Eigenfunction Estimates on Compact Manifolds with Boundary and Hörmander Multiplier Theorem

by

Xiangjin Xu

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Abstract

In this thesis, the L^∞ estimates and gradient estimates are proved for the eigenfunctions of either Dirichlet or Neumann Laplacian on compact Riemannian manifolds (M, g) with boundary. Applying these estimates, the Hörmander multiplier Theorem is proved under this setting. These extend the work that done previously by C. Sogge, A. Seeger and C. D. Sogge on compact manifolds without boundary. And the almost-everywhere convergence of eigenfunction expansions and Riesz means for function $f \in L_2^s(M)$ *or* $L^p(M)$ with respect to the Laplace-Beltrami operator on compact manifolds with boundary are also studied.

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Contents

Chapter 1

Introduction

The first part of my thesis studies the L^∞ estimates and gradient estimates for the eigenfunctions of either Dirichlet Laplacian or Neumann Laplacian on compact Riemannian manifolds (M, g) with boundary. Here we consider the Dirichlet eigenvalue problem

$$(\Delta + \lambda^2)u(x) = 0, \quad x \in M, \quad u(x) = 0, \quad x \in \partial M,$$

and the Neumann eigenvalue problem

$$(\Delta + \lambda^2)u(x) = 0, \quad x \in M, \quad \partial_n u(x) = 0, \quad x \in \partial M,$$

where $\Delta = \Delta_g$ is the Laplace-Beltrami operator associated to the Riemannian metric g, and ∂_n is the outward normal derivative on ∂M. Let $0 \le \lambda_1^2 \le \lambda_2^2 \le \lambda_3^2 \le \cdots$ denote the eigenvalues and let $\{e_j(x)\} \subset L^2(M)$ be an associated real orthogonal basis with unit L^2 norm. And define the unit band spectral projection operators,

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(f), \quad where \quad e_j(f)(x) = e_j(x) \int_M f(y) e_j(y) dy.$$

The study for L^p estimates for the eigenfunctions on compact manifolds has a long history. In the case of manifolds without boundary, the most general results of the form

$$||\chi_\lambda f||_p \leq C\lambda^{\sigma(p)} ||f||_2, \qquad \lambda \geq 1, \quad p \geq 2, \tag{1.1}$$

where

$$\sigma(p) = \max\{\frac{n-1}{2} - \frac{n}{p}, \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p})\}$$

were proved in Sogge [30]. These estimates cannot be improved since one can show that the operator norm satisfy $\limsup_{\lambda \to \infty} \lambda^{-\sigma(p)} ||\chi_\lambda||_{L^2 \to L^p} > 0$. (see Sogge [32]).

The special case of (1.1) where $p = \infty$ can be proved using the estimates of Hörmander [14] that proved the sharp Weyl formula for general self-adjoint elliptic operators on manifolds without boundary. In case of manifolds with boundary, Grieser [10], and Smith and Sogge [29] showed that the bounds (1.1) hold under the assumption that the manifold has geodesically concave boundary. The two dimensional case was handled in [10], and higher dimensional in [29]. Recently, Grieser [11] proved

$$||e_j(f)||_\infty \leq C\lambda_j^{(n-1)/2} ||f||_2.$$

And Sogge [33] proved the following L^∞ estimate for $\chi_\lambda f$ on compact manifolds with boundary,

$$||\chi_\lambda f||_\infty \leq C\lambda^{(n-1)/2} ||f||_2, \quad \lambda \geq 1,$$

which are sharp for instance when M is the upper hemisphere of S^n with standard metric. Here we have the L^∞ estimates and gradient estimates on the unit band spectral projection operators $\chi_\lambda f$ for both Dirichlet Laplacian and Neumann Laplacian,

Theorem 1. *Fix a compact Riemannian manifold (M, g) with boundary, for both Dirichlet Laplacian and Neumann Laplacian on M, there is a uniform constant C so that*

$$\begin{aligned} ||\chi_\lambda f||_\infty &\leq C\lambda^{(n-1)/2}||f||_2, \quad \lambda \geq 1, \\ ||\nabla \chi_\lambda f||_\infty &\leq C\lambda^{(n+1)/2}||f||_2, \quad \lambda \geq 1. \end{aligned}$$

The motivation to study the gradient estimates as Theorem 1 is that one cannot get the gradient estimates on the eigenfunctions of Dirichlet Laplacian from standard calculus of pseudo-differential operators as was done for the Laplacian on the manifolds without boundary, since $P = \sqrt{-\Delta_g}$ for the Dirichlet Laplacian is not a pseudo-differential operator any more and one cannot get good estimates on L^∞ bounds on χ_λ and $\nabla\chi_\lambda$ near the boundary only by studying the Hadamard parametrix of the wave kernel as in the case of manifolds without boundary in [27] and [32]. Here we shall prove Theorem 1 using the standard interior and boundary gradient estimates for second order elliptic operators in [8] and a maximum principle argument as used in [11] and [33].

The second part of my thesis studies the Hörmander multiplier Theorem for smooth compact Riemannian manifolds with boundary. Given a bounded function $m(\lambda) \in L^\infty(\mathbf{R})$ we can define operators, $m(P)$, by

$$m(P)f = \sum_{j=1}^{\infty} m(\lambda_j) e_j(f)$$

The Hörmander multiplier Theorem is that when one considers any space $L^p(M)$, for $1 < p < \infty$, what smoothness assumption on the function $m(\lambda)$ are needed to ensure that

$$m(P) : L^p(M) \to L^p(M).$$

Many authors studied the Hörmander multiplier Theorem under different setting. Specifically, one assume the following regularity assumption: suppose that $m \in L^\infty(\mathbf{R})$, let $L^2_s(\mathbf{R})$ denote the usual Sobolev space and fix $\beta \in C_0^\infty((1/2, 2))$ satisfying $\sum_{-\infty}^{\infty} \beta(2^j t) = 1$, $t > 0$, and suppose also that

$$\sup_{\lambda>0} \lambda^{-1+s} ||\beta(\cdot/\lambda)m(\cdot)||^2_{L^2_s} = \sup_{\lambda>0} ||\beta(\cdot)m(\lambda\cdot)||^2_{L^2_s} < \infty,$$

where real number $s > n/2$.

Hörmander [14] first proved the Hörmander multiplier Theorem for $\mathbf{R}^n$ under the above smoothness assumption. Stein [35] and Stein and Weiss [36] studied the Hörmander multiplier Theorem for multiple Fourier series, which can be regarded as the case for flat torus T^n. Seeger and Sogge [27] and Sogge [32] proved the boundedness of $m(P)$ on $L^p(M)$ for compact manifolds without boundary under the above smoothness assumption. Using the L^∞ estimates on $\chi_\lambda f$ and $\nabla\chi_\lambda f$ and the ideas in [27], [31]- [33], we have the following Hömander multiplier Theorem for both Dirichlet Laplacian and Neumann Laplacian on compact Riemannian manifolds with boundary:

Theorem 2. *Let $m \in L^\infty(\mathbf{R})$ satisfy the above smoothness assumption, then there are constants C_p such that*

$$||m(P)f||_{L^p(M)} \leq C_p ||f||_{L^p(M)}, \quad 1 < p < \infty.$$

The last part of my thesis study the problem on almost-everywhere convergent eigenfunction expansions and Riesz means of the Laplace-Beltrami operator Δ_g on a compact Riemannian manifold (M, g) with boundary. Denote the partial sums of eigenfunction expansions as

$$S_N(f; x) = \sum_{k=1}^{N} e(f)(x).$$

Let $S^*(f; x)$ denote the maximal function

$$S^*(f; x) = \sup_{N \geq 1} |S_N(f; x)|.$$

In [17], on a compact boundless manifold (M, g), Meaney proved the almost-everywhere convergence of eigenfunction expansions of the Laplace-Beltrami operator for function $f \in L^2_s(M)$, where $L^2_s(M)$ is the Sobolev space of order $s > 0$. Here we have the following result on almost-everywhere convergent eigenfunction expansions for both Dirichlet Laplacian and Neumann Laplacian on a compact manifold M with smooth boundary.

Theorem 3. *For $s > 0$, if $f \in L^2_s(M)$, we have*

$$\lim_{N \to \infty} S_N(f; x) = f(x), \quad \textit{almost everywhere on } M.$$

And for the maximal function $S^*(f;x)$, we have

$$||S^*(f)||_{L^2(M)} \leq C_s ||f||_{L^2_s(M)},$$

for some constant C_s.

When one study the almost-everywhere convergence of eigenfunction expansions of the Laplace-Beltrami operator for function $f \in L^p(M)$, we will consider an important class of special multiplier, Riesz means:

$$S^\delta_\lambda f(x) = \sum_{\lambda_j \leq \lambda} (1 - \frac{\lambda_j^2}{\lambda^2})^\delta e_j(f)(x),$$

of the spectral expansion. Stein and Weiss [36] studied the Riesz means for multiple Fourier series, which can be regarded as the case for flat torus T^n. Sogge [31] and Christ and Sogge [5] proved the sharp results for manifolds without boundary, which the Riesz means are uniformly bounded on all $L^p(M)$ spaces, provided that $\delta > \frac{n-1}{2}$, but no such result can hold when $\delta \leq \frac{n-1}{2}$. Recently, Sogge [33] proved the same results on Riesz means of the spectral expansion for Dirichlet Laplacian on manifolds with boundary. In [2], Alimov studied conditions for the convergence and Riesz summability of spectral expansions of piecewise smooth functions for self-adjoint elliptic operators on the compact subdomain of a n-dimensional domain.

Here we use the L^∞ estimates on χ_λ and a L^2 estimates on the normal derivative of eigenfunctions on the boundary to study the asymptotic behavior of spectral functions, and we obtain the following results:

Theorem 4. *Let f be a piecewise smooth function on a manifold M with boundary,* $\dim M = n$, *we have*

(1). $n = 2$*, on each compact subset of the smoothness domain of* f*, the spectral expansion is uniformly bounded and the Riesz means* $S_\lambda^\delta f(x)$ *of any positive order* $s > 0$ *uniformly converge to* f*.*

(2). $n > 2$*, on each compact subset of the smoothness domain of* f*, the Riesz means* $S_\lambda^\delta f(x)$ *of any positive order* $s \geq (n-1)/2$ *uniformly converge to* f*.*

Remark 1.0.1 *For* $n > 2$*, there are some simple examples, such as the characteristic function of unit ball* χ_B *in* $\mathbf{R}^n$*, see [23], show that the spectral expansion of a piecewise smooth function may diverge even at points far from the discontinuity surface, and, if* $n > 3$*, the divergence will be unbounded.*

Now applying the uniformly bounds for Riesz means on $L^p(M)$ in [33] and a density argument, from Theorem 4.1.2, we have the following almost everywhere convergence results for Riesz means on $L^p(M)$:

Theorem 5. *Fix a smooth compact Riemannian manifold with boundary of dimension* $n \geq 2$*, for any* $s > (n-1)/2$*, let* $f \in L^p(M)$*,* $1 \leq p \leq \infty$*, we have*

$$\lim_{\lambda\to\infty} S_\lambda^s f(x) = f(x), \quad almost\ everywhere\ for\ x \in M.$$

And for any $s > (n-1)/2$*, let* $f \in L^p(M)$*,* $1 \leq p \leq \infty$*, we have*

$$\lim_{\lambda\to\infty} S_\lambda^s f(x) = f(x), \quad in\ measure\ for\ x \in M.$$

Chapter 2

Eigenfunction Estimates

2.1 Introduction and Results

In this chapter we study the L^∞ estimates and gradient estimates for the eigenfunctions of either Dirichlet Laplacian or Neumann Laplacian on compact Riemannian manifolds (M, g) with boundary. Here we consider the Dirichlet eigenvalue problem

$$(\Delta + \lambda^2)u(x) = 0, \quad x \in M, \quad u(x) = 0, \quad x \in \partial M, \tag{2.1}$$

and the Neumann eigenvalue problem

$$(\Delta + \lambda^2)u(x) = 0, \quad x \in M, \quad \partial_n u(x) = 0, \quad x \in \partial M, \tag{2.2}$$

with $\Delta = \Delta_g$ being the Laplace-Beltrami operator associated to the Riemannian metric g, and ∂_n is the outward normal derivative on ∂M. Recall that the spectrum of (2.1)and (2.2) is discrete and tends to infinity. Let

$0 \le \lambda_1^2 \le \lambda_2^2 \le \lambda_3^2 \le \cdots$ denote the eigenvalues, and let $\{e_j(x)\}$ be an associated real orthogonal normalized basis in $L^2(M)$, and define spectral projection to $e_j(x)$ as

$$e_j(f)(x) = e_j(x) \int_M f(y) e_j(y) dy,$$

and the unit band spectral projection operators as

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(f),$$

In [33], Sogge proved that for a fixed compact Riemannian manifold (M, g) with boundary, there is a uniform constant C so that

$$||\chi_\lambda f||_\infty \le C\lambda^{(n-1)/2} ||f||_2, \quad \lambda \ge 1,$$

which is equivalent to

$$\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x)^2 \le C\lambda^{n-1}, \quad \forall x \in M.$$

And in [11], Grieser proved the L^∞ estimates

$$||e_j(f)||_{L^\infty} \le C\lambda_j^{(n-1)/2} ||f||_{L^2}$$

for both Dirichlet and Neumann Laplacian. Here we have the L^∞ estimates and gradient estimates on χ_λ for both Dirichlet Laplacian and Neumann Laplacian,

Theorem 2.1.1 *Fix a compact Riemannian manifold (M, g) with boundary, for both Dirichlet Laplacian and Neumann Laplacian on M, there is a uniform constant C so that*

$$\begin{aligned} ||\chi_\lambda f||_\infty &\le C\lambda^{(n-1)/2} ||f||_2, \quad \lambda \ge 1, \\ ||\nabla \chi_\lambda f||_\infty &\le C\lambda^{(n+1)/2} ||f||_2, \quad \lambda \ge 1. \end{aligned}$$

Remark 2.1.1 *Note that*

$$\begin{aligned}\chi_\lambda f(x) &= \int_M \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x) e_j(y) f(y) dy, \\ \nabla \chi_\lambda f(x) &= \int_M \sum_{\lambda_j \in [\lambda, \lambda+1)} \nabla e_j(x) e_j(y) f(y) dy,\end{aligned}$$

therefore, by the converse to Schwarz's inequality and orthogonality, one has the bounds

$$\begin{aligned}|\chi_\lambda f(x)| &\leq C\lambda^{(n-1)/2} ||f||_2, \\ |\nabla \chi_\lambda f(x)| &\leq C\lambda^{(n+1)/2} ||f||_2,\end{aligned}$$

at a given point x if and only if

$$\begin{aligned}\sum_{\lambda_j \in [\lambda, \lambda+1)} |e_j(x)|^2 &\leq C\lambda^{n-1}, \\ \sum_{\lambda_j \in [\lambda, \lambda+1)} |\nabla e_j(x)|^2 &\leq C\lambda^{n+1}.\end{aligned}$$

In the proof at section 2, we shall prove either one of these two kind gradient estimates.

In section 2.2, we prove the L^∞ estimates on eigenfunctions as did in Sogge [33], where the same estimate for Dirichlet Laplacian was proved. The only difference is that we need prove the interior estimates for Neumann Laplacian in addition.

In section 2.3, we prove the gradient estimates according to three cases: for the interior points with $\text{dist}(x, \partial M) \geq \epsilon\lambda^{-1}$, for the boundary points, and for the points on the strip near boundary with $\text{dist}(x, \partial M) \leq \epsilon\lambda^{-1}$. For interior points, we use the idea of gradient estimates for poisson equations, and apply this idea to manifolds without boundary, we obtain the gradient estimates directly. For boundary points, we show the gradient estimates for Dirichlet Laplacian, using a perturbation from a constant coefficients differential operators, on a small neighborhood at each boundary point. For the points in the λ^{-1} strip near boundary, we follow the idea as used in Sogge [33] and use the maximum principle to show the gradient estimates for both Dirichlet Laplacian and Neumann Laplacian, furthermore, we show the same estimates hold for Neumann Laplacian at boundary. Since the proof of gradient estimates only involve the L^∞ estimate of the eigenfunctions and didn't use any geometric property. Notice that for compact Riemannian manifolds without boundary, by Lemma 2.3.1 and the L^∞ estimates on the unit spectral projection operators χ_λ for a second-order elliptic differential operator on compact manifolds without boundary in Sogge [32], we have the gradient estimates for a second-order elliptic differential operator on compact manifolds without boundary.

