

B

Review of Trigonometry

Angles

Angles can be measured in degrees or radians (abbreviated as rad). The angle given by a complete revolution contains 360° , which is the same as 2π rad. Therefore

(1)

$$\pi \text{ rad} = 180^\circ$$

and

(2) $1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ$ $1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$

Example 1

(a) Find the radian measure of 60° . (b) Express $5\pi/4$ rad in degrees.

Solution

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60^\circ = 60 \left(\frac{\pi}{180} \right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4} \left(\frac{180}{\pi} \right) = 225^\circ$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

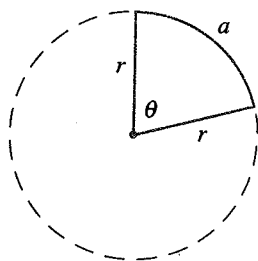


Figure 1

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a . Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π , we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a , we obtain

(3)

$$\theta = \frac{a}{r}$$

$$a = r\theta$$

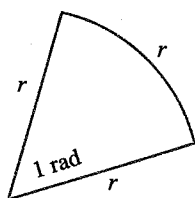


Figure 2

Remember that Equations 3 are valid only when θ is measured in radians.

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

Example 2

- (a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
 (b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

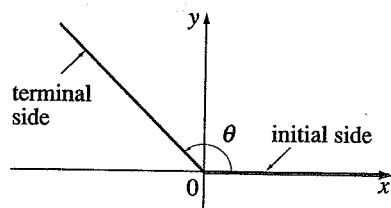
Solution

(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \frac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3 \left(\frac{3\pi}{8} \right) = \frac{9\pi}{8} \text{ cm}$$

Figure 3
 $\theta \geq 0$

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x -axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4. Figure 5 shows several examples of angles in standard position. Notice that different

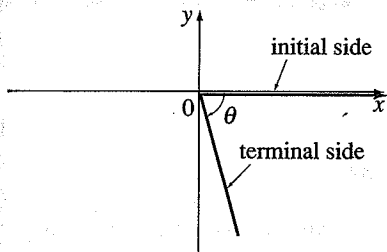
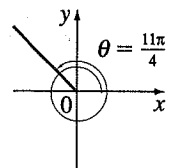
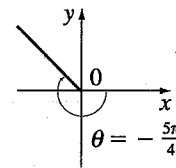
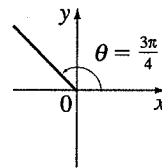
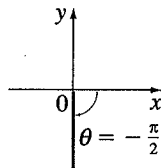
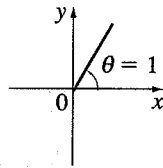


Figure 4
 $\theta < 0$

Figure 5
Angles in standard position



angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \quad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

The Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6):

(4)	$\sin \theta = \frac{\text{opp}}{\text{hyp}}$	$\csc \theta = \frac{\text{hyp}}{\text{opp}}$
	$\cos \theta = \frac{\text{adj}}{\text{hyp}}$	$\sec \theta = \frac{\text{hyp}}{\text{adj}}$
	$\tan \theta = \frac{\text{opp}}{\text{adj}}$	$\cot \theta = \frac{\text{adj}}{\text{opp}}$

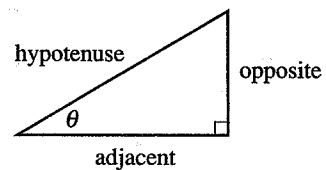


Figure 6

This definition does not apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

(5)	$\sin \theta = \frac{y}{r}$	$\csc \theta = \frac{r}{y}$
	$\cos \theta = \frac{x}{r}$	$\sec \theta = \frac{r}{x}$
	$\tan \theta = \frac{y}{x}$	$\cot \theta = \frac{x}{y}$

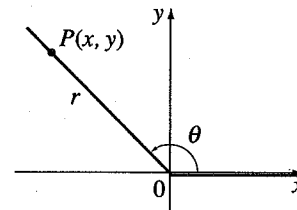


Figure 7

Since division by 0 is not defined, $\tan \theta$ and $\sec \theta$ are undefined when $x = 0$ and $\csc \theta$ and $\cot \theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin \theta$ means the sine of the angle whose radian measure is θ . For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number we must remember to set our calculator in radian mode and then we obtain



