Test 3 for Math 304-06, 11/25/19 Your name SOLUTIONS

Show **all the work or explanation** necessary to justify the answer. Remember standard notation: for example, in $\mathbb{R}^3$, $\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and so on, and $\mathbf{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$.

Please start each new numbered problem on a fresh page.

(1) (4 points) State the Cauchy–Schwarz inequality.
Solution: $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$. The absolute value sign on the inner product is essential. Half credit for omitting it, or assuming dot product or $<$.

(2) (4 points) State the triangle inequality for vectors.
Solution: $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$, or $\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

(3) (8 points) Is the function $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_2 + u_2v_1$ an inner product on $\mathbb{R}^2$? Prove your answer. (Read carefully.)
Solution: No. Check the fourth property: $\langle \mathbf{u}, \mathbf{u}\rangle \ge 0$. According to this inner product, $\langle \mathbf{u}, \mathbf{u}\rangle = 2u_1u_2$, which can be negative, and it can be zero for some nonzero vectors (that is two separate violations). It is sufficient if you found an example of a negative result, e.g., $\mathbf{u} = (1, -1)$ gives inner product -2.

If you checked the first three properties by actual calculation using this inner product and found they are satisfied, you got 4 points for that.

It is not enough to say this formula differs from the dot-product formula. There are many other inner products that can be defined on $\mathbb{R}^2$ (see the next question for one of them). You have to check the properties.

(4) (24 points) Consider the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + 8u_2v_2$ on $\mathbb{R}^2$.
Solution: I gave you this inner product for a reason. If you ignore it and use dot product instead, you get little credit.

(a) Find the norm of the vector $\mathbf{u} = (2, 1)$.
Solution: (4 pts) $\|(2,1)\| = \sqrt{\langle (2,1),(2,1)\rangle} = \sqrt{2\cdot 2\cdot 2 + 8\cdot 1\cdot 1} = 4$.

(b) Find a unit vector $\hat{\mathbf{u}}$ in the same direction as $\mathbf{u}$.
Solution: (2 pts) $\|\mathbf{u}\|^{-1}\mathbf{u} = \frac{1}{4}(2,1) = (\frac{1}{2}, \frac{1}{4})$.

(c) Find the distance between $\mathbf{u}$ and $\mathbf{e}_1$.
Solution: (2 pts) $\mathbf{u}-\mathbf{e}_1 = (1,1)$, so the answer is $\|(1,1)\| = \sqrt{2\cdot 1\cdot 1 + 8\cdot 1\cdot 1} = \sqrt{10}$.

(d) Find a basis for the orthogonal complement, $W^\perp$, of $W = \operatorname{span}\{\mathbf{u}\}$.
Solution: (6 pts) $W^\perp = \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{v}, \mathbf{u}\rangle = 0\}$. That means we have to solve the equation $\langle (v_1, v_2), (2, 1)\rangle = 0$. That is, $2v_1 2 + 8v_2 1 = 0$. The solution is $v_1 = -2v_2$, giving $\mathbf{v} = v_2\begin{bmatrix}-2\\1\end{bmatrix}$. A basis is therefore $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$, or instead you can use any (nonzero) scalar multiple of that basis vector.

(e) What is the orthogonal projection $\text{proj}_{W^\perp} \mathbf{u}$? If possible, solve without calculation.

Solution: (2 pts) The basis for $W^\perp$ is $\{(-2,1)\}$, so

$$\text{proj}_{W^\perp} \mathbf{u} = \frac{\langle (2,1),(-2,1)\rangle}{\langle (2,1),(2,1)\rangle} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{2\cdot 2\cdot(-2) + 8\cdot 1\cdot 1}{\langle (-2,1),(-2,1)\rangle} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \mathbf{0}.$$

The solution without calculation is this: Since $W^\perp$ is orthogonal to $\mathbf{u}$ by definition of the $\perp$ operation, the projection has to be the zero vector.

I gave full credit for either solution.

(f) What is the orthogonal projection $\text{proj}_W(1,0)$.

Solution: (4 pts) The basis for W is $\{\mathbf{u}\}$, so

$$\text{proj}_W(1,0) = \frac{\langle (1,0),(2,1)\rangle}{\langle (2,1),(2,1)\rangle} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2\cdot 1\cdot 2 + 8\cdot 0\cdot 1}{\langle (2,1),(2,1)\rangle} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{4}{16} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}.$$

(g) Use $(1,0)$ and the projection found in (4f) to find a nonzero vector that is orthogonal to $\text{proj}_W(1,0)$.