Finally we study the upper bounds for normal derivatives of the unit spectral projection operators Laplacian on $L^2(M)$. Let $0 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \cdots$ denote the eigenvalues for the Dirichlet Laplacian, and $\{e_j(x)\}$ be an associated real orthogonal normalized basis in $L^2(M)$, and define $e_j(f)(x)$ and the unit band spectral projection operators, $\chi_\lambda f$, as above. In [12], Hassell and Tao proved the following inequality for single Dirichlet eigenfunctions

$$c\lambda_j \leq ||\frac{\partial e_j}{\partial \nu}||_{L^2(\partial M)} \leq C\lambda_j,$$

where the upper bound holds for some constant C independent of j, and the lower bound holds provided that M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. Using the idea to show the Rellich-type estimates in [12] and the orthogonality of the eigenfunctions, we have the following results:

Theorem 2.1.2 *Let M be a smooth compact Riemannian manifold with boundary, and function $f \in L^2(M)$, then*

$$||\frac{\partial \chi_\lambda(f)}{\partial \nu}||_{L^2(\partial M)} \leq C\lambda^{3/2}||f||_{L^2(M)},$$

holds for some constant C independent of λ.

2.2 L^∞ Estimates of Eigenfunctions

In this section we shall prove the L^∞ Estimates of Eigenfunctions for either Dirichlet or Neumann Laplacian following the ideas in [11], where the same L^∞ estimates were done for single eigenfunctions, and in [33], where the L^∞ estimates were done for $\chi_\lambda f$ of Dirichlet Laplacian. Here we use the geodesic coordinates with respect to the boundary. We can find a small constant $c > 0$ so that the map $(x', x_n) \in \partial M \times [0, c) \to M$, sending (x', x_n) to the endpoint x, of the geodesic of length x_n which starts a $x' \in \partial M$ and is perpendicular to ∂M is a local diffeomorphism. We denote $d(x) = \text{dist}(x, \partial M)$ in whole thesis. In this local coordinates $x = (x_1, \cdots, x_{n-1}, x_n)$, the metric has the form

$$\sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j = (dx_n)^2 + \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j,$$

and the Laplacian can be written as

$$\Delta_g = \sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i},$$

Where $(g^{ij}(x))_{1\le i,j\le n}$ is the inverse matrix of $(g_{ij}(x))_{1\le i,j\le n}$, and $g^{nn} = 1$, and $g^{nk} = g^{kn} = 0$ for $k \neq n$. Also the $b_i(x)$ are C^∞ and real valued.

We show that one has the uniform bounds for either Dirichlet or Neumann boundary problem

$$|\chi_\lambda f(x)| \le C\lambda^{(n-1)/2}||f||_{L^2}, \quad \lambda \ge 1.$$

From Remark 2.1.1, we need only to should the following two results.

Proposition 2.2.1 *Fix the compact manifold (M, g) with boundary, for a given small constant $\epsilon > 0$, there is a uniform constant C so that for $\lambda \ge 1$*

$$\sum_{\lambda_j \in [\lambda, \lambda+1)} (e_j(x))^2 \le C_\epsilon \lambda^{n-1},$$

for any interior point x satisfying $d(x) \ge \epsilon\lambda^{-1}$.

Proposition 2.2.2 *Fix (M, g) as above, then for large λ we have*

$$\max_{\{x\,:\,d(x)\le\frac{1}{2}(\lambda+1)^{-1}\}} \sum_{\lambda_j\in[\lambda,\lambda+1)} (e_j(x))^2 \le 4 \max_{\{x\,:\,d(x)=\frac{1}{2}(\lambda+1)^{-1}\}} \sum_{\lambda_j\in[\lambda,\lambda+1)} (e_j(x))^2.$$

Proof of Proposition 2.2.1 For Dirichlet Laplacian case, one can see in Sogge [33]. As in Sogge [33], we shall see that the estimate for Neumann Laplacian in this Proposition is an immediate consequence of the similarly results as Theorem 17.5.10 in Hörmander [15], which in turn is based on earlier work of Seeley [28]. To state this result, we let

$$e(x, \lambda) = (2\pi)^{-n} \int_{\{\xi\in\mathbf{R}^n|\ |\xi|\le\lambda\}} (1 - e^{i2d(x)\xi_n})d\xi,$$

If we assume that the local coordinates have been chosen so that the Riemannian volume form is $dx_1 \cdots dx_n$, then the result just quoted says that there is a uniform constant C so that for $\lambda \geq 1$,

$$|\sum_{\lambda_j \leq \lambda} (e_j(x))^2 - e(x, \lambda)| \leq C\lambda(\lambda + d(x)^{-1})^{n-2}.$$

Since $\lambda(\lambda + d(x)^{-1})^{n-2} = O(\lambda^{n-1})$ for all x satisfying $d(x) \geq \epsilon\lambda^{-1}$. This yields Proposition 2.2.1 since

$$|e(x, \lambda + 1) - e(x, \lambda)| \leq C_\epsilon \lambda^{n-1}, \quad \forall x \; with \; d(x) \geq \epsilon\lambda^{-1}.$$

Next we show the above result for Neumann Laplacian. One may see that the proof is a straightforward modification of Theorem 17.5.10 in Hörmander [15] for Dirichlet Laplacian in his book line by line.

When $d(x) > c > 0$ for some fixed small constant c, this is a consequence of Theorem 7.2 in Seeley [28], where Seeley proved, there are constants C and $\alpha > 0$ such that for $\lambda > 1$

$$|\sum_{\lambda_j \leq \lambda} (e_j(x))^2 - e(x, \lambda)| \leq C[\lambda(d(x)^{\alpha-1}\lambda^{n-1} + d(x)^{-3/2}\lambda^{n-3/2}.$$

true for both Dirichlet Laplacian and Neumann Laplacian.

When $d(x) \leq c$, we will use the Tauberian Lemma (Lemma 17.5.6 in Hörmander [15])with $a = 1/c$ and

$$\mu(\lambda) = \frac{1}{2}\text{sign}(\lambda) \sum_{\lambda_j \leq \lambda} (e_j(x))^2,$$
$$\nu(\lambda) = \frac{1}{2}\text{sign}(\lambda) e(x, \lambda).$$

By the definition of $e(x,\lambda)$, we have

$$|de(x,\lambda)| \le C\lambda^{n-1}.$$

This proved the first part of assumption (17.5.13) in Lemma 17.5.6 in [15]. Our result will follow if we prove that

$$|(d\mu - d\nu) * \phi_a(\lambda)| \le C(|\lambda| + \mathrm{dist}(x, \partial M)^{-1})^{n-2},$$

where $\phi_a(\lambda) = \phi(\lambda/a)/a$ for $a > 0$, and ϕ is a positive function in $C^\infty(\mathbf{R})$ with $\int_{\mathbf{R}} \phi(\lambda) d\lambda = 1$ and the Fourier transform $\hat{\phi}$ with support in $(-1, 1)$. We can show the above estimate for Neumann Laplacian the same as Hörmander's argument for Dirichlet Laplacian in the proof of Theorem 17.5.10 in [15]. The only difference is that we replace those estimates for the Dirichlet wave kernel by those for the Neumann wave kernel wave kernel in the proof of Theorem 17.5.10 in [15]. The same estimates hold with some different constants. Then applying the Tauberian Lemma (Lemma 17.5.6 in Hörmander [15]), we show our assertion. Q.E.D.

Proof of Proposition 3.2. Here we use the geodesic coordinates with respect to the boundary. we use the same function $H(x)$ as in Proof of Proposition 2.4 in [33], where the estimate for Dirichlet Laplacian is discussed, to apply the maximum principle to get bound at boundary of the λ^{-1} strip, and furthermore, we show the outward normal derivatives of $H(x)$ on the boundary ∂M must be strictly positive as pointed in [11] for single eigenfunctions of Neumann Laplacian.

In what follows we shall assume that λ is large enough so that $\lambda \geq C/c$, where C is some universe constant and c is the uniform constant with respect to the local coordinates. Assume further that $spec(\sqrt{-\Delta_g}) \cap [\lambda, \lambda+1] \neq \emptyset$, and consider the function

$$H(x) = \frac{1}{(w(x))^2} \sum_{\lambda_j \in [\lambda, \lambda+1)} (e_j(x))^2,$$

where

$$w(x) = 1 - (\lambda+1)^2 x_n^2.$$

Support that in the strip $\{x \in M \, : \, 0 \leq x_n \leq \frac{1}{2}(\lambda+1)^{-1}\}$ the function $H(x)$ has a maximum at an interior point $x = x_0$. Then

$$v(x) = \frac{1}{w(x)} \sum_{\lambda_j \in [\lambda, \lambda+1)} \frac{e_j(x_0)}{w(x_0)} e_j(x)$$

must have a positive maximum at $x = x_0$. For because of our assumptions on the spectrum we then have $v(x_0) = \sum_{\lambda_j \in [\lambda, \lambda+1)} [\frac{e_j(x_0)}{w(x_0)}]^2 > 0$, while at other points in the strip

$$\begin{aligned}
|v(x)| &\leq \frac{1}{w(x)} (\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x)^2)^{1/2} \frac{1}{w(x_0)} (\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x_0)^2)^{1/2} \\
&= (H(x))^{1/2} (H(x_0))^{1/2} \\
&\leq H(x_0) = v(x_0)
\end{aligned}$$

Note that in the strip $\{x \in M \, : \, 0 \leq x_n \leq \frac{1}{2}(\lambda+1)^{-1}\}$ we have

$$\begin{aligned}
(\Delta + \lambda_j^2) w &= -2(\lambda+1)^2 - 2b_n(x) x_n (\lambda+1)^2 + \lambda_j (1 - (\lambda+1)^2 x_n^2) \\
&\leq -\frac{(\lambda+1)^2}{2}, \quad for \; \lambda_j \leq \lambda + 1,
\end{aligned}$$

assuming that λ is large enough so that $|2b_n(x)x_n| \leq 1/2$ in the strip. Also, in this strip we have that $\frac{1}{2} \leq w(x) \leq 1$.

Let us set

$$v_j(x) = \frac{e_j(x)}{w(x)} \frac{e_j(x_0)}{w(x_0)},$$

so that $v(x) = \sum \lambda_j \in [\lambda, \lambda+1) v_j(x)$. We also set

$$u_j(x) = \frac{e_j(x_0)}{w(x_0)} e_j(x),$$

and note that $\Delta + \lambda_j^2) u_j(x) = 0$.

A computation (one may see p.72, [26]) shows that for a given j we have

$$\begin{aligned} 0 &= \frac{1}{w(x)} (\Delta + \lambda_j^2) u_j^2(x) \\ &= \sum_{k,l=1}^{n} g^{kl}(x) \partial_k \partial_l v_j + \sum_{k=1}^{n} (\frac{2}{w} \sum_{l=1}^{n} g^{kl} \partial_l w + b_k) \partial_k v_j + \frac{v_j}{w} (\Delta + \lambda_j^2) w. \end{aligned}$$

Therefore, if we sum over $\lambda_j \in [\lambda, \lambda+1)$, we get

$$\sum_{k,l=1}^{n} g^{kl}(x) \partial_k \partial_l v + \sum_{k=1}^{n} (\frac{2}{w} \sum_{l=1}^{n} g^{kl} \partial_l w + b_k) \partial_k v = - \sum_{\lambda_j \in [\lambda, \lambda+1)} \frac{v_j}{w} (\Delta + \lambda_j^2) w.$$

In particular, at point $x = x_0$, we have

$$\begin{aligned} & \sum_{k,l=1}^{n} g^{kl}(x_0) \partial_k \partial_l v(x_0) + \sum_{k=1}^{n} (\frac{2}{w} \sum_{l=1}^{n} g^{kl}(x_0) \partial_l w(x_0) + b_k(x_0)) \partial_k v(x_0) \\ &= -\frac{1}{w(x_0)} \sum_{\lambda_j \in [\lambda, \lambda+1)} (\frac{e_j(x_0)}{w(x_0)})^2 (\Delta + \lambda_j^2) w(x_0) \\ &\geq \frac{(\lambda+1)^2}{2w(x_0)} \sum_{\lambda_j \in [\lambda, \lambda+1)} (\frac{e_j(x_0)}{w(x_0)})^2 > 0. \end{aligned} \tag{2.3}$$

But this is impossible since v have a positive maximum at x_0, which implies that $\partial_k v(x_0) = 0$ for every k, and $\sum_{k,l=1}^{n} g^{kl}(x_0) \partial_k \partial_l v(x_0) \leq 0$. Thus, we conclude that the function $H(x)$ cannot have a maximum at an interior point of the strip $\{x \in M \ : \ 0 \leq x_n \leq \frac{1}{2}(\lambda+1)^{-1}\}$.

Now we shall prove the function $H(x)$ cannot have a maximum value of the strip on ∂M. Suppose that there is a maximum at the boundary point $x_0 = (x', 0)$ on ∂M. Then by the same argument as above, we have $v(x)$ must have a positive maximum at $x_0 = (x', 0)$ and $v(x_0) > 0$. It implies v has a positive maximum at x_0 on ∂M, in our local coordinates,which means that $\partial_k v(x_0) = 0$ for every $k \leq n-1$, and $\sum_{k,l=1}^{n-1} g^{kl}(x_0)\partial_k\partial_l v(x_0) \leq 0$. And we also have

$$\begin{aligned} & \partial_n v(x) \\ = \ & (\partial_n \frac{1}{w(x)}) \sum_{\lambda_j \in [\lambda, \lambda+1)} \frac{e_j(x_0)}{w(x_0)} e_j(x) + \frac{1}{w(x)} \sum_{\lambda_j \in [\lambda, \lambda+1)} \frac{e_j(x_0)}{w(x_0)} \partial_n e_j(x) \\ = \ & -\frac{2(\lambda+1)^2 x_n}{w(x)^2} \sum_{\lambda_j \in [\lambda, \lambda+1)} \frac{e_j(x_0)}{w(x_0)} e_j(x) + \frac{1}{w(x)} \sum_{\lambda_j \in [\lambda, \lambda+1)} \frac{e_j(x_0)}{w(x_0)} \partial_n e_j(x). \end{aligned}$$

Since the Neumann boundary condition, we have $\partial_n v(x_0) = 0$. For our local coordinates, we have $g^{nn} = 1$, and $g^{nk} = g^{kn} = 0$ for $k \neq n$. Hence from (2.3), we have

$$\partial_n^2 v(x_0) = \frac{(\lambda+1)^2}{2} \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j^2(x_0) > 0.$$

But it cannot be true since in local coordinates, we have that $v(x_0', x_n)$ gets its maximum at $x_0 = (x_0', 0)$ from our assumption and $\partial_n v(x_0) = 0$, which implies $\partial_n^2 v(x_0) \leq 0$. It is a contradiction.

From above and our lower bound for w, we get that

$$\max_{\{x \,:\, d(x) \leq \frac{1}{2}(\lambda+1)^{-1}\}} \sum_{\lambda_j \in [\lambda, \lambda+1)} (e_j(x))^2 \leq 4 \max_{\{x \,:\, d(x) = \frac{1}{2}(\lambda+1)^{-1}\}} \sum_{\lambda_j \in [\lambda, \lambda+1)} (e_j(x))^2$$

As desired, which completes the proof of Proposition 3.2.

Q.E.D.

Combine Proposition 3.1 and Proposition 3.2, we proved the L^∞ estimates on χ_λ for Neumann Laplacian.

2.3 Gradient Estimates of Eigenfunctions

In this section, we study the gradient estimates of eigenfunctions for both Dirichlet Laplacian and Neumann Laplacian, and we shall prove the gradient estimates by using maximum principle according to three different cases: for the interior points with $d(x) \geq \epsilon\lambda^{-1}$, for the boundary points, and for the points on the strip near boundary with $d(x) \leq \epsilon\lambda^{-1}$. And for the eigenfunctions for the Dirichlet Laplacian on compact boundless manifolds, we also get the gradient estimates on $\chi_\lambda f$ applying the gradient estimates for interior points.

First we show the gradient estimates at the interior points with $d(x) \geq \epsilon(\lambda+1)^{-1}$ for both Dirichlet Laplacian and Neumann Laplacian,

Lemma 2.3.1 *For the Riemannian manifold (M, g) with boundary ∂M, we have the gradient estimate*

$$|\nabla\chi_\lambda f(x)| \leq C_\epsilon \lambda^{(n+1)/2} ||\chi_\lambda f||_2, \quad for\ d(x) \geq \epsilon(\lambda+1)^{-1}$$

Proof. We shall show this Lemma following the ideas in [8], where for the Poisson's equation $\Delta u = f$, there are gradient estimates for the interior point x_0 as

$$|\nabla u(x_0)| \leq \frac{C}{d} \sup_{\partial B} |u| + Cd \sup_{B} |f|,$$

by using maximum principle in a cube B centered at x_0 with length d.

