One purpose of this is to show you the difference between *finding the vector* given the coordinate vector (in 1, 3) and *finding the coordinate vector* given the vector (in 2, 4).

A basis for $\mathbb{R}^3$ is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(1) A vector $\mathbf{x} \in \mathbb{R}^3$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is $\mathbf{x}$?

Solution. The coordinate vector tells us that

$$\mathbf{x} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}.$$

(2) With the basis $\mathcal{B}$, what is the coordinate vector of $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$?

Solution 1. (Trick method!) Since $\mathbf{x} = \mathbf{y}$, the coordinate vector is $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution 2. Set $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and solve. The augmented matrix is $\begin{bmatrix} 1 & 3 & -1 & | & 3 \\ 2 & 2 & 0 & | & 4 \\ 3 & 0 & 1 & | & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$. So $c_1 = c_2 = c_3 = 1$ and the coordinate vector is $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

TURN OVER FOR MORE! MORE!

A basis for $\mathcal{P}_3$ is $\mathcal{D} = \{x^3, x^3+1, x^3+x, x^3+x^2\}$.

(3) A polynomial $p(x) \in \mathbb{P}_3$ has $[p(x)]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. What is $p(x)$?

Solution. From the coordinate vector we know that

$$p(x) = 1(x^3) + 0(x^3+1) + 1(x^3+x) + 0(x^3+x^2) = 2x^3 + x.$$

(4) What is the coordinate vector $[x^3+x^2+x+1]_{\mathcal{D}}$?

Solution 1. Set up the equation: $(x^3+x^2+x+1) = a_1(x^3) + a_2(x^3+1) + a_3(x^3+x) + a_4(x^3+x^2)$, which gives the polynomial equation

$$x^3+x^2+x+1 = (a_1+a_2+a_3+a_4)x^3 + a_4x^2 + a_3x + a_2.$$

Comparing coefficients,

$$\begin{aligned} a_1+a_2+a_3+a_4 &= 1, \\ a_4 &= 1, \\ a_3 &= 1, \\ a_2 &= 1, \end{aligned}$$

so $a_1 = -2$ and the coordinate vector is

$$[x^3+x^2+x+1]_{\mathcal{D}} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4.$$

Reminder: A coordinate vector is always in some $\mathbb{R}^n$, never in any other vector space.

Solution 2. I noticed that $x^3+x^2+x+1 = 1(x^3+x^2)+1(x^3+x)+1(x^3+1)-2(x^3)$, so the coordinate vector is

$$[x^3+x^2+x+1]_{\mathcal{D}} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4.$$