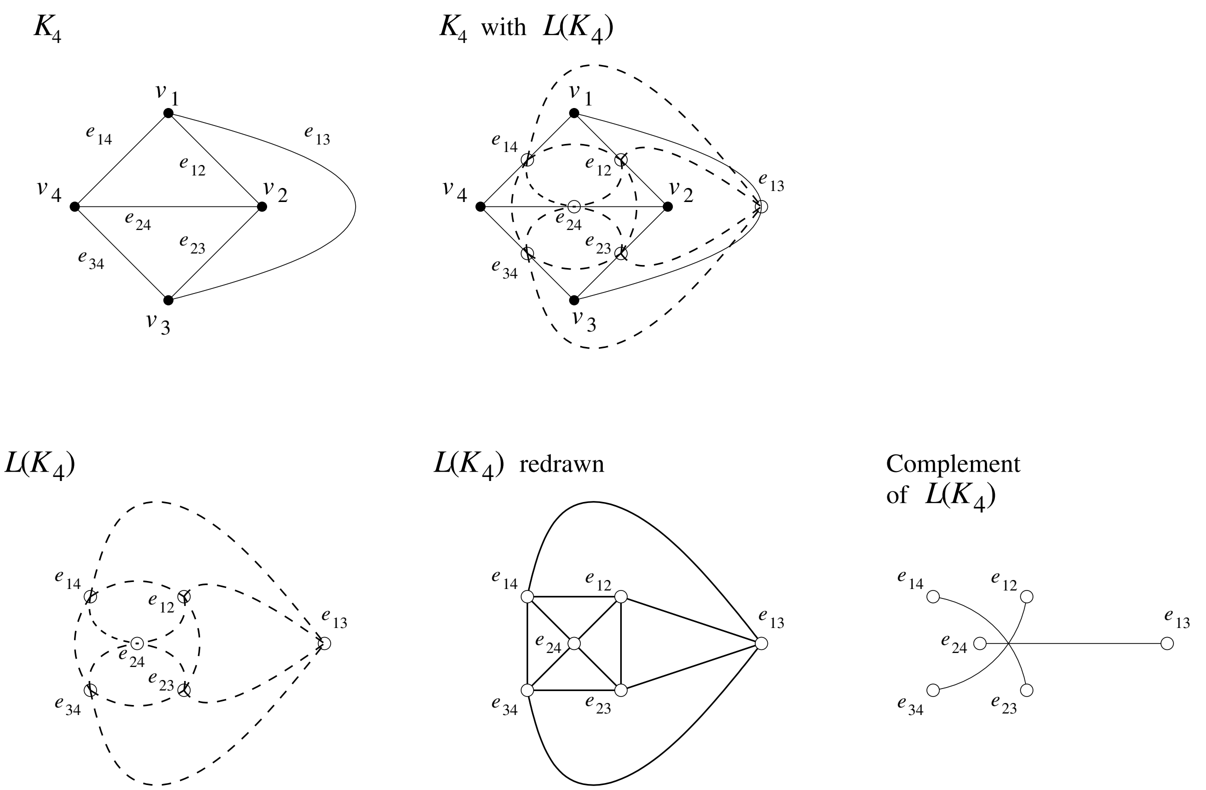
Line Graphs
Math 381
Version of March 25, 2021

Since edges are so important to a graph, sometimes we want to know how much of the graph is determined by its edges. There are a couple of ways to make this a precise question. The one we'll talk about is this: You know the edge set; you don't know the vertices of the edges; but you do know which edges are adjacent (that means they have a common vertex in G). That is, you know adjacency of edges of G. In other words, you have a graph whose vertices are the edges and whose edges tell you which edges (of the original graph G) are adjacent. This graph is the *line graph*, $L(G)$.

Formal definition: $L(G)$ is the graph such that $V(L(G)) := E(G)$ and in which $e, f \in E(G)$ are adjacent (as vertices of $L(G)$) if and only if they are adjacent edges in G. The name comes from the fact that "line" is another name for "edge"; the line graph is the graph of relations between edges.

Note that the vertices of G do not appear in $L(G)$.

Exercise LG.1. What is the line graph of K_4? What is $L(K_4)$? What is the complement $\overline{L(K_4)}$? Is either one of them a graph we've seen before? Here is the solution.



Notice that the complement of $L(K_4)$ is very simple: it's a 1-factor of K_6. So, $L(K_4) = K_6 \setminus F$ where F is a 1-factor. (It isn't always that easy to describe a line graph, so be grateful when it is possible!)

The line graph $L(K_4)$ is a familiar graph. Do you recognize it? (Hint: Figure 1.2.5.)

Exercise LG.2. What is the line graph of K_5? What is $\overline{L(K_5)}$? Do you recognize it?

Exercise LG.3. Find $L(K_{1,n})$. Do you recognize it? (!)

The line graph carries a lot of information about G. For instance, any time there are vertex and edge versions of some property, the edge version in G could be the same—or nearly the same—as the vertex version in $L(G)$. Two examples are vertex coloring versus edge coloring, or Hamilton cycles versus Eulerian circuits.

Theorem LG.1. *The chromatic number of $L(G)$ = the edge chromatic number of G. In symbols: $\chi(L(G)) = \chi'(G)$.*

Exercise LG.4. Write out a proof of Theorem LG.1.

