Clusterability in Signed Graphs

Selections from

**A Mathematical Bibliography of
Signed and Gain Graphs and Allied Areas**

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Annotations. I try to describe the relevant content in a consistent terminology and notation, in the language of signed graphs despite occasional clumsiness (hoping that this will suggest generalizations), and sometimes with my [bracketed] editorial comments. I sometimes try also to explain idiosyncratic terminology, in order to make it easier to read the original item. Several of the annotations incorporate open problems (of widely varying degrees of importance and difficulty).

I use these standard symbols:

- Γ is a graph (V, E) of order $n = |V|$, undirected, possibly allowing loops and multiple edges. It is normally finite unless otherwise indicated.
- Σ is a signed graph (V, E, σ) of order n. $|\Sigma|$ is its underlying graph. E_+, E_- are the sets of positive and negative edges and Σ_+, Σ_- are the corresponding spanning subgraphs (unsigned).
- $[\Sigma]$ is the switching class of Σ.
- $A(\ )$ is the adjacency matrix.
- $\sim$ means that two signed or gain graphs are switching equivalent (with the same underlying graph).
- $\simeq$ means that two signed or gain graphs are switching isomorphic (with isomorphic underlying graphs).
- $\cong$ denotes isomorphism.
- $l(\ )$ is the frustration index (= line index of imbalance).

Some standard terminology (much more will be found in the *Glossary* (Zaslavsky 1998c)):

circle: The graph of a simple closed path, or its edge set.

Adj Adjacency matrix, eigenvalues.
Alg Algorithms.
Bal Balance (mathematical), cobalance.
Clu Clusterability.
d Duality (graphs, matroids, or matrices).
Exp Expository.
Exr Interesting exercises (in an expository work).
Fr Frustration (imbalance); esp. frustration index (line index of imbalance).
Gen Generalization.
KG Signed complete graphs.
Phys Applications in physics (partial coverage).
PsS Psychological, sociological, and anthropological applications (partial coverage).
Ref Many references.
SD Signed digraphs: mathematical properties.
SG Signed graphs: mathematical properties.
VS Vertex-signed graphs ("marked graphs"); signed vertices and edges.
WD Weighted digraphs.
WG Weighted graphs.

Nir Ailon, Moses Charikar, and Alantha Newman

2005a Aggregating inconsistent information: Ranking and clustering. In: *STOC'05: Proc. 37th Annual ACM Symposium on the Theory of Computing* (Boston, 2005). ACM, New York, 2005. pp. 684–693. MR 2181673.

Conference version of (2008a). (**SG: WG: Clu: Alg**)

2008a Aggregating inconsistent information: ranking and clustering. *J. ACM* 55 (2008), no. 5, Art. 23, 27 pp. MR 2456548 (2009k:68280). (**SG: WG: Clu: Alg**)

Nikhil Bansal, Avrim Blum, and Shuchi Chawla

2002a Correlation clustering. In: *Proc. 43rd Ann. IEEE Sympos. Foundations of Computer Science (FOCS '02)*, pp. 238–247.

Preliminary version of (2004a). (**SG: KG: Clu: Alg**)

2004a Correlation clustering. *Machine Learning* 56 (2004), no. 1–3, 89–113.

Clusterability index Q [minimum number of inconsistent edges; see Doreian and Mrvar (1996a) for notation] in signed complete graphs is NP-hard. Polynomial-time algorithms for approximate optimal clustering: up to a constant factor from Q (§3); probably within $1-\varepsilon$ of $|E|-Q$ for any ε (i.e., maximizing consistent edges within $1-\varepsilon$) (§4). §3: A 2-clustering within $3Q_2$ (Thm. 2). A clustering within cQ where $c \approx 20000$ (Thm. 13). §4: A clustering within εn^2 of $|E|-Q$ with high probability but slow in terms of $1/\varepsilon$ (Thm. 15). Asymptotically faster in terms of $1/\varepsilon$ (Thm. 22). The $1-\varepsilon$ factor results from the fact that $|E|-Q = \binom{n}{2} - Q > \frac{1}{2}\binom{n}{2}$ [so is not strong]. §6: "Random noise". §7: "Extensions", considers edge weights in $[-1,1]$ (thus allowing incomplete graphs). Thm. 23: An unweighted approximation algorithm will also approximate this case, assuming "linear cost": e costs $(1-w(e))/2$ if within a cluster and $(1+w(e))/2$ if between clusters. Thm. 24: The problem for clustering that minimizes the total weight of $+$ edges outside clusters and $-$ edges within clusters ("minimizing disagreements") is APX-hard. [Improved in Charikar, Guruswami, and Wirth (2003a, 205a), Swamy (2004a). Generalized in Demaine *et al.* (2006a).] [Added: 22 Sept 2009.]

