UNGRADED ASSESSMENT 3, MATH 304-06, 11/18 YOUR NAME SOLUTIONS

(1) (0 points) What is the Solar System?

The Sun and all objects that are in orbit around it.

(2) (10 points) A basis for $\mathbb{R}^3$ is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$. Turn it into an orthogonal basis using Gram–Schmidt.

Solution: Call these vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, respectively. We want an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Step 1. Choose $\mathbf{u}_1 = \mathbf{b}_1$. The subspace $W_1 = \text{span}\{\mathbf{u}_1\}$.

Step 2. Use $\mathbf{b}_2$ to get $\mathbf{u}_2$. First we calculate the dot product $\mathbf{b}_2 \cdot \mathbf{u}_1 = 0$. Oh, $\mathbf{b}_2$ is already orthogonal to $\mathbf{u}_1$; what luck! We can set $\mathbf{u}_2 = \mathbf{b}_2$. The subspace $W_2 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Step 3. Use $\mathbf{b}_3$ to get $\mathbf{u}_3$. Calculate the dot products $\mathbf{b}_3 \cdot \mathbf{u}_1 = 0$ and $\mathbf{b}_3 \cdot \mathbf{u}_2 = -1$. Since at least one of them is not 0, $\mathbf{b}_3 \not\perp W_2$. Therefore, we can't use $\mathbf{b}_3$. We have to compute

$$\begin{aligned} \mathbf{u}_3 &= \mathbf{b}_3 - \left(\frac{\langle \mathbf{b}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} 1 + \frac{\langle \mathbf{b}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 \right) \\ &= \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} - \left(0\mathbf{u}_1 + \frac{-1}{11} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 13/11 \\ 10/11 \\ -3 \end{bmatrix}. \end{aligned}$$

This is $\mathbf{u}_3$ and it completes our orthogonal basis, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 13/11 \\ 10/11 \\ -3 \end{bmatrix} \right\}$.

TURN OVER FOR QUESTION 3

(3) (10 points) Diagonalize $\begin{bmatrix} 2 & 3 \\ 7 & -2 \end{bmatrix}$, if possible. Remember that this means either to find D (diagonal matrix) and P, or to show they do not exist. Do not do extra work.

Step 1. Find the eigenvalues.

$$\det \begin{bmatrix} 2-\lambda & 3 \\ 7 & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda) - 21 = \lambda^2 - 25.$$

Setting this $= 0$ we get $\lambda = \pm 5$.

Step 2. Find the eigenvectors.
For $\lambda = 5$:

$$\begin{bmatrix} 2-5 & 3 \\ 7 & -2-5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so $x_1 = x_2$ and an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
For $\lambda = -5$:

$$\begin{bmatrix} 2+5 & 3 \\ 7 & -2+5 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/7 \\ 0 & 0 \end{bmatrix}$$

so $x_1 = -\frac{3}{7}x_2$ and an eigenvector is $\begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}$. (Or you might prefer its scalar multiple $\begin{bmatrix} -3 \\ 7 \end{bmatrix}$.)

Step 3. Write the answer.

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -\frac{3}{7} \\ 1 & 1 \end{bmatrix}.$$