#5 here is corrected.

(1) (5 points) What is the Cauchy–Schwartz Inequality?
$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$.

(2) (0 points) What is the Cauchy–Schwartz Inequality used for?
Many things.

(3) (5 points) What is the Triangle Inequality (for vectors)?
$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|bfv\|$.

(4) (5 points) Prove that, if P is an orthogonal matrix and D is diagonal, then PDP^{-1} is symmetric.

(Explanation: A matrix A is symmetric when $A^T = A$. So, we test this with the matrix PDP^{-1}.)

Since P is orthogonal, $P^{-1} = P^T$. Thus, we want to know if $PDP^T = (PDP^T)^T$.

$$(PDP^T)^T = P^{TT}D^TP^T = PD^TP^T = PDP^T$$

since a diagonal matrix is symmetric.

(5) (20 points) In $\mathcal{P}_2$, there is an inner product (nicknamed "Betsy") $\langle p(t), q(t)\rangle := p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. The vectors $\mathbf{u}_1 = 1 = t^0$ and $\mathbf{u}_2 = t$ are orthogonal with respect to "Betsy". With respect to the inner product "Betsy",

(a) what is the orthogonal projection of $\mathbf{x} := t^2$ onto the subspace $\text{span}\{1, t\}$?

(b) what is the nearest point to $\mathbf{x}$ in the same subspace, $\text{span}\{1, t\}$?

(Explanation: Part of the question is not to be confused by writing vector notation for a polynomial. Keep in mind that we are in a vector space whose vectors are polynomials. E.g., $t^0 = 1$ is a constant polynomial.)

(a) Use the standard formula for orthogonal projection. For that, the basis for $W = \text{span}\{1, t\}$ must be an orthogonal set. (That's why I told you 1 and t are orthogonal.) Then,

$$\text{proj}_W \mathbf{x} = \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\langle t^2, t\rangle}{\langle t, t\rangle}t.$$

Now compute the inner products. For instance, in $\langle t^2, 1\rangle$ we have $p(t) = t^2$ and $q(t) = 1$.

$$\begin{aligned}
\langle t^2, 1\rangle &= (-1)^2(1) + (0^2)(1) + (1^2)(1) = 2,\\
\langle t^2, t\rangle &= (-1)^2(-1) + (0^2)(0) + (1^2)(1) = 0,\\
\langle 1, 1\rangle &= (1)(1) + (1)(1) + (1)(1) = 3,\\
\langle t, t\rangle &= (-1)(-1) + (0)(0) + (1)(1) = 2.
\end{aligned}$$

Substituting in the formula,

$$\text{proj}_W \mathbf{x} = \frac{2}{3}1 + \frac{0}{\langle t, t\rangle}t = \frac{2}{3}. \quad \text{(That is a polynomial of degree 0.)}$$

(b) The nearest point is the projection. Answer: $\frac{2}{3}$.

(6) (25 points) In $\mathbb{R}^3$, there is an inner product (nicknamed "Arthur") defined by $\langle \mathbf{u}, \mathbf{v} \rangle := 2u_1v_1 + 2u_2v_2 + (u_1 + u_3)(v_1 + v_3)$. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$.

(a) Which pairs of vectors in $\mathcal{E}$ are orthogonal with respect to the inner product "Arthur"?

(b) Use Gram–Schmidt orthogonalization to turn $\mathcal{E}$ into an orthogonal basis with respect to "Arthur".

(a) (You have to remember that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, etc.) Use Arthur:

$$\begin{aligned} \langle \mathbf{e}_1, \mathbf{e}_2 \rangle &= 2(1)(0) + 2(0)(1) + (1+0)(0+0) = 0, \\ \langle \mathbf{e}_1, \mathbf{e}_3 \rangle &= 2(1)(0) + 2(0)(0) + (1+0)(0+1) = 1, \\ \langle \mathbf{e}_2, \mathbf{e}_3 \rangle &= 2(0)(0) + 2(1)(0) + (0+0)(0+1) = 0. \end{aligned}$$

So $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal, $\mathbf{e}_2, \mathbf{e}_3$ are orthogonal, but bfe_1, bfe_3 are not orthogonal.

(b) We'll construct an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

First, we take $\mathbf{u}_1 = \mathbf{e}_1$.

Next, we consider $\mathbf{e}_2$. Since it is already orthogonal to $\mathbf{u}_1$ (which is $\mathbf{e}_1$) by part (a), we can use it as the next orthogonal basis vector: $bfu_2 = \mathbf{e}_2$.

Finally, we want $\mathbf{u}_3$ which is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. Since $\mathbf{e}_3$ is not orthogonal to both of them, we can't use it without modification. The vector we want for $\mathbf{u}_3$ is

$$\begin{aligned} &\mathbf{e}_3 - \left(\frac{\langle \mathbf{e}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} 1 + \frac{\langle \mathbf{e}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 \right) \\ &= \mathbf{e}_3 - \left(\frac{1}{3}\mathbf{u}_1 + \frac{0}{2}\mathbf{u}_2 \right) \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

This last vector is the answer for $\mathbf{u}_3$.

The basis we want is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}.$