(**SG: KG: Clu: Alg**)

Vladimir Batagelj

See also P. Doreian and W. de Nooy.

1990a [Closure of the graph value matrix.] (In Slovenian. English summary.) *Obzornik Mat. Fiz.* 37 (1990), 97–104. MR 91f:05058. Zbl 704.05035. (**SG: Adj, Bal, Clu**)

1994a Semirings for social networks analysis. *J. Math. Sociology* 19 (1994), 53–68. Zbl 827.92029. (**SG: Adj, Bal, Clu**)

M. Behzad and G. Chartrand

1969a Line-coloring of signed graphs. *Elem. Math.* 24 (1969), 49–52. MR 39 #5415. Zbl 175, 503 (e: 175.50302).

Λ_{BC} Their line graph $\Lambda_{BC}(\Sigma)$ of a signed simple graph Σ (not defined explicitly) is $L(|\Sigma|)$ with an edge negative when its two endpoints are negative edges in Σ. They "color" as in Cartwright and Harary (1968a) (i.e., clustering). Characterized: Σ with colorable line graphs. Found: the fewest colors for

line graphs of signed trees, K_n, and $K_{r,s}$. [For a more sophisticated kind of line graph see Vijayakumar (various) and Zaslavsky (1984c, 20xxa).]
(**SG: lg: Clu**)

John Bramsen

2002a Further algebraic results in the theory of balance. *J. Math. Sociology* 26 (2002), 309–319. Zbl 1014.05041.

Algorithmic ideas for estimating $l(\Sigma)$. Remarks on clusterability.
(**SG: Fr: Alg; Clu**)

Dorwin Cartwright and Frank Harary

1968a On the coloring of signed graphs. *Elem. Math.* 23 (1968), 85–89. MR 38 #2053. Zbl 155, 317 (e: 155.31703).

"Coloring" is clustering as in Davis (1967a). Thm. 1 adds a bit to Davis (1967a). Thm. 3: The clustering is unique $\iff$ all components of Σ^+ are adjacent. (**SG: Clu**)

1977a A graph theoretic approach to the investigation of system-environment relationships. *J. Math. Sociology* 5 (1977), 87–111. MR 56 #2477. Zbl 336.92026.
(**SD: Clu**)

1979a Balance and clusterability: an overview. In: Paul W. Holland and Samuel Leinhardt, eds., *Perspectives on Social Network Research* (Proc. Sympos., Dartmouth Coll., Hanover, N.H., 1975), Ch. 3, pp. 25–50. Academic Press, New York, 1979.
(**SG, SD, VS: Bal, Fr, Clu, Adj: Exp**)

Moses Charikar, Venkatesan Guruswami, and Anthony Wirth

2003a Clustering with qualitative information. In: *Proceedings of the 44th Annual Symposium on Foundations of Computer Science*, pp. 524–533. ACM?, 2003.

Conference version of (2005a). (**SG: WG: Clu: Alg**)

2005a Clustering with qualitative information. *J. Comput. System Sci.* 71 (2005), no. 3, 360–383. MR 2168358 (2006f:68141). (**SG: WG: Clu: Alg**)

Gary Chartrand

1977a *Graphs as Mathematical Models.* Prindle, Weber and Schmidt, Boston, 1977. MR 58 #9947. Zbl 384.05029.

[Reprinted (1985a).] (**SG: Bal, Clu**)

1985a *Introductory Graph Theory.* Dover Publications, New York, 1985. MR 86c:05001.

"Corrected reprint" of (1977a). (**SG: Bal, Clu**)

James A. Davis

1967a Clustering and structural balance in graphs. *Human Relations* 20 (1967), 181–187. Reprinted in: Samuel Leinhardt, ed., *Social Networks: A Developing Paradigm*, pp. 27–33. Academic Press, New York, 1977.

Σ is "clusterable" if its vertices can be partioned so that each positive edge is within a part and each negative edge joins different parts. Thm.: Σ is clusterable $\iff$ no circle has exactly one negative edge. [See Doreian and Mrvar (1996a).] (**SG: Clu**)

1979a The Davis/Holland/Leinhardt studies: An overview. In: Paul W. Holland and Samuel Leinhardt, eds., *Perspectives on Social Network Research* (Proc. Sympos., Dartmouth Coll., Hanover, N.H., 1975), Academic Press, New York, 1979.