Now we fix ϵ and $x_0 \in M$ with $\mathrm{dist}(x_0, \partial M) \geq \epsilon(\lambda+1)^{-1}$. We shall use maximum principle in the cube centered at x_0 with length $d = \epsilon(\lambda+1)^{-1}/\sqrt{n}$ to prove the same gradient estimates for $\chi_\lambda f$ as above for Poisson's equation.

Define the geodesic coordinates $x = (x_1, \cdots, x_n)$ centered at point x_0 as following, fixed an orthonormal basis $\{v_i\}_{i=1}^n \subset T_{x_0}M$, identity $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ with the point $\exp(\sum_{i=1}^n x_i v_i) \in M$. In small neighborhood of x_0 we take the metric with the form

$$\sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j,$$

and the Laplacian can be written as

$$\Delta_g = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},$$

Where $(g^{ij}(x))_{1 \leq i,j \leq n}$ is the inverse matrix of $(g_{ij}(x))_{1 \leq i,j \leq n}$, and $b_i(x)$ are in C^∞.

Now define the cube

$$Q = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ |x_i| < d,\ i = 1, \cdots, n\} \subset M,$$

where we can choose $d = \epsilon(\lambda+1)^{-1}/\sqrt{n} \leq \mathrm{dist}(x_0, \partial M)/\sqrt{n}$.

Denote $u(x; f) = \chi_\lambda f(x)$, we have $u \in C^2(Q) \cap C^0(\bar{Q})$, and

$$\Delta_g u(x; f) = - \sum_{\lambda_j \in [\lambda, \lambda+1)} \lambda_j^2 e_j(f) := h(x; f).$$

From the L^∞ estimate in [33], and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|h(x;f)|^2 &= \Big(\sum_{\lambda_j\in[\lambda,\lambda+1)}(\lambda_j^2 e_j(x))\Big(\int_M e_j(y)f(y)dy\Big)\Big)^2\\
&\le \sum_{\lambda_j\in[\lambda,\lambda+1)}\lambda_j^4 e_j^2(x)\sum_{\lambda_j\in[\lambda,\lambda+1)}\Big(\int_M e_j(y)f(y)dy\Big)^2\\
&\le (\lambda+1)^4\Big(\sum_{\lambda_j\in[\lambda,\lambda+1)} e_j^2(x)\Big)||\chi_\lambda f||^2_{L^2(M)}\\
&\le C(\lambda+1)^{n+3}||\chi_\lambda f||^2_{L^2(M)}
\end{aligned}$$

We estimate $|D_n u(0;f)| = |\frac{\partial}{\partial x_n}u(0;f)|$ first, the same estimate holds for $|D_i u(0;f)|$ with $i = 1,\cdots,n-1$ also. Now in the half-cube

$$Q' = \{x = (x_1,\cdots,x_n)\in \mathbf{R}^n|\ |x_i| < d,\ i = 1,\cdots,n-1,\ 0 < x_n < d.\}\subset M,$$

Consider the function

$$\varphi(x',x_n;f) = \frac{1}{2}[u(x',x_n;f) - u(x',-x_n;f)],$$

where we write $x = (x',x_n) = (x_1,\cdots,x_{n-1},x_n)$. One see that $\varphi(x',0;f) = 0$, $\sup_{\partial Q'}|\varphi| \le \sup_{\partial Q}|u| := A$, and $|\Delta_g\varphi| \le \sup_Q |h| := N$ in Q'. Consider also the function

$$\psi(x',x_n) = \frac{A}{d^2}[|x'|^2 + \alpha x_n(nd - (n-1)x_n)] + \beta N x_n(d - x_n)$$

defined on the half-cube Q' and $\alpha \ge 1$ and $\beta \ge 1$ will be determined below. Obviously $\psi(x',x_n) \ge 0$ on $x_n = 0$ and $\psi(x',x_n) \ge A$ in the remaining portion of $\partial Q'$.

$$\begin{aligned}
\Delta_g\psi(x) &= \frac{A}{d^2}[2\mathrm{tr}(g^{ij}(x)) - (2n\alpha - 2\alpha + 1) + 2\sum_{i=1}^n b_i(x)x_i + b_n(x)(n\alpha d\\
&\quad -(2n\alpha - 2\alpha + 1)x_n)] + N\beta[-2g^{nn}(x) + b_n(x)(d - 2x_n)]
\end{aligned}$$

Since in M, $\mathrm{tr}(g^{ij}(x))$ and $b_i(x)$ are bounded uniformly and $g^{nn}(x)$ is positive, then for a large α, we can make

$$2\mathrm{tr}(g^{ij}(x)) - (2n\alpha - 2\alpha + 1) \leq -1,$$

Fix such a α, since $d = \epsilon(\lambda+1)^{-1}/\sqrt{n}$, for large λ, we have

$$2\sum_{i=1}^{n} b_i(x)x_i + b_n(x)(n\alpha d - (2n\alpha - 2\alpha + 1)x_n) < 1.$$

Then the first term is negative. For second term, let β large enough, we have

$$\beta[-2g^{nn}(x) + b_n(x)(d - 2x_n)] < -1.$$

Hence we have $\Delta_g\psi(x) \leq -N$ in Q'.

Now we have $\Delta_g(\psi \pm \varphi) \leq 0$ in Q' and $\psi \pm \varphi \geq 0$ on $\partial Q'$, from which it follows by the maximum principle that $|\varphi(x', x_n; f)| \leq |\psi(x', x_n)|$ in Q'. Letting $x' = 0$ in the expressions for ψ and φ, then dividing by x_n and letting x_n tend to zero, we obtain

$$|D_n u(0; f)| = \lim_{x_n \to 0} |\frac{\varphi(0, x_n; f)}{x_n}| \leq \frac{\alpha n A}{d} + \beta d N.$$

Note that $d = \epsilon(\lambda+1)^{-1}/\sqrt{n}$, $A \leq C(\lambda+1)^{(n-1)/2}$, and $N \leq (\lambda+1)^{(n+3)/2}$, then we have the estimate

$$|D_n u(0; f)| \leq C_\epsilon(\lambda+1)^{(n+1)/2}.$$

The same estimate holds for $|D_i u(0)|$, $i = 1, \cdots, n-1$. Hence we have

$$|\nabla u(0)| \leq C_\epsilon(\lambda+1)^{(n+1)/2}.$$

Since the estimate is for any $x_0 \in M$ with $\text{dist}(x_0, \partial M) \geq \epsilon(\lambda + 1)^{-1}$, the Lemma is proved.

Q.E.D.

Remark 2.3.1 *One can also use the parametrix for wave kernel to show local Weyl estimates for interior points, then obtain the gradient estimates as our Lemma directly by estimating the gradient of the parametrix for wave kernel. Here since we use the* L^∞ *estimates on* $\chi_\lambda f$ *in [33], where have used the parametrix method already, we can use basic method as maximum principle to get gradient estimates here.*

Next we show the following gradient estimates for the points on boundary ∂M for Dirichlet Laplacian.

Lemma 2.3.2 *Assume* $|u(x)| \leq C_1 \lambda^{(n-1)/2}$ *in* M *and* $u(x) = 0$ *on* $\partial M \in C^2$, *and*

$$\Delta_g u(x) \geq -C_2 \lambda^{(n+3)/2},$$

then we have

$$|\nabla u(x)| \leq C \lambda^{(n+1)/2}, \quad \forall x \in \partial M.$$

Proof. As did in proof of Proposition 2.2 in [33], we use the geodesic coordinates with respect to the boundary. We can find a small constant $c > 0$ so that the map $(x', x_n) \in \partial M \times [0, c) \to M$, sending (x', x_n) to the endpoint , x, of the geodesic of length x_n which starts a $x' \in \partial M$ and is perpendicular to ∂M is a local diffeomorphism. In this local coordinates

$x = (x_1, \cdots, x_{n-1}, x_n)$, the metric has the form

$$\sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j = (dx_n)^2 + \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j,$$

and the Laplacian can be written as

$$\Delta_g = \sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i},$$

Where $(g^{ij}(x))_{1\le i,j\le n}$ is the inverse matrix of $(g_{ij}(x))_{1\le i,j\le n}$, and $g^{nn} = 1$, and $g^{nk} = g^{kn} = 0$ for $k \neq n$, and $b_i(x)$ are in C^∞ and real valued.

Fix a point $x_0 \in \partial M$, choose a local coordinate so that $x_0 = (0, \cdots, 0, R)$, with $R = \lambda^{-1}$. Without loss the generality, we may assume that at the point x_0, $\Delta_g(x_0) = \Delta$, the Euclidean Laplacian, since we can transfer $\Delta_g(x_0)$ to Δ by a suitable nonsingular linear transformation as did in Chapter 6 at [8]. Since $g^{ij}(x)$ and $b_i(x)$ are C^∞, we have a constant $\Lambda > 0$ such that $|\nabla g^{ij}(x)| < \Lambda$ and $|\nabla b_i(x)| < \Lambda$ hold for all x is the λ^{-1} strip of the boundary.

We first assume $n = \dim M \geq 3$. For $n = 2$, we need define another comparing function $v(x)$. Define a function

$$v(x) = \alpha\lambda^{(3-n)/2}(\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}) + \beta\lambda^{(n+3)/2}(R^2 - r^2),$$

on $A_R = B_{2R}(0) - B_R(0)$, where $r = ||x||_{\mathbf{R}^n}$, $B_r(0) = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ |x| < r\}$, and $\alpha \geq 1$ and $\beta \geq 1$ will be determined below. Here we assume $B_R(0)$ is tangent to ∂M at point x_0 from outside. We will compare $v(x)$ and $u(x)$ in $A_R \cap M$. For any $x \in A_R \cap M$, we have

$$\begin{aligned}\Delta_g v(x) &= [\frac{(n-2)\mathrm{tr}(g^{ij}(x))}{r^n} - \frac{n(n-2)\sum g^{ij}(x)x_i x_j}{r^{n+2}} + \frac{(n-2)\sum b_i(x)x_i}{r^n}] \\ &\times\ \alpha\lambda^{(3-n)/2} - [\mathrm{tr}(g^{ij}(x)) + 2\sum b_i(x)x_i]\beta\lambda^{(n+3)/2},\end{aligned}$$

Since M is a compact manifolds, we have $0 < \theta < \Theta < \infty$ such that $|b_i(x)| < \Theta$, $\theta < \mathrm{tr}(g^{ij}(x)) < \Theta$ and $\theta|y|^2 < \sum g^{ij}(x)y_iy_j < \Theta|y|^2$ hold for all $x \in M$. And since $x \in B_R$ and $R = \lambda^{-1}$, we have $\lambda^{-1} < |x| < 2\lambda^{-1}$ and $|\sum b_i(x)x_i| < \theta/4$ for large λ. Then we have

$$\begin{aligned}\Delta_g v(x) &\leq \frac{n(n+1)\Lambda|x-x_0|}{r^n}\alpha\lambda^{(3-n)/2} - (\theta - 2\cdot\frac{\theta}{4})\beta\lambda^{(n+3)/2} \\ &\leq \frac{3n(n+1)\Lambda R}{r^n}\alpha\lambda^{(3-n)/2} - \frac{\theta}{2}\beta\lambda^{(n+3)/2} \\ &\leq 3n(n+1)\Lambda\alpha\lambda^{(n+1)/2} - \frac{\theta}{2}\beta\lambda^{(n+3)/2}\end{aligned}$$

Now let

$$3n(n+1)\Lambda\alpha\lambda^{-1} - \frac{\theta}{2}\beta \leq -C_2, \tag{2.4}$$

where C_2 is the constant in the assumptions of this Lemma, then we have $\Delta_g v(x) \leq \Delta u(x)$ in $A_R \cap M$.

Next we compare the values of v and u on $\partial(A_R \cap M)$.

Case I, $x \in \partial(A_R \cap M) \cap \partial A_R$.

$$v(x) = \alpha\lambda^{\frac{3-n}{2}}(\frac{1}{R^{n-2}} - \frac{1}{(2R)^{n-2}}) + \beta\lambda^{\frac{n+3}{2}}(R^2 - (2R)^2) \geq (\frac{\alpha}{2} - 3\beta)\lambda^{\frac{n-1}{2}},$$

Now let

$$\frac{\alpha}{2} - 3\beta \geq C_1, \tag{2.5}$$

then we have $v(x) \geq u(x)$ for $x \in \partial(A_R \cap M) \cap \partial A_R$.

Case II, $x \in \partial(A_R \cap M) \cap \partial M$.

$$v(x) = \alpha\lambda^{(3-n)/2}(\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}) + \beta\lambda^{(n+3)/2}(R^2 - r^2) := h(r),$$

Since $B_R(0)$ is tangent to ∂M at point x_0 from outside, we know the range of $h(r)$ is $[R, 2R]$ and $f(R) = 0$,

$$f'(r) = (n-2)\alpha\lambda^{\frac{3-n}{2}}\frac{1}{r^{n-1}} - 2\beta\lambda^{\frac{n+3}{2}}r \geq (2^{1-n}(n-2)\alpha - 2\beta)\lambda^{\frac{n+1}{2}}.$$

Now let

$$2^{1-n}(n-2)\alpha - 2\beta > 0, \tag{2.6}$$

Then $f(r) \geq f(R) = 0$, and we have $v(x) \geq u(x)$ for $x \in \partial(A_R \cap M) \cap \partial M$.

Finally we need determine the values of α and β. Let $\beta = 4C_2/\theta$, then from (2.4), (2.5), (2.6), we have the range of α as

$$\max\{\frac{2^n\beta}{n-2}, 6\beta + 2C_1\} \leq \alpha \leq \frac{C_2\lambda}{3n(n+1)\Lambda}.$$

For large λ, the range of α is not empty set.

Hence for large λ, we can find a function $v(x)$ such that $\Delta_g(v(x) \pm u(x)) \leq 0$ in $A_R \cap M$ and $v(x) \pm u(x) \geq 0$ on $\partial(A_R \cap M)$, from which it follows by the maximum principle that $v(x) \pm u(x) \geq 0$ in $A_R \cap M$. On the other hand, $v(x_0) - u(x_0) = 0$. Hence we have

$$|\nabla u(x_0)| \leq \frac{\partial v}{\partial r}(x_0) = [(n-2)\alpha - 2\beta]\lambda^{(n+1)/2} := C'\lambda^{(n+1)/2}$$

Since x_0 is an arbitrary point on ∂M, the Lemma is proved for $n \geq 3$.

For $n = 2$, we define the function

$$v(x) = \alpha\lambda^{1/2}(lnr - lnR) + \beta\lambda^{5/2}(R^2 - r^2),$$

for $x \in A_R \cap M$. By the same computation as above, we can show the Lemma holds also.

Q.E.D.

Note that from L^∞ estimates of eigenfunctions, we know $\chi_\lambda f(x)$ satisfies the conditions of above Lemma, since

$$\begin{aligned} |\chi_\lambda f(x)|^2 &= |\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(f)|^2 \\ &\leq (\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x)^2)^2 ||\chi_\lambda f||_{L^2}^2 \\ &\leq C\lambda^{n-1} ||\chi_\lambda f||_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} \Delta_g \chi_\lambda f(x) &= -\sum_{\lambda_j \in [\lambda, \lambda+1)} \lambda_j^2 e_j(f) \\ &\geq -(\lambda+1)^2 (\sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(x)^2)^{\frac{1}{2}} ||\chi_\lambda f||_{L^2} \\ &\geq -C\lambda^{(n+3)/2} ||\chi_\lambda f||_{L^2} \end{aligned}$$

Apply that above Lemma to $\chi_\lambda f$, we have the gradient estimate on the boundary points.

Lemma 2.3.3 *For the Riemannian manifold* (M, g) *with boundary* $\partial M \in C^2$*, we have the gradient estimate*

$$|\nabla \chi_\lambda f(x)| \leq C\lambda^{(n+1)/2} ||\chi_\lambda f||_2, \quad \forall\, x \in \partial M.$$

Now we deal with the gradient estimate in the λ^{-1} strip of the boundary for both Dirichlet Laplacian and Neumann Laplacian. Here we also get the gradient estimates on boundary for Neumann Laplacian.

