TEST 3 FOR MATH 304-06, 11/25/19 YOUR NAME ______________________

Show **all the work or explanation** necessary to justify the answer. Remember standard notation: for example, in $\mathbb{R}^3$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and so on, and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

Please start each new numbered problem on a fresh page.

(1) (4 points) State the Cauchy–Schwarz inequality.

(2) (4 points) State the triangle inequality for vectors.

(3) (8 points) Is the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_2 + u_2 v_1$ an inner product on $\mathbb{R}^2$? Prove your answer. (Read carefully.)

(4) (24 points) Consider the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1 v_1 + 8u_2 v_2$ on $\mathbb{R}^2$.
  (a) Find the norm of the vector $\mathbf{u} = (2, 1)$.
  (b) Find a unit vector $\hat{\mathbf{u}}$ in the same direction as $\mathbf{u}$.
  (c) Find the distance between $\mathbf{u}$ and $\mathbf{e}_1$.
  (d) Find a basis for the orthogonal complement, $W^\perp$, of $W = \operatorname{span}\{\mathbf{u}\}$.
  (e) What is the orthogonal projection $\operatorname{proj}_{W^\perp} \mathbf{u}$? If possible, solve without calculation.
  (f) What is the orthogonal projection $\operatorname{proj}_W (1, 0)$.
  (g) Use $(1, 0)$ and the projection found in (4f) to find a nonzero vector that is orthogonal to $\operatorname{proj}_W (1, 0)$.

(5) (10 points) Let V be an inner product space (that is, a vector space with inner product $\langle\ ,\ \rangle$). Prove that $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ for vectors in V if and only if $\mathbf{u} \perp \mathbf{v}$.

(6) (30 points) Consider the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$.
  (a) Can A be orthogonally diagonalized? Explain why (or why not) in few words.
  (b) Find the eigenvalues of A and their multiplicities.
  (c) Find a basis for the eigenspace of each eigenvalue.
  (d) Find the diagonal matrix D and the matrix P for diagonalization.
  (e) Is A similar to D?
  (f) Produce an orthogonal diagonalization of A. That means find the matrices D and P for an orthogonal diagonalization.
  (g) Invert the matrix P from part (6f) *without doing any computation.*
  (h) (0 pts) Have you been to Diagon Alley? Did you see an eigenvector there?

PLEASE TURN OVER FOR MORE! MORE! MORE!

(7) (10 points) Consider $\mathbb{P}_2$, the polynomial vector space with basis $\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$, where

$$p_1(t) = t^2, \quad p_2(t) = t^2 + t, \quad p_3(t) = t^2 + t + 1.$$

Also, the vector space $\mathbb{R}^2$ with the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Find the coordinate vector, $[p(t)]_{\mathcal{B}}$, of $p(t) = t - 2$ with respect to the basis $\mathcal{B}$.

(b) For the transformation $T : \mathbb{R}^2 \to \mathbb{P}_2$ given by the rule $T(a, b) = at^2 + (a+b)t - 2b$, find the matrix M of T with respect to the bases $\mathcal{E}$ and $\mathcal{B}$.

(8) (10 points) A basis for $\mathbb{R}^3$ is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Turn it into an orthonormal basis using Gram–Schmidt.