Survey of triad analysis in signed complete digraphs; cf. e.g. Davis and Leinhardt (1972a), Wasserman and Faust (1994a). [Added: 28 April 2009.] **(PsS, SD: Clu(Gen): Exp)**

James A. Davis and Samuel Leinhardt

1972a The structure of positive interpersonal relations in small groups. In: Joseph Berger, Morris Zelditch, Jr., and Bo Anderson, eds., *Sociological Theories in Progress, Vol. Two*, Ch. 10, pp. 218–251. Houghton Mifflin, Boston, 1972.

In "ranked clusterability" the vertices of a signed complete, symmetric digraph are divided into levels. The set of levels is totally ordered. A symmetric pair, $\{+vw, +wv\}$ or $\{-vw, -wv\}$, should be within a level. For an asymmetric pair, $\{+vw, -wv\}$, w should be at a higher level than v. Analysis in relation to both randomly generated and observational data. [Added: 28 April 2009.] **(PsS, SD: Clu(Gen))**

Erik D. Demaine, Dotan Emanuel, Amos Fiat, and Nicole Immorlica

2006a Correlation clustering in general weighted graphs. Approximation and Online Algorithms. *Theoretical Computer Sci.* 361 (2006), no. 2–3, 172–187.

Weighted signed-graph clustering; *cf.* Bansal, Blum, and Chawla (2002a, 2004a). An $O(\log n)$-approximation algorithm for the weighted case based on linear-programming rounding and region growing. [Added: 13 Sept 2009.] **(SG: WG: Clu: Alg)**

Erik Demaine and Nicole Immorlica

2003a Correlation clustering with partial information. In: *Proc. 6th Int. Workshop on Approximation Algorithms for Combinatorial Optimization Problems and 7th Int. Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM-APPROX 2003)* (Princeton, N.J., 2003), pp. 1–13. 2003.

Conference version of Demaine, Emanuel, Fiat, and Immorlica (2006a). [Added: 13 Sept 2009.] **(SG: WG: Clu: Alg)**

Patrick Doreian

2004a Evolution of signed human networks. *Metodoloŷski Zvezki* 1 (2004), 277–293.

Reviews the development of balance and clustering theory for signed (di)graphs in social psychology, mainly Doreian and Mrvar (1996a), Doreian and Krackhardt (2001a), and especially Hummon and Doreian (2003a). The difference between Heider's (1946a) and Cartwright and Harary's (1956a) models, and the need to combine them. [Added: 24 April 2009.] **(PsS: Exp: SD, Bal, Clu, Alg)**

2006a Book review: W. de Nooy, A. Mrvar, and V. Batagelj, *Exploratory Social Network Analysis with Pajek*. *Social Networks* 28 (2006), 269–274.

Review of de Nooy, Mrvar, and Batagelj (2005a). **(SG, SD: Bal, Clu, PsS)**

2008a A multiple indicator approach to blockmodeling signed networks. *Social Networks* 30 (2008), 247–258.

Signed graphs $\Sigma_1, \ldots$ ("multiple indicators") may be approximations of a hidden signed graph Σ. Goals: detect whether Σ exists, and find an optimal clustering of Σ. Methods: (1) Examine the Σ_j for compatibility via statistical tests. (2) Estimate Σ by $\sum_j \sigma_j$. (3) Applies the clusterability index and algorithm of Doreian and Mrvar (1996a). ((2) implies using weighted signed graphs.) This article treats examples, with analysis of the methods' success. [Added: 27 April 2009.] **(PsS, SD: sg: Clu)**

Patrick Doreian, Vladimir Batagelj, and Anuška Ferligoj

2005a *Generalized Blockmodeling*. Structural Analysis in the Social Sciences, No. 25. Cambridge Univ. Press, Cambridge, Eng., 2005.

Ch. 10: "Balance theory and blockmodeling signed networks". Thm. (pp. 305–306; proof by Martin Everett): The sizes of the partitions of V that minimize the clustering index (Doreian and Mrvar 1996a) are consecutive integers. **(PsS, SD: sg: Clu, Bal)**

Patrick Doreian, Roman Kapuscinski, David Krackhardt, and Janusz Szczypula

1996a A brief history of balance through time. *J. Math. Sociology* 21 (1996), 113–131. Reprinted in Patrick Doreian and Frans N. Stokman, eds., *Evolution of Social Networks*, pp. 129–147. Gordon and Breach, Australia, Amsterdam, etc., 1997.