Lemma 2.3.4 *For the Riemannian manifold (M,g) with boundary, we have the gradient estimate*

$$|\nabla\chi_\lambda f(x)| \le C_\epsilon \lambda^{(n+1)/2}||\chi_\lambda f||_2, \quad for\ 0\le d(x)\le \epsilon(\lambda+1)^{-1}.$$

where ϵ is the same as in Lemma 2.1 and we will determine its value here.

Proof. We shall apply the maximum principle to $\sum_{\lambda_l\in[\lambda,\lambda+1)}|\nabla e_j(x)|^2$ on the λ^{-1} boundary strip as in [33], where dealt with the L^∞ estimates for $\chi_\lambda f$ on the λ^{-1} boundary strip. As Lemma 2.2, we also use the geodesic coordinates with respect to the boundary. First we estimate

$$\begin{aligned}\frac{1}{2}\Delta_g \sum_{\lambda_l\in[\lambda,\lambda+1)}|\nabla e_j(x)|^2 &= \sum_{\lambda_l\in[\lambda,\lambda+1)}[\sum_{i,j,k=1}^n g_{ij}(x)\frac{\partial^2 e_l(x)}{\partial x_i\partial x_k}\frac{\partial^2 e_l(x)}{\partial x_j\partial x_k}\\ &\quad +(\nabla e_l(x),\nabla(\Delta e_l(x))) + Ric(\nabla e_l(x),\nabla e_l(x))]\end{aligned}$$

Since $(g_{ij}(x))$ is metric tensor and M is a compact manifolds, we have a constant $\theta>0$ such that

$$g_{ij}(x)y_iy_j \ge \theta|y|^2$$

holds for all $y=(y_1,\cdots,y_n)\in\mathbf{R}^n$ and all $x\in M$. And $Ric > -B$ for some positive constant B in whole M. Then for large λ, we have $B\le 2\lambda+3$. Hence we have

$$\begin{aligned}\frac{1}{2}\Delta_g \sum_{\lambda_l\in[\lambda,\lambda+1)}|\nabla e_j(x)|^2 &\ge -\sum_{\lambda_l\in[\lambda,\lambda+1)}[\lambda_l^2|\nabla e_j(x)|^2 + Ric(\nabla e_l(x),\nabla e_l(x))]\\ &\ge -(\lambda+2)^2\sum_{\lambda_l\in[\lambda,\lambda+1)}|\nabla e_j(x)|^2\end{aligned}$$

Now we define a function $w(x) = 1-a(\lambda+1)^2x_n^2$ for the strip $\{x\in M|\ 0\le x_n\le \epsilon(\lambda+1)^{-1}\}$, and the constants a and ϵ will be determined below. We have

$$\frac{1}{2} \leq 1 - a\epsilon^2 \leq w(x) \leq 1, \tag{2.7}$$

and following the computation in proof of Proposition 2.2 in [33], we have

$$\Delta w(x) = -2a(\lambda+1)^2 - 2ab_n(x)x_n(\lambda+1)^2 \leq -a(\lambda+1)^2,$$

for all point in the strip, here assuming that λ is large enough so that $|b_n(x)x_n| \leq 1/2$. Define $h(x) = \sum_{\lambda_l \in [\lambda,\lambda+1)} |\nabla e_j(x)|^2/w(x)$, we have

$$\begin{aligned} \Delta_g \sum_{\lambda_l \in [\lambda,\lambda+1)} |\nabla e_j(x)|^2 &= w(x)\Delta_g h(x) + h(x)\Delta w(x) + 2(\nabla h(x), \nabla w(x)) \\ &\geq -2(\lambda+2)^2 \sum_{\lambda_l \in [\lambda,\lambda+1)} |\nabla e_j(x)|^2 \\ &= -2(\lambda+2)^2 w(x)h(x). \end{aligned}$$

Divide $w(x)$ both sides, and apply the estimate of $\Delta_g w(x)$, we have

$$\Delta h(x) + 2(\nabla h(x), \frac{\nabla w(x)}{w(x)}) + (4-a)(\lambda+1)^2 h(x) \geq 0.$$

If we let $a > 4$, $h(x)$ achieves its maximum on $\partial\{x \in M|\ 0 \leq x_n \leq \epsilon(\lambda+1)^{-1}\}$. Now we need only to show that $h(x)$ can not achieve its non-zero maximum on ∂M.

Assume that $h(x)$ achieve its non-zero maximum at $x_0 \in \partial M$. Since we have $\nabla w(x) = 0$ for all $x \in \partial M$, then at the point $x_0 \in \partial M$ where $h(x)$ achieve its maximum, we have

$$\Delta h(x_0) + (4-a)(\lambda+1)^2 h(x_0) \geq 0.$$

Since $\partial_k^2 h(x_0) \leq 0$ for $k = 1, 2, \cdots, n-1$, and $h(x_0) > 0$, we have $\partial_n^2 h(x_0) > 0$. On the other hand, since $w(x) = 1$, $\nabla w(x) = 0$ and $\partial_n e_j(x) = 0$ for all $x \in \partial M$, we have

$$\begin{aligned}
\partial_n h(x_0) &= \partial_n [\sum_{\lambda_l \in [\lambda, \lambda+1)} |\nabla e_j(x)|^2 / w(x)]|_{x=x_0} \\
&= \sum_{\lambda_l \in [\lambda, \lambda+1)} [(2w(x_0) \sum_{k=1}^{n-1} (\partial_k e_j(x_0), \partial_k \partial_n e_j(x_0)) \\
&+ 2w(x_0)(\partial_n e_j(x_0), \partial_k \partial_n e_j(x_0)) - |\nabla e_j(x)|^2 \partial_n w(x_0))/w(x_0)^2] \\
&= 0
\end{aligned}$$

hence from $h(x)$ achieving its maximum at $x_0 \in \partial M$, we have $\partial_n^2 h(x_0) \leq 0$, which is contradicted with above.

Hence we have

$$\sup_{\{x \in M |\, 0 \leq x_n \leq \epsilon(\lambda+1)^{-1}\}} \sum_{\lambda_l \in [\lambda, \lambda+1)} |\nabla e_j(x)|^2 \leq C\lambda^{n+1}.$$

Finally we determine the constant a and ϵ. From proof, we need $a > 4$ and $a\epsilon^2 \leq 1/2$, which is easy to satisfy, for example, we may let $a = 8$ and $\epsilon = 1/4$.

Q.E.D.

Combine above Lemmas, we get the gradient estimates for eigenfunctions of both Dirichlet Laplacian and Neumann Laplacian on compact Riemannian manifolds with boundary.

Since here our proof only involve the L^∞ estimate of the eigenfunctions and didn't use any geometric property. Notice that for compact Riemannian manifolds without boundary, by Lemma 2.3.1 and the L^∞ estimates on the unit spectral projection operators χ_λ for a second-order elliptic differential operator on compact manifolds without boundary Sogge [32], we have the following gradient estimates

Theorem 2.3.1 *Fix a compact Riemannian manifold* (M, g) *of dimension* $n \geq 2$ *without boundary, there is a uniform constant* C *so that*

$$||\nabla \chi_\lambda f||_\infty \leq C\lambda^{(n+1)/2}||f||_2, \quad \lambda \geq 1.$$

And the bounds are uniform if there is a uniformly bound on the norm of $\mathrm{tr}(g^{ij}(x))$ *for a class metrics* g *on* M*.*

For Riemannian manifolds without boundary, in [34], the authors proved that for generic metrics on any manifold one has the bounds $||e_j||_{L^\infty(M)} = o(\lambda_j^{(n-1)/2})$ for L^2 normalized eigenfunctions. As Lemma 2.1, we can show the L^∞ gradient estimates $||\nabla e_j||_\infty = o(\lambda_j^{(n+1)/2})$ for generic metrics on any manifold. For Laplace-Beltrami operator Δ_g, there are eigenvalues $\{-\lambda_j^2\}$, where $0 \leq \lambda_0^2 \leq \lambda_1^2 \leq \cdots \to \infty$ are counted with multiplicity. Let $\{e_j(x)\}$ be an associated orthogonal basis of L^2 normalized eigenfunctions. If λ^2 is in the spectrum of $-\Delta_g$, let $V_\lambda = \{u \mid \Delta_g u = -\lambda^2 u\}$ denote the corresponding eigenspace. We define the eigenfunction growth rate in term of

$$L^\infty(\lambda, g) = \sup_{u \in V_\lambda; ||u||_{L^2}=1} ||u||_{L^\infty}.$$

and the gradient growth rate in term of

$$L^\infty(\nabla, \lambda, g) = \sup_{u \in V_\lambda; ||u||_{L^2}=1} ||\nabla u||_{L^\infty}.$$

In [34], Sogge and Zelditch proved the following results

$$L^\infty(\lambda, g) = o(\lambda_j^{(n-1)/2})$$

for a generic metric on any manifold.

Here we apply our Lemma 2.1, we have the gradient estimates

Theorem 2.3.2 *$L^\infty(\nabla, \lambda, g) = o(\lambda_j^{(n+1)/2})$ for a generic metric on any manifold. And the bounds are uniform if there is a uniformly bound on the norm of $\mathrm{tr}(g^{ij}(x))$ for (M, g).*

Proof. here the manifold has no boundary, we can apply our Lemma 2.1 to any point in M. From Theorem 1.4 in [34], we have $L^\infty(\lambda, g) = o(\lambda_j^{(n-1)/2})$ for a generic metric on any manifold. Fix that metric on the manifold and a L^2 normalized eigenfunction $u(x)$, apply our Lemma 2.1 to $u(x)$ at each point $x_0 \in M$, we have

$$|\nabla u(x_0)| \leq \frac{\alpha n A}{d} + \beta d N.$$

where α and β are constants depending on the norm of $\mathrm{tr}(g^{ij}(x))$ at M only, and $A = \sup_{\partial Q} |u| = o(\lambda_j^{(n-1)/2})$, and $N = \sup_Q |\lambda^2 u| = (\lambda_j^{(n+3)/2})$, where the cube

$$Q = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n |\ |x_i| < d,\ i = 1, \cdots, n\} \subset M,$$

we choose $d = (\lambda + 1)^{-1}/\sqrt{n}$.

Hence we have

$$|\nabla u(x_0)| = o(\lambda_j^{(n+1)/2})$$

holds for all L^2 normalized eigenfunction $u(x) \in V_\lambda$, furthermore, the bounds are uniform when those metrics of (M, g) have a uniformly bound on the norm of $\mathrm{tr}(g^{ij}(x))$ from the proof. Hence we have our Theorem. Q.E.D.

2.4 Upper bounds for normal derivatives

In this section, we study the upper bounds for normal derivatives of the unit spectral projection operators of Dirichlet Laplacian on $L^2(M)$, where (M, g) is a compact Riemannian manifold of dimension $n \geq 2$ with C^2 boundary. Let $0 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \cdots$ denote the eigenvalues for the Dirichlet Laplacian, and $\{e_j(x)\}$ be an associated real orthogonal normalized basis in $L^2(M)$, and define $e_j(f)(x)$ and the unit band spectral projection operators, $\chi_\lambda f$, for function $f \in L^2(M)$, and we also denote

$$(\chi_\lambda^{(1)} f) = -\Delta_g \chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1)} \lambda_j^2 e_j(f).$$

Following the idea using in the proof of the Rellich-type estimates in [12], we have the following Rellich-type Lemma for $\chi_\lambda f$.

Lemma 2.4.1 *Let $\chi_\lambda f$ be the operators on $f \in L^2(M)$ defined as above, then for any differential operator A,*

$$\begin{aligned}\int_{\partial M} \frac{\partial(\chi_\lambda f)}{\partial \nu} A((\chi_\lambda f)) d\sigma &= \int_M ((\chi_\lambda f), [-\Delta_g, A]((\chi_\lambda f)))dx \\ &+ \int_M ((\chi_\lambda f), A((\chi_\lambda^{(1)} f)))dx - \int_M ((\chi_\lambda^{(1)} f), A((\chi_\lambda f)))dx, \qquad (2.8)\end{aligned}$$

where Δ_g is the Laplace-Beltrami operator associated to the Riemannian metric g, ν is the outward normal derivative on the boundary ∂M, $d\sigma$ is the area element on the boundary ∂M, and dx is the volume element on M.

Proof. Since $\chi_\lambda f$ vanishes at the boundary ∂M, we apply Green's formula to get

$$\begin{aligned}
& \int_{\partial M} \frac{\partial(\chi_\lambda f)}{\partial \nu} A((\chi_\lambda f)) d\sigma \\
= & \int_M ((\chi_\lambda f), -\Delta_g(A\chi_\lambda f)))dx - \int_M (-\Delta_g((\chi_\lambda f)), A((\chi_\lambda f)))dx \\
= & \int_M ((\chi_\lambda f), -\Delta_g(A(\chi_\lambda f)))dx - \int_M (\chi_\lambda f, A((\chi_\lambda f)))dx \\
= & \int_M ((\chi_\lambda f), [-\Delta_g, A]((\chi_\lambda f)))dx \\
& + \int_M ((\chi_\lambda f), A((\chi_\lambda^{(1)} f)))dx - \int_M ((\chi_\lambda^{(1)} f), A((\chi_\lambda f)))dx
\end{aligned}$$

Here we use the fact that $e_j(f)$ satisfying the following Dirichlet boundary problem

$$(\Delta_g + \lambda_j^2)u(x) = 0, \quad x \in M; \quad u(x) = 0, \quad x \in \partial M.$$

Q.E.D.

To prove an upper bound for the $L^2(\partial M)$ norm of $\partial_\nu(\chi_\lambda f)$ on the boundary ∂M, we shall choose an operator A so that the left hand side of (2.8) is a positive form in $\partial_\nu(\chi_\lambda f)$. As before, we use the geodesic coordinates with respect to the boundary. We can find a small constant $c > 0$ so that the map $(x', x_n) \in \partial M \times [0, c) \to M$, sending (x', x_n) to the endpoint, x, of the geodesic of length x_n which starts a $x' \in \partial M$ and is perpendicular to ∂M is a local diffeomorphism. In this local coordinates $x = (x', x_n) = (x_1, \cdots, x_{n-1}, x_n)$, the metric has the form

$$\sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j = (dx_n)^2 + \sum_{i,j=1}^{n-1} g_{ij}(x') dx^i dx^j,$$

and the Laplacian can be written as

$$\begin{aligned}\Delta_g &= \sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} \\ &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij}(x)\frac{\partial}{\partial x_j},\end{aligned}$$

Where $(g^{ij}(x))_{1\le i,j\le n}$ is the inverse matrix of $(g_{ij}(x))_{1\le i,j\le n}$, and $g^{nn} = 1$, and $g^{nk} = g^{kn} = 0$ for $k \neq n$. Also the $b_i(x)$ are C^∞ and real valued.

We use the above local coordinates $x = (x', x_n)$, then we choose the differential operator $A = \eta(x_n)\partial_{x_n}$ on this local coordinates for each $\chi_\lambda f$, where $\eta \in C_c^\infty(\mathbf{R})$ is identically 1 for x_n close to 0, and vanishes for $x_n \ge \delta$, satisfying $|\eta'(x_n)| \le C/\delta$ and $|\eta''(x_n)| \le C/\delta^2$. Here $0 < \delta < c$, where c is the constant above associated with the local coordinates, since the manifold is compact, the constant c has uniformly lower bound respect to the local coordinates. We show the following result:

Lemma 2.4.2 *For above differential operator A, we have*

$$||\frac{\partial(\chi_\lambda f)}{\partial\nu}||^2_{L^2(\partial M)} \;\le\; C\delta^{-1}\lambda^2||\chi_\lambda f||^2_{L^2(M)} + C\lambda^3||\chi_\lambda f||^2_{L^2(M)},$$

holds as $\lambda \to \infty$ for some constant C independent of $||f||_{L^2}$ and λ.

Proof. From Lemma 4.1, we have the identity (2.8). Now plug the above differential operator A in (2.8). The left hand side of (2.8) is precisely the square of the L^2 norm of $\partial_\nu(\chi_\lambda f)$. Here we estimate the right hand side of (2.8) in two cases:

Case I: Estimate on the first term of the right hand side of (2.8).