$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write $\sin 3^\circ$ and, with our calculator

in degree mode, we find that

$$\sin 3^\circ \approx 0.05234$$

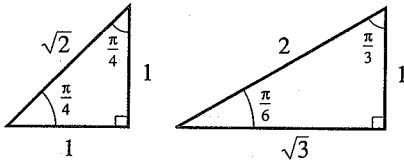


Figure 8

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 8. For instance,

$$\begin{aligned} \sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \sin \frac{\pi}{6} &= \frac{1}{2} & \sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} \\ \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} & \cos \frac{\pi}{3} &= \frac{1}{2} \\ \tan \frac{\pi}{4} &= 1 & \tan \frac{\pi}{6} &= \frac{1}{\sqrt{3}} & \tan \frac{\pi}{3} &= \sqrt{3} \end{aligned}$$

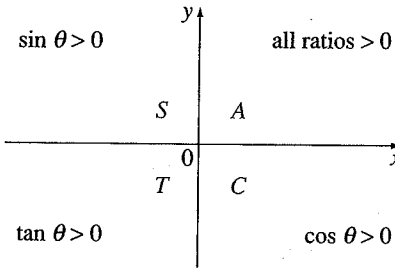


Figure 9

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule “All Students Take Calculus” in Figure 9.

Example 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

Solution From Figure 10 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \quad y = \sqrt{3} \quad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\begin{aligned} \sin \frac{2\pi}{3} &= \frac{\sqrt{3}}{2} & \cos \frac{2\pi}{3} &= -\frac{1}{2} & \tan \frac{2\pi}{3} &= -\sqrt{3} \\ \csc \frac{2\pi}{3} &= \frac{2}{\sqrt{3}} & \sec \frac{2\pi}{3} &= -2 & \cot \frac{2\pi}{3} &= -\frac{1}{\sqrt{3}} \end{aligned}$$

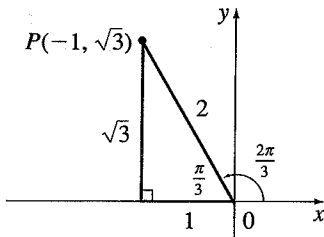


Figure 10

The following table gives some values of $\sin \theta$ and $\cos \theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

Example 4 If $\cos \theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ .

Solution Since $\cos \theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 11. If the opposite side has length x , then the Pythagorean theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

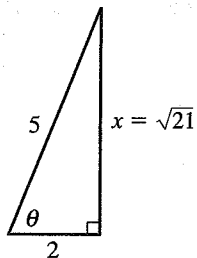


Figure 11

$$\sin \theta = \frac{\sqrt{21}}{5} \quad \tan \theta = \frac{\sqrt{21}}{2}$$

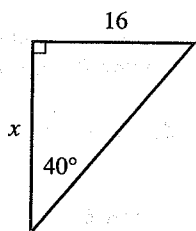


Figure 12

$$\csc \theta = \frac{5}{\sqrt{21}} \quad \sec \theta = \frac{5}{2} \quad \cot \theta = \frac{2}{\sqrt{21}}$$

Example 5 Use a calculator to approximate the value of x in Figure 12.

Solution From the diagram we see that

$$\tan 40^\circ = \frac{16}{x}$$

Therefore
$$x = \frac{16}{\tan 40^\circ} \approx 19.07$$

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

(6)

$$\begin{aligned} \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

For the next identity we refer to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

(7)

$$\sin^2 \theta + \cos^2 \theta = 1$$

If we now divide both sides of Equation 7 by $\cos^2 \theta$ and use Equations 6, we get

(8)

$$\tan^2 \theta + 1 = \sec^2 \theta$$

Similarly, if we divide both sides of Equation 7 by $\sin^2 \theta$, we get

(9)

$$1 + \cot^2 \theta = \csc^2 \theta$$

(10a) The identities

$$\sin(-\theta) = -\sin \theta$$

(10b)

$$\cos(-\theta) = \cos \theta$$

show that \sin is an odd function and \cos is an even function. (See Section 4 of Review and Preview.) They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 39).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

$$(11) \quad \sin(\theta + 2\pi) = \sin \theta \quad \cos(\theta + 2\pi) = \cos \theta$$

These identities show that the sine and cosine functions are periodic with period 2π .