§ 2.3: "A method for group balance". Describes the negation-minimal index of clusterability (generalized imbalance) from Doreian and Mrvar (1996a). **(SG: Bal, Clu: Fr(Gen): Exp)**

§ 3.3: "Results for group balance". Describes results from analysis of data on a small (social) group, in terms of frustration index l and a clusterability index $\min_{k>2} 2P_{k,.5}$ (slightly different from the index in Doreian and Mrvar (1996a)), finding both measures (but more so the latter) decreasing with time. **(PsS: Bal, Clu: Fr(Gen))**

Patrick Doreian and Andrej Mrvar

1996a A partitioning approach to structural balance. *Social Networks* 18 (1996), 149–168.

They propose indices for clusterability (as in Davis (1967a)) that generalize the frustration index of Σ. Fix $k \geq 2$ and $\alpha \in [0,1]$. For a partition π of V into k "clusters", they define $P(\pi) := \alpha n_- + (1-\alpha)n_+$, where $n_+ :=$ number of positive edges between clusters, $n_- :=$ number of negative edges within clusters, and $0 \leq \alpha \leq 1$ is fixed. The first proposed measure is $P_k := \min P(\pi)$, minimized over k-partitions. A second suggestion is the "negation-minimal index of generalized imbalance" [i.e., of clusterability], the smallest number of edges whose negation (equivalently, deletion) makes Σ clusterable. [Call it the 'clusterability index' $Q(\Sigma)$.] [Note that $P(\pi)$ effectively generalizes the Potts Hamiltonian as given by Welsh (1993a). *Question.* Does $P(\pi)$ fit into an interesting generalized Potts model?] [$P(\pi)$ also resembles the Potts Hamiltonian in Fischer and Hertz (1991a) (*q.v.* for a related research question).]

They employ a local optimization algorithm to evaluate $P_{k,\alpha}$ and find an optimal partition: random descent from partition to neighboring partition, where π and π' are neighbors if they differ by transfer of one vertex or exchange of two vertices between two clusters. This was found to work well if repeated many times. [A minimizing partition into at most k clusters is equivalent to a ground state of the k-spin Potts model in the form given by Welsh (1993a), but not quite in that of Fischer and Hertz (1991a).]

Terminology: $P(\pi)$ is called the "criterion function". [More explicitly, one might call $Q(\Sigma, \pi; \alpha) := 2P(\pi)$ the 'α-weighted clusterability index of π', so the clusterability index $Q(\Sigma) = \min_\pi Q(\Sigma, \pi; .5)$; and call $Q_k := 2P_k$ the 'k-clusterability index' of Σ.] Clusterability is "k-balance" or "generalized balance". Clusters are "plus-sets". Signed digraphs are employed in the notation but direction is ignored.

[Further developments in Doreian *et al.* (various), Hummon and Doreian (2003a), Bansal *et al.* (2004a), Demaine *et al.* (2006a), Mrvar and Doreian (2009a).]

[The data in Doreian (2008a), and common sense, suggest that clusters should be allowed to overlap. This is an unplumbed direction.] [Added: 22 Sept 2009.] (**SD: sg: Bal, Clu: Fr(Gen), Alg, PsS**)

1996b Structural balance and partitioning signed graphs. In: A. Ferligoj and A. Kramberger, eds., *Developments in Data Analysis*, pp. 195–208. Metodološki zvezki, Vol. 12. FDV, Ljubljana, Slovenia, 1996.

Similar to (1996a). Some lesser theoretical detail; some new examples. The k-clusterability index $P_{k,\alpha}$ (1996a) is compared for different values of k, seeking the minimum. [But for which value(s) of α is not stated.] Interesting observation: optimal values of k were small. It is said that positive edges between parts are far more acceptable socially than negative edges within parts [thus, in the criterion function α should be rather near 1].

(**SD: sg: Bal, Clu: Fr(Gen), Alg, PsS**)

2009a Partitioning signed social networks. *Social Networks* 31 (2009), no. 1, 1–11.