In the local coordinates, we have the representation of $[-\Delta_g, A]$ as

$$\begin{aligned}
[-\Delta_g, A] &= -(\partial_n^2 + \sum_{i,j=1}^{n-1} g^{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i)(\eta(x_n)\partial_n) \\
&\quad +\eta(x_n)\partial_n(\partial_n^2 + \sum_{i,j=1}^{n-1} g^{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i) \\
&= -2\eta'(x_n)\partial_n^2 - \eta''(x_n)\partial_n - b_n(x)\eta'(x_n)\partial_n + \sum_{i=1}^{n}(\partial_n b_i(x))\partial_i \\
&= -2\eta'(x_n)\partial_n^2 - (\eta''(x_n) + b_n(x)\eta'(x_n) - \partial_n b_n)\partial_n + \sum_{i=1}^{n-1}(\partial_n b_i)\partial_i
\end{aligned}$$

Denote $C = \max_M\{|b_n(x)|, |\partial_n b_i(x)|\}$. Plus the above representation of $[-\Delta_g, A]$ in the first term of the right hand side of (2.8), and integral by parts for first term, we have

$$\begin{aligned}
& |\int_M \{2\eta'(\partial_n(\chi_\lambda f))^2 - (\eta'' + b_n\eta' - \partial_n b_n)(\chi_\lambda f)\partial_n(\chi_\lambda f) \\
& + \sum_{i=1}^{n-1}(\partial_n b_i)(\chi_\lambda f)\partial_i(\chi_\lambda f)\}dx| \\
\leq\ & C\int_M \{|\eta'|(\partial_n(\chi_\lambda f))^2 + (|\eta''| + |\eta'| + 1)|(\chi_\lambda f)||\partial_n(\chi_\lambda f)| \\
& + \sum_{i=1}^{n-1} |(\chi_\lambda f)||\partial_i(\chi_\lambda f)|\}dx \\
\leq\ & C\int_M \{\delta_N^{-1}(\partial_n(\chi_\lambda f))^2 + (\delta^{-2} + \delta^{-1} + 1)|(\chi_\lambda f)||\partial_n(\chi_\lambda f)| \\
& + \sum_{i=1}^{n-1} |(\chi_\lambda f)||\partial_i(\chi_\lambda f)|\}dx \\
\leq\ & C\int_M \{(\delta^{-1} + \delta^{-2}\epsilon)(\partial_n(\chi_\lambda f))^2 + \sum_{i=1}^{n-1}|\partial_i(\chi_\lambda f)|^2 + (\epsilon^{-1}+1)|(\chi_\lambda f)|^2\}dx \\
\leq\ & C\int_M \{(\delta^{-1} + \delta^{-2}\epsilon + 1)(\nabla(\chi_\lambda f))^2 + (\epsilon^{-1}+1)|(\chi_\lambda f)|^2\}dx
\end{aligned}$$

Here we use the geometric mean inequality and $\epsilon > 0$ is an arbitrary constant.

On the other hand, we have the following estimates

$$\begin{aligned}||\nabla \chi_\lambda f||^2_{L^2(M)} &= \int_M ((\chi_\lambda f), -\Delta_g((\chi_\lambda f)))dx \\ &= \sum_{\lambda_j \in [\lambda, \lambda+1)} \lambda_j^2 \int_M (((\chi_\lambda f)), e_j(f))dx \\ &\leq (\lambda+1)^2 \sum_{\lambda_j \in [\lambda, \lambda+1)} ||e_j(f)||^2_{L^2(M)} \\ &= (\lambda+1)^2 ||\chi_\lambda f||^2_{L^2(M)}\end{aligned}$$

Hence we know that the first term is bounded by

$$C\{(\delta^{-1} + \delta^{-2}\epsilon)(\lambda+1)^2 + \epsilon^{-1}\}||f||^2_{L^2(M)} \leq C\delta^{-1}(\lambda+1)^2||f||^2_{L^2(M)},$$

Here we let $\epsilon = \delta$.

Case II: Estimate on the other two terms of the right hand side of (2.8).

For our differential operator $A = \eta(x_n)\partial_n$, define two index sets as

$$\begin{aligned}I^+ &= \{(j,k) \ : \ \int_M (e_j(f), A(e_k(f)))dx > 0\}, \\ I^- &= \{(j,k) \ : \ \int_M (e_j(f), A(e_k(f)))dx < 0\}.\end{aligned}$$

we have

$$\begin{aligned}&\int_M (\chi_\lambda f, A(\chi_\lambda^{(1)} f))dx - \int_M (\chi_\lambda^{(1)} f, A(\chi_\lambda f))dx \\ = &\sum_{\lambda_j \in [\lambda, \lambda+1)} \sum_{\lambda_k \in [\lambda, \lambda+1)} (\lambda_j^2 - \lambda_k^2) \int_M (e_j(f), A(e_k(f)))dx \\ \leq &\sum_{\{\lambda_j, \lambda_k \in [\lambda, \lambda+1); (j,k) \in I^+\}} (\lambda_j^2 - \lambda^2) \int_M (e_j(f), A(e_k(f)))dx \\ &+ \sum_{\{\lambda_j, \lambda_k \in [\lambda, \lambda+1); (j,k) \in I^-\}} (\lambda_j^2 - (\lambda+1)^2) \int_M (e_j(f), A(e_k(f)))dx \\ = &\sum_{\lambda_j \in [\lambda, \lambda+1)} \sum_{\lambda_k \in [\lambda, \lambda+1)} (\lambda_j^2 - \frac{\lambda^2 + (\lambda+1)^2}{2}) \int_M (e_j(f), A(e_k(f)))dx \\ &+(\lambda + \frac{1}{2})[\sum_{(j,k) \in I^+} - \sum_{(j,k) \in I^-}] \int_M (e_j(f), A(e_k(f)))dx\end{aligned}$$

by integrating by parts, we have

$$\begin{aligned}
& \int_M (\chi_\lambda f, A(\chi_\lambda^{(1)} f))dx - \int_M (\chi_\lambda^{(1)} f, A(\chi_\lambda f))dx \\
= & -\int_M (\chi_\lambda^{(1)} f, \partial_n(\eta_N \chi_\lambda f))dx - \int_M (\chi_\lambda^{(1)} f, \eta\partial_n(\chi_\lambda f))dx \\
= & -\int_M \eta'(\chi_\lambda^{(1)} f, \chi_\lambda f)dx - 2\int_M ((\chi_\lambda^{(1)} f), \eta\partial_n(\chi_\lambda f))dx
\end{aligned}$$

Notice that $||\chi_\lambda^{(1)} f||_{L^2} \leq (\lambda+1)^2||f||_{L^2}$ and $||\nabla(\chi_\lambda^{(1)} f))||_{L^2} \leq (\lambda+1)^3||f||_{L^2}$, then there are the following estimates

$$\begin{aligned}
|\int_M \eta'(\chi_\lambda^{(1)} f, \chi_\lambda f)dx| & \leq C\delta^{-1}\int_M |\chi_\lambda^{(1)} f| \cdot |\chi_\lambda f|dx \\
& \leq C\delta^{-1}||(\chi_\lambda^{(1)} f)||_{L^2}||(\chi_\lambda f)||_{L^2} \\
& \leq C\delta^{-1}(\lambda+1)^2||\chi_\lambda f||_{L^2}^2;
\end{aligned}$$

$$\begin{aligned}
|\int_M (\chi_\lambda^{(1)} f, \eta\partial_n(\chi_\lambda f))dx| & \leq \int_M |\eta\partial_n(\chi_\lambda f)| \cdot |\chi_\lambda f|dx \\
& \leq ||\chi_\lambda^{(1)} f||_{L^2}||\eta\partial_n(\chi_\lambda f)||_{L^2} \\
& \leq ||\chi_\lambda^{(1)} f||_{L^2}||\nabla(\chi_\lambda f)||_{L^2} \\
& \leq (\lambda+1)^3||\chi_\lambda f||_{L^2}^2.
\end{aligned}$$

Combine above two Cases, we have the estimates as

$$\begin{aligned}
||\frac{\partial(\chi_\lambda f)}{\partial\nu}||_{L^2(\partial M)}^2 & \leq C\delta^{-1}(\lambda+1)^2||f||_{L^2}^2 + ||\chi_\lambda^{(1)} f||_{L^2}||\partial_n(\chi_\lambda f)||_{L^2} \\
& \leq C\delta^{-1}(\lambda+1)^2||\chi_\lambda f||_{L^2}^2 + (\lambda+1)^3||\chi_\lambda f||_{L^2}^2. \quad (2.9)
\end{aligned}$$

Q.E.D.

Now let $\delta_0 = \lambda^{-1}$ in Lemma 2.4.2, we prove Theorem 2.1.2.

2.5 Further Study

In this section, we discuss some further studies related to eigenfunction estimates. First I would like to study the estimates of Laplacian eigenfunctions and spectrum for domains in $\mathbf{R}^n$ and compact manifolds with respect to some global geometry property, such as curvatures and geodesic flows, which relates to the inverse spectral problems (see [34], [39] and [43]). I plan to study whether the results in [34] and [39] are true for manifolds with boundary. Furthermore, I'd like to study the wave kernel of Laplacian on compact manifolds with boundary and with singularity. Second I would like to study the bilinear and multilinear eigenfunction estimates for Laplacian spectral projectors on manifolds with boundary, and their application to nonlinear Schrödinger equations and nonlinear wave equations on manifolds. In [6] and [7], these estimates were proved for manifolds without boundary and were applied to study nonlinear Schrödinger equations. I'd like to study these problems for manifolds with boundary.

Chapter 3

Hörmander Multiplier Theorem

3.1 Introduction and Results

In this chapter we prove the Hörmander multiplier Theorem for smooth compact Riemannian manifolds with boundary. Given a bounded function $m(\lambda) \in L^\infty(\mathbf{R})$ we can define operators, $m(P)$, by

$$m(P)f = \sum_{j=1}^{\infty} m(\lambda_j) e_j(f) \tag{3.1}$$

such operators are always bounded on $L^2(M)$. However, if one considers any other space $L^p(M)$, it is known that some smoothness assumption on the function $m(\lambda)$ are needed to ensure the boundedness of

$$m(P) : L^p(M) \to L^p(M). \tag{3.2}$$

When $m(\lambda)$ is C^∞ and, moreover, in the symbol class S^0, i.e.,

$$|(\frac{d}{d\lambda})^\alpha m(\lambda)| \leq C_\alpha (1 + |\lambda|)^{-\alpha}, \qquad \alpha = 0, 1, 2, \cdots$$

It has been known for some time that (3.2) holds for all $1 < p < \infty$ on compact manifolds (see [37]). Many authors studied the Hörmander multiplier Theorem under different setting. Specifically, one assume the following regularity assumption: suppose that $m \in L^\infty(\mathbf{R})$, let $L^2_s(\mathbf{R})$ denote the usual Sobolev space and fix $\beta \in C_0^\infty((1/2, 2))$ satisfying $\sum_{-\infty}^{\infty} \beta(2^j t) = 1,\ t > 0$, and suppose also that

$$\sup_{\lambda>0} \lambda^{-1+s} ||\beta(\cdot/\lambda)m(\cdot)||^2_{L^2_s} = \sup_{\lambda>0} ||\beta(\cdot)m(\lambda\cdot)||^2_{L^2_s} < \infty, \tag{3.3}$$

where real number $s > n/2$.

Hörmander [14] first proved the Hörmander multiplier Theorem for $\mathbf{R}^n$ under the assumption (3.3), using the Calderón-Zygmund decomposition and the estimates on the kernel of the multiplier. Stein [35] and Stein and Weiss [36] studied the Hörmander multiplier Theorem for multiple Fourier series, which can be regarded as the case for flat torus T^n. Seeger and Sogge [27] and Sogge [32] proved the boundedness of $m(P)$ on $L^p(M)$ for compact manifolds without boundary under the assumption (3.3), by studying a local multiplier Theorem for multipliers supported in dyadic intervals first, and using the estimates for the unit band spectral projection operators χ_λ and the local multiplier Theorem to obtain the Hörmander multiplier Theorem for compact manifolds without boundary.

Using the L^∞ estimates on $\chi_\lambda f$ and $\nabla\chi_\lambda f$ and the ideas in [27], [31]- [33], we have the following Hömander multiplier Theorem for compact manifolds with boundary:

Theorem 3.1.1 *Let $m \in L^\infty(\mathbf{R})$ satisfy (3.3), then there are constants C_p*

such that

$$||m(P)f||_{L^p(M)} \le C_p||f||_{L^p(M)}, \quad 1 < p < \infty. \tag{3.4}$$

In section 3.2, we give the outline of the proof of Theorem 3.1.1 and reduce the Theorem to show the weak-type $(1,1)$ estimates on $m(P)$ by the Marcinkiewicz interpolation Theorem. In section 3.3, we prove the strong $(1,1)$ estimates for Remainder $r(P)$. In section 3.4, we prove the weak-type $(1,1)$ estimates on the main term $\tilde{m}(P)$.

3.2 Outline of Proof

In this section, we give the outline of the proof of Hörmander Multiplier Theorem. Since the complex conjugate of m satisfies the same hypotheses (3.4), we need only to prove Theorem 3.1.1 for exponents $1 < p \le 2$. This will allow us to exploit orthogonality and also reduce Theorem to show that $m(P)$ is weak-type $(1,1)$,

$$\mu\{x : |m(P)f(x)| > \alpha\} \le \alpha^{-1}||f||_{L^1}. \tag{3.5}$$

Here $\mu(E)$ denotes the dx measure of $E \subset M$. Since $m(P)$ is bounded on $L^2(M)$, (3.5) implies Theorem 3.1.1 by the Marcinkiewicz interpolation Theorem.

To study the weak-type $(1,1)$ estimate of the operator $m(P)$, we need to relate the operator $m(P)$ to wave equation. Since the all eigenvalues of Dirichlet Laplacian are positive, we may assume $m(t)$ is an even function on

$\mathbf{R}$, otherwise we need only replace $m(t)$ by $\bar{m}(t)$, where the even function $\bar{m}(t) = m(t)$ for $t > 0$. Then we have

$$\begin{aligned} m(P)f(x) &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{m}(t)e^{itP}f(x)dt \\ &= \frac{1}{\pi}\int_{\mathbf{R}_+} \hat{m}(t)\cos(tP)f(x)dt \\ &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{m}(t)\sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)\int_M e_{\lambda_k}(y)f(y)dydt \\ &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{m}(t)\{\int_M \sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)e_{\lambda_k}(y)f(y)dy\}dt \end{aligned}$$

Here $P = \sqrt{-\Delta}$, and

$$u(t,x) = \cos(tP)f(x) = \int_M \sum_{k\geq 1}\cos(t\lambda_k)e_{\lambda_k}(x)e_{\lambda_k}(y)f(y)dy$$

is the cosine transform of f. Thus, it is the solution of the following Dirichlet-Cauchy problem:

$$(\frac{\partial^2}{\partial t^2} - \Delta_g)u(t,x) = 0, \quad u(0,x) = f(x),\ u_t(0,x) = 0.$$

We shall use the finite propagation speed for solutions to the wave equation. Specifically, if f is supported inside a geodesic ball $B(x_0, R)$ centered at x_0 with radius R, then $x \to \cos(tP)f$ vanishes outside of $B(x_0, 2R)$ if $0 \leq t \leq R$. We will use this property in section 3.4 of proof to show the estimates on the terms with cancellation property in Calderón-Zygmund decomposition of L^1 function f.

The proof of the weak-type (1,1) estimates of $m(P)$ will involve a splitting of $m(P)$ into two pieces: a main piece to which the Euclidean arguments can apply, plus a remainder which can be show to satisfy much better bounds than what are needed by using the estimates for the unit spectral projection operators, which is the same idea as proof of Hömander multiplier Theorem for compact manifolds without boundary in [27] and [32]. Specifically, let $\rho \in C_0^\infty(\mathbf{R})$ define as

$$\rho(t) = 1, \quad for\ |t| \le \frac{\epsilon}{2}, \qquad \rho(t) = 0, \quad for\ |t| \ge \epsilon. \tag{3.6}$$

where $\epsilon > 0$ is a given small constant related to the manifold, which will be specified later. Write $m(P) = \tilde{m}(P) + r(P)$, where

$$\begin{aligned} \tilde{m}(P) &= (m * \check{\rho})(P) = \frac{1}{2\pi}\int e^{itP}\rho(t)\hat{m}(t)dt \\ r(P) &= (m * (1-\rho)\check{})(P) = \frac{1}{2\pi}\int e^{itP}(1-\rho(t))\hat{m}(t)dt \end{aligned}$$

To estimate the main term and remainder, we define for $\lambda = 2^j$, $j = 1, 2, \cdots$,

$$m_\lambda(\tau) = \beta(\frac{\tau}{\lambda})m(\tau). \tag{3.7}$$

The proof of the weak-type (1,1) estimates of $m(P)$ will follow the following two parts:

Part 1: Estimate on the remainder

$$||r(P)f||_{L^1} \le C||f||_{L^1}.$$

Part 2: weak-type $(1, 1)$ estimate on the main term

$$\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \le \alpha^{-1}||f||_{L^1}.$$

We will show these two parts in following two sections.

