4.1, we can use the same argument in Felmer [6] and Rabinowitz [10], where got the same convergence results as Theorem 4.1 for the potential H having growth rate $|u|^\mu$ at infinity, since all the subharmonics $\{u_k\}$ are obtained from some modified system $(\mathrm{HS})^n$ for some large n, whose potential grows as $|u|^\mu$ at infinity. We omit the details here. Hence Theorem 4.1 holds.

For the convergent result in Theorem 4.2, since we have had uniform C^1 upper bound of $\{u_k(t)\}_{k\in\mathbf{N}}$, we need only follow the argument of Sections 2.2 and 2.3 in Tanaka [11], then we have Theorem 4.2 holds. Here we omit the details also.

Remark 4.3. In Xu [15], the author deals with similar problem on subharmonics and their asymptotic behaviors of the first order Hamiltonian systems without assuming the convexity on u-variable.

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