Generalizes the ideas of (1996a) (*q.v.*). Given: A signed digraph $(\vec{\Gamma}, \sigma)$; a "criterion function" $P(\pi, \rho) := \alpha n^+ + (1-\alpha)n^-$, where $\pi := \{B_1, \ldots, B_k\}$ partitions V into "clusters", $\rho : \pi \times \pi \to \{+, -\}$, $0 \leq \alpha \leq 1$ is fixed, and $n^\varepsilon :=$ number of edges $\overrightarrow{B_i B_j}$ with sign ε for which $\rho(B_i, B_j) = -\varepsilon$ (over all i, j). Objective: (π, ρ), or simply $k := |\pi|$, that minimizes $P(\pi, \rho)$. What is new and most interesting is ρ; also new is using the edge directions.

Call $(\vec{\Gamma}, \sigma)$ "sign clusterable" if $\exists$ (π, ρ) with $P(\pi, \rho) = 0$. Clusterability is sign clusterability with $\rho = \rho_+$, where $\rho_+(B_i, B_j) := +$ if $i = j$, $-$ if $i \neq j$. Let $P(k) := \min\{P(\pi, \rho) : |\pi| = k\}$. Thm. 4: $P(k)$ is monotonically decreasing. [Thus, there is always an optimum π with singleton clusters. Why this does not vitiate the model is not addressed.] Thm. 5: If $(\vec{\Gamma}, \sigma)$ is sign clusterable, then $(\vec{\Gamma}, -\sigma)$ also is. If $(\vec{\Gamma}, \sigma)$ is clusterable, then $(\vec{\Gamma}, -\sigma)$ is not clusterable with the same π [provided $E \neq \varnothing$]. If $(\vec{\Gamma}, \sigma)$ is sign clusterable with $\rho = -\rho_+$, then $(\vec{\Gamma}, -\sigma)$ is clusterable with the same π. "Relocation": Shift one vertex, or exchange two vertices, between blocks so as to decrease P, as in (1996a). This is said (but not proved) to minimize P.

Refinements discussed: partially prespecified blocks; null blocks (without outgoing edges); criterion functions with special treatment of null blocks.

Applications to standard test examples of social psychology.

Dictionary: "balanced" = clusterable; "relaxed balanced" = sign clusterable; "k-balanced" = clusterable with $|\pi| = k$; "relaxed structural balance blockmodel" = this whole system. [Added: 7 Feb. 2009.]

(**SG: Bal, Clu, PsS**)

D. Emanuel and A. Fiat

See also E. Demaine.

2003a Correlation clustering—Minimizing disagreements on arbitrary weighted graphs. In: *Proceedings of ESA*, 2003.

Conference version of Demaine, Emanuel, Fiat, and Immorlica (2006a). [Added: 13 Sept 2009.] (**SG: WG: Clu: Alg**)

Joseph Fiksel

1980a Dynamic evolution in societal networks. *J. Math. Sociology* 7 (1980), 27–46. MR 81g:92023(*q.v.*). Zbl 434.92022. (**SG: Clu, VS**)

K.H. Fischer and J.A. Hertz

1991a *Spin Glasses.* Cambridge Studies in Magnetism: 1. Cambridge Univ. Press, Cambridge, Eng., 1991. MR 93m:82019.

Sect. 3.7: "The Potts glass". The Hamiltonian (without edge weights) is $H = -\frac{1}{2}\sum \sigma(e_{ij})(k\delta(s_i, s_j) - 1)$. [It is not clear that the authors intend to permit negative edges. If they are allowed, H is rather like Doreian and Mrvar's (1996a) $P(\pi)$. *Question.* Is there a worthwhile generalized signed and weighted Potts model with Hamiltonian that specializes both to this form of H and to P?] [Also cf. Welsh (1993a) on the Ashkin–Teller–Potts model.] (**Phys: sg, clu: Exp**)

Ioannis Giotis and Venkatesan Guruswami

2006a Correlation clustering with a fixed number of clusters. In: *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 1167–1176, ACM, New York, 2006. MR 2373844 (2009f:62098). (**SG: WG: Clu: Alg**)

2006b Correlation clustering with a fixed number of clusters. *Theory Comput.* 2 (2006), 249–266. MR 2322880 (2009e:68118). (**SG: WG: Clu: Alg**)

Terry C. Gleason and Dorwin Cartwright

1967a A note on a matrix criterion for unique colorability of a signed graph. *Psychometrika* 32 (1967), 291–296. MR 35 #989. Zbl 184, 492 (e: 184.49202).