3.3 Strong (1, 1) Estimates for Remainder r(P)

In this section we will prove:

Part 1: Estimate on the remainder

$$||r(P)f||_{L^1} \leq C||f||_{L^1}.$$

We first show

$$||r(P)f||_{L^2} \leq C||f||_{L^1}.$$

Here we follow the first part in proof of Theorem 5.3.1 in [32] to estimate the remainder. Define

$$r_\lambda(P) = (m_\lambda * (1-\rho)\check{)}(P) = \frac{1}{2\pi}\int e^{itP}(1-\rho(t))\hat{m}_\lambda(t)dt$$

Notice that $r_0(P) = r(P) - \sum_{j\geq 1} r_{2^j}(P)$ is a bounded and rapidly decreasing function of P. Hence $r_0(P)$ is bounded from L^1 to any L^p space. We need only to show

$$||r_\lambda(P)f||_{L^2} \leq C\lambda^{n/2-s}||f||_{L^1}, \quad \lambda = 2^j, j = 1, 2, \cdots.$$

Using the L^∞ asymptotic estimate for the unit spectral projection operator χ_k on compact manifold (M, g) with boundary, see [33], we have

$$\begin{aligned}
||r_\lambda(P)f||_{L^2}^2 &\leq \sum_{k=1}^{\infty} ||r_\lambda(P)\chi_k f||_{L^2}^2 \\
&\leq C\sum_{k=1}^{\infty} \sup_{\tau\in[k,k+1]} |r_\lambda(\tau)|^2(1+k)^{n-1}||f||_{L^1}^2.
\end{aligned}$$

Hence we need only to show

$$\sum_{k=1}^{\infty} \sup_{\tau\in[k,k+1]} |r_\lambda(\tau)|^2(1+k)^{n-1} \le C\lambda^{n-2s}$$

Notice since $m_\lambda(\tau) = 0$, for $\tau \notin [\lambda/2, 2\lambda]$, we have

$$\begin{aligned} \tilde{m}_\lambda(\tau) &= O((1+|\tau|+|\lambda|)^{-N}) \\ r_\lambda(\tau) &= O((1+|\tau|+|\lambda|)^{-N}) \end{aligned}$$

for any N when $\tau \notin [\lambda/4, 4\lambda]$. Hence we need only to show

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau\in[k,k+1]} |r_\lambda(\tau)|^2(1+k)^{n-1} \le C\lambda^{n-2s}$$

that is

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau\in[k,k+1]} |r_\lambda(\tau)|^2 \le C\lambda^{1-2s}$$

Using the fundamental theorem of calculus and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \sum_{k=\lambda/4}^{4\lambda} \sup_{\tau\in[k,k+1]} |r_\lambda(\tau)|^2 \\ \le\ & C\left(\int_{\mathbf{R}} |r_\lambda(\tau)|^2 d\tau + \int_{\mathbf{R}} |r'_\lambda(\tau)|^2 d\tau\right) \\ =\ & C\left(\int_{\mathbf{R}} |\hat{m}_\lambda(t)(1-\rho(t))|^2 dt + \int_{\mathbf{R}} |t\hat{m}_\lambda(t)(1-\rho(t))|^2 dt\right) \end{aligned}$$

Recall that $\rho(t) = 1$, for $|t| \le \frac{\epsilon}{2}$, by a change variables shows that this is dominated by

$$\begin{aligned} & \lambda^{-1-2s} \int_{\mathbf{R}} |t^s \hat{m}_\lambda(t/\lambda)|^2 dt \\ =\ & \lambda^{-1-2s} ||\lambda\beta(\cdot)m(\lambda\cdot)||^2_{L^2_s} \\ =\ & \lambda^{1-2s} ||\beta(\cdot)m(\lambda\cdot)||^2_{L^2_s} \\ \le\ & C\lambda^{1-2s} \end{aligned}$$

Here the first equality comes from a change variables, the second equality comes from the definition of Sobolev norm of $L_s^2(M)$ and the third inequality comes from our condition (3.4).

Hence we have the estimate for the remainder

$$||r(P)f||_{L^2} \leq C||f||_{L^1}.$$

And since our manifold is compact, we have

$$||r(P)f||_{L^1} \leq Vol(M)^{1/2}||r(P)f||_{L^2} \leq C||f||_{L^1}.$$

that is, we have the strong-type $(1, 1)$ estimate on the remainder $r(P)$.

3.4 Weak-type $(1, 1)$ Estimates on Main Term $\tilde{m}(P)$

In this section we will prove:

Part 2: weak-type $(1, 1)$ estimate on the main term

$$\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \leq \alpha^{-1}||f||_{L^1}.$$

In [27] and [32], for compact manifold without boundary, the above estimate on $\tilde{m}(P)$ could be estimated by computing its kernel explicitly via the Hadamard parametrix and then estimating the resulting integral operator using straightforward adaptations of the arguments for the Euclidean case. But now for manifolds with boundary, this approach does not seem to work since the known parametrix for the wave equation do not seem strong enough unless one assumes that the boundary is geodesically concave. Here we shall get around this fact by following the ideas in [33], where deals with the Riesz means on compact manifolds with smooth boundary by using the finite propagation speed of solutions of the Dirichlet wave equation.

Now if we argue as [27], [32] and [33], the weak-type $(1,1)$ estimate on $\tilde{m}(P)$ would follow from that the integral operator

$$\begin{aligned}
\tilde{m}(P)f(x) &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{m}(t)\rho(t)e^{itP}f(x)dt \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{m}(t)\rho(t)\sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)\int_M e_{\lambda_k}(y)f(y)dydt \\
&= \frac{1}{2\pi}\int_M \{\int_{\mathbf{R}} \hat{m}(t)\rho(t)\sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)e_{\lambda_k}(y)dt\}f(y)dy
\end{aligned}$$

with the kernel

$$\begin{aligned}
K(x,y) &= \int_{\mathbf{R}} \hat{m}(t)\rho(t)\sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)e_{\lambda_k}(y)dt \\
&= \sum_{k\geq 1}(m*\check{\rho})(\lambda_k)e_{\lambda_k}(x)e_{\lambda_k}(y)
\end{aligned}$$

is weak-type $(1,1)$. Now define the dyadic decomposition

$$K_\lambda(x,y) = \int_{\mathbf{R}} \hat{m}_\lambda(t)\rho(t)\sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)e_{\lambda_k}(y)dt$$

We have

$$K(x,y) = \sum_{j=1}^{\infty} K_{2^j}(x,y) + K_0(x,y)$$

where K_0 is bounded and vanishes when $\mathrm{dist}(x,y)$ is larger than a fixed constant. In order to estimate $K_\lambda(x,y)$, we make a second dyadic decomposition as follows

$$K_{\lambda,l}(x,y) = \int_{\mathbf{R}} \hat{m}_\lambda(t)\beta(2^{-l}\lambda|t|)\rho(t)\sum_{k\geq 1} e^{it\lambda_k}e_{\lambda_k}(x)e_{\lambda_k}(y)dt$$

We have

$$K_\lambda(x,y) = \sum_{l=-\infty}^{\infty} K_{\lambda,l}(x,y)$$

Define

$$T_{\lambda,l}(P)f(x) = \int_M K_{\lambda,l}(x,y)f(y)dy,$$

we have

$$T_{\lambda,l}(P) = \int_{\mathbf{R}} \hat{m}_\lambda(t)\beta(2^{-l}\lambda|t|)\rho(t)e^{itP}dt.$$

From above two dyadic decompositions, we have

$$\tilde{m}(P)f(x) = \sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty} T_{2^k,l}(P)f(x). \tag{3.8}$$

Note that, because of the support properties of $\rho(t)$, $K_{\lambda,l}(x,y)$ vanishes if l is larger than a fixed multiple of $log\lambda$. Now we exploit the fact that the finite propagation speed of the wave equation mentioned before implies that the kernels of the operators $T_{\lambda,l}$, $K_{\lambda,l}$ must satisfy

$$K_{\lambda,l}(x,y) = 0, \quad if \ \mathrm{dist}(x,y) \geq C(2^l\lambda^{-1}),$$

since $\cos(tP)$ will have a kernel that vanishes on this set when t belongs to the support of the integral defining $K_{\lambda,l}(x,y)$. Hence in each of the second sum of (3.8), there are uniform constants $c, C' > 0$ such that

$$c\lambda\mathrm{dist}(x,y) \leq 2^l \leq C\lambda \tag{3.9}$$

must be satisfied for each $\lambda = 2^k$, we will use this key observation when we estimate each $||T_{\lambda,l}||_{L^1\to L^2}$ and sum up these estimates on $T_{\lambda,l}$s.

Now for $T_{\lambda,l}(P)$s, we have the following estimates:

$$(a). \qquad ||T_{\lambda,l}(P)f||_{L^2(M)} \leq C(2^l)^{-s}\lambda^{n/2}||f||_{L^1(M)}$$

$$(b). \qquad ||T_{\lambda,l}(P)g||_{L^2(M)} \leq C(2^l)^{-s_0}\lambda^{n/2}[\lambda \max_{y,y_0\in\Omega} \text{dist}(y,y_0)]||g||_{L^1(\Omega)}$$

where $\Omega = support(g)$, $\int_\Omega g(y)dy = 0$ and $n/2 < s_0 < \min\{s, n/2+1\}$.

Now we first show estimate (a). Notice that $\beta(2^{-l}\lambda|t|)\rho(t) = 0$ when $|t| \leq 2^{l-1}\lambda^{-1}$, we can use the same idea to prove estimate (a) as we prove the estimate on the remainder $r(P)$ in Part 1. Now we use orthogonality of χ_k for $k \in \mathbf{N}$, and the L^∞ estimates on χ_k in [33], we have

$$\begin{aligned} ||T_{\lambda,l}(P)f||_{L^2}^2 &\leq \sum_{k=1}^\infty ||T_{\lambda,l}(P)\chi_k f||_{L^2}^2 \\ &\leq C\sum_{k=1}^\infty \sup_{\tau\in[k,k+1]} |T_{\lambda,l}(\tau)|^2(1+k)^{n-1}||f||_{L^1}^2 \end{aligned}$$

Hence we need only to show

$$\sum_{k=1}^\infty \sup_{\tau\in[k,k+1]} |T_{\lambda,l}(\tau)|^2(1+k)^{n-1} \leq C(2^l)^{-2s}\lambda^n$$

Notice since $m_\lambda(\tau) = 0$, for $\tau \notin [\lambda/2, 2\lambda]$, we have

$$T_{\lambda,l}(\tau) = O((1+|\tau|+|\lambda|)^{-N})$$

for any N when $\tau \notin [\lambda/4, 4\lambda]$. Then we have

$$\begin{aligned} &\sum_{k\notin[\lambda/4,4\lambda]} \sup_{\tau\in[k,k+1]} |T_{\lambda,l}(\tau)|^2(1+k)^{n-1} \\ \leq\ & C\sum_{k\notin[\lambda/4,4\lambda]} (1+k+\lambda)^{-2N}(1+k)^{n-1} \\ \leq\ & C\int_{x>1, x\notin[\lambda/4,4\lambda]} \frac{x^{n-1}}{(x+\lambda)^{2N}}dx \\ \leq\ & C(1+\lambda)^{n-2N} \end{aligned}$$

Since $2^l \leq C\lambda$ from our observation (3.9) above, we need only to show

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau\in[k,k+1]} |T_{\lambda,l}(\tau)|^2(1+k)^{n-1} \leq C(2^l)^{-2s}\lambda^n$$

that is

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau\in[k,k+1]} |T_{\lambda,l}(\tau)|^2 \leq C(2^l)^{-2s}\lambda$$

Using the fundamental theorem of calculus and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \sum_{k=\lambda/4}^{4\lambda} \sup_{\tau\in[k,k+1]} |T_{\lambda,l}(\tau)|^2 \\
\leq\ & C\left(\int_{\mathbf{R}} |T_{\lambda,l}(\tau)|^2 d\tau + \int_{\mathbf{R}} |T'_{\lambda,l}(\tau)|^2 d\tau\right) \\
=\ & C\left(\int_{\mathbf{R}} |\hat{m}_\lambda(t)\beta(2^{-l}\lambda|t|)\rho(t)|^2 dt + \int_{\mathbf{R}} |t\hat{m}_\lambda(t)\beta(2^{-l}\lambda|t|)\rho(t)|^2 dt\right)
\end{aligned}$$

Recall that $\beta(2^{-l}\lambda|t|)\rho(t) = 0$ when $|t| \leq 2^{l-1}\lambda^{-1}$, by a change variables shows that this is dominated by

$$\begin{aligned}
& (2^l)^{-2s}\lambda^{-1}\int_{\mathbf{R}} |t^s\hat{m}_\lambda(t/\lambda)|^2 dt + (2^l)^{-2s+2}\lambda^{-2}\int_{\mathbf{R}} |t^s\hat{m}_\lambda(t/\lambda)|^2 dt \\
=\ & (2^l)^{-2s}(1+\lambda^{-2}2^{2l})\lambda^{-1}||\lambda\beta(\cdot)m(\lambda\cdot)||^2_{L^2_s} \\
=\ & (2^l)^{-2s}\lambda(1+\lambda^{-2}2^{2l})||\beta(\cdot)m(\lambda\cdot)||^2_{L^2_s} \\
\leq\ & C(2^l)^{-2s}\lambda(1+\lambda^{-2}2^{2l}) \\
\leq\ & C(2^l)^{-2s}\lambda
\end{aligned}$$

Here the first equality comes from a change variables, the second equality comes from the definition of Sobolev norm of $L^2_s(M)$, the third inequality comes from our condition (3.4), and the last inequality comes from the observation (3.9). Hence we proved the estimate (a),

$$||T_{\lambda,l}(P)f||_{L^2(M)} \leq C(2^l)^{-s}\lambda^{n/2}||f||_{L^1(M)}$$

Next we prove the estimate (b). We will use the orthogonality of $\{e_j\}_{j\in\mathbf{N}}$,

$$\int_M e_{\lambda_k}(x)e_{\lambda_j}(x)dx = \delta_{kj},$$

and the L^∞ estimates on $\nabla\chi_k$ for all $k \in \mathbf{N}$ as in Theorem 1.1. Now for function $g \in L^1(M)$ such that $\Omega = support(g)$ and $\int_\Omega g(y)dy = 0$. For some fixed point $y_0 \in \Omega$, we have

$$\begin{aligned}
& ||T_{\lambda,l}(P)g||^2_{L^2} \\
= \; & \int_M |\int_\Omega K_{\lambda,l}(x,y)g(y)dy|^2 dx \\
= \; & \int_M |\int_\Omega [K_{\lambda,l}(x,y) - K_{\lambda,l}(x,y_0)]g(y)dy|^2 dx \\
& (\textit{here use the cancellation of } g) \\
= \; & \int_M |\int_\Omega \sum_{k\geq 1} T_{\lambda,l}(\lambda_k)e_{\lambda_k}(x)[e_{\lambda_k}(y) - e_{\lambda_k}(y_0)]g(y)dy|^2 dx \\
= \; & \sum_{k\geq 1}\int_M |\int_\Omega \sum_{\lambda_j\in[k,k+1)} \{T_{\lambda,l}(\lambda_j)e_{\lambda_j}(x)[e_{\lambda_j}(y) - e_{\lambda_j}(y_0)]\}g(y)dy|^2 dx \\
& (\textit{here use the orthogonality}) \\
\leq \; & \sum_{k\geq 1}\int_M \max_{y\in\Omega}|\sum_{\lambda_j\in[k,k+1)} \{T_{\lambda,l}(\lambda_j)e_{\lambda_j}(x)[e_{\lambda_j}(y) - e_{\lambda_j}(y_0)]\}|^2 dx| \int |g(y)|dy|^2 \\
= \; & ||g||^2_{L^1}\sum_{k\geq 1}\int_M |\sum_{\lambda_j\in[k,k+1)} \{T_{\lambda,l}(\lambda_j)e_{\lambda_j}(x)[e_{\lambda_j}(y_1) - e_{\lambda_j}(y_0)]\}|^2 dx \\
& (\textit{where the maximum achieves at } y_1) \\
= \; & ||g||^2_{L^1}\sum_{k\geq 1}\int_M |(\nabla_y \sum_{\lambda_j\in[k,k+1)} T_{\lambda,l}(\lambda_j)e_{\lambda_j}(x)e_{\lambda_j}(\bar{y}), y_1 - y_0)|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&= ||g||_{L^1}^2 \sum_{k\geq 1} \int_M |\{ \sum_{\lambda_j\in[k,k+1)} T_{\lambda,l}(\lambda_j) e_{\lambda_j}(x)(\nabla e_{\lambda_j}(\bar{y}), y_1 - y_0)\}|^2 dx \\
&= ||g||_{L^1}^2 \sum_{k\geq 1} \sum_{\lambda_j\in[k,k+1)} |T_{\lambda,l}(\lambda_j)(\nabla e_{\lambda_j}(\bar{y}), y_1 - y_0)|^2 \\
&\leq ||g||_{L^1}^2 \sum_{k\geq 1} \max_{\tau\in[k,k+1)} |T_{\lambda,l}(\tau)|^2 \{ \sum_{\lambda_j\in[k,k+1)} |\nabla e_{\lambda_j}(\bar{y})|^2 \mathrm{dist}(y_1, y_0)^2 \} \\
&\quad (\textit{here use the orthogonality}) \\
&\leq ||g||_{L^1}^2 [\max_{y,y_0\in\Omega} \mathrm{dist}(y,y_0)]^2 \sum_{k\geq 1} \max_{\tau\in[k,k+1)} |T_{\lambda,l}(\tau)|^2 \{ \sum_{\lambda_j\in[k,k+1)} |\nabla e_{\lambda_j}(\bar{y})|^2 \} \\
&\leq C||g||_{L^1}^2 [\max_{y,y_0\in\Omega} \mathrm{dist}(y,y_0)]^2 \sum_{k\geq 1} \max_{\tau\in[k,k+1)} |T_{\lambda,l}(\tau)|^2 (1+k)^{n+1}
\end{aligned}
$$