Solution: (2 pts) What I was looking for is $(1,0) - \text{proj}_W(1,0) = (\frac{1}{2}, -\frac{1}{4})$, which is always orthogonal to $\text{proj}_W(1,0)$ by general theory.

A second solution is to project $(1,0)$ onto $W^\perp$. We know a basis for $W^\perp$ from part (d). This method gives

$$\text{proj}_{W^\perp} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\langle (1,0),(-2,1)\rangle}{\langle (-2,1),(-2,1)\rangle} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{-4+0}{8+8} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/4 \end{bmatrix}$$

(the same answer—not a coincidence). This is not exactly what I was looking for, but I accepted it.

In fact, I accepted any correct answer with correct work.

(5) (10 points) Let V be an inner product space (that is, a vector space with inner product $\langle\ ,\ \rangle$). Prove that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ for vectors in V if and only if $\mathbf{u} \perp \mathbf{v}$.

Solution: Rewrite the left side of the equation as

$$\begin{aligned} \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle &= \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle, \end{aligned}$$

using the basic properties of an inner product. This last $= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\langle \mathbf{u},\mathbf{v}\rangle = 0$, which is the criterion for orthogonality of $\mathbf{u}$ and $\mathbf{v}$ (i.e., $\mathbf{u} \perp \mathbf{v}$) with respect to the inner product.

This is not a question about dot product.

One example is not a proof. It is only one example and tells you nothing about other examples.

(6) (30 points) Consider the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$.

(a) Can A be orthogonally diagonalized? Explain why (or why not) in few words.
Solution: (2 pts) Yes, because it is symmetric, or you could say $A = A^T$. There is no other valid reason.

(b) Find the eigenvalues of A and their multiplicities.
Solution: (8 pts) We find the determinant of $A - \lambda I$, i.e.,

$$\det \begin{bmatrix} \lambda & 0 & 0 & 4 \\ 0 & \lambda & 4 & 0 \\ 0 & 4 & \lambda & 0 \\ 4 & 0 & 0 & \lambda \end{bmatrix} = (\lambda^2 - 4^2)^2$$

so $\lambda = 4, 4, -4, -4$. You should use a valid method of evaluating the determinant, or you will get wrong answers and it will spoil the rest of the problem. Note that we can do row operations on $A - \lambda I$ to get the determinant, but we cannot do row operations on A because that will change all the eigenvalues.

(c) Find a basis for the eigenspace of each eigenvalue.
Solution: (5 pts each) First, I'll do $\lambda = 4$:

$$A - 4I = \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \\ 4 & 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have two equations and two free variables. Specifically, $x_1 - x_4 = 0$ and $x_2 - x_3 = 0$ so (watch out for the sign changes) $x_1 = x_4$ and $x_2 = x_3$. This gives the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

So, a basis for the eigenspace of $\lambda = 4$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Second, I'll do $\lambda = -4$:

$$A - (-4)I = \begin{bmatrix} 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have two equations and two free variables. Specifically, $x_1 + x_4 = 0$ and $x_2 + x_3 = 0$ so (watch out for the sign changes) $x_1 = -x_4$ and $x_2 = -x_3$. This gives the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

So, a basis for the eigenspace of $\lambda = -4$ is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(d) Find the diagonal matrix D and the matrix P for diagonalization.

Solution: (3 pts) The diagonal matrix is easy: $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ (from the eigenvalues). Some people prefer to put the -4's first. The order doesn't matter, as long as you form P in the same order.

The matrix P is formed by stacking up the eigenvectors: $P = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Use the same order of eigenvectors as the order you used for eigenvalues in D.

(e) Is A similar to D?

Solution: (1 pt) Yes, because $A = PDP^{-1}$.

(f) Produce an orthogonal diagonalization of A. That means find the matrices D and P for an orthogonal diagonalization.

Solution: (3 pts) D is always the diagonal eigenvalue matrix, the same as in (d).

For the new P we have to check whether the two eigenvalue bases in part (c) are orthonormal bases. Fortunately for us, the two eigenvectors in the $\lambda = 4$ basis are orthogonal and the two vectors in the $\lambda = -4$ basis are orthogonal. (If not, we have to do Gram–Schmidt on each set separately to get an orthogonal basis.) The eigenvectors for $\lambda = 4$ are orthogonal to those for $\lambda = -4$ by a theorem (or by checking; if they are not, you have a numerical error). Thus, we already have an orthogonal basis.