"Colorable" = clusterable. The adjacency matrices of Σ^+ and Σ^- are employed separately. The arithmetic is mostly "Boolean", i.e., $1+1=0$. A certain integral matrix T shows whether or not Σ is clusterable. [Added: 11 Nov. 2008.] (**SG: Clu, Adj**)

Norman P. Hummon and Patrick Doreian

†2003a Some dynamics of social balance processes: bringing Heider back into balance theory. *Social Networks* 25 (2003), 17–49.

Presents a model for evolution of balance and clusterability (as in Davis 1967a) of a signed digraph and explores it via computer simulations.

Definitions: Given a signed digraph $\vec{\Sigma}$ and a partition π of V, define the 'clusterability' $c(\vec{\Sigma}, \pi) :=$ (# negative edges within blocks of π) + (# positive edges between blocks). Define $\pi(\vec{\Sigma}) :=$ any π that minimizes $c(\vec{\Sigma}, \pi)$. Define $\vec{\Sigma}(v_i) := \{v_i\vec{v}_j \in \vec{E}(\vec{\Sigma})\}$ with signs. ($\vec{\Sigma}$ models relations in a social group V. $\vec{\Sigma}_i$ is the graph of relations perceived by v_i.)

Initial conditions: Fixed $|V|$, fixed "contentiousness" $p :=$ the probability that an initial edge is negative, a fixed "communication" rule, random $\vec{\Sigma}^0$

and, for each $v_i \in V$, $\vec{\Sigma}_i^0 := \vec{\Sigma}^0$. At time $t+1$, $\vec{\Sigma}_i^t(v_i)$ changes to $\vec{\Sigma}_i^{t+1}(v_i)$ to minimize $d(d(\vec{\Sigma}_i^{t+1}, \pi(\vec{\Sigma}^t))$. Then $\vec{\Sigma}_j^{t+1}(i)$ changes to $\vec{\Sigma}_i^{t+1}(v_i)$ for some v_j (depending on $\vec{\Sigma}_i$ and the communication rule).

Computer simulations examined the types of changes and emerging clusterability of $\vec{\Sigma}^t$ or $\vec{\Sigma}_i^t$ as t increases, under four different communication rules, random initial conditions with various p, and $|V| = 3, 5, 7, 10$. The outcomes are highly suggestive (see §4; p seems influential). [*Problem.* Predict the outcomes in terms of initial conditions through a mathematical analysis.] [Added: 26 April 2009.] (**SD, sg: Bal, Clu: Alg**)(**PsS**)

John E. Hunter

1978a Dynamic sociometry. *J. Math. Sociology* 6 (1978), 87–138. MR 58 #20631. (**SG: Bal, Clu**)

Takehiro Inohara

2002a Characterization of clusterability of signed graph in terms of Newcomb's balance of sentiments. *Appl. Math. Comput.* 133 (2002), no. 1, 93–104. MR 2003i:05064. Zbl 1023.05072.

Assumption: all $\sigma(i,i) = +$. Thm. 3: A signed complete digraph is clusterable if and only if $\sigma(i,j) = -$ or $\sigma(j,k) = \sigma(i,k)$ for every triple $\{i,j,k\}$ of vertices (not necessarily distinct). [The notation is unnecessarily complicated.] (**SD: Clu, PsS**)

2003a Clusterability of groups and information exchange in group decision making with approval voting system. *Appl. Math. Comput.* 136 (2003), no. 1, 1–15. MR 2004b:91059. (**SD: KG: Bal, Clu, PsS**)

2004a Quasi-clusterability of signed graphs with negative self evaluation. *Appl. Math. Comput.* 158 (2004), no. 1, 201–215. MR 2005f:05072. (**SD: Clu, PsS**)

2004b Signed graphs with negative self evaluation and clusterability of graphs. *Appl. Math. Comput.* 158 (2004), no. 2, 477–487. MR 2005f:05073. (**SD: Clu, PsS**)

Eugene C. Johnsen and H. Gilman McCann

1982a Acyclic triplets and social structure in complete signed digraphs. *Social Networks* 3 (1982), 251–272.

Balance and clustering analyzed via triples rather than edges. [Possible because the digraph is complete. A later analysis via triples is in Doreian and Krackhardt (2001a).] (**SD: Bal, Clu**)

Jérôme Kunegis, Andreas Lommatzsch, and Christian Bauckhage

20xxa The slashdot zoo: mining a social network with negative edges. In: *Proc. 18th Int. Conf. on World Wide Web* (Madrid, 2009), pp. 741–750. Association for Computing Machinery, New York, 2009. (**SG: WG: Clu: Alg**)

T.A. McKee

1987a A local analogy between directed and signed graphs. *Utilitas Math.* 32 (1987), 175–180. MR 89a:05075. Zbl 642.05023. (**SG: d, Clu, Bal**)

Andrej Mrvar and Patrick Doreian

2009a Partitioning signed two-mode networks. *J. Math. Sociology* 33 (2009), no. 3, 196–221.