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

$$(12a) \quad \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$(12b) \quad \cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are indicated in Exercises 85, 86, and 87.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b we obtain the following **subtraction formulas**:

$$(13a) \quad \sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$(13b) \quad \cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for $\tan(x \pm y)$:

$$(14a) \quad \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$(14b) \quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas (12) we get the **double angle formulas**:

$$(15a) \quad \sin 2x = 2 \sin x \cos x$$

$$(15b) \quad \cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double angle formulas for $\cos 2x$:

$$(16a) \quad \cos 2x = 2 \cos^2 x - 1$$

$$(16b) \quad \cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$ we get the following **half-angle formulas**, which are useful in integral calculus:

(17a)

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

(17b)

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally we state the **product formulas**, which can be deduced from Equations 12 and 13.

(18a)

$$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$$

(18b)

$$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$$

(18c)

$$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$$

There are many other trigonometric identities but those we have stated are the ones used most often in calculus. If you forget any of them, remember that they can all be deduced from Equations 12a and 12b.

Example 6 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

Solution Using the double angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \quad \text{or} \quad \sin x(1 - 2 \cos x) = 0$$

Therefore there are two possibilities:

$$\begin{aligned} \sin x = 0 & \quad \text{or} \quad 1 - 2 \cos x = 0 \\ x = 0, \pi, 2\pi & \quad \cos x = \frac{1}{2} \\ & \quad x = \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

There are five solutions to the given equation: $0, \pi/3, \pi, 5\pi/3, \text{ and } 2\pi$. ■

Graphs of the Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 13(a), is obtained by plotting points for $0 \leq x \leq 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the integer multiples of π , that is,

$$\sin x = 0 \quad \text{when } x = n\pi \quad n \text{ an integer}$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a) the graph of cosine is obtained by shifting

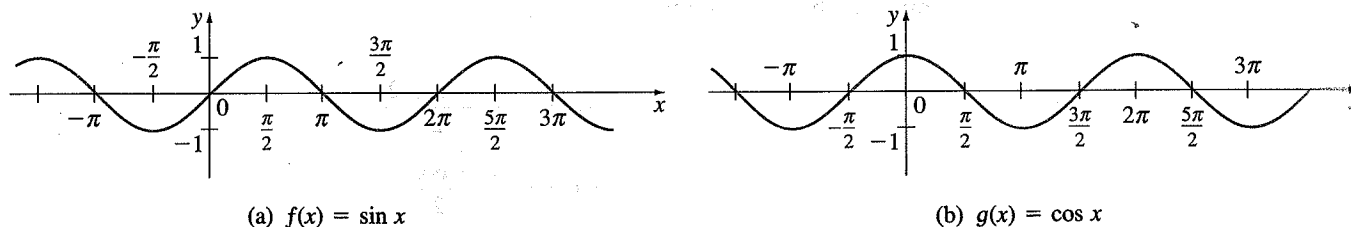


Figure 13

the graph of sine by an amount $\pi/2$ to the left [see Figure 13(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

The graphs of the remaining four trigonometric functions are shown in Figure 14 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π , whereas cosecant and secant have period 2π .

