Quiz List for Math 304-04, Fall 2018
Solutions

Quiz 1. (M 8/27)

Qn 1. What is my name?

Answer: Thomas Zaslavsky

Qn 2. Let $a, b \in \mathbb{R}$. What is the rank of the matrix $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$?

Answer: It depends on the values of a and b.
If both $a, b \neq 0$, the rank is 2 because M has two pivot columns.
If one is 0 and the other is not, the rank is 1 because M has one pivot column.
If $a, b = 0$, the rank is 0 because $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the zero matrix.
(Note that it is not important whether a or b equals 1.)

Quiz 2. (F 8/31)

Qn 1. What is your name?

Answer: (You know best.)

Qn 2. Assume C is a 4×6 matrix with rank 4. Let L_C be the function whose equation is $L_C(\mathbf{x}) = C \cdot \mathbf{x}$.

(i) What is the domain of L_C?
Answer: $\mathbb{R}^6$ because $\mathbf{x}$ has to be a 1×6 matrix, i.e., $\mathbf{x} \in \mathbb{R}^6$, for the multiplication to be valid.

(ii) What is the codomain of L_C?
Answer: $\mathbb{R}^4$ because the product has 4 rows and one column; that is, $C\mathbf{x} \in \mathbb{R}^4$.

(iii) Is L_C one-to-one (injective)? Why?
Answer: No, because we have a test (Theorem 1.8), which is that L_C is injective if and only if $r = n$ (rank = # of columns).
Answer: (Alternative.) No, because there are two vectors $\mathbf{x}$ with the same value of $C\mathbf{x}$ (I'm not producing them here, but this answer is complete only if you produce two such vectors; we'll do that soon).

(iv) Is L_C onto (surjective)? Why?
Answer: Yes, because according to our test, L_C is surjective if and only if $r = m$ (rank = # of rows).
Answer: (Alternative.) Yes, because for every vector $\mathbf{y} \in \mathbb{R}^4$, I can produce an $\mathbf{x} \in \mathbb{R}^6$ such that $C\mathbf{x} = \mathbf{y}$ (I'm not actually producing $\mathbf{x}$ here; we'll study this soon).

Quiz 3. (M 9/17)

Qn 1. Write $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 9 \\ 4 \end{bmatrix}$ as a linear combination of the standard basis vectors in $\mathbb{R}^4$.

Answer: $\mathbf{x} = 1\mathbf{e}_1 + 3\mathbf{e}_2 + 9\mathbf{e}_3 + 4\mathbf{e}_4$.

Qn 2. A linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ has values

$$L(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \text{ and } L(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

What is the matrix of L?

Answer: The matrix is $\begin{bmatrix} 2 & 3 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$.

(The values of $L(\mathbf{e}_j)$ are the columns of the matrix.)

Quiz 4. (W 10/3)

Qn 1. Suppose A and B are symmetric $n \times n$ matrices. Prove that AB is symmetric if and only if $AB = BA$.

Answer: AB is symmetric $\iff$ $(AB)^T = AB$ $\iff$ $B^T A^T = AB$ (by the transpose of a product) $\iff$ $BA = AB$ (because A, B are symmetric). Done!

Quiz 5. (F 10/5)

Qn 1. Write the definition of when a matrix A is symmetric.

Answer: A is symmetric if $A^T = A$.

Qn 2. Write the vector $\begin{bmatrix} 1 \\ -1 \\ \pi \\ e \end{bmatrix}$ as a linear combination of the standard basis vectors in $\mathbb{R}^4$.

Answer: $\begin{bmatrix} 1 \\ -1 \\ \pi \\ e \end{bmatrix} = 1\mathbf{e}_1 - 1\mathbf{e}_2 + \pi\mathbf{e}_3 + e\mathbf{e}_4.$

Qn 3. Write $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, if possible. If not, show it is impossible.

Answer: The first step is to set up the linear combination with unknown coefficients:

$$\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This is a linear system

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}.$$

You solve it the same way as any linear system, by setting up the augmented matrix and reducing to RREF.

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & \frac{7}{2} \end{bmatrix}$$

It follows that $c_1 = 5$, $c_2 = -\frac{9}{2}$, $c_3 = \frac{7}{2}$, so the answer is

$$\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{9}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Quiz 6. (M 10/29)

Two bases of $\mathbb{R}^3$ are $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Find the coordinate vector of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with respect to $\mathcal{B}$.