Theorem LG.2. *$L(G)$ has a Hamilton cycle if (but not only if) G has an Eulerian circuit.*

Proof. Suppose G has an Eulerian circuit, $v_0e_1v_1e_2v_2 \cdots v_{q-1}e_qv_q$, where $e_1, \ldots, e_q$ are all distinct and are all the edges of G, but $v_0, v_1, \ldots, v_q$ are not necessarily distinct and furthermore $v_0 = v_q$. (The definition of a circuit requires that $v_0 = v_q$. It does not require the vertices v_i in the circuit to be distinct.) That implies e_1, e_2 are adjacent (at v_1), e_2, e_3 are adjacent (at v_2), …, e_{q-1}, e_q are adjacent (at v_{q-1}), and finally e_q, e_1 are adjacent (since $v_q = v_0$).

Now, in $L(G)$ the entire vertex set is $V(L(G)) = \{e_1, e_2, \ldots, e_q\}$, so if we make a walk $e_1e_2 \cdots e_qe_1$, it is a closed walk that doesn't repeat any vertex (except at the end) and does contain every vertex of $L(G)$. That is, it's a Hamilton cycle of $L(G)$. This proves that, if G has an Eulerian circuit, $L(G)$ has a Hamilton cycle.

What remains is to prove the converse is false. See Exercise LG.5. □

Corollary LG.3. *If G is a graph that is connected and has all positive even degrees, then $L(G)$ has a Hamilton cycle.*

Proof. By the Euler–Hierholzer Theorem, G has an Eulerian circuit. Then Theorem LG.2 implies $L(G)$ has a Hamilton cycle. □

Exercise LG.5. To complete the proof of Theorem LG.2, find an example of a graph such that $L(G)$ has a Hamilton cycle but G has no Eulerian circuit. Can you find many examples? Can you find different kinds of examples? (If you do, you'll know what I mean.)

Theorem LG.4. *Assume G has no isolated vertices. Then $L(G)$ is connected if and only if G is connected.*

Exercise LG.6. Prove Theorem LG.4.

Theorem LG.5. *Assume G is a connected graph. The largest size of a clique in $L(G)$ equals the maximum degree in G, $\Delta(G)$, except when G is ____________.*

Exercise LG.7. (a) Complete the statement of Theorem LG.5. (b) Then give a proof.

The strongest connection of all between $L(G)$ and G is Whitney's Theorem.

Theorem LG.6 (Whitney's Theorem)**.** *If G and H are connected graphs and $L(G)$ is isomorphic to $L(H)$, then G and H are isomorphic, or else $G = K_{1,3}$ and $H = K_3$.*

Hassler Whitney worked on graph theory for a few years in the 1930's before becoming a famous topologist. Whitney's Theorem is much too difficult for us to prove in the time available. But it is amazing that you can deduce all the vertices from just the edge adjacencies, with one tiny exception.

There are lots and lots of good problems about line graphs—interesting, challenging, but solvable. Some of them are some of the exercises.

Exercise LG.8. What is the line graph of O, the octahedral graph? What is $\overline{L(O)}$? Compare these graphs to $L(Q_3)$ and its complement, to see what you can see.

Exercise LG.9. What is $\overline{L(C_n)}$ for $n = 4, 5$? Is it a recognizable graph?

Exercise LG.10. Find $L(K_{2,n})$.

Exercise LG.11. Find $L(K_{3,3})$.

Exercise LG.12. Find all graphs G such that $L(G)$ is a complete graph.

Exercise LG.13. Some graphs are not line graphs.
(a) Is W_4 a line graph?
(b) Is W_5 a line graph?
(c) Which wheels are line graphs?

In class we proved Proposition LG.7:

Proposition LG.7. *If G contains $K_{1,3}$ as an induced subgraph, then it is not a line graph.*

Proposition LG.8. *$K_{m,n}$ is a line graph if and only if $m, n \leq 2$.*

Exercise LG.14. Which bipartite graphs are line graphs?

Exercise LG.15. Which trees are line graphs?

Exercise LG.16. (Research Problems)
Exercise LG.5 just asks you to find a few counterexamples to the converse of Theorem LG.2, i.e., graphs such that G has no Euler circuit and yet $L(G)$ has a Hamilton cycle. There are *much* more challenging problems in relation to the converse. Here are a few. If you want to work on them, I'll like to see what you come up with.

(a) One hopes that there are not that many counterexamples (compared with all graphs). It would be good to classify all counterexamples.
  (i) As a first step toward such a classification, come up with infinite families of counterexamples.
  (ii) Find general properties that imply G is a counterexample, or is not a counterexample. For instance, is there a degree property that implies G is a counterexample? Or, that implies G is not a counterexample?
  This is a good research problem. If you find something on your own, or something published, and write it up, I'll be very interested.
(b) A very strong property of a graph would be that every Hamilton cycle of $L(G)$ is obtained from an Eulerian circuit in G. Can you find such graphs? I suspect there are very few.
(c) The strongest possible result would tell you, for any graph G, exactly which Hamilton cycles of $L(G)$ correspond to Eulerian circuits of G. This is probably very hard.