But we need unit vectors. Each basis vector in this example has norm $\sqrt{2}$. So, we divide each vector by $\sqrt{2}$, giving the orthogonal matrix

$$P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}.$$

(g) Invert the matrix P from part (6f) *without doing any computation.*

Solution: (3 pts) If you have a correct (i.e., orthogonal) matrix P in (f), then $P^{-1} = P^T$ so you only have to transpose.

(h) (0 pts) Have you been to Diagon Alley? Did you see an eigenvector there?

Solution: No, and no (speaking personally), but I hope to, one day.

(7) (10 points) Consider $\mathbb{P}_2$, the polynomial vector space with basis $\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$, where

$$p_1(t) = t^2, \quad p_2(t) = t^2 + t, \quad p_3(t) = t^2 + t + 1.$$

Also, the vector space $\mathbb{R}^2$ with the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Find the coordinate vector, $[p(t)]_{\mathcal{B}}$, of $p(t) = t - 2$ with respect to the basis $\mathcal{B}$.
Solution: (4 pts) You want to solve $t - 2 = c_1p_1(t) + c_2p_2(t) + c_3p_3(t)$. The solution is

$$\begin{aligned} t - 2 &= c_1p_1(t) + c_2p_2(t) + c_3p_3(t) = c_1(t^2) + c_2(t^2 + t) + c_3(t^2 + t + 1) \\ &= (c_1 + c_2 + c_3)t^2 + (c_2 + c_3)t + c_3, \end{aligned}$$

so $c_1 + c_2 + c_3 = 0$, $c_2 + c_3 = 1$, and $c_3 = -2$. The solution is $c_1 = -1$, $c_2 = 3$, $c_3 = -2$, so the answer is $[p(t)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

(b) For the transformation $T : \mathbb{R}^2 \to \mathbb{P}_2$ given by the rule $T(a, b) = at^2 + (a+b)t - 2b$, find the matrix M of T with respect to the bases $\mathcal{E}$ and $\mathcal{B}$.
Solution: (3pts) Theory says that $M = \big[[T(\mathbf{e}_1)]_{\mathcal{B}} \ [T(\mathbf{e}_2)]_{\mathcal{B}}\big]$ so you have to find $T(\mathbf{e}_1) = t^2 + t$ and $T(\mathbf{e}_2) = t - 2$. (3 pts) Then $M = \big[[t^2 + t]_{\mathcal{B}} \ [t - 2]_{\mathcal{B}}\big]$. Since $t^2 + t = p_2(t)$, we know $[t^2 + t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. We know $[t - 2]_{\mathcal{B}}$ from part (a). So,

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 3 \\ 0 & -1 \end{bmatrix}.$$

(8) (10 points) A basis for $\mathbb{R}^3$ is $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix} \right\}$.

Turn it into an orthonormal basis using Gram–Schmidt.

Solution: (Orthogonal basis: 9 points. Orthonormal basis: 10 points.) Let's call the vectors $\mathbf{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix}2\\1\\1\end{bmatrix}$.

We start by constructing an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

First step: $\mathbf{u}_1 = \mathbf{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

Second step: $\mathbf{u}_2 = \mathbf{b}_2 - \dfrac{\mathbf{b}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix} - \dfrac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix}$. (You can simplify this by multiplying by 2; but I won't in this solution.)

Third step: $\mathbf{u}_3 = \mathbf{b}_3 - \dfrac{\mathbf{b}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \dfrac{\mathbf{b}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix}2\\1\\1\end{bmatrix} - \dfrac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \dfrac{-1/2}{1/2}\begin{bmatrix}-1/2\\1/2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$. (Watch the signs. A common error is to lose track of the three negative signs in the last vector.)

This gives an orthogonal basis. For the orthonormal basis we calculate norms: $\|\mathbf{u}_1\| = \sqrt{2}$, $\|\mathbf{u}_2\| = \frac{1}{\sqrt{2}}$, $\|\mathbf{u}_3\| = 1$. Thus, the orthonormal basis is

$$\left\{ \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}.$$

Check (optional): Are the vectors orthogonal? Yes, all dot products are 0. Are they unit vectors? Yes. Yay!