Answer:
We solve $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{v}$. That becomes the augmented matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

so the coordinate vector is $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \end{bmatrix}^T$.
It's also possible to guess this particular answer by noticing that $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}^T = 2\mathbf{v}$ so $\mathbf{v} = \frac{1}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2 + \frac{1}{2}\mathbf{b}_3$, which tells you the components of the coordinate vector.

(b) If $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$, what is $\mathbf{w}$?

Answer: By the definition of a coordinate vector, $\mathbf{w} = 5\mathbf{b}_1 + 0\mathbf{b}_2 - 1\mathbf{b}_3 = \begin{bmatrix} 4 & 5 & -1 \end{bmatrix}^T$.

(c) Find the transition matrix (change of basis matrix) ${}_{\mathcal{E}}P_{\mathcal{B}}$.
Answer: The transition matrix has the form

$$\begin{bmatrix} [\mathbf{b}_1]_{\mathcal{E}} & [\mathbf{b}_2]_{\mathcal{E}} & [\mathbf{b}_3]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$$

since $\mathcal{E}$ is the standard basis. Therefore, the answer is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(d) How is ${}_{\mathcal{E}}P_{\mathcal{B}}$ related to ${}_{\mathcal{B}}P_{\mathcal{E}}$? (Do not compute.)
Answer: They are inverse matrices. I.e., ${}_{\mathcal{E}}P_{\mathcal{B}} = ({}_{\mathcal{B}}P_{\mathcal{E}})^{-1}$, or if you prefer, ${}_{\mathcal{B}}P_{\mathcal{E}} = ({}_{\mathcal{E}}P_{\mathcal{B}})^{-1}$.

Quiz 7. (F 11/9)

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Qn 1. Find all eigenvalues of A and their eigenspaces.
Answer:
Step 1: Find the eigenvalues. We need $\det(A - \lambda I) = 0$. First, compute the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix}$$
$$= \Big[(1-\lambda)^3 + (-1) + 0\Big] - \Big[0 + 0 + 0\Big]$$
$$= -\lambda^3 + 3\lambda^2 - 3\lambda = -\lambda(\lambda^2 - 3\lambda + 3).$$

Next, solve $-\lambda(\lambda^2 - 3\lambda + 3) = 0$. One solution is $\lambda_1 = 0$, from the first factor. For the second factor, apply the quadratic formula: $\lambda^2 - 3\lambda + 3 = 0$ gives $\lambda_2, \lambda_3 = \dfrac{3 \pm \sqrt{9 - 12}}{2}$, which are not real numbers, so we ignore them. (Confession: I made a little mistake in choosing A.)
Step 2: Find the eigenspaces. There is only one eigenvalue to work with here, namely $\lambda_1 = 0$. Its eigenspace is $\mathrm{Nul}(A - \lambda_1 I) = \mathrm{Nul}(A)$. Thus, I will find the null space of A:

$$A \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

That gives $\mathbf{x} = x_3 \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$, where x_3 is a free variable. The eigenspace is

$$\mathrm{Nul}(A - \lambda_1 I) = \mathrm{span}\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$$

and a basis for it is $\{ \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T \}$. (Both of those answers for the eigenspace were accepted on this quiz.)

Qn 2. Compare their algebraic and geometric multiplicities.
Answer: This only makes sense for $\lambda_1 = 0$. The algebraic multiplicity is 1, since λ_1 is a zero just once of the characteristic polynomial. The geometric multiplicity is the dimension of the eigenspace, which is also 1. Thus, the algebraic and geometric multiplicities are equal.

Quiz 8. (M 11/19)

Diagonalize $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. That means

1. Find the eigenvalues.
   Answer: $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 1 = \lambda^2 - 3\lambda + 1.$
   This is the characteristic polynomial. Set it equal to 0 and solve by the quadratic formula: $\lambda = \frac{3 \pm \sqrt{9-4}}{2}$, so the eigenvalues are
   $$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$
2. Find the eigenvectors.
   Answer: For eigenvectors corresponding to λ_1 we set up the matrix
   $$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & 2-\lambda_2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix},$$
   which gives one free variable so the eigenspace basis has one vector. Solving the system, $x_1 = -\frac{1-\sqrt{5}}{2} x_2$ with x_2 free. That gives us an eigenvector
   $$\mathbf{x}_1 = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$
   Similarly, for λ_2 we get an eigenvector
   $$\mathbf{x}_2 = \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$
3. Find the matrices P and D such that $A = PDP^{-1}$.
   Answer: The general rule is that D is the eigenvalue matrix and, if we have a basis of eigenvectors, $\{\mathbf{x}_1, \mathbf{x}_2\}$, then $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$. Thus,
   $$D = \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}.$$