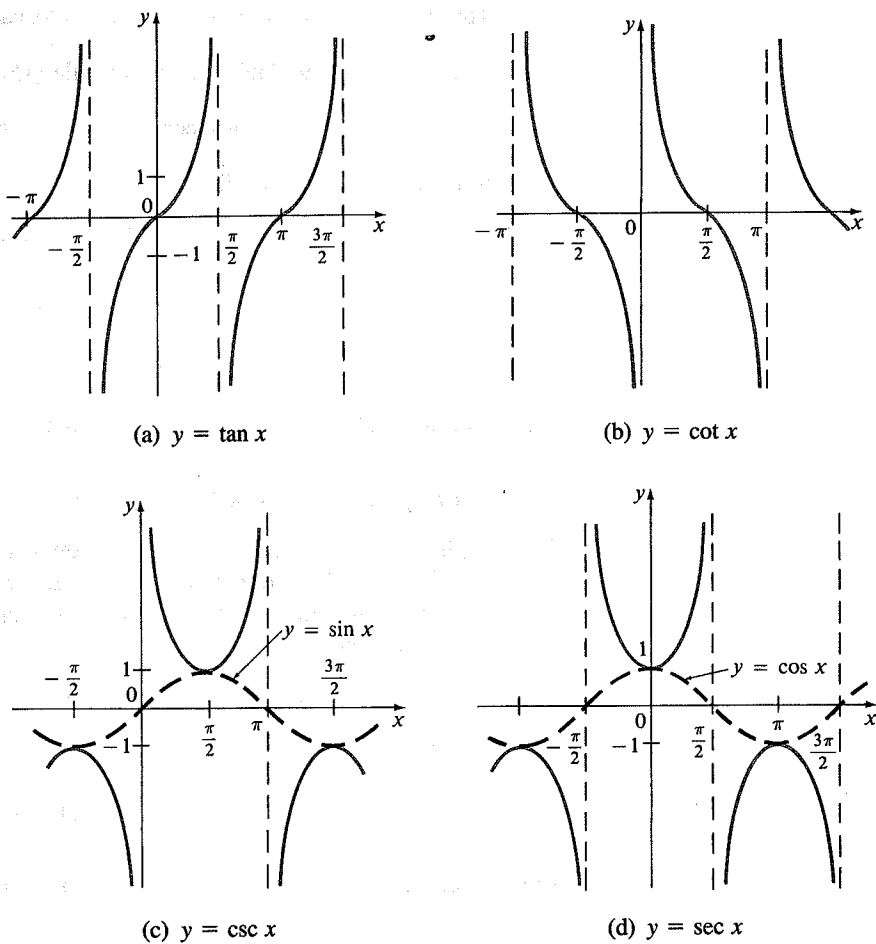


Figure 14

APPENDIX B Exercises

In Exercises 1–6 convert from degrees to radians.

1. 210° 2. 300° 3. 9°
 4. -315° 5. 900° 6. 36°

In Exercises 7–12 convert from radians to degrees.

7. 4π 8. $-\frac{7\pi}{2}$ 9. $\frac{5\pi}{12}$
 10. $\frac{8\pi}{3}$ 11. $-\frac{3\pi}{8}$ 12. 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.
 14. If a circle has radius 10 cm, what is the length of the arc subtended by a central angle of 72° ?
 15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?
 16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

In Exercises 17–22 draw, in standard position, the angle whose measure is given.

17. 315° 18. -150° 19. $-\frac{3\pi}{4}$ rad
 20. $\frac{7\pi}{3}$ rad 21. 2 rad 22. -3 rad

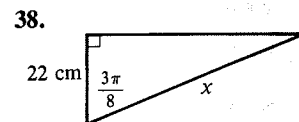
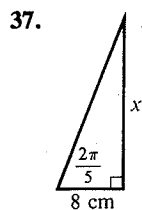
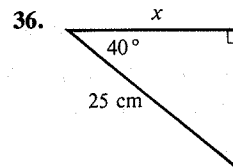
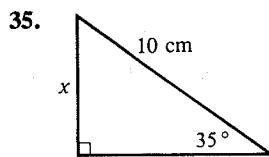
In Exercises 23–28 find the exact trigonometric ratios for the angles whose radian measures are given.

23. $\frac{3\pi}{4}$ 24. $\frac{4\pi}{3}$ 25. $\frac{9\pi}{2}$
 26. -5π 27. $\frac{5\pi}{6}$ 28. $\frac{11\pi}{4}$

In Exercises 29–34 find the remaining trigonometric ratios.

29. $\sin \theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$
 30. $\tan \alpha = 2$, $0 < \alpha < \frac{\pi}{2}$
 31. $\sec \phi = -1.5$, $\frac{\pi}{2} < \phi < \pi$
 32. $\cos x = -\frac{1}{3}$, $\pi < x < \frac{3\pi}{2}$
 33. $\cot \beta = 3$, $\pi < \beta < 2\pi$
 34. $\csc \theta = -\frac{4}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$

In Exercises 35–38 find, correct to five decimal places, the length of the side labeled x .