Now using the same computation as to the estimate (a), for some constant s_0 satisfying $n/2 < s_0 < \min\{s, n/2+1\}$, we have

$$\sum_{k\geq 1} \max_{\tau\in[k,k+1)} |T_{\lambda,l}(\tau)|^2 (1+k)^{n+1} \leq C(2^l)^{-2s_0} \lambda^{n+2}.$$

Combine above two estimates, we proved the estimate (b),

$$||T_{\lambda,l}(P)g||_{L^2(M)} \leq C(2^l)^{-s_0} \lambda^{n/2} [\lambda \max_{y,y_0\in\Omega} \mathrm{dist}(y,y_0)] ||g||_{L^1(\Omega)}$$

Now we use the estimates (a) and (b) to show

$$\tilde{m}(P)f(x) = \int_M K(x,y)f(y)dy$$

is weak-type (1,1). We let $f(x) = g(x) + \sum_{k=1}^{\infty} b_k(x) := g(x) + b(x)$ be the Calderón-Zygmund decomposition of $f \in L^1(M)$ at the level α using the same idea as Lemma 0.2.7 in [32]. Let $Q_k \supset supp(b_k)$ be the cube associated to b_k on M, and we have

$$
\begin{aligned}
&||g||_{L^1} + \sum_{k=1}^{\infty} ||b_k||_{L^1} \leq 3||f||_{L^1} \\
&|g(x)| \leq 2^n \alpha \qquad \textit{almost everywhere,}
\end{aligned}
$$

and for certain non-overlapping cubes Q_k,

$$b_k(x) = 0 \quad for\ x \notin Q_k \quad and \quad \int_M b_k(x)dx = 0$$
$$\sum_{k=1}^{\infty} \mu|Q_k| \leq \alpha^{-1}||f||_{L^1}.$$

Now we show the weak-type $(1,1)$ estimate for $\tilde{m}(P)$. Since

$$\{x : |\tilde{m}(P)f(x)| > \alpha\} \subset \{x : |\tilde{m}(P)g(x)| > \frac{\alpha}{2}\} \cup \{x : |\tilde{m}(P)b(x)| > \frac{\alpha}{2}\}$$

Notice

$$\int_M |g|^2 dx \leq 2^n \alpha \int_M |g| dx.$$

Hence we use the L^2 boundedness of $\tilde{m}(P)$ and Tchebyshev's inequality to get

$$\mu\{x : |\tilde{m}(P)g(x)| > \alpha/2\} \leq C\alpha^{-2}||g||_{L^2}^2 \leq C'\alpha^{-1}||f||_{L^1}.$$

Let Q_k^* be the cube with the same center as Q_k but twice the side-length. After possibly making a translation, we may assume that

$$Q_k = \{x : \max |x_j| \leq R\}.$$

Let $\mathcal{O}^* = \cup Q_k^*$, we have

$$\mu|\mathcal{O}^*| \leq 2^n \alpha^{-1}||f||_{L^1}.$$

and

$$\begin{aligned} & \mu\{x \notin \mathcal{O}^* : |\tilde{m}(P)b(x)| > \alpha/2\} \\ \leq \ & 2\alpha^{-1} \int_{x \notin \mathcal{O}^*} |\tilde{m}(P)b(x)| dx \\ \leq \ & 2\alpha^{-1} \sum_{k=1}^{\infty} \int_{x \notin Q_k^*} |\tilde{m}(P)b_k(x)| dx \end{aligned}$$

Hence we need only to show

$$\begin{aligned}
& \int_{x\notin Q_k^*} |\tilde{m}(P)b_k(x)|dx \\
= & \int_{x\notin Q_k^*} |\int_{Q_k} K(x,y)b_k(y)dy|dx \\
\leq & \; C\int_M |b_k|dx.
\end{aligned}$$

From the double dyadic decomposition (3.8), we show two estimates of $T_{\lambda,l}(P)b_k(x)$ on set $\mathcal{O}^c = \{x \in M \ : \ x \notin \mathcal{O}^*\}$,

$$\begin{aligned}
(I) \quad ||T_{\lambda,l}(P)b_k||_{L^1(\mathcal{O}^c)} & \leq C(2^l)^{n/2-s}||b_k||_{L^1(Q_k)} \\
(II) \quad ||T_{\lambda,l}(P)b_k||_{L^1(\mathcal{O}^c)} & \leq C(2^l)^{n/2-s_0}[\lambda \max_{y,y_0\in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\end{aligned}$$

Since our observation (3.9), as was done in [33], in order to prove (I), (II), it suffices to show that for all geodesic balls $B_{R_{\lambda,l}}$ of radius $R_{\lambda,l} = 2^l\lambda^{-1}$, one has the bounds

$$\begin{aligned}
(I)' \quad ||T_{\lambda,l}(P)b_k||_{L^1(\mathcal{O}^c\cap B_{R_{\lambda,l}})} & \leq C(2^l)^{n/2-s}||b_k||_{L^1(Q_k)} \\
(II)' \quad ||T_{\lambda,l}(P)b_k||_{L^1(\mathcal{O}^c\cap B_{R_{\lambda,l}})} & \leq C(2^l)^{\frac{n}{2}-s_0}[\lambda \max_{y,y_0\in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\end{aligned}$$

To show $(I)'$, using the estimate (a), and Hölder inequality, we get

$$\begin{aligned}
& ||T_{\lambda,l}(P)b_k||_{L^1(\{x\notin\mathcal{O}^*\}\cap B_{R_{\lambda,l}})} \\
\leq & \; Vol(B_{R_{\lambda,l}})^{1/2}||T_{\lambda,l}(P)b_k||_{L^2} \\
\leq & \; C(2^l\lambda^{-1})^{n/2}(2^l)^{-s}\lambda^{n/2}||b_k||_{L^1} \\
= & \; C(2^l)^{n/2-s}||b_k||_{L^1}
\end{aligned}$$

To show $(II)'$, using the cancellation property $\int_{Q_k} b_k(y)dy = 0$, the estimate (b), and Hölder inequality, we have

$$
\begin{aligned}
& ||T_{\lambda,l}(P)b_k||_{L^1(\{x\notin\mathcal{O}^*\}\cap B_{R_{\lambda,l}})} \\
\leq\ & Vol(B_{R_{\lambda,l}})^{1/2}||T_{\lambda,l}(P)b_k||_{L^2} \\
\leq\ & C(2^l\lambda^{-1})^{n/2}(2^l)^{-s_0}\lambda^{n/2}[\lambda \max_{y,y_0\in Q_k} \text{dist}(y,y_0)]||bk||_{L^1(Q_k)} \\
=\ & C(2^l)^{n/2-s_0}[\lambda \max_{y,y_0\in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)}
\end{aligned}
$$

From our observation (3.9), and estimates (I), we have

$$
\begin{aligned}
& \sum_{l=-\infty}^{\infty} ||T_{\lambda,l}(P)b_k||_{L^1(x\notin\mathcal{O}^*)} \\
\leq\ & C \sum_{2^l\geq c\lambda\text{dist}(x,y)} (2^l)^{n/2-s}||b_k||_{L^1(Q_k)} \\
\leq\ & C_s(\lambda\text{dist}(x,y))^{n/2-s}||b_k||_{L^1(Q_k)} \\
\leq\ & C_s(\lambda R)^{n/2-s}||b_k||_{L^1(Q_k)},
\end{aligned}
$$

and from $\max_{y,y_0\in Q_k}\text{dist}(y,y_0) \leq CR$, estimate (II), and $n/2 < s_0 < \min\{s, n/2+1\}$, we have

$$
\begin{aligned}
& \sum_{l=-\infty}^{\infty} ||T_{\lambda,l}(P)b_k||_{L^1(x\notin\mathcal{O}^*)} \\
\leq\ & C \sum_{2^l\geq c\lambda\text{dist}(x,y)} (2^l)^{n/2-s_0}[\lambda \max_{y,y_0\in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)} \\
\leq\ & C_{s_0}(\lambda\text{dist}(x,y))^{n/2-s_0}[\lambda \max_{y,y_0\in Q_k} \text{dist}(y,y_0)]||b_k||_{L^1(Q_k)} \\
\leq\ & C_{s_0}(\lambda R)^{n/2+1-s_0}||b_k||_{L^1(Q_k)}.
\end{aligned}
$$

Therefore, we combine the above two estimate we conclude that

$$
\int_{x\notin Q_k^*} |\tilde{m}(P)b_k(x)|dx
$$

$$\begin{aligned}
&\leq \sum_{j=0}^{\infty}\sum_{l=-\infty}^{\infty} ||T_{2^j,l}(P)b_k||_{L^1(x\notin\mathcal{O}^*)}\\
&\leq C_s(\sum_{2^jR>1}(\lambda R)^{n/2-s}||b_k||_{L^1} + C_{s_0}\sum_{2^jR\leq 1}(\lambda R)^{n/2+1-s_0}||b_k||_{L^1}\\
&\leq C_s||b_k||_{L^1}
\end{aligned}$$

Hence we have the weak-type $(1,1)$ estimate on the main term

$$\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \leq \alpha^{-1}||f||_{L^1}.$$

Combine Case 1 and Case 2, we have the weak-type estimate of $m(P)$ and we finish the proof of Theorem 1.2.

Using the same argument as above, we will give a new proof of Hörmander Multiplier Theorem for any second-order self-adjoint elliptic differential operator P on boundless compact manifolds, which was proved in Seeger and Sogge [29] and Sogge [32], since we have the gradient estimates for P (Theorem 2.3.1). In our new proof, we avoid to construct the paramatrix of the wave kernel when we try to show the weak-type $(1,1)$ estimates of the multiplier $m(P)$.

3.5 Further Study

In this section, we discuss some problems for future research. I'd like to generalize the harmonic analysis on Euclidean spaces (cf. [35], [36]) to general Riemannian manifolds settings. More precisely, we will study the L^2 restriction theorem, Riesz means, and general multiplier problems on Riemannian manifolds. In [32], those problems were solved by studying the wave kernel using the paramatrix construction. In this chapter, we studied the multiplier problems on manifolds with boundary. I plan to study some problems related to the L^2 restriction theorem on manifolds with boundary and Bochner-Riesz type multipliers for cones. I also like to study the multiplier problems for domains on $\mathbf{R}^n$ or compact manifolds with rough boundary, and the elliptic operators with irregular coefficients. What I study here is in the C^∞ category.

Chapter 4

Convergence of Eigenfunction expansion and Riesz Means

4.1 Introduction and Results

In this chapter we study the problem on almost-everywhere convergent eigenfunction expansions and Riesz means of the Laplace-Beltrami operator Δ_g on a compact Riemannian manifold (M, g) with boundary. Denote the partial sums of eigenfunction expansions as

$$S_N(f;x) = \sum_{k=1}^{N} e(f)(x).$$

Let $S^*(f;x)$ denote the maximal function

$$S^*(f;x) = \sup_{N \geq 1} |S_N(f;x)|.$$

Let E_t is the corresponding expansion of the identity to the spectrum

$$E_t f(x) = \int_M e(x,y,t) f(y) dy, \quad where \quad e(x,y,\lambda) = \sum_{\lambda_j \leq \lambda} e_j(x) e_j(y).$$

is the spectral function of the Laplacian.

In [17], Meaney proved some results on almost-everywhere convergent eigenfunction expansions of the Laplace-Beltrami operator for function $f \in L^2_s(M)$, where $L^2_s(M)$ is the Sobolev space of order $s > 0$, on a compact boundless manifold. Here we have the following result on almost-everywhere convergent eigenfunction expansions on a compact manifold M with smooth boundary.

Theorem 4.1.1 *For $s > 0$, if $f \in L^2_s(M)$, we have*

$$\lim_{N\to\infty} S_N(f;x) = f(x), \quad \textit{almost everywhere on } M.$$

And for the maximal function $S^*(f;x)$, we have

$$||S^*(f)||_{L^2_s(M)} \leq C_s ||f||_{L^2_s(M)},$$

for some constant C_s.

For each $s > 0$, we introduce an important class of special multiplier, Riesz means

$$S^s_\lambda f(x) = \int_0^\lambda (1 - \frac{t}{\lambda})^s dE_t f(x), \tag{4.1}$$

of the spectral expansion. Stein and Weiss [36] studied the Riesz means for multiple Fourier series, which can be regarded as the case for flat torus T^n.

Sogge [31] and Christ and Sogge [5] proved the sharp results for manifolds without boundary, which the Riesz means (4.1) are uniformly bounded on all $L^p(M)$ spaces, provided that $\delta > \frac{n-1}{2}$, but no such result can hold when $\delta \le \frac{n-1}{2}$. Recently, Sogge [33] proved the same results on Riesz means of the spectral expansion for Dirichlet Laplacian on manifolds with boundary.

In [2], Alimov studied conditions for the convergence and Riesz summability of spectral expansions of piecewise smooth functions for self-adjoint elliptic operators on the compact subdomain of a n-dimensional domain. Here we say that a function $f(x)$ on M is piecewise smooth if it is uniformly continuous in M, and has uniformly derivatives in M up to order $l \ge 0$. In [23] and [24], Pinsky, Stanton and Trapa obtained some necessary and sufficient conditions for the convergence of Fourier inversion and spectral expansion of the Laplace operator of a rotationally invariant Riemannian manifold by using the asymptotic properties of corresponding special functions. And in [25], Pinsky and Taylor use a wave equation approach to study point-wise Fourier inversion and point-wise convergence or divergence of spectral expansion of Laplace operator of Riemannian manifolds with some symmetry, including spheres, hyperbolic spaces and other compact and noncompact rank-one symmetric space, and on strongly scattering manifolds.

Now for general compact Riemannian manifolds, one can't use the asymptotic properties of special functions to study the asymptotic behavior of spectral functions any more. Here we use the L^∞ estimates on χ_λ and a L^2 estimates on the normal derivative of eigenfunctions on the boundary, instead of the asymptotic properties of special functions, to study the asymptotic behavior of spectral functions, and we obtain the following results,

Theorem 4.1.2 *Let f be a piecewise smooth function on a manifold M with boundary,* $\dim M = n$, *we have*

(1). $n = 2$, on each compact subset of the smoothness domain of f, the spectral expansion is uniformly bounded and the Riesz means $S_\lambda^s f(x)$ of any positive order $s > 0$ uniformly converge to f.

(2). $n > 2$, on each compact subset of the smoothness domain of f, the Riesz means $S_\lambda^s f(x)$ of any positive order $s \geq (n-1)/2$ uniformly converge to f.

Remark 4.1.1 *For $n > 2$, there are some simple examples, such as the characteristic function of unit ball χ_B in $\mathbf{R}^n$, see [23], show that the spectral expansion of a piecewise smooth function may diverge even at points far from the discontinuity surface, and, if $n > 3$, the divergence will be unbounded.*

Now applying the uniformly bounds for Riesz means on $L^p(M)$ in [33] and a density argument, from Theorem 4.1.2, we have the following almost everywhere convergence results for Riesz means on $L^p(M)$.