§2, "Formalization of block-modeling signed two-mode data": A signed two-mode network is a bipartite signed simple graph with color classes V_1, V_2. The objective is partitions π_1, π_2 of V_1, V_2 that minimize a "criterion function" $P := \alpha i_- + (1-\alpha)i_+$; usually $\alpha = .5$. $k_1 := |\pi_1|$ and $k_2 := |\pi_2|$, or other restrictions, may be specified. Definitions: $\pi_i := \{V_{i1}, \dots, V_{ik_i}\}$. A "block" is a nonvoid set $E(V_{1i}, V_{2j})$. Its sign is the sign of the majority of edges, + if a draw. e is "consistent" with (π_1, π_2) if it is in a block of sign $\sigma(e)$. $i_\varepsilon :=$ number of inconsistent edges of sign ε. [Added: 17 August 2009.] **(SG: Clu, PsS)**

Wouter de Nooy, Andrej Mrvar, and Vladimir Batagelj

2005a *Exploratory Social Network Analysis with Pajek.* Structural Anal. Soc. Sci., No. 27. Cambridge Univ. Press, Cambridge, Eng., 2005.

Pajek is a computer package that analyzes networks, i.e., graphs, including signed graphs. Ch. 4: "Sentiments and friendship." Computation of balance and clusterability of signed (di)graphs. §4.2: "Balance theory." Introductory. §4.4: "Detecting structural balance and clusterability." How to use Pajek to optimize clustering. §4.5: "Development in time." Pajek can look for evolution towards balance or clusterability.

§10.3: "Triadic analysis." Types of balance and clusterability, with the triads (order-3 induced subgraphs) that do or do not occur in each. Table 16, p. 209, "Balance-theoretic models", is a chart of 6 related models. §§10.7, 10.10: "Questions" and "Answers." Some are on balance models. §10.9: "Further reading." [Added: 28 April 2009.] **(SG, SD, PsS: Bal, Clu, Alg: Exp)**

Philippa Pattison

1993a *Algebraic Models for Social Networks.* Structural Analysis in the Social Sciences, 7. Cambridge Univ. Press, Cambridge, 1993.

Ch. 8, pp. 258–9: "The balance model. The complete clustering model." Embedded in a more general framework. **(SG, Sgnd: Adj, Bal, Clu: Exp)**

Edmund R. Peay

1977b Indices for consistency in qualitative and quantitative structures. *Human Relations* 30 (1977), 343–361.

Proposes an index of nonclusterability for signed graphs and generalizes to edges weighted by a linearly ordered set. **(SG, Gen: Clu: Fr(Gen))**

1982a Structural models with qualitative values. *J. Math. Sociology* 8 (1982), 161–192. MR 83d:92107. Zbl 486.05060.

See mainly §3: "Structural consistency." **(sd: Gen: Bal, Clu)**

1976b *Discrete Mathematical Models, With Applications to Social, Biological, and Environmental Problems.* Prentice-Hall, Englewood Cliffs, N.J., 1976. Zbl 363.90002.

§3.1: "Signed graphs and the theory of structural balance." Many topics are developed in the exercises. Exercise 4.2.7 (from Phillips (1967a)).
(SG, SD: Bal, Alg, Adj, Clu, Fr, PsS: Exp, Exr)

Tadeusz Sozański

1976a Processus d'équilibration et sous-graphes équilibrés d'un graphe signé complet. *Math. Sci. Humaines*, No. 55 (1976), 25–36, 83. MR 58 #27613.

Σ denotes a signed K_n. The "level of balance" ("indice du niveau d'équilibre") $\rho(\Sigma) :=$ maximum order of a balanced subgraph. [Complement of the vertex deletion number.] Define distance $d(\Sigma_1, \Sigma_2) := |E_{1+} \triangle E_{2+}|$. Say

Σ is p-clusterable if Σ^+ consists of p disjoint cliques [its "clusters"]. Thm. 1 evaluates the frustration index of a p-clusterable Σ. Thm. 2 bounds $l(\Sigma)$ in terms of n and $\rho(\Sigma)$. A negation set U for Σ "conserves" a balanced induced subgraph if they are edge-disjoint; it is "(strongly) conservative" if it conserves some (resp., every) maximum-order balanced induced subgraph. Thm. 3: Every minimum negation set conserves every balanced induced subgraph of order $> \frac{2}{3}n$. Thm. 4: A minimum negation set can be ordered so that, successively negating its edges one by one, ρ never decreases.

