Quiz 10 for Math 304-06, 11/20 Your name SOLUTIONS

(1) (5 points) Complete the sentence: An orthogonal matrix is defined as a square matrix B such that

Answer 1: the columns are orthogonal unit vectors.
Answer 2: $B^T B = I$.

(2) (5 points) Complete the sentence: A matrix A is called orthogonally diagonalizable if

it is diagonalizable in the form $A = PDP^{-1}$ where P is an orthogonal matrix.

(3) (5 points) Complete the sentence: If A is symmetric, if λ_1 and λ_2 are two different eigenvalues, and if λ_1 has an eigenvector $\mathbf{u}_1$ and λ_2 has an eigenvector $\mathbf{u}_2$, then $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal.

(4) (5 points) Complete the sentence: Suppose A is an $n \times n$ matrix. Then $\mathbb{R}^n$ has a basis that consists of orthogonal eigenvectors of A, if and only if A is symmetric.

(5) (10 points) Let $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$. Find an ortbogonal matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix. Also find D.

Step 1: Find the eigenvalues.

$$\begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = (9-\lambda)(16-\lambda) - 12^2 = \lambda^2 - 25\lambda.$$

Setting this equal to 0, we get $\lambda = 0, 25$. They are the eigenvalues.

Step 2: Find the eigenvectors.
For $\lambda_1 = 0$,

$$A - \lambda I = A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}$$

so the eigenvector satisfies $x_1 = -\frac{4}{3}x_2$, thus we can use as the eigenvector either $\mathbf{u}_1 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ or $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ (or other scalar multiples), as you choose. But we want a unit vector, so we we should use $\mathbf{u}_1 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$, since $\| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \| = 5$.

For $\lambda_2 = 25$,

$$A - \lambda I = A - 25I = \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$$

so the eigenvector satisfies $x_1 = \frac{3}{4}x_2$, thus we can use as the eigenvector either $\mathbf{u}_2 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ or $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (or other scalar multiples), as you choose. Since we want a unit vector, we we should use $\mathbf{u}_2 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, since $\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \| = 5$.

Step 3. State the answer, which is

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}, \quad P = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}.$$