Theorem 4.1.3 *Fix a smooth compact Riemannian manifold with boundary of dimension $n \geq 2$, for any $s > (n-1)/2$, let $f \in L^p(M)$, $1 \leq p \leq \infty$, we have*

$$\lim_{\lambda\to\infty} S_\lambda^s f(x) = f(x), \quad \textit{almost everywhere for } x \in M.$$

And for any $s > (n-1)/2$, let $f \in L^p(M)$, $1 \leq p \leq \infty$, we have

$$\lim_{\lambda\to\infty} S_\lambda^s f(x) = f(x), \quad \textit{in measure for } x \in M.$$

4.2 Pointwise Convergence of S_N on $L_s^2(M)$

In this section we study the problem on almost-everywhere convergence of eigenfunction expansions on $L_s^2(M)$ of both Dirichlet Laplacian and Neumann Laplacian on a compact Riemannian manifold (M, g) with boundary. In order to prove Theorem 4.1.1, we need the following Lemmas.

Lemma 4.2.1 (Rademacher-Menchoff Theorem) *Let* (Ω, μ) *be a positive measure space and let* $\{\phi_n\}_{n=1}^{\infty}$ *be an orthonormal basis in* $L^2(\Omega, \mu)$*. The series* $\sum_{n=0}^{\infty} c_n \phi_n(x)$ *is convergent almost everywhere on* Ω *provided*

$$\sum n = 0^{\infty} |c_n|^2 (\log n)^2 < \infty.$$

This Theorem can be found in [1] (see (2.3.2) in [1]).

Lemma 4.2.2 *There is a constant* $K > 0$*, such that*

$$||S^*(f)||_{L^2(M)}^2 \leq K \sum_{n=1}^{\infty} ||\chi_n(f)||_{L^2(M)}^2 (\log(n+2))^2.$$

This result is proved in Chapter II in [18].

Proof of Theorem 4.1.1. Fix $f \in L_s^2(M)$ with $s > 0$. Let $c_n = ||\chi_n f||_{L^2(M)}$, for $n > 0$, set

$$\phi_n(x) = c_n^{-1} \chi_n f(x), \quad x \in M,$$

provided $c_n \neq 0$. Otherwise let ϕ_n be an arbitrary element in

$$\mathrm{H}_n = \mathrm{span}\{e_\lambda(x) \ : \lambda \in [n, n+1)\}, \ \textit{with} \ ||\phi_n||_{L^2(M)} = 1.$$

Then we have

$$\sum_{n=1}^{\infty} \chi_n f(x) = \sum_{n=1}^{\infty} \phi_n(x).$$

From Lemma 4.2.1, to prove the almost everywhere convergence for the spectral expansion of f on M, we need only to show that

$$\sum_{j=2}^{\infty} |c_j|^2 (\log j)^2 < \infty.$$

Since $f \in L_s^2(M)$, we have

$$||f||^2_{L_s^2(M)} = \sum_{j=1}^{\infty} (1+\lambda_j^2)^{s/2} ||f||^2_{L^2(M)} \sim \sum_{n=1}^{\infty} (1+n^2)^{s/2} ||\chi_n f||^2_{L^2(M)} < \infty,$$

Now we have

$$\begin{aligned} \sum_{j=2}^{\infty} |c_j|^2 (\log j)^2 &= \sum_{j=2}^{\infty} |c_j|^2 (1+j^2)^{s/2} [(\log j)^2 (1+j^2)^{-s/2}] \\ &\leq C_s \sum_{j=2}^{\infty} |c_j|^2 (1+j^2)^{s/2} < \infty. \end{aligned}$$

Hence we have the almost everywhere convergence for the spectral expansion of f on M.

From Lemma 4.2.2, we have

$$\begin{aligned} ||S^*(f)||^2_{L^2(M)} &\leq K \sum_{j=1}^{\infty} ||\chi_j(f)||^2_{L^2(M)} (\log(j+2))^2 \\ &\leq K \sum_{j=1}^{\infty} ||\chi_j(f)||^2_{L^2(M)} (1+j^2)^{s/2} [(1+j^2)^{-s/2} (\log(j+2))^2] \\ &\leq C_s \sum_{j=1}^{\infty} ||\chi_j(f)||^2_{L^2(M)} (1+j^2)^{s/2} \\ &= C_s ||f||^2_{L_s^2(M)}. \end{aligned}$$

Here $C_s = K \sup_j [(1+j^2)^{-s/2} (\log(j+2))^2] < \infty$. We show the second assertion of Theorem 4.1.1.

Q.E.D.

4.3 Convergent Results of Riesz Means

In this section, we shall prove Theorem 4.1.2. For each $\tau \geq 0$, we introduce the kernel

$$G_\tau(x,y) = \int_0^\infty \lambda^{-\tau} d_\lambda e(x,y,\lambda)$$

of fractional order. In this notation, we can see $G_1(x,y)$ is the Green function of the Laplacian for the eigenvalue problem. And for any smooth function defined on M, we have the following equality

$$\int_{\partial M} (\partial_\nu G_\tau(x,y)) f(y) ds(y) = \sum_{j=1}^{\infty} e_j(x) \lambda_j^{-\tau} \int_S (\partial_\nu (e_j(y)) f(y) d\sigma, \tag{4.2}$$

where σ is the area element on surface ∂M, and ν is the outward normal direction on the boundary. In [12], the authors have the following results: for the inequality

$$c\lambda_j \leq ||\partial_\nu e_j||^2_{L^2(\partial M)} \leq C\lambda_j,$$

the upper bound holds for some constant C independent of λ_j, and the lower bound holds provided that M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. In particular, the lower boundary holds if M is a sub-domain of Euclidean space. Here we use the idea of proving the upper bounds in [12], we obtain the following estimates:

Lemma 4.3.1 *For any smooth function f on M, the estimates*

$$(a) \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1)} \left| \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma \right|^2 \leq C\lambda^{n+1} ||f||^2_{L^2(\partial M)},$$

$$(b) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma|^2 \leq 2(\lambda+1)^4 ||\chi_\lambda f||^2_{L^2(M)} + 2||\chi_\lambda(\Delta f)||^2_{L^2(M)},$$

hold both as $\lambda \to \infty$.

Proof. For estimate (a), we use the upper bound for $||\partial_\nu e_j||_{L^2(\partial M)}$ in [12], and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma|^2 \\ \leq & \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} \int_{\partial M} (\partial_\nu e_j(y))^2 d\sigma \int_{\partial M} f(y)^2 d\sigma \\ \leq & ||f||^2_{L^2(\partial M)} \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} ||\partial_\nu e_j||^2_{L^2(\partial M)} \\ \leq & C\lambda^{n+1} ||f||^2_{L^2(\partial M)} \end{aligned}$$

For the last inequality we use the Weyl formula

$$\sharp\{\lambda_j \ : \ \sqrt{\lambda_j} \in [\lambda, \lambda+1)\} = C\lambda^{n-1} + o(\lambda^{n-1}).$$

For estimate (b), by the Green's formula, we have

$$\begin{aligned} \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma &= \int_M \Delta e_j(y) \cdot f(y) dy - \int_M e_j(y) \cdot \Delta f(y) dy \\ &= -\lambda_j \int_M e_j(y) \cdot f(y) dy - \int_M e_j(y) \cdot (\Delta f(y)) dy. \end{aligned}$$

Notice that $\int_M e_j(y) \cdot f(y) dy$ is the Fourier coefficient of f with respect to the spectral decomposition, then we have

$$\sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma|^2$$

$$\leq \; 2 \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} [\lambda_j^2(\int_M e_j(y)\cdot f(y)dy)^2 + (\int_M e_j(y)\cdot(\Delta f(y))dy)^2]$$
$$\leq \; 2(\lambda+1)^4||\chi_\lambda f||^2_{L^2(\partial M)} + 2||\chi_\lambda(\Delta f)||^2_{L^2(\partial M)},$$

Q.E.D.

Applying the L^∞ estimates on χ_λ in [33], we have the following Lemma:

Lemma 4.3.2 *For any given smooth function f on M, the estimates*

$$(a) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |e_j(x)\int_{\partial M}\partial_\nu e_j(y)f(y)ds(y)| \;\leq\; C\lambda^n||f||_{L^2(\partial M)}$$
$$(b) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |e_j(x)\int_{\partial M}\partial_\nu e_j(y)f(y)ds(y)| \;\leq\; C\lambda^{\frac{n+3}{2}}||\chi_\lambda f||_{L^2(M)} + C\lambda^{\frac{n-1}{2}}||\chi_\lambda(\Delta f)||_{L^2(M)}$$

hold both as $\lambda \to \infty$.

Proof. From [33], we have estimates

$$\sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} e_j(x)^2 \;\leq\; C\lambda^{n-1},$$

then by the Cauchy-Schwarz inequality, we get the results from Lemma 2.1.

Q.E.D.

Lemma 4.3.3 *Let $\tau > 0$, then for any smooth function f and any constant $h \in (0,1)$, the estimates*

$$(a) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\lambda_j^{-2\tau}e_j(x)\int_{\partial M}\partial_\nu e_j(y)f(y)d\sigma| \;\leq\; C\lambda^{n-2\tau}||f||_{L^2(\partial M)}$$
$$(b) \sum_{\sqrt{\lambda_j}\in[\lambda,\lambda+1)} |\lambda_j^{-2\tau}e_j(x)\int_{\partial M}\partial_\nu e_j(y)f(y)d\sigma| \;\leq\; C\lambda^{\frac{n+3}{2}-2\tau}||\chi_\lambda f||_{L^2} + C\lambda^{\frac{n-1}{2}-2\tau}||\chi_\lambda(\Delta f)||_{L^2}$$

hold both as $\lambda \to \infty$.

Proof of Theorem 4.1.2 Given a piecewise smooth function f on M. Let us fix an arbitrary compact set $K \subset M - \partial M$ and consider a smooth function $f_\tau(x)$ with compact support in M such that in some neighborhood U of the compact subset K, we have

$$f_\tau(x) = \int_{\partial M} \partial_\nu G_\tau(x, y) f(y) d\sigma, \quad x \in U.$$

From (4.2), the right hand side has the spectral expansion

$$\int_{\partial M} \partial_\nu G_\tau(x, y) f(y) d\sigma = \sum_{j=1}^{\infty} \lambda_j^{-\tau} e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma. \tag{4.3}$$

Consider the function

$$\phi_\tau(\lambda) = \phi_\tau(\lambda, x) = \sum_{\lambda_j \le \lambda} \lambda_j^{-\tau} e_j(x) \int_{\partial M} \partial_\nu e_j(y) f(y) d\sigma - E_\lambda f_\tau(x).$$

We can see that the Fourier-Stieltjes transform

$$\Phi(\xi) = \int_0^\infty e^{-it\xi} d\phi_\tau(t^2)$$

is bounded in a neighborhood of zero and vanishes at $\xi = 0$ together with all its derivatives of even order.

Lemma 4.3.3 implies that if $x \in K$, we have

$$(a) \quad |\phi_\tau((t+h)^2) - \phi_\tau(t^2)| \le C t^{n-2\tau} ||f||_{L^2(\partial M)}, \quad as\ t \to \infty,$$

$$(b) \quad |\phi_\tau((t+h)^2) - \phi_\tau(t^2)| \le C(t+1)^{\frac{n+3}{2}-2\tau} ||\chi_t f||_{L^2(M)}$$
$$+ C(t+1)^{\frac{n-1}{2}-2\tau} ||\chi_t(\Delta f)||_{L^2(M)}, \quad as\ t \to \infty,$$

for any $h \in [0,1]$. According to the Tauberian Theorem of Hörmander (Lemma 17.5.6 in [15]), for Riesz means, we have the following estimates

$$
\begin{aligned}
&(a) \quad |\int_0^\lambda (1-\frac{t}{\lambda})^s d\phi_\tau(t)| \le C\lambda^{n-2\tau-s}||f||_{L^2(\partial M)}, \quad as\ t\to\infty,\\
&(b) \quad |\int_0^\lambda (1-\frac{t}{\lambda})^s d\phi_\tau(t)| \le C\lambda^{\frac{n+3}{2}-2\tau-s}||E_\lambda f||_{L^2(M)}\\
&\qquad +C\lambda^{\frac{n-1}{2}-2\tau-s}||E_\lambda(\Delta f)||_{L^2(M)}, \quad as\ \lambda\to\infty,
\end{aligned}
$$

for all $s \ge 0$.

Now we set $\tau = 1$, since $G_1(x,y)$ is the Green's function of the Laplacian for the eigenvalue problem, we have $f_1(x) = f(x)$ for all $x \in U$. The left term in the last inequalities (a) and (b) is the error term for the spectral expansion $E_\lambda f(x)$ and the Riesz means $S^s_\lambda f(x)$ to the $f(x)$.

For $n = 2$, the estimate (a) is better than the estimate (b), and implies the assertion (1) of Theorem 4.1.2.

For $n > 2$, the estimate (b) is better than the estimates (a), and implies the assertion (2) of Theorem 4.1.2, here we need use the properties, for $f \in L^2_{n-1-2s}(M)$, which ensures $\Delta f \in L^2_{n-5-2s}(M)$,

$$
\begin{aligned}
\lambda^{n-1-2s}||E_\lambda f||^2_{L^2(M)} &\to 0, \quad as\ \lambda\to\infty\\
\lambda^{n-5-2s}||E_\lambda(\Delta f)||^2_{L^2(M)} &\to 0, \quad as\ \lambda\to\infty.
\end{aligned}
$$

It follows from when $f \in L^2_{n-1-2s}(M)$, one has

$$||f||^2_{L^{n-1-2s}(M)} = \sum_{k=1}^{\infty} k^{n-1-2s}||\chi_k f||^2_{L^2(M)} < \infty.$$

The same reason for $\Delta f \in L^2_{n-5-2s}(M)$. Q.E.D.

For Theorem 4.1.3, we know that for smooth functions we have the almost everywhere convergence on M from Theorem 4.1.2. Notice $C^\infty(M)$ dense on $L^p(M)$ for any $1 \le p \le \infty$, approximated any $f \in L^p(M)$ by smooth functions $\{f_k\}$, using the uniform bound results of Riesz means on $L^p(M)$ in [33], we have convergence for Riesz means $S^s_\lambda f(x)$ for any $f \in L^p(M)$ in measure. When we further assume that $f \in L^2_{n-1-2s}(M)$, we can let smooth functions $\{f_k\}$ approximate f in $L^2_{n-1-2s}(M)$, the from proof Theorem 1.1, we know that for all f_k, the Riesz means $S^s_\lambda f_k(x)$ uniformly converge to $f_k(x)$ as $\lambda \to \infty$, in any compact subset of $M - \partial M$ for all $k \in \mathbf{N}$, which ensures the almost everywhere convergence for $S^s_\lambda f(x)$ to $f(x)$ in M.

4.4 Further Study

In this section, we discuss some problems for future research. I'd like to study the degenerate Fourier integral operators (FIOs), Gibbs' phenomenon, Pinsky's phenomenon and the decay rates for Fourier inversion on domains on $\mathbf{R}^n$ and spectral expansions on Riemannian manifolds. For degenerate FIOs, there are many studies already (cf. [9], [21], [22], and [35]). I'd like to find the decay estimates for FIOs with C^∞ degenerate phase functions and study the generalized Radon transforms and Hilbert transforms. In [23], [25] and [38], Gibbs' phenomenon and Pinsky's phenomenon were studied for Fourier inversions and spectral expansions on some special manifolds. I'd like to study Gibbs' phenomenon and Pinsky's phenomenon for general Riemannian manifolds.

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Vita

Xiangjin Xu was born in November 1974 and was raised in Dongyang, Zhejiang, P. R. China. He began his undergraduate career in September 1992 at Nankai University, Tianjin, P. R. China, and in July 1996, he received Bachelor of Science degree in Mathematics from Nankai University. That fall, he enrolled in the Master program at Nankai Institute of Mathematics, and in June 1999, he received Master of Science degree in Mathematics, advised by Dr. Yiming Long, from Nankai Institute of Mathematics. That summer, he came to Department of Mathematics at University of Pennsylvania as a Ph.D. student. In fall 2000, he transferred to Department of Mathematics in Johns Hopkins University. He studied harmonic analysis on manifolds and nonlinear partial differential equations advised by Dr. Christopher Sogge. He defended this thesis on March 18, 2004.