(**SG: KG: Fr, Clu**)

1982a Model rownowagi strukturalnej. Teoria grafow oznakowanych i jej zastosowania w naukach spotecznych. [The structural balance model. The theory of signed graphs and its applications in the social sciences.] (In Polish.) Ph.D. thesis, Jagellonian Univ., Krakow, 1982. (**SG, PsS: Bal, Fr, Clu, Adj, Ref**)

Chaitanya Swamy

2004a Correlation clustering: Maximizing agreements via semidefinite programming. In: *Proc. Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (New Orleans, 2004), pp. 526–527 (electronic). ACM, New York, and SIAM, Philadelphia, 2004. MR 2291092. (**SG: WG: Clu: Alg**)

Ioan Tomescu

1974a La réduction minimale d'un graphe à une réunion de cliques. *Discrete Math.* 10 (1974), 173–179. MR 51 #247. Zbl 288.05127. (**SG: Bal, Clu**)

Stanley Wasserman and Katherine Faust

1994a *Social Network Analysis: Methods and Applications.* Structural Anal. Soc. Sci., 8. Cambridge Univ. Press, Cambridge, 1994. Zbl 980.24676.

§1.2: "Historical and theoretical foundations." A brief summary of various network methods in sociometry, signed graphs and digraphs among them. §4.4: "Signed graphs and signed directed graphs." Mathematical basics. §4.5: "Valued graphs and valued directed graphs." Mentions unweighted and positively weighted signed (di)graphs. Ch. 6: "Structural balance and transitivity." Application of balance of signed (di)graphs and of ensuing notions like clusterability, historically evolving into transitivity of unsigned digraphs. History and evaluation. §6.1: "Structural balance." Balance, indices of imbalance. §6.2: "Clusterability." All graphs, and complete graphs, as in Davis (1967a). §6.3: "Generalizations of clusterability." §6.3.2: "Ranked clusterability." As in Davis and Leinhardt (1972a). [Added: 28 April 2009.]

(**PsS, SG, SD: Bal, Fr, Clu, Gen: Exp, Ref**)

D.J.A. Welsh [Dominic Welsh]

1993a *Complexity: Knots, Colourings and Counting.* London Math. Soc. Lecture Note Ser., 186. Cambridge Univ. Press, Cambridge, Eng., 1993. MR 94m:57027. Zbl. 799.68008.

Includes very brief treatments of some appearances of signed graphs.

Sect. 4.4: "The Ashkin–Teller–Potts model". This treatment of the Potts model has a different Hamiltonian from that of Fischer and Hertz (1991a). [It does not seem that Welsh intends to admit edge signs but if they are allowed then the Hamiltonian (without edge weights) is $-\sum \sigma(e_{ij})(\delta(s_i, s_j) - 1)$. Up to halving and a constant term, this is Doreian and Mrvar's (1996a) clusterability measure $P(\pi)$, with $\alpha = .5$, of the vertex partition induced by

the state.] [Also cf. Fischer and Hertz (1991a).] (**clu, Phys**)

Bo Yang, William K. Cheung, and Jiming Liu

2007a Community mining from signed social networks. *IEEE Transactions on Knowledge and Data Engineering* 19 (2007), no. 10, 1333–1348.

Given a (positively weighted) signed (di)graph, the authors provide an algorithm for an approximate clustering. Input: The graph and a length parameter l. Step 1: Construct transition probabilities $pij := [\sigma ij w_{ij}]^+/d(v_i)$. Step 2: Apply the probabilities in a random walk of length $\leq l$; the matrix of l-step probabilities is $(p_{ij})^l$. Combine in a cluster the vertices that are detect reasonable clusters.

Also, a cut algorithm for approximate clustering. A cluster is $X \subset V$ such that the total net degree $d^{\pm}(\Sigma{:}X) \geq d^{\pm}(X, X^c)$ and $d^{\pm}(X^c, X) \leq d^{\pm}(\Sigma{:}X^c)$. [Added: 11 Feb. 2009.] (**SG: WG: Clu: Alg**)