39. (a) Prove Equation 10a.
 (b) Prove Equation 10b.
 40. (a) Prove Equation 14a.
 (b) Prove Equation 14b.
 41. (a) Prove Equation 18a.
 (b) Prove Equation 18b.
 (a) Prove Equation 18c.

Prove the identities in Exercises 42–58.

42. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ 43. $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
 44. $\sin(\pi - x) = \sin x$ 45. $\sin \theta \cot \theta = \cos \theta$
 46. $(\sin x + \cos x)^2 = 1 + \sin 2x$
 47. $\sec y - \cos y = \tan y \sin y$
 48. $\tan^2 \alpha - \sin^2 \alpha = \tan^2 \alpha \sin^2 \alpha$
 49. $\cot^2 \theta + \sec^2 \theta = \tan^2 \theta + \csc^2 \theta$
 50. $2 \csc 2t = \sec t \csc t$ 51. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
 52. $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta$
 53. $\sin x \sin 2x + \cos x \cos 2x = \cos x$
 54. $\sin^2 x - \sin^2 y = \sin(x + y) \sin(x - y)$
 55. $\frac{\sin \phi}{1 - \cos \phi} = \csc \phi + \cot \phi$
 56. $\tan x + \tan y = \frac{\sin(x + y)}{\cos x \cos y}$
 57. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$
 58. $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expressions in Exercises 59–64.

59. $\sin(x + y)$ 60. $\cos(x + y)$ 61. $\cos(x - y)$

62. $\sin(x - y)$ 63. $\sin 2y$ 64. $\cos 2y$

In Exercises 65–72 find all values of x in the interval $[0, 2\pi]$ that satisfy the given equation.

65. $2 \cos x - 1 = 0$ 66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$ 68. $|\tan x| = 1$

69. $\sin 2x = \cos x$ 70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$ 72. $2 + \cos 2x = 3 \cos x$

In Exercises 73–76 find all values of x in the interval $[0, 2\pi]$ that satisfy the given inequality.

73. $\sin x \leq \frac{1}{2}$ 74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$ 76. $\sin x > \cos x$

In Exercises 77–82 graph the given functions by starting with the graphs in Figures 12 and 13 and applying the transformations of Section 6 of Review and Preview where appropriate.

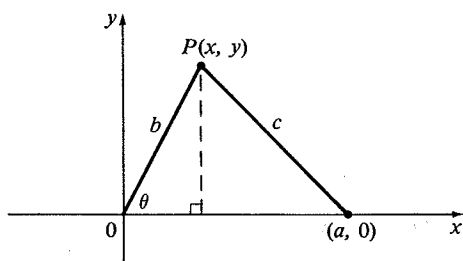
77. $y = \cos\left(x - \frac{\pi}{3}\right)$ 78. $y = \tan 2x$

79. $y = \frac{1}{3}\tan\left(x - \frac{\pi}{2}\right)$ 80. $y = 1 + \sec x$

81. $y = |\sin x|$ 82. $y = 2 + \sin\left(x + \frac{\pi}{4}\right)$

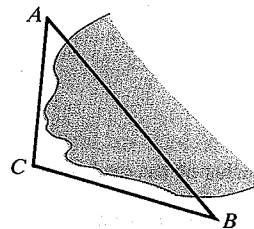
83. Prove the **Law of Cosines**: If a triangle has sides with lengths a , b , and c , and θ is the angle between the sides with lengths a and b , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



(Hint: Introduce a coordinate system so that θ is in standard position as in the figure. Express x and y in terms of θ and then use the distance formula to compute c .)

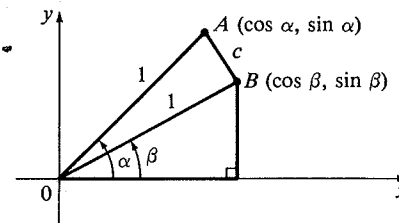
84. In order to find the distance $|AB|$ across a small inlet, a point C is located as in the figure and the following measurements were recorded: $\angle C = 103^\circ$, $|AC| = 820$ m, $|BC| = 910$ m. Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

[Hint: Compute c^2 in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the subtraction formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

to prove the subtraction formula for the sine function.