Quiz 6 for Math 304-06, 10/25 Your name SOLUTIONS

(1) (5 points) What do we mean by saying a square matrix A is *diagonalizable*?

We mean there exist square matrices P and D such that D is diagonal and $A = PDP^{-1}$.

(2) (15 points) Is $C = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ diagonalizable? (Hint: Yes.) Diagonalize it!

Step 1. Find the eigenvalues.

$$\det(C - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 2 & 0-\lambda \end{vmatrix} = \lambda^2 - 2.$$

Setting this $= 0$, we find the eigenvalues $\lambda = \pm\sqrt{2}$.

Step 2. Find the eigenvectors.
For $\lambda = \sqrt{2}$, we want the null space of

$$\begin{bmatrix} 0-\sqrt{2} & 1 \\ 2 & 0-\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -1/\sqrt{2} \\ 0 & 0 \end{bmatrix},$$

so $x_1 - \frac{1}{\sqrt{2}}x_2 = 0$, i.e., $x_1 = \frac{1}{\sqrt{2}}x_2$. A suitable eigenvector is $\begin{bmatrix} 1/\sqrt{2} \\ 1 \end{bmatrix}$, or if you prefer, $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ or any other scalar multiple, *except the zero vector*, which cannot ever be an eigenvector.
For $\lambda = -\sqrt{2}$, we want the null space of

$$\begin{bmatrix} 0+\sqrt{2} & 1 \\ 2 & 0+\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix},$$

so $x_1 + \frac{1}{\sqrt{2}}x_2 = 0$, i.e., $x_1 = -\frac{1}{\sqrt{2}}x_2$. (Notice how the sign changes from the first equation when you solve for x_1; this is important.) A suitable eigenvector is $\begin{bmatrix} -1/\sqrt{2} \\ 1 \end{bmatrix}$, or if you prefer, $\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}$ or any other scalar multiple *except* the zero vector.

Step 3. Write the answer:
Yes, C is diagonalizable. (You should say this! How else do I know what you think?)

The matrices are $D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$ and $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 1 \end{bmatrix}$, or you may prefer $P = \begin{bmatrix} 1 & -1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$, etc.